

LIMIT CYCLES OF PIECEWISE LINEAR DYNAMICAL SYSTEMS WITH THREE ZONES AND LATERAL SYSTEMS*

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Abstract In this paper, we give some evidences what cause more limit cycles for piecewise dynamical systems. We say, the angles or the number of zones are critical points. We study an example of linear lateral systems and an example of linear Y-shape systems, and prove that they have five and four crossing limit cycles by using Newton-Kantorovich Theorem, respectively.

Keywords Y-shape systems, Newton-Kantorovich Theorem, limit cycle, lateral systems.

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1. Introduction

In recent years, the research of discontinuous dynamical systems has been developing rapidly, especially the planar piecewise linear systems with two pieces separated by a straight line, see [4]- [16], [21] and [22]. They are widely used to model many real processes in actual projects, see [1] and [9]. An important problem related to their dynamical behavior is about the number and distribution of limit cycles. However, it has not been solved even for the planar piecewise linear dynamical systems with two pieces separated by a straight line. The research of piecewise linear dynamical systems could go back to [2] and still continues to attract the attention of many researchers. Lum and Chua in [23] conjectured that a continuous piecewise linear vector field in the plane with two pieces has at most one limit cycle. In 1998, this conjecture was proved by Freire et al. in [5], and recently a new and shorter proof has been done by Llibre et al. in [17] in 2013. At the same time, there are many researchers trying to study the maximum number of limit cycles for planar piecewise linear systems with two pieces separated by a straight line. Han and Zhang showed systems with two limit cycles in [10]. Huan and Yang provided numerical evidence on the existence of three limit cycles in [11]. Llibre and Ponce proved analytically in [15] that the systems in [11] have three limit cycles. Freire et al. showed some other systems with three limit cycles in [7] and [8]. They also claimed in [8] that the existence of a focus in one zone is sufficient to get three nested limit cycles, independently on the dynamics of the other linear zone. Llibre et al. summarized the lower bounds for the maximum number of limit cycles of planar piecewise linear

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systems with two pieces separated by a straight line in [16] (see Table 1), where F, N and S represent focus type, node type and saddle type, respectively. It shows that the lower bound is three. Euzébio and Llibre estimated in [4] that the upper bound is four for the case that one of the singular points of the subsystems is on the separation line. Later, Llibre et al. proved that the exact upper bound of that case is two in [19] and [20].

Table 1. Lower bounds for the maximum number of limit cycles of planar piecewise linear dynamical systems with two pieces separated by a straight line.

| | | | |
|---|---|---|---|
| | F | N | S |
| F | 3 | 3 | 3 |
| N | – | 2 | 2 |
| S | – | – | 2 |

In recent years, there are some conclusions about *lateral systems* which are planar piecewise dynamical systems with two pieces separated by two rays starting from the same point and forming an angle $\theta \in (0, \pi)$. Llibre et al. provided in [18] that the maximum number of limit cycles for linear lateral systems is two when the origin is a singular point of one of subsystems with associated nonzero eigenvalues. Using Melnikov Method of higher order, Cardin and Torregrosa showed that there exist five limit cycles in linear lateral systems with angle $\frac{\pi}{2}$ in [3].

From the above, we know that the discontinuousness and the separation lines may affect dynamical behaviors of piecewise dynamical systems, esp. the number of limit cycles. But what causes emerging more limit cycles? In [25] we proved that the angles or the number of pieces of piecewise dynamical systems are critical factors. And showed that the angles 0 and π of zones are the critical values for the planar piecewise linear dynamical systems. Most simply, the graph of Poincaré map (see Definition 3.1) of the subsystem with focus type in the zone with an angle $\theta \neq 0$ or π has at most one inflection point and can have one. On the other hand, the graph of the Poincaré map is a straight line or convex or concave if the angle is equal to π . The compound function of the Poincaré maps from various zones shall have more intersections in the case with an angle $\theta \neq 0$ or π , which implies that systems have more limit cycles.

In this paper, we pay more attention to study the crossing limit cycles of planar piecewise linear dynamical systems, which should go through every zones.

The main results of this paper are shown in the following.

Theorem 1.1. *The linear lateral systems (3.1) with focus-focus type possess at least five crossing limit cycles.*

Remark 1.1. Cardin and Torregrosa showed that there exist five limit cycles in lateral systems with the angle $\frac{\pi}{2}$ in [3]. The five limit cycles come from the perturbation of a linear center under piecewise linear perturbation of higher order by studying Melnikov functions of up to sixth order. We also found the lateral systems with at least five limit cycles in Theorem 1.1 in a different way by studying the graphs of Poincaré maps, and they cross the two separated rays.

Theorem 1.2. *The linear Y-shape systems (4.1) with focus-focus-focus type possess at least four crossing limit cycles.*

Remark 1.2. Notice that the origin is the singular point of one of three subsystems of systems (4.1), so the graph of the Poincaré map of one subsystem has one inflection point, and the graph of the compound Poincaré maps of the other two subsystems has no inflection point, see [25]. On the other hand, both graphs of Poincaré maps of two subsystems of lateral systems (3.1) have one inflection point. Therefore, systems (3.1) may have more limit cycles than systems (4.1). In fact, if we do not limit that the singular point of one subsystem of Y-shape systems is the origin, then the lateral systems can be considered as a special case of Y-shape systems. Hence, it is easy from Theorem 1.1 to find a linear Y-shape system with five limit cycles.

The paper is organized as follows. We shall give some preliminaries in the second section. In the third section, we study the linear lateral systems (3.1) with focus-focus type. The graphs of Poincaré maps of both subsystems have one inflection point. Five limit cycles are found firstly using Poincaré map and then their existence is proved by Newton-Kantorovich Theorem. Similarly, in the fourth section, we study the linear Y-shape systems (4.1) with focus-focus-focus type which are given in [25]. We shall prove that the systems have four crossing limit cycles, which could support the Theorem 1.2. Some precise proof which is similar to Theorem 1.1 is omitted.

2. Preliminaries

We list the following Theorem and Lemmas that shall be used to prove Theorem 1.2, and refer to Theorem 2.1 and Lemma 2.2 of [24] and Lemma 2.1 of [15] in detail.

Let $B_r(x_0)$ be the set of points $x \in \mathbb{R}^n$ such that $|x - x_0| < r$, and denote the closure of $B_r(x_0)$ as $\overline{B_r(x_0)}$.

Theorem 2.1 (Newton-Kantorovich Theorem). *Given a function $\mathbf{f} : C \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ and a convex $C_0 \subset C$, assume that \mathbf{f} is C^1 in C_0 and that the following assumptions hold:*

- (a) $|\mathbf{Df}(z) - \mathbf{Df}(z')| \leq \gamma|z - z'|$ for all $z, z' \in C_0$,
- (b) $|\mathbf{Df}(z_0)^{-1}\mathbf{f}(z_0)| \leq \alpha$,
- (c) $|\mathbf{Df}(z_0)^{-1}| \leq \beta$,

for some $z_0 \in C_0$. Consider

$$h = \alpha\beta\gamma, \quad r_{1,2} = \frac{1 \pm \sqrt{1 - 2h}}{h}\alpha.$$

If $h \leq \frac{1}{2}$ and $\overline{B_{r_1}(z_0)} \subset C_0$, then the sequence $\{z_k\}$ defined by

$$z_{k+1} = z_k - \mathbf{Df}(z_k)^{-1}\mathbf{f}(z_k) \quad \text{for } k = 0, 1, 2, \dots$$

is contained in $B_{r_1}(z_0)$ and converges to the unique zero of $\mathbf{f}(z)$ contained in $C_0 \cap B_{r_2}(z_0)$.

We shall take the norm $|\cdot|_\infty$ in this paper, that is

$$|z|_\infty = \max_i |z_i|, \quad \text{for } z = (z_1, z_2, \dots, z_n)^T,$$

$$|A|_\infty = \max_i \left\{ \sum_j |a_{ij}| \right\}, \quad \text{for } A = (a_{ij}).$$

In order to estimate the values α , β and γ in Theorem 2.1, we need the following two Lemmas.

Lemma 2.1. *Given a function $\mathbf{g} = (g_1, g_2, g_3)^T : C \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and a convex $C_0 \subset C$, if \mathbf{g} is C^1 in C_0 and $z, z' \in C_0$, then*

$$|D\mathbf{g}(z) - D\mathbf{g}(z')|_\infty \leq 9 \max_{1 \leq i, j, k \leq 3} \left[\frac{\partial^2 g_i}{\partial z_j \partial z_k} \right] |z - z'|_\infty,$$

where $z = (z_1, z_2, z_3)^T$, $z' = (z'_1, z'_2, z'_3)^T$ and $[\frac{\partial^2 g_i}{\partial z_j \partial z_k}]$ denotes the maximum of $|\frac{\partial^2 g_i}{\partial z_j \partial z_k}|$ on C_0 .

Lemma 2.2. *Let A be an $n \times n$ real matrix and B an approximation of A^{-1} . Then*

$$|A^{-1}|_\infty \leq \frac{|B|_\infty}{1 - |Id - AB|_\infty}.$$

3. Proof of Theorem 1.1

We consider the following systems

$$\dot{X} = \begin{cases} MX + m, & \text{if } X \in \Sigma^-, \\ NX + n, & \text{if } X \in \Sigma^+, \end{cases} \tag{3.1}$$

where

$$M = \begin{pmatrix} -2 & -1 \\ 2 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} -202 & -1 \\ 40402 & 200 \end{pmatrix}, \quad m = \begin{pmatrix} -\frac{1}{4} \\ -400 \end{pmatrix}, \quad n = \begin{pmatrix} \frac{1}{10} \\ \frac{1}{5} \end{pmatrix},$$

and

$$\begin{aligned} \Sigma^- &= \{(x, y) : x < 0, y > 0\}, \\ \Sigma^+ &= \{(x, y) : x > 0, y \geq 0\} \cup \{(x, y) : x \geq 0, y < 0\} \cup \{(x, y) : x < 0, y < 0\}. \end{aligned}$$

The two separation rays are

$$\begin{aligned} \Gamma_1 &= \{(x, y) : x = 0, y \geq 0\}, \\ \Gamma_2 &= \{(x, y) : y = 0, x \leq 0\}. \end{aligned}$$

We use X_H and X_L to represent subsystems in the zone Σ^- and Σ^+ , respectively. See Figure 1. According to [14], we define the **crossing sets** Γ_1^c and Γ_2^c as

$$\begin{aligned} \Gamma_1^c &= \{(0, y) : H^+(0, y)H^-(0, y) > 0, y \geq 0\}, \\ \Gamma_2^c &= \{(x, 0) : H^+(x, 0)H^-(x, 0) > 0, x \leq 0\}. \end{aligned}$$

and the **sliding sets** Γ_1^s and Γ_2^s as the complements to Γ_1^c and Γ_2^c , i.e.

$$\begin{aligned} \Gamma_1^s &= \{(0, y) : H^+(0, y)H^-(0, y) \leq 0, y \geq 0\}, \\ \Gamma_2^s &= \{(x, 0) : H^+(x, 0)H^-(x, 0) \leq 0, x \leq 0\}. \end{aligned} \tag{3.2}$$

It is clear that the systems (3.1) are focus-focus type.

We first use Poincaré map to find the limit cycles and then prove rigorously their existence. The definition of Poincaré map is shown for better understanding.

Definition 3.1. **Poincaré map of system X_H** is a function $x_1 = P_H(y_0)$, where $(0, y_0) \in \Gamma_1$ and $(x_1, 0) \in \Gamma_2$ are the two points connected by one orbit of system X_H . Similarly, **Poincaré map of system X_L** is a function $y_1 = P_L(x_1)$, where $(x_1, 0) \in \Gamma_2$ and $(0, y_1) \in \Gamma_1$ are the two points connected by one orbit of system X_L . See Figure 1.

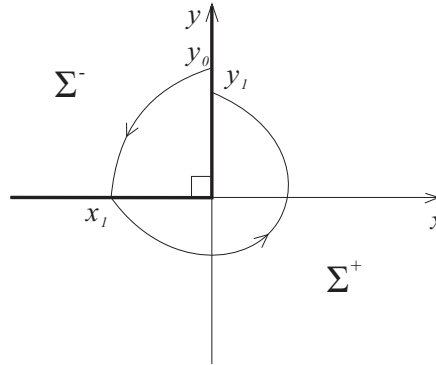


Figure 1. The illustration of Poincaré maps $y_1 = P_H(x_1)$ and $x_1 = P_L(y_0)$.

Obviously, if there exists a crossing limit cycles of systems (3.1), then there exists y_0 such that $P_H(y_0) = P_L^{-1}(y_0)$. In another word, we can determine the number of limit cycles by determining the number of intersections of curves $x_1 = P_H(y_0)$ and $x_1 = P_L^{-1}(y_0)$.

By calculation, we get that the Poincaré map $x_1 = P_H(y_0)$ is

$$y_0 = \frac{400 \sin t_h + \frac{1601}{4} e^{t_h} [1 - e^{-t_h} (\cos t_h + \sin t_h)]}{\cos t_h + \sin t_h},$$

$$x_1 = \frac{-\frac{1}{4} \sin t_h - 200 e^{-t_h} [1 - e^{t_h} (\cos t_h - \sin t_h)]}{\cos t_h + \sin t_h},$$

and the Poincaré map $y_1 = P_L(x_1)$ is

$$x_1 = \frac{\frac{1}{10} \sin t_r + \frac{101}{10} e^{-t_r} [1 - e^{t_r} (\cos t_h - \sin t_h)]}{\cos t_h + 201 \sin t_h},$$

$$y_1 = \frac{-\frac{1}{5} \sin t_h - \frac{20403}{10} e^{t_r} [1 - e^{-t_r} (\cos t_h + \sin t_h)]}{\cos t_h + 201 \sin t_h}.$$

The graphs of $x_1 = P_H(y_0)$ and $x_1 = P_L^{-1}(y_0)$ have 5 intersections A_1, A_1, A_3, A_4 and A_5 , see Figure 2, where the red curve is the graph of $x_1 = P_H(y_0)$ and the black curve is the graph of $x_1 = P_L^{-1}(y_0)$. In order to see more clearly, we enlarged some of five intersections. The five limit cycles of systems (3.1) corresponding the points A_i ($i = 1, 2, \dots, 5$) are showed in Figure 3 and 4, respectively.

Next, We rigorously prove the existence of 5 limit cycles. First, we change the problem of finding the limit cycles of systems (3.1) into the problem of finding the

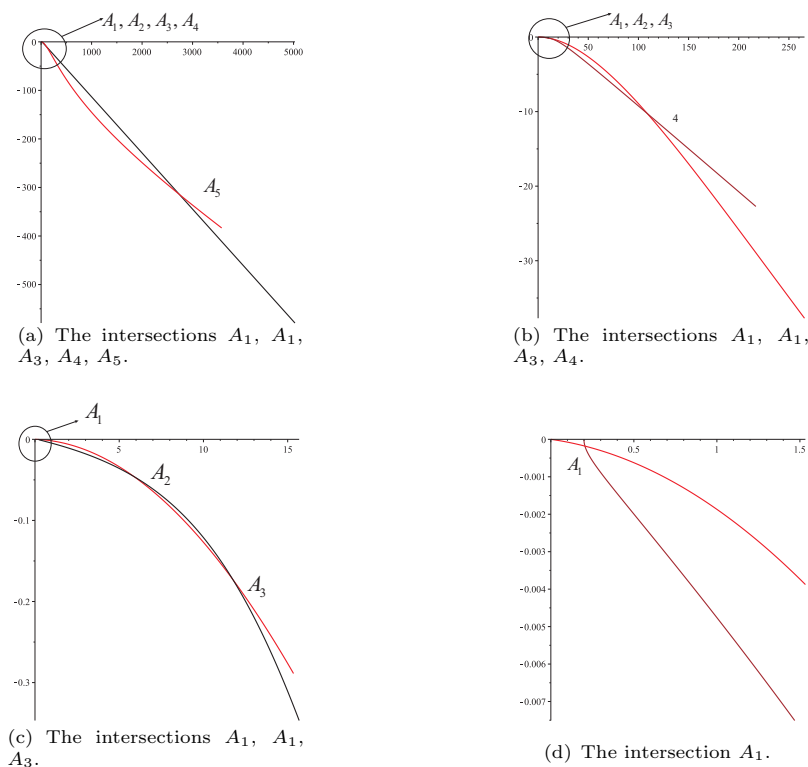


Figure 2. The intersections of $x_1 = P_H(y_0)$ and $x_1 = P_L^{-1}(y_1)$.

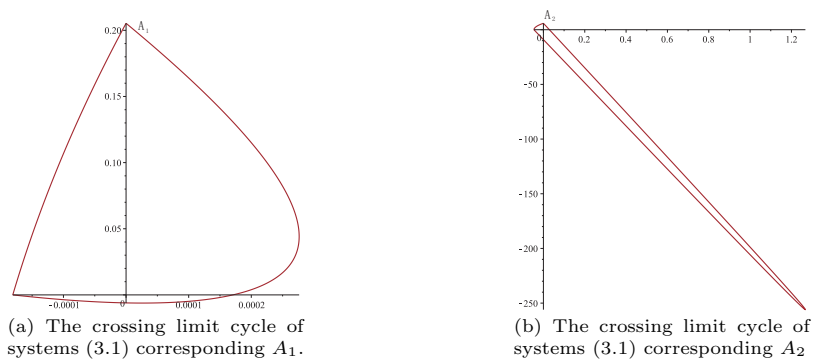
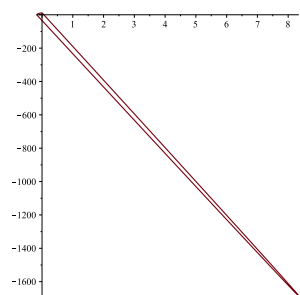


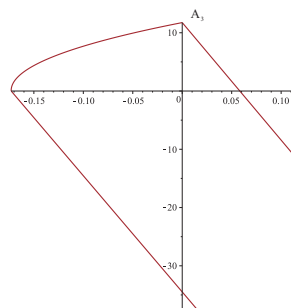
Figure 3. The crossing limit cycles of systems (3.1) corresponding A_1 and A_2 .

isolated solutions of system (3.3). Secondly, we estimate the values α, β and γ in Theorem 2.1, respectively, and lastly we prove Theorem 1.1 by using Newton-Kantorovich Theorem.

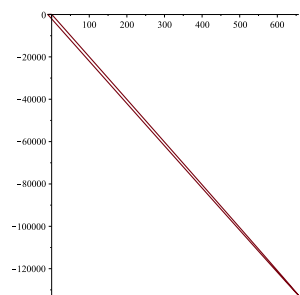
We from (3.2) get that the sliding set of systems (3.1) in Γ_1 is the segment $\{(0, y) : 0 \leq y \leq \frac{1}{10}\}$. The orbit of systems (3.1) starting from $(0, y_0)$ with $y_0 > \frac{1}{10}$ enters into Σ^- caused by the direction of vector field at $(0, y_0)$, and exits Σ^- through Γ_2 and enters into Σ^+ , and finally reach $(0, y_1)$ with $y_1 > \frac{1}{10}$. Therefore,



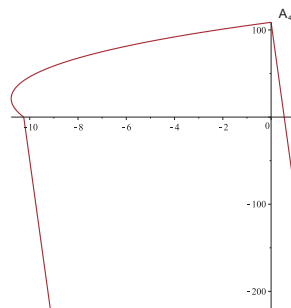
(a) The crossing limit cycle of systems (3.1) corresponding A_3 .



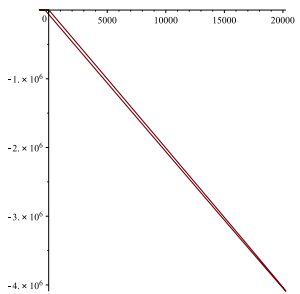
(b) The enlarge graph of crossing limit cycle of systems (3.1) corresponding A_3 near the origin.



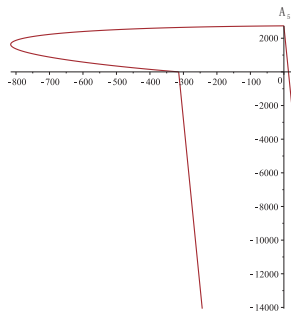
(c) The crossing limit cycle of systems (3.1) corresponding A_4 .



(d) The enlarge graph of crossing limit cycle of systems (3.1) corresponding A_4 near the origin.



(e) The crossing limit cycle of systems (3.1) corresponding A_5 .



(f) The enlarge graph of crossing limit cycle of systems (3.1) corresponding A_5 near the origin.

Figure 4. The crossing limit cycles of systems (3.1) corresponding A_3 , A_4 and A_5 .

if systems (3.1) have crossing limit cycles, then they must surround the segment $\{(0, y) : 0 \leq y \leq \frac{1}{10}\}$.

We get from (3.1) that the solution of system X_H passing through the point

$(0, Y)$ with $Y > \frac{1}{10}$ at the time $t = 0$ is

$$\begin{aligned} x_h(t_h) &= 200 + e^{-t_h}[-200 \cos t_h + 200 \sin t_h - \sin t_h(Y + \frac{1601}{4})], \\ y_h(t_h) &= -\frac{1601}{4} + e^{-t_h}[-400 \sin t_h + (\cos t_h + \sin t_h)(Y + \frac{1601}{4})], \end{aligned}$$

and the solution of system X_L passing through the point $(0, Y)$ with $Y > \frac{1}{10}$ at the time $t = 0$ is

$$\begin{aligned} x_l(t_r) &= -\frac{101}{10} + e^{-t_r}[\frac{101}{10} \cos t_r - \frac{20301}{10} \sin t_r - \sin t_r(Y - \frac{20403}{10})], \\ y_l(t_r) &= \frac{20403}{10} + e^{-t_r}[\frac{2040301}{5} \sin t_r + (\cos t_h + 201 \sin t_h)(y_0 - \frac{20403}{10})]. \end{aligned}$$

In the zone Σ^- , the solution of systems (3.1) starting at the point $(0, Y)$ with $Y > \frac{1}{10}$ reaches Γ_2 anticlockwise at first time after the time $t_h > 0$. In the zone Σ^+ , the solution of systems (3.1) starting at the point $(0, Y)$ with $Y > \frac{1}{10}$ reaches Γ_2 clockwise at first time after the time $t_r < 0$, see Figure 1. Thus, the periodic solutions of systems (3.1) are characterized by the solutions of system

$$\begin{aligned} f_1(t_h, t_r, Y) &:= y_h(t_h) = 0, \\ f_2(t_h, t_r, Y) &:= x_r(t_r) = 0, \\ f_3(t_h, t_r, Y) &:= Y_0(x_h(t_h)) - y_r(t_r) = 0, \end{aligned} \tag{3.3}$$

where $t_h > 0, t_r < 0$ and $Y > \frac{1}{10}$.

It is easy to get the five isolated numerical solutions of system (3.3) by the software Maple. We denote them by $z_0^1 = (t_h^1, t_r^1, Y^1), z_0^2 = (t_h^2, t_r^2, Y^2), z_0^3 = (t_h^3, t_r^3, Y^3), z_0^4 = (t_h^4, t_r^4, Y^4)$ and $z_0^5 = (t_h^5, t_r^5, Y^5)$, where

$$\begin{aligned} t_h^1 &= 0.000513290373638641697454460782 \dots, \\ t_r^1 &= -0.11955221783175832815893090222 \dots, \\ Y^1 &= 0.205316251348505754951839443786 \dots, \\ t_h^2 &= 0.014675552793706057903441902056 \dots, \\ t_r^2 &= -0.688127444068533526392903994451 \dots, \\ Y^2 &= 5.871108237677659838212061270152 \dots, \\ t_h^3 &= 0.029348756191102401394123551060 \dots, \\ t_r^3 &= -1.505392075875461173587800519945 \dots, \\ Y^3 &= 11.746313370382026370282077467858 \dots, \\ t_h^4 &= 0.261492966943715224144155680486 \dots, \\ t_r^4 &= -3.039531323800416571394230522282 \dots, \\ Y^4 &= 108.746926757392770008325179115844 \dots, \\ t_h^5 &= 1.720874199275308820298191960149 \dots, \\ t_r^5 &= -3.142702665184348910894824498210 \dots, \\ Y^5 &= 2736.698490214051925132348730921498 \dots. \end{aligned}$$

In the following, using Theorem 2.1 (Newton-Kantorovich Theorem), we shall prove analytically that there are five isolated solutions of system (3.3) near $z_0^1, z_0^2, z_0^3, z_0^4$ and z_0^5 . Obviously, These five solutions correspond to five limit cycles of systems (3.1).

Here $\mathbf{f} = (f_1, f_2, f_3)^T$, $n = 3$ and $C = \mathbb{R}^3$. Let C_0 take C_0^i , respectively, where

$$\begin{aligned} C_0^1 &= [0.0005, 0.0006] \times [-0.1196, -0.1195] \times [0.2053, 0.2054], \\ C_0^2 &= [0.0146, 0.0147] \times [-0.6882, -0.6881] \times [5.8711, 5.8712], \\ C_0^3 &= [0.0293, 0.0294] \times [-1.5054, -1.5053] \times [11.7463, 11.7464], \\ C_0^4 &= [0.2614, 0.2615] \times [-3.0394, -3.0395] \times [108.7469, 108.7470], \\ C_0^5 &= [1.7208, 1.7209] \times [-3.1428, -3.1427] \times [2736.6984, 2736.6985], \end{aligned}$$

and z_0^i is the interior point of C_0^i . We apply Theorem 2.1 to function $\mathbf{f} = (f_1, f_2, f_3)^T$.

Next, we estimate the values γ , α and β of Theorem 2.1 in C_0^i , respectively.

According to Theorem 2.1 and Lemma 2.1, we need compute the second partial derivative of the functions f_i in C_0^i for estimating the value γ . The second partial derivative of the functions f_i are shown as follows,

$$\begin{aligned} \frac{\partial^2 f_1}{(\partial t_h)^2} &= \frac{1}{2} e^{-t_h} (4Y \sin t_h - 4Y \cos t_h + 1601 \sin t_h - \cos t_h), \\ \frac{\partial^2 f_1}{\partial t_h \partial Y} &= -2e^{-t_h} \sin t_h, \\ \frac{\partial^2 f_2}{(\partial t_r)^2} &= \frac{1}{5} e^{-t_r} (10Y \sin t_r - 2010Y \cos t_r - 20403 \sin t_r + 20401 \cos t_r), \\ \frac{\partial^2 f_2}{\partial t_r \partial Y} &= -2e^{-t_r} (101 \sin t_r - 100 \cos t_r), \\ \frac{\partial^2 f_3}{(\partial t_h)^2} &= \frac{1}{2} e^{-t_h} (4Y \cos t_h - 800 \sin t_h + 801 \cos t_h), \\ \frac{\partial^2 f_3}{\partial t_h \partial Y} &= -e^{-t_h} (\cos t_h - \sin t_h), \\ \frac{\partial^2 f_3}{(\partial t_r)^2} &= -\frac{1}{5} e^{-t_r} (10Y \cos t_r + 101 \sin t_r - 102 \cos t_r), \\ \frac{\partial^2 f_3}{\partial t_r \partial Y} &= e^{-t_r} (\cos t_r - \sin t_r). \end{aligned}$$

Other second partial derivatives of f_i are equal to zero. Since $|\sin t| \leq 1$, $|\cos t| \leq 1$ and $Y \geq \frac{1}{10}$, we deduce that

$$\begin{aligned} \left| \frac{\partial^2 f_1}{(\partial t_h)^2} \right| &\leq e^{-t_h} (4Y + 801), \\ \left| \frac{\partial^2 f_1}{\partial t_h \partial Y} \right| &\leq 2e^{-t_h}, \\ \left| \frac{\partial^2 f_2}{(\partial t_r)^2} \right| &\leq \frac{1}{5} e^{-t_r} (2020Y + 40804), \\ \left| \frac{\partial^2 f_2}{\partial t_r \partial Y} \right| &\leq 402e^{-t_r}, \end{aligned} \tag{3.4}$$

$$\begin{aligned} \left| \frac{\partial^2 f_3}{(\partial t_h)^2} \right| &\leq \frac{1}{2} e^{-t_h} (4Y + 1601), \\ \left| \frac{\partial^2 f_3}{\partial t_h \partial Y} \right| &\leq 2e^{-t_h}, \\ \left| \frac{\partial^2 f_3}{(\partial t_r)^2} \right| &\leq \frac{1}{5} e^{-t_r} (10Y + 203), \\ \left| \frac{\partial^2 f_3}{\partial t_r \partial Y} \right| &\leq 2e^{-t_r}. \end{aligned}$$

We shall apply Theorem 2.1 to prove that there is an unique solution sufficiently close to z_0^2 of system (3.3) in C_0^2 . Noticing that the upper bound functions of second partial derivative of the functions f_i (see the right functions of (3.4)) are decreasing in the variable t_h and increasing in the variables t_r and Y , we obtain that they reach their maximum values at $(t_h, t_r, Y) = (0.0146, -0.6881, 5.8712)$ on the boundary of C_0^2 , that is,

$$\begin{aligned} \left| \frac{\partial^2 f_1}{(\partial t_h)^2} \right| &\leq 813, & \left| \frac{\partial^2 f_1}{\partial t_h \partial Y} \right| &\leq 2, \\ \left| \frac{\partial^2 f_2}{(\partial t_r)^2} \right| &\leq 20979, & \left| \frac{\partial^2 f_2}{\partial t_r \partial Y} \right| &\leq 801, \\ \left| \frac{\partial^2 f_3}{(\partial t_h)^2} \right| &\leq 1618, & \left| \frac{\partial^2 f_3}{\partial t_h \partial Y} \right| &\leq 2, \\ \left| \frac{\partial^2 f_3}{(\partial t_r)^2} \right| &\leq 105, & \left| \frac{\partial^2 f_3}{\partial t_r \partial Y} \right| &\leq 4. \end{aligned}$$

Hence, we get

$$\max_{1 \leq i, j, k \leq 3} \left[\frac{\partial^2 f_i}{\partial z_j \partial z_k} \right] \leq 20979,$$

where $z_1 = t_h$, $z_2 = t_r$ and $z_3 = Y$. Thus, we deduce from Theorem 2.1 that

$$|Df(z) - Df(z')|_\infty \leq 9 \times 20979 |z - z'|_\infty = 188811 |z - z'|_\infty,$$

which means $\gamma = 188811$.

Next, we compute $Df(z_0^2)^{-1}$ and $f(z_0^2)$ to estimate α and β in C_0^2 . We directly compute $Df(z_0^2)$ and $f(z_0^2)$ as follows

$$Df(z_0^2) = \begin{pmatrix} -400.091728 \dots & 0 & 0.9997867 \dots \\ 0 & -1852.80913 \dots & -252.49142 \dots \\ -0.158271 \dots & -9.36458 \dots & -1.278284 \dots \end{pmatrix},$$

and

$$f(z_0^2) = \begin{pmatrix} -4.567 \times 10^{-29} \\ -3.574 \times 10^{-28} \\ -1.84 \times 10^{-30} \end{pmatrix}.$$

The approximation of inverse of $D\mathbf{f}(z_0^2)$ is

$$D\mathbf{f}(z_0^2)^{-1} \approx B = \begin{pmatrix} -0.002107 \cdots & 0.005007 \cdots & -.990821 \cdots \\ -0.021375 \cdots & -0.273639 \cdots & 54.033562 \cdots \\ 0.156852 \cdots & 2.004036 \cdots & -396.504065 \cdots \end{pmatrix}.$$

Using Lemma 2.2, we estimate $|D\mathbf{f}(z_0^2)^{-1}|_\infty$ as follows

$$\begin{aligned} |D\mathbf{f}(z_0^2)^{-1}|_\infty &\leq \frac{|B|_\infty}{1 - |Id - BD\mathbf{f}(z_0^1)|_\infty} \\ &\leq \frac{400}{1 - 2 \times 10^{-32}} \\ &\leq 401. \end{aligned}$$

Hence,

$$\begin{aligned} |D\mathbf{f}(z_0^2)^{-1}f(z_0^2)|_\infty &\leq |D\mathbf{f}(z_0^2)^{-1}|_\infty |\mathbf{f}(z_0^2)|_\infty \\ &\leq 401 \times 4 \times 10^{-28} \\ &= 1.604 \times 10^{-25}. \end{aligned}$$

Take $\alpha = 1.604 \times 10^{-25}$ and $\beta = 401$, then

$$\begin{aligned} h &= \alpha\beta\gamma = 1.21443 \cdots \times 10^{-17}, \\ r_1 &= \frac{1 + \sqrt{1 - 2h}}{h} \alpha = 2.6415 \cdots \times 10^{-8}, \\ r_2 &= \frac{1 - \sqrt{1 - 2h}}{h} \alpha = 1.6040 \cdots \times 10^{-25}. \end{aligned}$$

Obviously, $h \leq \frac{1}{2}$ and $\overline{B_{r_1}(z_0)} \subset C_0^2$. Therefore, we from Theorem 2.1 get that the function $\mathbf{f}(z)$ has an unique zero \bar{z}_0^2 in $C_0^2 \cap B_{r_2}(z_0)$, so systems (3.1) have a limit cycle to be approximation \bar{z}_0^2 .

Similarly, we can prove that there is unique solution of system (3.3) in C_0^1 , C_0^3 , C_0^4 and C_0^5 sufficiently close to z_0^1 , z_0^3 , z_0^4 and z_0^5 , respectively. Here, we only provide the values (α, β, γ) in C_0^1 , C_0^3 , C_0^4 and C_0^5 as follows:

$$\begin{aligned} C_0^1 &: (9 \times 10^{-26}, 402, 83619), \\ C_0^3 &: (4 \times 10^{-25}, 181, 523350), \\ C_0^4 &: (6 \times 10^{-24}, 29, 1.8 \times 10^7) \\ C_0^5 &: (4 \times 10^{-23}, 55, 2.7 \times 10^8). \end{aligned}$$

Thus, Theorem 1.1 holds.

4. Proof of Theorem 1.2

The linear Y -shape systems showed in [25] are

$$\dot{X} = \begin{cases} (H_1(X), H_2(X))^T = AX + h, & \text{if } X \in S_H, \\ (L_1(X), L_2(X))^T = BX, & \text{if } X \in S_L, \\ (R_1(X), R_2(X))^T = CX + r, & \text{if } X \in S_R, \end{cases} \quad (4.1)$$

where

$$A = \begin{pmatrix} -3 & -1 \\ 5 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} \frac{7}{5} & -1 \\ \frac{26}{25} & 1 \end{pmatrix}, \quad C = \begin{pmatrix} \frac{7}{5} & -1 \\ \frac{149}{100} & 0 \end{pmatrix},$$

$$h = \begin{pmatrix} 0 \\ -50 \end{pmatrix}, \quad r = \begin{pmatrix} 1 \\ 500 \end{pmatrix},$$

and

$$S_H = \{(x, y) : x < 0, y > 0\},$$

$$S_L = \{(x, y) : x < 0, y < 0\},$$

$$S_R = \{(x, y) : x > 0\}.$$

The three separation rays are

$$\Sigma = \{(x, y) : x = 0, y \geq 0\},$$

$$\Pi = \{(x, y) : x = 0, y \leq 0\},$$

$$\Gamma = \{(x, y) : y = 0, x \leq 0\}.$$

See Figure 5.

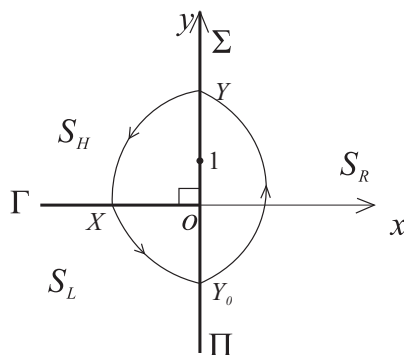


Figure 5. The illustration of the systems (4.1).

The sliding set of systems (4.1) is the segment $\{(0, y) : 0 \leq y \leq 1\}$. The orbit starting from $(0, y_0)$ with $y_0 > 1$ enters into S_H caused by the direction of vector field at $(0, y_0)$ and exits S_H through Γ and enters into S_L , then exits S_L through Π and enters S_R , and finally reach $(0, y_1)$ with $y_1 > 1$. Therefore, if systems (4.1) have crossing limit cycles, then they must surround the segment $\{(0, y) : 0 \leq y \leq 1\}$.

We get from (4.1) that the solution of system X_H passing through the point $(0, Y)$ with $Y > 1$ at the time $t = 0$ is

$$x_h(t) = 25 - e^{-t}[25 \cos t + (Y + 25) \sin t],$$

$$y_h(t) = -75 + e^{-t}[(Y + 75) \cos t + (2Y + 25) \sin t],$$

the solution of system X_L passing through the point $(X, 0)$ with $X < 0$ at the time $t = 0$ is

$$\begin{aligned}x_l(t) &= e^{\frac{6}{5}t} X (\cos t + \frac{1}{5} \sin t), \\y_l(t) &= \frac{26}{25} e^{\frac{6}{5}t} X \sin t,\end{aligned}$$

and the solution of system X_R passing through the point $(0, Y)$ with $Y > 1$ at the time $t = 0$ is

$$\begin{aligned}x_r(t) &= -\frac{50000}{149} + e^{\frac{7}{10}t} [\frac{50000}{149} \cos t - (Y + \frac{34851}{149}) \sin t], \\y_r(t) &= -\frac{69851}{149} + e^{\frac{7}{10}t} [(Y + \frac{69851}{149}) \cos t - (\frac{7}{10}Y - \frac{256043}{1490}) \sin t].\end{aligned}$$

When the solution of system X_L starting the point $(X, 0)$ reaches $(0, Y_0)$ in Π at first time after the time t_l , we have

$$\begin{aligned}x_l(t_l) &= e^{\frac{6}{5}t_l} X (\cos t_l + \frac{1}{5} \sin t_l) = 0, \\y_l(t_l) &= \frac{26}{25} e^{\frac{6}{5}t_l} X \sin t_l = Y_0,\end{aligned}$$

where $(0, Y_0)$ with $Y_0 < 0$ is the intersection of the solution and Π . Noticing that $e^{\frac{6}{5}t_l} X < 0$, we get

$$\cos t_l + \frac{1}{5} \sin t_l = \frac{\sqrt{26}}{5} \sin(t_l + \phi) = 0,$$

where

$$\sin \phi = \frac{5}{\sqrt{26}}, \quad \cos \phi = \frac{1}{\sqrt{26}},$$

which implies that $t_l + \phi = \pi$. Thus,

$$Y_0(X) = \frac{26}{25} e^{\frac{6}{5}t_l} X \sin t_l = \frac{\sqrt{26}}{5} e^{\frac{6}{5}(\pi-\phi)} X.$$

Hence, in the left half plane, the solution of systems (4.1) starting from the point $(0, Y)$ with $Y > 1$ reaches Γ anticlockwise at first time after the time $t_h > 0$, then it reaches Π at first time after the time $t_l = \pi - \phi$. In the right half plane, the solution of systems (4.1) starting from the point $(0, Y)$ with $Y > 1$ reaches Π clockwise at first time after the time $t_r < 0$, see Figure 5. Thus, the periodic solutions of systems (4.1) are characterized by the solutions of system

$$\begin{aligned}f_1(t_h, t_r, Y) &:= y_h(t_h) = 0, \\f_2(t_h, t_r, Y) &:= x_r(t_r) = 0, \\f_3(t_h, t_r, Y) &:= Y_0(x_h(t_h)) - y_r(t_r) = 0,\end{aligned}\tag{4.2}$$

where $t_h > 0$, $t_r < 0$ and $Y > 1$.

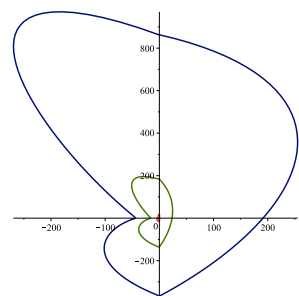
It is easy to get the four isolated numerical solutions of system (4.2) by the software Maple denoted by $z_0^1 = (t_h^1, t_r^1, Y^1)$, $z_0^2 = (t_h^2, t_r^2, Y^2)$, $z_0^3 = (t_h^3, t_r^3, Y^3)$ and $z_0^4 = (t_h^4, t_r^4, Y^4)$, where

$$\begin{aligned}t_h^1 &= 0.051082485537090654027478391699 \dots, \\t_r^1 &= -0.005991641144842323606659692949 \dots, \\Y^1 &= 2.500011126744488998265267149580 \dots,\end{aligned}$$

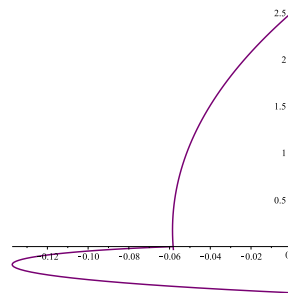
$$\begin{aligned}
 t_h^2 &= 0.366352369760846977711095707901 \dots, \\
 t_r^2 &= -0.065755878802056887120229960471 \dots, \\
 Y^2 &= 17.700177606150772018324516693546 \dots, \\
 \\
 t_h^3 &= 1.626000830767912176084155886824 \dots, \\
 t_r^3 &= -0.615075468396652934815516392986 \dots, \\
 Y^3 &= 185.623364172831585830597833605051 \dots, \\
 \\
 t_h^4 &= 2.278351664758820050509738185822 \dots, \\
 t_r^4 &= -1.662276099415349659753656001746 \dots, \\
 Y^4 &= 875.696131205034226078862459530872 \dots.
 \end{aligned}$$

The precise proof of the existence of four crossing limit cycles is similar to the proof of Theorem 1.1, so we omitted.

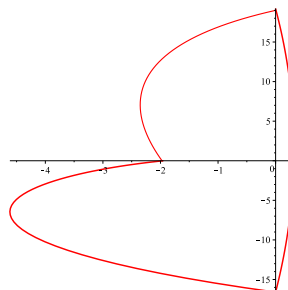
The four limit cycles of system (4.1) are showed in Figure 6. We enlarged the two limit cycles nearest to the origin in Figure 6 (b) and (c), so we could see them more clearly.



(a) Four crossing limit cycles of systems (4.1)



(b) The crossing limit cycle nearest the origin.



(c) The crossing limit cycle secondly nearest the origin.

Figure 6. Crossing limit cycles of systems (4.1).

Thus, Theorem 1.2 holds.

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