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STABILITY RESULTS AND EXISTENCE THEOREMS FOR NONLINEAR DELAY-FRACTIONAL DIFFERENTIAL EQUATIONS WITH φ_P^*-OPERATOR

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Abstract The study of delay-fractional differential equations (fractional DEs) have recently attracted a lot of attention from scientists working on many different subjects dealing with mathematically modeling. In the study of fractional DEs the first question one might raise is whether the problem has a solution or not. Also, whether the problem is stable or not? In order to ensure the answer to these questions, we discuss the existence and uniqueness of solutions (EUS) and Hyers-Ulam stability (HUS) for our proposed problem, a nonlinear fractional DE with p-Laplacian operator and a non zero delay $\tau > 0$ of order $n-1 < \nu^*$, $\epsilon < n$, for $n \geq 3$ in Banach space $\mathcal{A}$. We use the Caputo's definition for the fractional differential operators $\mathcal{D}^{\nu^*}$, $\mathcal{D}^{\epsilon}$. The assumed fractional DE with p-Laplacian operator is more general and complex than that studied by *Khan et al. Eur Phys J Plus, (2018);133:26.*

Keywords Hybrid fractional differential equations, Hyers-Ulam stability, Caputo's fractional derivative, existence and uniqueness, topological degree theory.

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1. Introduction

Recently, mathematical modeling with the help of fractional DEs have caught the attention of researchers in several applied scientific fields in the previous two decades. These models can be studied in real life in the fields like signals, biology, viscoelastic theory, computer networking, control theory, set theory, fluid dynamics, hydrodynamics, image processing, and many others [7, 13, 24, 40, 41].

Different research aspects of FDEs have been considered by scientists through numerous mathematical procedures. Cabada et al. [3] studied EUS of nonlinear fractional DEs and gave applications. Hu *et al.* [16] investigated a system of FDEs involving nonlinear φ_p^*-operator at resonance for the existence theorems and gave some applications. Zhang et al. [47] considered existence results for fractional DE with φ_p^*-operator and multi points boundary conditions at resonance with the help of

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Mawhin's theorem and provided application. Mahmudov and Unul [35] considered a FDE where the order $\epsilon \in (2,3]$ having integral conditions for existence criteria. They also studied existence theorems for an impulsive FDE [37] and nonlinear FDE involving φ_p^* as a p-Laplacian operator [36]. Jiang et al. [21] studied existence theorems for fractional DEs with φ_p^* operator at resonance by using Mawhin's coincidence theory and provided applications.

Recently, some scientists worked on fractional DEs with singularities for the analysis of existence results and applications. For example, Zhang and Liu [48] studied existence theorems for fractional DEs with multi point boundary conditions with the help of Mawhin's theorems and present an application of their results. Liu et al. [31] investigated EUS for a class of singular fractional DEs using index theory. Guo et al. [11] proved uniqueness theorems for position solutions of a singular fractional DE with nonlinear φ_p^*-operator and Riemann-Stieltjes type boundary conditions. Vong [45] used the upper lower solution method with fixed point theorem for the study of a fractional DE with a singularity and integral boundary conditions. For some more related results, we suggest the readers [26, 28].

The study of fractional DEs with delay have also been studied by several scientists. For instance, Zhang et al. [49] considered fractional order stochastic differential equations with multiple delays. Thanh et al. [43] investigated stability results for fractional DE in Caputo's sence with time-delay. Cong and Tuan [6] studied global solutions, exponential boundedness, existence and uniqueness for a class of fractional DEs with delay. Deng et al. [9] proved stability results for a system of fractional DEs with delays and provided applications. Haristova and Tunc [14] studied fractional order integro-differential equations with delays using the Caputo's definition of fractional derivative.

To the best of our study in the field, no one considered delay fractional DEs of higher order with singularity and p-Laplacian operator for the EUS and stability analysis. Motivated by the above cited works, we use fixed point theorems for the study of EUS and HU-stability of fractional DE with singularity and φ_p^*-operator of the kind:

$$\begin{cases} \mathcal{D}^{\nu^*}\left[\varphi_p^*[\mathcal{D}^{\epsilon}x(t)]\right] = -\Omega(t)\xi^*(t, x(t-\eta^*)), \\ \left(\varphi_p^*[\mathcal{D}^{\epsilon}x(t)]\right)^{(i)}|_{t=0} = 0,\ i = 0,1,2,\dots,n-1, \\ x'(1) = 0,\ x(1) = x'(0),\ x^{(j)}(0) = 0,\ j = 2,3,\dots,n-1, \end{cases} \tag{1.1}$$

where $n-1 < \epsilon, \epsilon \leq n$, $n \geq 3$ and $\xi^*(t, x(t-\eta^*))$, $\Omega(t)$ are continuous while singular at some points. The fractional derivatives $\mathcal{D}^{\epsilon}$, $\mathcal{D}^{\epsilon}$ are in the Caputo's sense, $\varphi_p^*(r) = |r|^{p-2}r$ is nonlinear φ_p^*-operator satisfying that $1/q + 1/p = 1$ and $\varphi_p^{*-1} = \varphi_q^*$. The suggested fractional DE with φ_p^* is more extended and complicated than the problems considered in [34, 42].

In literature, no valuable consideration has been given to the subject area of the paper. Therefore, our results are based on the importance of the study, i.e., EUS and stability of the problem (1.1). For these aims, we will convert the problem to an alternate integral form of fractional order with the help of the corresponding Green function $\mathcal{G}^{\epsilon}(t,s)$ and the Green function will be examined to know its nature for the increase or decrease of the function in the assumed interval $(0,1]$. After these, with the use of fixed point theorems the EUS will be proved and HUS will be derived. For the application of the results, we give an illustrative example.

Definition 1.1 ([13, 40, 41]). For $\epsilon > 0$ and a function $\psi : (0, +\infty) \to \mathbb{R}$, the integral of arbitrary order is defined as

$$\mathcal{I}^{\epsilon}\psi(t) = \frac{1}{\Gamma(\epsilon)} \int_0^t (t-\eta)^{\epsilon-1}\psi(\eta)\, d\eta,$$

provided that the integral is defined on interval $(0, +\infty)$ and

$$\Gamma(\epsilon) = \int_0^{+\infty} e^{-s}s^{\epsilon-1}ds.$$

Definition 1.2 ([13, 40, 41]). Let $\psi(t) : (0, +\infty) \to \mathbb{R}$, be a continuous function. Then the fractional order Caputo's derivative of order $\epsilon > 0$ is given by

$$\mathcal{D}^{\epsilon}\psi(t) = \frac{1}{\Gamma(r-\epsilon)} \int_0^t (t-s)^{k-\epsilon-1}\psi^{(k)}(s)ds,$$

for $[\epsilon] + 1 = r$, where $[\epsilon]$ represents the integeral part of ϵ, such that, the integral is defined on $(0, +\infty)$.

Lemma 1.1 ([13, 40, 41]). *For and$\psi \in C^{n-1}$, $\epsilon \in (n-1, n]$, we have*

$$\mathcal{I}^{\epsilon}\mathcal{D}^{\epsilon}\psi(t) = \psi(t) + m_0 + m_1 t + m_2 t^2 + \ldots + m_{n-1}t^{n-1},$$

for $m_k \in \mathbb{R}$ *for* $k = 0, 1, 2, ..., n-1$.

Theorem 1.2 ([10]). *Let $\mathcal{A}$ be a Banach space and assume $Q \subset \mathcal{A}$ be a cone. Suppose that $\mathcal{V}_1, \mathcal{V}_2$ are two bounded subsets of $\mathcal{A}$ such that $0 \in \mathcal{V}_1$, $\overline{\mathcal{V}_1} \subset \mathcal{V}_2$, and the operator $\mathcal{F}_0^* : Q \cap (\overline{\mathcal{V}_2} \backslash \mathcal{V}_1) \to Q$ is continuous and such that*

$(\mathcal{N}_1)$ $\|\mathcal{F}_0^* z\| \leq \|z\|$ if $z \in Q \cap \partial\mathcal{V}_1$ and $\|\mathcal{F}_0^* z\| \geq \|z\|$ if $z \in Q \cap \partial\mathcal{V}_2$, or

$(\mathcal{N}_2)$ $\|\mathcal{F}_0^* z_0\| \geq \|z\|$ if $z \in Q \cap \partial\mathcal{V}_1$ and $\|\mathcal{F}_0^* z\| \leq \|z\|$ if $z \in Q \cap \partial\mathcal{V}_2$.
Then $\mathcal{F}_0^*$ has fixed point in $Q \cap (\overline{\mathcal{V}_2} \backslash \mathcal{V}_1)$.

Lemma 1.3 ([27]). *For the φ_p^*, we have*
(1) If $\gamma_1^*\gamma_2 > 0$, $1 < p \leq 2$, and $|\gamma_1^*|, |\gamma_2^*| \geq \rho > 0$, then

$$|\varphi_p^*(\gamma_1^*) - \varphi_p^*(\gamma_2^*)| \leq (p-1)\rho^{p-2}|\gamma_1^* - \gamma_2^*|.$$

(2) If $|\gamma_1^*|, |\gamma_2^*| \leq \rho^*$, $p > 2$, then

$$|\varphi_p^*(\gamma_1^*) - \varphi_p^*(\gamma_2^*)| \leq \rho^{*p-2}(p-1)|\gamma_1^* - \gamma_2^*|.$$

2. Green function and properties

Theorem 2.1. *The* (1.1) *is equivalent to*

$$x(t) = \int_0^1 \mathcal{G}^{\epsilon}(t,s)\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\Omega(\zeta_0)\xi^*(\zeta_0, x(\zeta_0))\Big)d\zeta_0 ds, \tag{2.1}$$

where $\mathcal{G}^{\epsilon}(t,s)$ is defined as

$$\mathcal{G}^{\epsilon}(t,s) = \begin{cases} \frac{-(t-s)^{\epsilon-1}}{\Gamma(\epsilon)} + \frac{(1-s)^{\epsilon-1}}{\Gamma(\epsilon)} + \frac{t(1-s)^{\epsilon-3}}{\Gamma(\epsilon-1)}, & s \leq t, \\ \frac{(1-s)^{\epsilon-1}}{\Gamma(\epsilon)} + \frac{t(1-s)^{\epsilon-3}}{\Gamma(\epsilon-1)}, & s \geq t. \end{cases} \tag{2.2}$$

Proof. Applying the fractional integral operator $\mathcal{I}^{\nu^*}$ to (1.1) and by the virtue of Lemma 1.1, we have the alternate form of the problem (1.1) as below

$$\varphi_p^*[\mathcal{D}^\epsilon x(t)] = -\mathcal{I}^{\nu^*}\big[\Omega(t)\xi^*(t, x(t-\eta^*))\big] + c_1 + c_2 t + c_3 t^2 + \ldots + c_n t^{n-1}. \tag{2.3}$$

By $\Big(\varphi_p^*(\mathcal{D}^\epsilon x(t))\Big)^{(i)}|_{t=0} = 0$, we imply $c_1 = c_2 = \ldots = c_n = 0$. Then (2.3), implies

$$\varphi_p^*\Big(\mathcal{D}^\epsilon x(t)\Big) = -\mathcal{I}^{\nu^*}\big[\Omega(t)\xi^*(t, x(t-\eta^*))\big]. \tag{2.4}$$

With the help of (2.4), we have

$$\mathcal{D}^\epsilon x(t) = -\varphi_q^*\Big(\mathcal{I}^{\nu^*}\big[\Omega(t)\xi^*(t, x(t-\eta^*))dt\big]\Big). \tag{2.5}$$

Using integral operator of fractional order $\mathcal{I}^\epsilon$ to (2.5) and using Lemma 1.1 again, we have

$$x(t) = -\mathcal{I}^\epsilon\Big(\varphi_q^*(\mathcal{I}^{\nu^*}\big[\Omega(t)\xi^*(t, x(t-\eta^*)))\big]\Big) + m_1 + m_2 t + m_3 t^2 + \ldots + m_n t^{n-1}. \tag{2.6}$$

By $x^{(j)}(0) = 0$ for $j = 2, 3, \ldots, n-1$ in (2.6), we get $m_1 = m_2 = m_4 = \ldots = m_n = 0$. From condition $x'(1) = 0$, we have

$$k_2 = \mathcal{I}^{\epsilon-1}\Big(\varphi_q^*(\mathcal{I}^{\nu^*}\big[\Omega(t)\xi^*(t, x(t)))\big]\Big)|_{t=1}. \tag{2.7}$$

Now with the help of the boundary condition $x(1) = x'(0)$, we have $m_1 = \mathcal{I}^{\epsilon-1}\Big(\varphi_q^*$ $(\mathcal{I}^{\nu^*}\big[\Omega(t)\xi^*(t, x(t-\eta^*)))\big]\Big)|_{t=1}$. Thus, using the values of m_i $(i = 1, 2, \ldots, n)$ in (2.6), we obtain

$$\begin{aligned}
x(t) =& -\mathcal{I}^\epsilon\Big(\varphi_q^*(\mathcal{I}^{\nu^*}\big[\Omega(t)\xi^*(t, x(t-\eta^*)))\big]\Big) + \mathcal{I}^{\epsilon-1}\Big(\varphi_q^*(\mathcal{I}^{\nu^*}\big[\Omega(t)\xi^*(t, x(t-\eta^*)))\big]\Big)|_{t=1} \\
&+ t\mathcal{I}^{\epsilon-1}\Big(\varphi_q^*(\mathcal{I}^{\nu^*}\big[\Omega(t)\xi^*(t, x(t-\eta^*))\big]\Big)|_{t=1} \\
=& \Big[-\int_0^t \frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)} + \int_0^1 \frac{(1-s)^{\epsilon-1}}{\Gamma(\epsilon)} + t\int_0^t \frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)}\Big] \\
&\varphi_q^*\big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(\zeta_0)\xi^*(\zeta_0, x(\zeta_0-\eta^*))\big]d\zeta_0\big)ds \\
=& \int_0^1 \mathcal{G}^\epsilon(t,s)\varphi_q^*\big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(t)\xi^*(\zeta_0, x(\zeta_0-\eta^*))\big]d\zeta_0\big)ds,
\end{aligned} \tag{2.8}$$

where $\mathcal{G}^\epsilon(t,s)$ is well defined by (2.2). □

Lemma 2.2. *The function $\mathcal{G}^\epsilon(t,s)$ defined by the equation (2.2) satisfies the following relations:*

($\mathcal{N}_1$) $0 < \mathcal{G}^\epsilon(t,s)$ *for all* $s, t \in (0,1)$;

($\mathcal{N}_2$) *the function* $\mathcal{G}^\epsilon(t,s)$ *is increasing in* t *and* $\mathcal{G}^\epsilon(1,s) = \max_{t\in[0,1]} \mathcal{G}^\epsilon(t,s)$;

($\mathcal{N}_3$) $\mathcal{G}^\epsilon(t,s) \geq t^{\epsilon-1}\max_{t\in[0,1]} \mathcal{G}^\epsilon(t,s)$ *for* $s, t \in (0,1)$.

Proof. In order to prove $(\mathcal{N}_1)$, we consider two cases.
Case 1: For $s \leq t$, consider

$$\begin{aligned}\mathcal{G}^{\epsilon}(t,s) &= -\frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)} + \frac{(1-s)^{\epsilon-1}}{\Gamma(\epsilon)} + t\frac{(1-s)^{\epsilon-3}}{\Gamma(\epsilon-2)} \\ &= -t^{\epsilon-1}\frac{(1-\frac{s}{t})^{\epsilon-1}}{\Gamma(\epsilon)} + \frac{(1-s)^{\epsilon-1}}{\Gamma(\epsilon)} + t\frac{(1-s)^{\epsilon-3}}{\Gamma(\epsilon-2)} \\ &\geq -t^{\epsilon-1}\frac{(1-s)^{\epsilon-1}}{\Gamma(\epsilon)} + t^{\epsilon-1}\frac{(1-s)^{\epsilon-1}}{\Gamma(\epsilon)} + t^{\epsilon}\frac{(1-s)^{\epsilon-2}}{\Gamma(\epsilon-1)} \\ &\geq 0.\end{aligned} \tag{2.9}$$

Case 2: For $0 < t \leq s$, we have

$$\mathcal{G}^{\epsilon}(t,s) = \frac{(1-s)^{\epsilon-1}}{\Gamma(\epsilon)} + t\frac{(1-s)^{\epsilon-3}}{\Gamma(\epsilon-2)} > 0. \tag{2.10}$$

With (2.9) and (2.10), it is proved that $\mathcal{G}^{\epsilon}(t,s) > 0$ for all $0 < s,t < 1$.

For $(\mathcal{N}_2)$, we consider the following cases.
Case 1: For $s \leq t$,

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{G}^{\epsilon}(t,s) &= -\frac{(t-s)^{\epsilon-2}}{\Gamma(\epsilon-1)} + \frac{(1-s)^{\epsilon-2}}{\Gamma(\epsilon-1)} \\ &\geq -t^{\epsilon-2}\frac{(1-\frac{s}{t})^{\epsilon-2}}{\Gamma(\epsilon-1)} + t^{\epsilon-2}\frac{(1-s)^{\epsilon-2}}{\Gamma(\epsilon-1)} \\ &\geq -t^{\epsilon-2}\frac{(1-s)^{\epsilon-2}}{\Gamma(\epsilon-1)} + t^{\epsilon-2}\frac{(1-s)^{\epsilon-2}}{\Gamma(\epsilon-1)} > 0.\end{aligned} \tag{2.11}$$

Case 2: For $0 < t \leq s < 1$, we obtain

$$\frac{\partial}{\partial t}\mathcal{G}^{\epsilon}(t,s) = \frac{(1-s)^{\epsilon-2}}{\Gamma(\epsilon-1)} > 0. \tag{2.12}$$

By (2.11) and (2.12), we have $\frac{\partial}{\partial t}\mathcal{G}^{\epsilon}(t,s) > 0$ for $s,t \in (0,1)$ which implies the $\mathcal{G}^{\epsilon}(t,s)$ is increasing with respect to t. Therefore, for $t \geq s$, we get

$$\max_{t\in[0,1]} \mathcal{G}^{\epsilon}(t,s) = \frac{(1-s)^{\epsilon-1}}{\Gamma(\epsilon-1)} = \mathcal{G}^{\epsilon}(1,s). \tag{2.13}$$

Similarly, for $s \geq t$, we have

$$\max_{t\in[0,1]} \mathcal{G}^{\epsilon}(t,s) = \frac{(1-s)^{\epsilon-1}}{\Gamma(\epsilon)} + \frac{(1-s)^{\epsilon-2}}{\Gamma(\epsilon-1)} = \mathcal{G}^{\epsilon}(1,s). \tag{2.14}$$

For $(\mathcal{N}_3)$, we presume the following two cases.
Case 1: For $t \geq s$, we have

$$\begin{aligned}\mathcal{G}^{\epsilon}(t,s) &= -\frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)} + \frac{(1-s)^{\epsilon-1}}{\Gamma(\epsilon)} + t\frac{(1-s)^{\epsilon-3}}{\Gamma(\epsilon-2)} \\ &\geq -t^{\epsilon-1}\frac{(1-\frac{s}{t})^{\epsilon-1}}{\Gamma(\epsilon)} + \frac{t^{\epsilon-1}}{\Gamma(\epsilon)}(1-s)^{\epsilon-1} + \frac{t^{\epsilon-1}}{\Gamma(\epsilon-2)}(1-s)^{\epsilon-3} \\ &\geq -t^{\epsilon-1}\frac{(1-s)^{\epsilon-1}}{\Gamma(\epsilon)} + t^{\epsilon-1}\frac{(1-s)^{\epsilon-1}}{\Gamma(\epsilon)} + t^{\epsilon-1}\frac{(1-s)^{\epsilon-3}}{\Gamma(\epsilon-2)} \\ &= t^{\epsilon-1}\frac{(1-s)^{\epsilon-3}}{\Gamma(\epsilon-2)} = t^{\epsilon-1}\max_{t\in[0,1]}\mathcal{G}^{\epsilon}(t,s) = t^{\epsilon-1}\mathcal{G}^{\epsilon}(1,s).\end{aligned} \tag{2.15}$$

Case 2: Consider $s \geq t$. Then

$$\begin{aligned}\mathcal{G}^{\epsilon}(t,s) &= \frac{(1-s)^{\epsilon-1}}{\Gamma(\epsilon)} + t\frac{(1-s)^{\epsilon-3}}{\Gamma(\epsilon-2)} \\ &\geq t^{\epsilon-1}\frac{(1-s)^{\epsilon-1}}{\Gamma(\epsilon)} + t^{\epsilon-1}\frac{(1-s)^{\epsilon-3}}{\Gamma(\epsilon-2)} \\ &= t^{\epsilon-1}\max_{t\in[0,1]}\mathcal{G}^{\epsilon}(t,s) = t^{\epsilon-1}\mathcal{G}^{\epsilon}(1,s).\end{aligned} \tag{2.16}$$

By (2.15) and (2.16), the proof of $(\mathcal{N}_3)$ is completed. □

3. Existence results

We assume the space $\mathcal{A} = \mathcal{C}[0,1]$ with a norm $\|x\| = \max_{t\in[0,1]}\{|x(t)| : x \in \mathcal{A}\}$ and a cone P of non-negative functions of $\mathcal{A}$, where $P = \{x \in \mathcal{A} : x(t) \geq t^{\epsilon}\|x\|,\ t \in [0,1]\}$. Let $\mathcal{Z}(r) = \{x \in P : \|x\| < r\}$ and having boundary $\partial\mathcal{Z}(r) = \{x \in P : \|x\| = r\}$.

By Theorem 2.1, the system (1.1) is equivalent to

$$x(t) = \int_0^1 \mathcal{G}^{\epsilon}(t,s)\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(\zeta_0)\xi^*(\zeta_0, x(\zeta_0-\eta^*))\big]d\zeta_0 ds\Big). \tag{3.1}$$

Let $\mathcal{F}_0^* : P\backslash\{0\} \to \mathcal{A}$ by

$$\mathcal{F}_0^* x(t) = \int_0^1 \mathcal{G}^{\epsilon}(t,s)\varphi_q^*\Big(\int_0^1 \mathcal{G}^{\nu^*}(s,\zeta_0)\big[\Omega(\zeta_0)\xi^*(\zeta_0, x(\zeta_0-\eta^*))\big]d\zeta_0\Big)ds. \tag{3.2}$$

With the Theorem 2.1, a solution $x(t)$ of (1.1), is equivalent to fixed point of $\mathcal{F}_0$:

$$x(t) = \mathcal{F}_0^* x(t). \tag{3.3}$$

We assume the following conditions:

- $(\mathcal{P}_1)$ $\xi^* : ((0,1)\times(0,+\infty)) \to [0,+\infty)$ is continuous;
- $(\mathcal{P}_2)$ $\Omega : (0,1) \to [0,+\infty)$ is continuous on $(0,1)$ and non vanishing and $\|\Omega\| = \max_{t\in[0,1]}|\Omega(t)| < +\infty$;
- $(\mathcal{P}_3)$ For of $a_1, \mathbb{M}^*_{\xi^*}$ positive constants and $k_1 \in [0,1]$, the function ξ^* satisfies
$$|\xi^*(t, x(t-\eta^*))| \leq \varphi_p^*\big(a_1|x(t)|^{k_1} + \mathbb{M}^*_{\xi^*}\big);$$
- $(\mathcal{P}_4)$ For a constant $\lambda_{\xi^*} > 0$ and all $u, v \in \mathcal{A}$,
$$|\xi^*(t, x(t-\eta^*)) - \xi^*(t, v(t-\eta^*))| \leq \lambda_{\xi^*}|x(t) - v(t)|.$$

Theorem 3.1. *Assume that conditions $(\mathcal{P}_1)-(\mathcal{P}_3)$ are fulfilled. Then the operator $\mathcal{F}_0^*$ is completely continuous operator.*

Proof. Let $x \in \overline{\mathcal{Z}(r_2)}\backslash\mathcal{Z}(r_1)$, then by Lemma 2.2 and (3.2), we have

$$\begin{aligned}\mathcal{F}_0^* x(t) &= \int_0^1 \mathcal{G}^{\epsilon}(t,s)\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(t)\xi^*(\zeta_0, x(\zeta_0-\eta^*))\big]d\zeta_0\Big)ds \\ &\leq \int_0^1 \mathcal{G}^{\epsilon}(1,s)\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(\zeta_0)\xi^*(\zeta_0, x(\zeta_0-\eta^*))\big]d\zeta_0\Big)ds,\end{aligned} \tag{3.4}$$

$$
\begin{aligned}
\mathcal{F}_0^* x(t) &= \int_0^1 \mathcal{G}^\epsilon(t,s)\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(\zeta_0)\xi^*(\zeta_0, x(\zeta_0-\eta^*))\big]d\zeta_0\Big)ds \\
&\geq t^{\epsilon-1}\int_0^1 \mathcal{G}^\epsilon(1,s)\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(\zeta_0)\xi^*(\zeta_0, x(\zeta_0-\eta^*))\big]d\zeta_0\Big)ds.
\end{aligned} \tag{3.5}
$$

From (3.4) and (3.5)

$$
\mathcal{F}_0^* x(t) \geq t^{\epsilon-1}\|\mathcal{F}_0^* u\|, \quad t \in [0,1]. \tag{3.6}
$$

This implies $\mathcal{F}_0^* : \overline{\mathcal{Z}(r_2)}\backslash\mathcal{Z}(r_1) \to P$. Now, we show that $\mathcal{F}_0^*$ is continuous, for this we need $\|\mathcal{F}_0^*(u_n) - \mathcal{F}_0^*(u)\| \to 0$ as $n \to \infty$, to see this, consider

$$
\begin{aligned}
&|\mathcal{F}_0^* u_n(t) - \mathcal{F}_0^* x(t)| \\
=&\Big|\int_0^1 \mathcal{G}^\epsilon(t,s)\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(\zeta_0)\xi^*(\zeta_0, u_n(\zeta_0-\eta^*))\big]d\zeta_0\Big)ds \\
&-\int_0^1 \mathcal{G}^\epsilon(t,s)\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(\zeta_0)\xi^*(\zeta_0, x(\zeta_0-\eta^*))\big]d\zeta_0)ds\Big| \qquad (3.7)\\
\leq&\int_0^1 \big|\mathcal{G}^\epsilon(t,s)\big|\Big|\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(\zeta_0)\xi^*(\zeta_0, u_n(\zeta_0-\eta^*))\big]d\zeta_0\Big)ds \\
&-\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(\zeta_0)\xi^*(\zeta_0, x(\zeta_0-\eta^*))\big]d\zeta_0\Big)\Big|ds.
\end{aligned}
$$

With the help of (3.7) and the continuity of ξ^*, we have $|\mathcal{F}_0^* u_n(t) - \mathcal{F}_0^* x(t)| \to 0$ as $n \to +\infty$, which shows that $\mathcal{F}_0^*$ is continuous.

Now, for the uniformly boundedness of $\mathcal{F}_0^*$, by (3.2) and (Ω_1), we get

$$
\begin{aligned}
|\mathcal{F}_0^* x(t)| &= \Big|\int_0^1 \mathcal{G}^\epsilon(t,s)\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(\zeta_0)\xi^*(\zeta_0, x(\zeta_0-\eta^*))\big]d\zeta_0\Big)ds\Big| \\
&= \int_0^1 |\mathcal{G}^\epsilon(t,s)|\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[|\Omega(\zeta_0)\xi^*(\zeta_0, x(\zeta_0-\eta^*))|\big]d\zeta_0\Big)ds \\
&\leq \int_0^1 |\mathcal{G}^\epsilon(1,s)|\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\|\Omega\|\varphi_p^*\big(a_1\|x\|^{k_1} + \mathbb{M}_{\xi^*}^*\big)d\zeta_0\Big)ds \\
&\leq \Big(\frac{2}{\Gamma(\epsilon+1)} + \frac{1}{\Gamma(\epsilon)}\Big)\Big[\frac{1}{\Gamma(\nu^*+1)}\Big]^{q-1}\|\Omega\|^{q-1}\Big(a_1\|x\|^{k_1} + \mathbb{M}_{\xi^*}^*\Big) \\
&= \Delta_1^*\|\Omega\|^{q-1}\Big(a_1\|x\| + \mathbb{M}_{\xi^*}^*\Big),
\end{aligned} \tag{3.8}
$$

where $\Delta_1^* = \big(\frac{2}{\Gamma(\epsilon+1)} + \frac{1}{\Gamma(\epsilon)}\big)\big[\frac{1}{\Gamma(\nu^*+1)}\big]^{q-1}$. By (3.8), the operator $\mathcal{F}_0^* : \overline{\mathcal{Z}(r_2)}\backslash\mathcal{Z}(r_1)$ is uniformly bounded.

Now for the equicontinuity of the operator $\mathcal{F}_0^*$, by $(\mathcal{P}_3)$, Theorem 2.1 and (3.2), for any $t_1, t_2 \in [0,1]$, we have

$$
\begin{aligned}
&|\mathcal{F}_0^* x(t_1) - \mathcal{F}_0^* x(t_2)| \\
=&\Big|\int_0^1 \mathcal{G}^\epsilon(t_1,s)\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(\zeta_0)\xi^*(\zeta_0, x(\zeta_0-\eta^*))\big]d\zeta_0)ds\Big) \\
&-\int_0^1 \mathcal{G}^\epsilon(t_2,s)\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(\zeta_0)\xi^*(\zeta_0, x(\zeta_0-\eta^*))\big]d\zeta_0\Big)ds\Big|
\end{aligned} \tag{3.9}
$$

$$\leq \int_0^1 \big|\mathcal{G}^\epsilon(t_1,s)-\mathcal{G}^\epsilon(t_2,s)\big|\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\|\Omega\|\varphi_p^*\big(a_1\|x\|^{k_1}+\mathbb{M}^*_{\xi^*}\big)d\zeta_0\Big)ds$$
$$\leq \Big(\frac{|t_1^\epsilon - t_2^\epsilon|}{\Gamma(\epsilon+1)}+\frac{|t_1^2-t_2^2|}{\Gamma(\epsilon)}\Big)\Big[\frac{1}{\Gamma(\nu^*+1)}\Big]^{q-1}\|\Omega\|^{q-1}(a_1\|x\|^{k_1}+\mathbb{M}^*_{\xi^*}).$$

As $t_1 \to t_2$, we have that (3.9) tends to zero. Hence, the operator $\mathcal{F}_0^*(\overline{\mathcal{Z}(r_2)}\backslash\mathcal{Z}(r_1))$ is an equicontinuous and with Arzela'-Ascoli theorem, we have $\mathcal{F}_0^*(\overline{\mathcal{Z}(r_2)}\backslash\mathcal{Z}(r_1))$ is compact. This implies $\mathcal{F}_0^*$ is compact in $\overline{\mathcal{Z}(r_2)}\backslash\mathcal{Z}(r_1)$ and consequently; $\mathcal{F}_0^*:\overline{\mathcal{Z}(r_2)}\backslash\mathcal{Z}(r_1)\to P$ is completely continuous. □

Here, we define height for $\xi^*(t,x(t))$ for $r>0$, and

$$\begin{cases} \varphi^*_{\max}(t,r)=\max\limits_{t\in(0,1)}\{\xi^*(t,x(t-\eta^*)) : t^{\epsilon-1}r\leq x\leq r\}, \\ \varphi^*_{\min}(t,r)=\min\limits_{t\in(0,1)}\{\xi^*(t,x(t-\eta^*)) : t^{\epsilon-1}r\leq x\leq r\} \ . \end{cases} \tag{3.10}$$

Theorem 3.2. *Assume that $(\mathcal{P}_1)-(\mathcal{P}_3)$ hold and there exist $a,b\in\mathbb{R}^+$ such that*

$(\mathcal{Z}_1)$ $a\leq \int_0^1 \mathcal{G}^\epsilon(1,s)\varphi_q^*\big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\varphi^*_{\min}(\zeta_0,a)d\zeta_0\big)ds<+\infty$ *and*
$\int_0^1 \mathcal{G}^\epsilon(1,s)\varphi_q^*\big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(\zeta_0)\varphi^*_{\max}(\zeta_0,b)\big]d\zeta_0\big)ds\leq b$ *or*

$(\mathcal{Z}_2)$ $\int_0^1 \mathcal{G}^\epsilon(1,s)\varphi_q^*\big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(\zeta_0)\varphi^*_{\max}(\zeta_0,a)\big]d\zeta_0\big)ds<a$ *and*
$b\leq \int_0^1 \mathcal{G}^\epsilon(1,s)\varphi_q^*\big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(\zeta_0)\varphi^*_{\min}(\zeta_0,b)\big]d\zeta_0\big)ds<+\infty.$

Then, the fractional DE with φ_p^*-operator (1.1) *has a positive solution $x\in P$ and $a\leq\|x\|\leq b$.*

Proof. We consider $(\mathcal{Z}_1)$.
For $x\in\partial\mathcal{Z}(a)$, then we have $\|x\|=a$ and $t^{\epsilon-1}a\leq x(t)\leq a$, $t\in[0,1]$. Then by (3.10), we have $\varphi^*_{\min}(t,u)\leq\xi^*(t,u)$, where $0<t<1$. Hence, we can get

$$\begin{aligned}\|\mathcal{F}_0^*x(t)\| &= \max_{t\in[0,1]}\int_0^1 \mathcal{G}^\epsilon(t,s)\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(\zeta_0)\xi^*(\zeta_0,x(\zeta_0-\eta^*))\big]d\zeta_0\Big)ds \\ &\geq t^{\epsilon-1}\int_0^1 \mathcal{G}^\epsilon(1,s)\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(\zeta_0)\xi^*(\zeta_0,x(\zeta_0-\eta^*))\big]d\zeta_0\Big)ds \\ &\geq \int_0^1 \mathcal{G}^\epsilon(1,s)\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(\zeta_0)\varphi^*_{\min}(\zeta_0,a)\big]d\zeta_0\Big)ds\geq a=\|x\|.\end{aligned} \tag{3.11}$$

If $x(t)\in\partial\mathcal{Z}(b)$, then $\|x\|=b$ and $t^{\epsilon-1}b\leq u\leq b$, for $0\leq t\leq 1$. By (3.10), we get $\varphi^*_{\max}(t,u)\geq\xi^*(t,u)$. This implies

$$\begin{aligned}\|\mathcal{F}_0^*x(t)\| &= \max_{t\in[0,1]}\int_0^1 \mathcal{G}^\epsilon(t,s)\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(\zeta_0)\xi^*(\zeta_0,x(\zeta_0-\eta^*))\big]d\zeta_0\Big)ds \\ &\leq t^{\epsilon-1}\int_0^1 \mathcal{G}^\epsilon(1,s)\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(\zeta_0)\xi^*(\zeta_0,x(\zeta_0-\eta^*))\big]d\zeta_0\Big)ds \\ &\leq \int_0^1 \mathcal{G}^\epsilon(1,s)\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(\zeta_0)\varphi^*_{\max}(\zeta_0,b)\big]d\zeta_0\Big)ds\leq b=\|x\|.\end{aligned} \tag{3.12}$$

By Lemma 1.2, $\mathcal{F}_0^*(x) = x \in \overline{\mathcal{Z}(b)}\backslash\mathcal{Z}(a)$, which means $a \leq \|x^*\| \leq b$, and by Lemma 2.2, Theorem 2.1, we have $x^*(t) \geq t^{\epsilon-1}\|x^*\| \geq at^{\epsilon-1} > 0$, for $t \in (0,1)$. Therefore, x^* is an increasing positive solution. Furthermore, we derive that

$$\begin{aligned}\frac{\partial}{\partial t}x^*(t) &= \frac{\partial}{\partial t}\mathcal{F}_0^* x(t)\\ &= \int_0^1 \frac{\partial}{\partial t}\mathcal{G}^\epsilon(t,s)\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(\zeta_0)\xi^*(\zeta_0, x^*(\zeta_0-\eta^*))\big]d\zeta_0\Big)ds\\ &> 0. \end{aligned} \tag{3.13}$$

□

4. Stability

In this section, we discuss the HUS of the singular fractional DE with φ_p^*-operator (1.1) with the help of our results in [23, 29] and the related references therein.

Definition 4.1. We say that (3.1) is HUS if for every $\lambda > 0$, there exists a constant $\mathcal{D}^* > 0$ such that:
If

$$\Big|x(t) - \int_0^1 \mathcal{G}^\epsilon(t,s)\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(\zeta_0)\xi^*(\zeta_0, x(\zeta_0-\eta^*))\big]d\zeta_0\Big)ds\Big| \leq \lambda, \tag{4.1}$$

then there exists $y(t)$ satisfying

$$y(t) = \int_0^1 \mathcal{G}^\epsilon(t,s)\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(\zeta_0)\xi^*(\zeta_0, y(\zeta_0-\eta^*))\big]d\zeta_0\Big)ds, \tag{4.2}$$

such that

$$|x(t) - y(t)| \leq \mathcal{D}^*\lambda^*. \tag{4.3}$$

Theorem 4.1. *The problem* (1.1), *is HUS provided that* $(\mathcal{P}_1)$, $(\mathcal{P}_2)$ *and* $(\mathcal{P}_4)$ *are satisfied.*

Proof. With the help of Definition 4.1 and Theorem 3.2, we assume that $x(t)$ is a positive solution of the Singular fractional DE (3.1) and $y(t)$ is an another approximate solution which is satisfying (4.2). Then

$$\big|x(t) - y(t)\big| \tag{4.4}$$

$$\begin{aligned}=&\Big|\int_0^1 \mathcal{G}^\epsilon(t,s)\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(\zeta_0)\xi^*(\zeta_0, x(\zeta_0-\eta^*))\big]d\zeta_0\Big)ds\\ &- \int_0^1 \mathcal{G}^\epsilon(t,s)\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(\zeta_0)\xi^*(\zeta_0, y(\zeta_0-\eta^*))\big]d\zeta_0\Big)ds\Big| \end{aligned} \tag{4.5}$$

$$\begin{aligned}\leq&(p-1)\rho^{p-2}\|\Omega\|^{q-1}\Big(\int_0^1\big|\mathcal{G}^\epsilon(t,s)\big|\Big|\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(\zeta_0)\xi^*(\zeta_0, x(\zeta_0-\eta^*))\big]d\zeta_0\Big)ds\\ &- \varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(\zeta_0)\xi^*(\zeta_0, y(\zeta_0-\eta^*))\big]d\zeta_0\Big)ds\Big|\\ \leq&(p-1)\rho^{p-2}\lambda_{\xi^*}\Big(\frac{1}{\Gamma(\epsilon+1)} + \frac{1}{\Gamma(\epsilon)}\Big)\Big[\frac{1}{\Gamma(\nu^*+1)}\Big]^{q-1}\|\Omega\|^{q-1}\|x-y\| \end{aligned}$$

where $\mathcal{D}^* = \rho^{p-2}(p-1)\lambda_{\xi^*}\left(\frac{2}{\Gamma(\epsilon+1)}+\frac{1}{\Gamma(\epsilon)}\right)\left[\frac{1}{\Gamma(\nu^*+1)}\right]^{q-1}\|\Omega\|^{q-1}$. Hence (4.4) is Hyers-Ulam stable. Consequently, the HUS of (1.1) is proved. □

5. Example

In this section, an example is presented to illustrate the results in Sections 3 and 4.

Example 5.1. For $\psi_1(t,x(t-\eta^*)) = x^3(t) + \frac{1-\eta^*}{\sqrt{x(t)}}$, $t \in [0,1]$, $p = 3, \nu^* = \epsilon = 3.5, \eta^* = \frac{2}{3}$, $\Omega = \frac{1}{\sqrt{1-t}}$, $\psi_1(t,x(t)) = x^3(t) + \frac{1}{3\sqrt{x(t)}}$, we consider the following singular fractional DE with φ_p^*-operator:

$$\begin{cases} \mathcal{D}^{\nu^*}\left[\varphi_p^*[\mathcal{D}^\epsilon x(t)]\right] + \left[x^3(t) + \dfrac{1}{\sqrt{1-t}}\dfrac{1-\eta^*}{\sqrt{x(t)}}\right] = 0, \\ \left(\varphi_p^*[\mathcal{D}^\epsilon x(t)]\right)^{(i)}|_{t=0} = 0, \\ x'(1) = 0,\ x^{(j)}(0) = 0,\ x'(0) = x(1), \end{cases} \tag{5.1}$$

where $i = 0,1,2$ and $j = 2,3$. Clearly $\Omega \in C((0,1),[0,+\infty))$, $\xi^* \in C((0,1)\times(0,+\infty),[0,+\infty)$. We consider the following cases:

$$\varphi^*_{\max}(t,r) = \max\{x^3 + \frac{1-\eta^*}{x^{\frac{1}{5}}} : t^{\frac{5}{2}}r \le x \le r\} \le r^3 + \frac{1}{3}t^{\frac{1}{3}}r^{\frac{1}{5}},$$

$$\varphi^*_{\min}(t,r) = \min\{x^3 + \frac{1-\eta^*}{x^{\frac{1}{5}}} : t^{\frac{5}{2}}r \le x \le r\} \ge t^{\frac{15}{2}}r^3 + \frac{1}{3r^{\frac{1}{5}}},$$

as height functions. Then, we have

$$\begin{aligned} &\int_0^1 \mathcal{G}^\epsilon(1,s)\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(\zeta_0)\varphi^*_{\max}(\zeta_0,b)\big]d\zeta_0\Big)ds \\ &= \int_0^1 \mathcal{G}^\epsilon(1,s)\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(\zeta_0)\varphi^*_{\max}(\zeta_0,1)\big]d\zeta_0\Big)ds \\ &\le \int_0^1 \mathcal{G}^\epsilon(1,s)\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\frac{1}{\sqrt{1-\zeta_0}}\Big(1+\frac{1}{3\sqrt{\zeta_0}}\Big)\big]d\zeta_0\Big)ds \\ &= 0.0626232 < 1, \end{aligned} \tag{5.2}$$

$$\begin{aligned} &\int_0^1 \mathcal{G}^\epsilon(t,s)\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(\zeta_0)\psi_{\min}(\zeta_0,a)\big]d\zeta_0\Big)ds \\ &= \int_0^1 \mathcal{G}^\epsilon(1,s)\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\Omega(\zeta_0)\psi_{\min}(\zeta_0,\frac{1}{1000})\big]d\zeta_0\Big)ds \\ &\ge \int_0^1 \mathcal{G}^\epsilon(1,s)\varphi_q^*\Big(\frac{1}{\Gamma(\nu^*)}\int_0^s (s-\zeta_0)^{\nu^*-1}\big[\frac{1}{\sqrt{1-\zeta_0}}\Big(\zeta_0^{\frac{15}{2}}\frac{1}{1000^3}+\frac{1000^{\frac{1}{5}}}{3}\Big)\big]d\zeta_0\Big)ds \\ &= 0.0040495 > \frac{1}{1000}. \end{aligned} \tag{5.3}$$

By the help of Theorem 3.2, (5.1) has a solution $x*$ and satisfying $\frac{1}{1000} \le \|x^*\| \le 1$.

6. Conclusion

This paper is related to the study of EUS and Hyers-Ulam stability of (1.1). In the literature, there is no any relative paper which may cover the subject area of the paper. Therefore, our results were based on the importance of study. For the EUS and HUS of (1.1), we have converted the problem to fractional integral form with the help of Green function $\mathcal{G}^{\epsilon}(t,s)$. For our suggested problem, it was proved that $\mathcal{G}^{\epsilon}(t,s)$ is an increasing positive function in t on the interval $[0,1]$. Then, by the help of fixed point theorems, theorems for the EUS were obtained and Hyers-Ulam stability was also examined. For the application of the results, we have suggested an example. For our problem, the readers may work on multiplicity results considering different fractional order derivatives.

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Authors' contributions. The first author initiated the study and produced the existence and uniqueness of solution for the suggested problem (1.1). The second author helped in the Hyers-Ulam stability result and produced an application. The third author worked in the writeup of the paper and examined the nature of the Green's function. Thus, all the authors of the paper claim equal contributions.

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