

# RECOVERING A SPACE-DEPENDENT SOURCE TERM IN A TIME-FRACTIONAL DIFFUSION WAVE EQUATION\*

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**Abstract** This work is concerned with identifying a space-dependent source function from noisy final time measured data in a time-fractional diffusion wave equation by a variational regularization approach. We provide a regularity of direct problem as well as the existence and uniqueness of adjoint problem. The uniqueness of the inverse source problem is discussed. Using the Tikhonov regularization method, the inverse source problem is formulated into a variational problem and a conjugate gradient algorithm is proposed to solve it. The efficiency and robust of the proposed method are supported by some numerical experiments.

**Keywords** Inverse source problem, Tikhonov regularization, conjugate gradient algorithm.

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## 1. Introduction

Time fractional diffusion equations and diffusion wave equations arise when using time fractional derivatives instead of the standard time derivatives and can be used to describe sub-diffusion and super-diffusion phenomena in [2, 23]. Direct problems for time fractional diffusion equations, i.e initial value problems and initial boundary value problems have received a lot of attentions in recent years, such as some uniqueness and existence results [22, 28], the maximum principle [19, 21], numerical solution by finite element and difference methods in [12, 13, 24, 36].

However, there are a number of parameters may not be measured directly in some practical situations, and have to be inferred indirectly from some given measured data. The inverse source problems of anomalous diffusion process aim at determining the source function of physical field from some known measurement. For the time-fractional diffusion equations, the inverse source problems have attracted immense interest in recent years, see, e.g, [4, 10, 27, 29, 32–34, 37].

Recently, the research for time-fractional diffusion-wave equations becomes a popular topic. Direct problems for time-fractional diffusion-wave equations have been investigated, see, e.g, [1, 3, 5, 6, 11, 28]. However, the inverse problems for

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fractional diffusion-wave equations have only a few papers such as inverse time-dependant source problems on an unbounded domain [20] and on a bounded domain [31], the uniqueness of inverse coefficients [14]. In this paper, we investigate an inverse space-dependent source problem for a time fractional diffusion wave equation in a bounded domain.

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^d$  with sufficient smooth boundary  $\partial\Omega$ . We consider the following the time-fractional diffusion wave problem.

$$\begin{cases} \partial_{0+}^{\alpha} u(x, t) + Lu(x, t) = f(x)g(t), & x \in \Omega, 0 < t \leq T, \\ u(x, 0) = a(x), & x \in \bar{\Omega}, \\ \partial_t u(x, 0) = b(x), & x \in \bar{\Omega}, \\ u(x, t) = 0, & x \in \partial\Omega, 0 < t \leq T, \end{cases} \quad (1.1)$$

where  $1 < \alpha < 2$  and  $\partial_{0+}^{\alpha} u(x, t)$  is the Caputo left-sided derivative defined by

$$\partial_{0+}^{\alpha} u(x, t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} \frac{\partial^2 u}{\partial s^2}(x, s) ds, \quad t > 0,$$

and  $L$  is a symmetric uniformly elliptic operator defined on  $D(L) = H^2(\Omega) \cap H_0^1(\Omega)$  given by

$$Lu(x) = - \sum_{i,j=1}^d \frac{\partial}{\partial x_j} (A_{ij}(x) \frac{\partial}{\partial x_i} u(x)) + c(x)u(x), \quad x \in \Omega, \quad (1.2)$$

in which the coefficients satisfy

$$A_{ij} = A_{ji}, \quad 1 \leq i, j \leq d, \quad A_{ij} \in C^1(\bar{\Omega}), \quad (1.3)$$

$$\sigma \sum_{i=1}^d \xi_i^2 \leq \sum_{i,j=1}^d A_{ij}(x) \xi_i \xi_j, \quad \forall x \in \bar{\Omega}, \quad \forall (\xi_1, \dots, \xi_d) \in \mathbb{R}^d, \quad \sigma > 0, \quad (1.4)$$

$$c(x) \geq 0, \quad x \in \bar{\Omega}, \quad c(x) \in C(\bar{\Omega}). \quad (1.5)$$

If all functions  $f(x)$ ,  $g(t)$ ,  $a(x)$ ,  $b(x)$  are given, then the problem (1.1) is a direct problem. The inverse problem considered here is to determine the spatial source  $f(x)$  in problem (1.1) from the additional final data

$$u(x, T) = h(x), \quad x \in \Omega. \quad (1.6)$$

Unless otherwise specified, we always assume  $f(x) \in L^2(\Omega)$ ,  $a(x) \in H^2(\Omega) \cap H_0^1(\Omega)$ ,  $b(x) \in D(L^{\frac{1}{\alpha}})$ ,  $h(x) \in H^2(\Omega) \cap H_0^1(\Omega)$ ,  $g \in C^1[0, T]$ ,  $g''(t) \in L^1(0, T)$ , where  $L^1(0, T)$  denote the Lebesgue integrable function space.

The inverse source problem for the time-fractional diffusion wave equation mentioned above is an ill-posed problem (refer to Section 4). As we know, the investigations for inverse source problems of the time-fractional diffusion-wave equation are very few. Šišková in [31] considered an inverse time-dependant source problem for a time fractional wave equation from an additional integration condition. The purpose of this work is to determine the space-dependent source problem in a time fractional diffusion wave equation by the final time measured data.

The main difficulty here is to obtain the gradient of the Tikhonov regularization functional. To get it, we need an integration formula by parts for fractional derivatives which is unknown. As one of the main contributions of our work, we provide a

new integration by parts for fractional derivatives from which we can obtain a new-type adjoint problem related to the Riemann-Liouville fractional derivative. The existence and uniqueness of a weak solution for the adjoint problem are investigated. Moreover, the uniqueness of inverse source problem is discussed. This paper offers a generalized approach to solve the inverse space-dependent source problem in an irregular domain for a multi-dimensional case. Although the Tikhonov regularization method is well known, we have to face new difficulty on its application in solving inverse problems of fractional partial differential equations.

The paper is organized as follows. In Section 2, we present some preliminaries. The regularity of the direct problem as well as the existence and uniqueness for the adjoint problem are provided in Section 3. We discuss the uniqueness of inverse source problem in Section 4. In Section 5, we formulate the inverse problem into a variational problem and obtain the gradient of the Tikhonov regularization functional by an adjoint problem. In Section 6, we will show a conjugate gradient algorithm. In Section 7, the numerical results for two examples in one dimensional case and one example in two dimensional case are presented. In Section 8, we give a conclusion on the paper.

## 2. Preliminaries

Throughout this paper, we use the following definitions given in [16] and some lemmas.

**Definition 2.1.** The Mittag-Leffler function is

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}, \quad (2.1)$$

where  $\alpha > 0$  and  $\beta \in \mathbb{C}$  are arbitrary constants.

**Definition 2.2.** Let  $\Omega = [0, T]$  be finite interval on the real axis  $\mathbb{R}$ . The Riemann-Liouville left-sided and right-sided fractional integrals  $I_{0+}^{\alpha}f$  and  $I_{T-}^{\alpha}f$  of order  $\alpha \in \mathbb{C}(\Re(\alpha) > 0)$  are defined by

$$I_{0+}^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0, \quad (2.2)$$

$$I_{T-}^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_t^T \frac{f(s)}{(s-t)^{1-\alpha}} ds, \quad t < T. \quad (2.3)$$

**Definition 2.3.** Let  $\Omega = [0, T]$  be finite interval on the real axis  $\mathbb{R}$ . The Riemann-Liouville left-sided and right-sided fractional derivative  $D_{0+}^{\alpha}f$  and  $D_{T-}^{\alpha}f$  of order  $\alpha \in \mathbb{C}(\Re(\alpha) > 0)$  are defined by

$$D_{0+}^{\alpha}f(t) = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dt^2} \int_0^t \frac{f(s)}{(t-s)^{\alpha-1}} ds, \quad 1 < \Re(\alpha) < 2, \quad t > 0,$$

$$D_{T-}^{\alpha}f(t) = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dt^2} \int_t^T \frac{f(s)}{(s-t)^{\alpha-1}} ds, \quad 1 < \Re(\alpha) < 2, \quad t < T,$$

and

$$D_{0+}^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^{\alpha}} ds, \quad 0 < \Re(\alpha) < 1, \quad t > 0,$$

$$D_{T-}^{\alpha} f(t) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T \frac{f(s)}{(s-t)^{\alpha}} ds, \quad 0 < \Re(\alpha) < 1, \quad t < T.$$

**Lemma 2.1** (see Lemma 3.2 in [28]). For  $\lambda > 0$ ,  $\alpha > 0$  and positive integer  $m \in \mathbb{N}$ , we have

$$\frac{d^m}{dt^m} E_{\alpha,1}(-\lambda t^{\alpha}) = -\lambda t^{\alpha-m} E_{\alpha,\alpha-m+1}(-\lambda t^{\alpha}), \quad t > 0. \quad (2.4)$$

**Lemma 2.2** (see (1.10.7), (2.1.56), (2.2.52) in [16]). For  $\lambda \in \mathbb{R}$ ,  $\alpha > 0$ ,  $\gamma > 0$ , we have

$$\begin{aligned} \frac{d}{dt}(t^{\gamma-1} E_{\alpha,\gamma}(-\lambda t^{\alpha})) &= t^{\gamma-2} E_{\alpha,\gamma-1}(-\lambda t^{\alpha}), \quad t > 0, \\ D_{0+}^{\alpha}(t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^{\alpha})) &= -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^{\alpha}), \quad t > 0, \\ D_{0+}^{\gamma}(t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^{\alpha})) &= t^{\alpha-\gamma-1} E_{\alpha,\alpha-\gamma}(-\lambda t^{\alpha}), \quad t > 0. \end{aligned}$$

**Lemma 2.3** (see Theorem 1.6 in [26]). Let  $0 < \alpha < 2$  and  $\beta \in \mathbb{R}$  be arbitrary. We suppose that  $\mu$  is such that  $\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}$ . Then there exists a constant  $C_1 = C_1(\alpha, \beta, \mu) > 0$  such that

$$|E_{\alpha,\beta}(z)| \leq \frac{C_1}{1+|z|}, \quad \mu \leq |\arg(z)| \leq \pi. \quad (2.5)$$

**Lemma 2.4.** Let  $\lambda > 0$ ,  $\alpha > 0$ , we have

$$\frac{d}{dt} E_{\alpha,\alpha-1}(-\lambda t^{\alpha}) = -\lambda t^{\alpha-1} [E_{\alpha,2\alpha-2}(-\lambda t^{\alpha}) + (2-\alpha)E_{\alpha,2\alpha-1}(-\lambda t^{\alpha})], \quad t > 0.$$

**Proof.** By Definition 2.1, it is easy to prove

$$\begin{aligned} \frac{d}{dt} E_{\alpha,\alpha-1}(-\lambda t^{\alpha}) &= \sum_{k=1}^{\infty} \frac{\alpha k (-\lambda t^{\alpha})^{k-1} (-\lambda t^{\alpha-1})}{\Gamma(\alpha k + \alpha - 1)} \\ &= -\lambda t^{\alpha-1} \left[ \sum_{k=0}^{\infty} \frac{(-\lambda t^{\alpha})^k}{\Gamma(\alpha k + 2\alpha - 2)} + (2-\alpha) \sum_{k=0}^{\infty} \frac{(-\lambda t^{\alpha})^k}{\Gamma(\alpha k + 2\alpha - 1)} \right] \\ &= -\lambda t^{\alpha-1} [E_{\alpha,2\alpha-2}(-\lambda t^{\alpha}) + (2-\alpha)E_{\alpha,2\alpha-1}(-\lambda t^{\alpha})]. \end{aligned}$$

□

**Definition 2.4.** Define  $AC^n[0, T] = \{f(t) | f \in C^{n-1}[0, T], f^{n-1}(t) \in AC[0, T]\}$ , where  $AC[0, T]$  is the space of functions which are absolutely continuous on  $[0, T]$ .

**Definition 2.5.** Define the function space

$$\begin{aligned} AC_{2-\alpha}[0, T] &= \{f(t) | t^{2-\alpha} f(t) \in AC[0, T]\}, \\ AC_{2-\alpha}^T[0, T] &= \{f(t) | (T-t)^{2-\alpha} f(t) \in AC[0, T]\}. \end{aligned}$$

In order to deduce the adjoint problem, we have to use an integration by parts for fractional derivatives. The following new formula is important in the study of fractional diffusion-wave equations.

**Lemma 2.5.** Let  $u(t) \in AC^2[0, T]$ ,  $v(t) \in C[0, T]$ ,  $v(T) = 0$ ,  $v'(t) \in AC_{2-\alpha}^T[0, T]$ , then we have

$$\int_0^T \partial_{0+}^{\alpha} u(t) v(t) dt$$

$$= -u'(0)I_{T-}^{2-\alpha}v(0) + u(T)D_{T-}^{\alpha-1}v(T) - u(0)D_{T-}^{\alpha-1}v(0) + \int_0^T u(t)D_{T-}^{\alpha}v(t)dt.$$

**Proof.** Since  $v(t) \in C[0, T]$ ,  $v'(t) \in AC_{2-\alpha}^T[0, T]$ ,  $1 < \alpha < 2$ , we know  $v' \in L^1(0, T) \cap C_{2-\alpha}^T[0, T]$ , noting that  $v(T) = 0$ , by Lemmas 2.2 and 2.8 in [16], we have  $\frac{d}{dt}I_{T-}^{2-\alpha}v(t) = D_{T-}^{\alpha-1}v = \partial_{T-}^{\alpha-1}v = I_{T-}^{2-\alpha}v'(t) \in C[0, T]$ . By Lemma 2.10 in [35] and the above equality, we have  $D_{T-}^{\alpha}v(t) = \frac{d^2}{dt^2}I_{T-}^{2-\alpha}v(t) = \frac{d}{dt}I_{T-}^{2-\alpha}v'(t) = D_{T-}^{\alpha-1}v' \in L^1(0, T)$ . From  $v(t) \in C[0, T]$ , it is easy to know  $I_{T-}^{2-\alpha}v(T) = 0$ . By Lemma 2.7 in [16] and using integration by parts for two times, we have

$$\begin{aligned} \int_0^T \partial_{0+}^{\alpha}u(t)v(t)dt &= \int_0^T I_{0+}^{2-\alpha}u''(s)v(s)ds = \int_0^T u''(s)I_{T-}^{2-\alpha}v(s)ds \\ &= -u'(0)I_{T-}^{2-\alpha}v(0) - \int_0^T u'(s)\frac{d}{ds}I_{T-}^{2-\alpha}v(s)ds \\ &= -u'(0)I_{T-}^{2-\alpha}v(0) + \int_0^T u'(s)D_{T-}^{\alpha-1}v(s)ds \\ &= -u'(0)I_{T-}^{2-\alpha}v(0) + u(T)D_{T-}^{\alpha-1}v(T) - u(0)D_{T-}^{\alpha-1}v(0) \\ &\quad + \int_0^T u(s)D_{T-}^{\alpha}v(s)ds. \end{aligned}$$

□

**Lemma 2.6** (see [8]). *Let  $f \in L^p(0, T)$  and  $g \in L^q(0, T)$ , with  $1 \leq p, q \leq \infty$ , and  $\frac{1}{p} + \frac{1}{q} = 1$ . Then the function  $f * g$  define by*

$$f * g(t) = \int_0^t f(t-s)g(s)ds,$$

*belong to  $C[0, T]$  and satisfies*

$$|f * g(t)| \leq \|f\|_{L^p(0,t)}\|g\|_{L^q(0,t)}, \quad 0 \leq t \leq T.$$

### 3. Regularity of a weak solution for the direct problem and the existence and uniqueness for the adjoint problem

Denote the eigenvalues of  $L$  with homogeneous Dirichlet boundary condition as  $\lambda_n$  and the corresponding eigenfunctions as  $\varphi_n \in H^2(\Omega) \cap H_0^1(\Omega)$ , which means  $L\varphi_n = \lambda_n\varphi_n$ ,  $\varphi_n|_{\partial\Omega} = 0$ , counting according to the multiplicities, we can set  $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_n \leq \dots$  and  $\{\varphi_n\}_{n=0}^{\infty}$  is an orthonormal basis of  $L^2(\Omega)$ .

Henceforth  $(\cdot, \cdot)$  denotes the scalar product in  $L^2(\Omega)$ . Now we define a fractional power  $L^{\gamma}$  of  $L$  with  $\gamma \geq 0$  as (see Section 2.6 in [25])

$$D(L^{\gamma}) = \{\psi \in L^2(\Omega) | \sum_{n=1}^{\infty} \lambda_n^{2\gamma} |(\psi, \varphi_n)|^2 < \infty\},$$

then  $D(L^{\gamma})$  is a Hilbert space with the norm

$$\|\psi\|_{D(L^{\gamma})} = \left\{ \sum_{n=1}^{\infty} \lambda_n^{2\gamma} |(\psi, \varphi_n)|^2 \right\}^{\frac{1}{2}}.$$

Note that  $D(L^\gamma) \subset L^2(\Omega)$ , for  $\gamma \geq 0$  and by Theorem 1 in [9], we know  $D(L^{\frac{1}{2}}) = H_0^1(\Omega)$ ,  $D(L) = H^2(\Omega) \cap H_0^1(\Omega)$ .

On the existence and uniqueness of weak solution for problem (1.1), one can refer to [15, 28]. In the following, based on the methods in [28], we improve the regularity of the weak solution such that  $u(x, t) \in AC^2([0, T]; L^2(\Omega))$  while enhancing the smoothness of  $g(t)$ .

**Theorem 3.1.** *Let  $1 < \alpha < 2$ ,  $f(x) \in L^2(\Omega)$ ,  $a(x) \in H^2(\Omega) \cap H_0^1(\Omega)$ ,  $b(x) \in D(L^{\frac{1}{\alpha}})$ ,  $g(t) \in AC^2[0, T]$ , then the solution of problem (1.1) holds  $u(x, t) \in C([0, T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap AC^2([0, T]; L^2(\Omega))$ .*

**Proof.** By the result in [28], we know that the weak solution of the direct problem (1.1) is given by

$$\begin{aligned} u(x, t) &= \sum_{n=1}^{\infty} E_{\alpha,1}(-\lambda_n t^\alpha)(a, \varphi_n)\varphi_n(x) + \sum_{n=1}^{\infty} tE_{\alpha,2}(-\lambda_n t^\alpha)(b, \varphi_n)\varphi_n(x) \\ &\quad + \sum_{n=1}^{\infty} (f, \varphi_n) \int_0^t g(t-\tau)\tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \tau^\alpha) d\tau \varphi_n(x) \\ &= u_1(x, t) + u_2(x, t) + u_3(x, t). \end{aligned} \tag{3.1}$$

It follows that

$$\begin{aligned} Lu(x, t) &= \sum_{n=1}^{\infty} \lambda_n E_{\alpha,1}(-\lambda_n t^\alpha)(a, \varphi_n)\varphi_n(x) + \sum_{n=1}^{\infty} tE_{\alpha,2}(-\lambda_n t^\alpha)\lambda_n(b, \varphi_n)\varphi_n(x) \\ &\quad + \sum_{n=1}^{\infty} \lambda_n (f, \varphi_n) \int_0^t g(t-\tau)\tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \tau^\alpha) d\tau \varphi_n(x) \\ &= Lu_1(x, t) + Lu_2(x, t) + Lu_3(x, t). \end{aligned} \tag{3.2}$$

Based on the method in [28], we can see  $Lu \in C([0, T]; L^2(\Omega))$ , and it deduce that  $u(x, t) \in C([0, T]; H^2(\Omega) \cap H_0^1(\Omega))$  by the regularity of the second-order elliptic operator.

By (3.1) and Lemmas 2.1-2.2, we have

$$\frac{\partial^2}{\partial t^2} u_1(x, t) = \sum_{n=1}^{\infty} (-\lambda_n) t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n t^\alpha)(a, \varphi_n)\varphi_n(x), \quad t > 0,$$

and

$$\frac{\partial^2}{\partial t^2} u_2(x, t) = \sum_{n=1}^{\infty} (b, \varphi_n) (-\lambda_n t^{\alpha-1}) E_{\alpha,\alpha}(-\lambda_n t^\alpha)\varphi_n(x), \quad t > 0.$$

By (3.1) and Lemma 2.2, it is not hard to prove

$$\begin{aligned} \frac{\partial^2}{\partial t^2} u_3(x, t) &= \sum_{n=1}^{\infty} (f, \varphi_n) [g(0)t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n t^\alpha) + g'(0)t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) \\ &\quad + \int_0^t g''(t-\tau)\tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \tau^\alpha) d\tau] \varphi_n(x), \quad t > 0. \end{aligned}$$

By Lemma 2.3, we have

$$\left\| \frac{\partial^2}{\partial t^2} u_1(x, t) \right\|_{L^2(\Omega)}^2 = t^{2(\alpha-2)} \sum_{n=1}^{\infty} |\lambda_n (a, \varphi_n)|^2 |E_{\alpha,\alpha-1}(-\lambda_n t^\alpha)|^2$$

$$\leq Ct^{2(\alpha-2)} \|a(x)\|_{D(L)}^2, \quad t > 0,$$

and

$$\begin{aligned} \left\| \frac{\partial^2}{\partial t^2} u_2(x, t) \right\|_{L^2(\Omega)}^2 &= \sum_{n=1}^{\infty} |\lambda_n(b, \varphi_n)|^2 |t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n t^\alpha)|^2 \\ &= \sum_{n=1}^{\infty} |\lambda_n^{\frac{1}{\alpha}}(b, \varphi_n)|^2 \left| \frac{(\lambda_n t^\alpha)^{\frac{\alpha-1}{\alpha}}}{1 + \lambda_n t^\alpha} \right|^2 \leq C \|b(x)\|_{D(L^{\frac{1}{\alpha}})}^2. \end{aligned}$$

By Lemma 2.6, we have

$$\begin{aligned} \left\| \frac{\partial^2}{\partial t^2} u_3(x, t) \right\|_{L^2(\Omega)}^2 &\leq C \sum_{n=1}^{\infty} |(f, \varphi_n)|^2 |t^{\alpha-2} E_{\alpha, \alpha-1}(-\lambda_n t^\alpha)|^2 \\ &\quad + C \sum_{n=1}^{\infty} |(f, \varphi_n)|^2 |t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n t^\alpha)|^2 \\ &\quad + C \sum_{n=1}^{\infty} |(f, \varphi_n)|^2 \left| \int_0^t g''(t-\tau) \tau^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n \tau^\alpha) d\tau \right|^2 \\ &\leq Ct^{2(\alpha-2)} \|f(x)\|_{L^2(\Omega)}^2, \quad t > 0. \end{aligned}$$

From the above estimates, we can see that  $\frac{\partial^2}{\partial t^2} u(x, t) \in L^1(0, T; L^2(\Omega))$ . It is not hard to prove  $u(x, t) \in C^1([0, T]; L^2(\Omega))$ , then we know  $u(x, t) \in AC^2([0, T]; L^2(\Omega))$ .  $\square$

In the following, we consider an initial boundary value problem for the Riemann-Liouville fractional derivative equation as follow,

$$\begin{cases} D_{0+}^\alpha \bar{v}(x, \tau) + L\bar{v}(x, \tau) = 0, & x \in \Omega, \quad 0 < \tau \leq T, \\ D_{0+}^{\alpha-1} \bar{v}(x, \tau)|_{\tau=0} = \psi(x), & x \in \bar{\Omega}, \\ \bar{v}(x, \tau)|_{\tau=0} = 0, & x \in \bar{\Omega}, \\ \bar{v}(x, \tau) = 0, & x \in \partial\Omega, \quad 0 < \tau \leq T. \end{cases} \quad (3.3)$$

For the new issued problem (3.3) given above, we can obtain the existence and uniqueness of a weak solution by using the method of separation of variables and the properties of the Mittag-Leffler functions.

**Theorem 3.2.** *Let  $\psi \in L^2(\Omega)$ , then there exists a unique solution for problem (3.3) and the solution holds  $\bar{v}(x, \tau) \in C([0, T]; L^2(\Omega)) \cap C((0, T]; H^2(\Omega) \cap H_0^1(\Omega))$  and the estimate*

$$\|\bar{v}\|_{C([0, T]; L^2(\Omega))} \leq C \|\psi\|_{L^2(\Omega)}. \quad (3.4)$$

Moreover, if  $\psi \in D(L^{\frac{1}{\alpha}})$ , one has  $\bar{v}_\tau(x, \tau) \in AC_{2-\alpha}([0, T]; L^2(\Omega))$ .

**Proof.** By the separation of variables, we suppose that the solution is of form  $\bar{v}(x, \tau) = \sum_{n=1}^{\infty} \bar{v}_n(\tau) \varphi_n(x)$ , it is easy to see that  $\bar{v}_n(\tau)$  satisfy

$$\begin{cases} D_{0+}^\alpha \bar{v}_n(\tau) + \lambda_n \bar{v}_n(\tau) = 0, & 0 < \tau \leq T, \\ D_{0+}^{\alpha-1} \bar{v}_n(\tau)|_{\tau=0} = (\psi, \varphi_n), \\ \bar{v}_n(\tau)|_{\tau=0} = 0. \end{cases} \quad (3.5)$$

By the second and third equalities in Lemma 2.2, it is easy to verify that one solution of problem (3.5) is

$$\bar{v}_n(\tau) = (\psi, \varphi_n) \tau^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n \tau^\alpha).$$

Then, we know the formal solution for (3.3) is

$$\bar{v}(x, \tau) = \sum_{n=1}^{\infty} (\psi, \varphi_n) \tau^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n \tau^\alpha) \varphi_n(x). \quad (3.6)$$

Since

$$\|\bar{v}(x, \tau)\|_{L^2(\Omega)}^2 = \sum_{n=1}^{\infty} |\tau^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n \tau^\alpha)|^2 |(\psi, \varphi_n)|^2 \leq C \|\psi\|_{L^2(\Omega)}^2, \quad \tau \in [0, T],$$

then we have  $\|\bar{v}(x, \tau)\|_{C([0, T]; L^2(\Omega))} \leq C \|\psi\|_{L^2(\Omega)}$ .

By (3.6), we have

$$L\bar{v}(x, \tau) = \sum_{n=1}^{\infty} (\psi, \varphi_n) \tau^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n \tau^\alpha) \lambda_n \varphi_n(x). \quad (3.7)$$

For any  $0 < t_0 < T$ ,  $\tau \in [t_0, T]$ , by Lemma 2.3, we have

$$\begin{aligned} \|L\bar{v}(\cdot, \tau)\|_{L^2(\Omega)}^2 &= \sum_{n=1}^{\infty} |(\psi, \varphi_n)|^2 |\lambda_n \tau^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n \tau^\alpha)|^2 \\ &\leq C \sum_{n=1}^{\infty} |(\psi, \varphi_n)|^2 \left| \frac{\lambda_n \tau^{\alpha-1}}{1 + \lambda_n \tau^\alpha} \right|^2 \leq \frac{C}{t_0^2} \|\psi\|_{L^2(\Omega)}^2. \end{aligned}$$

It follows from the continuous of  $E_{\alpha, \alpha}(-\lambda_n \tau^\alpha)$  on  $\tau > 0$ , we know  $L\bar{v}(x, \tau) \in C((0, T]; L^2(\Omega))$ , that deduce  $\bar{v}(x, \tau) \in C((0, T]; H^2(\Omega) \cap H_0^1(\Omega))$ .

Next, if  $\psi \in D(L^{\frac{1}{\alpha}})$  we prove  $\bar{v}_\tau(x, \tau) \in AC_{2-\alpha}([0, T]; L^2(\Omega))$ . By the first equality in Lemma 2.2, we have

$$\bar{v}_\tau(x, \tau) = \sum_{n=1}^{\infty} (\psi, \varphi_n) \tau^{\alpha-2} E_{\alpha, \alpha-1}(-\lambda_n \tau^\alpha) \varphi_n(x), \quad \tau > 0,$$

then it follows that

$$\tau^{2-\alpha} \bar{v}_\tau(x, \tau) = \sum_{n=1}^{\infty} (\psi, \varphi_n) E_{\alpha, \alpha-1}(-\lambda_n \tau^\alpha) \varphi_n(x).$$

By continuous of  $E_{\alpha, \alpha-1}(-\lambda_n \tau^\alpha)$ , it prove that  $\tau^{2-\alpha} \bar{v}_\tau(x, \tau) \in C([0, T]; L^2(\Omega))$ .

By Lemma 2.4, we have

$$\begin{aligned} \frac{\partial}{\partial \tau} (\tau^{2-\alpha} \bar{v}_\tau(x, \tau)) &= \sum_{n=1}^{\infty} (\psi, \varphi_n) [-\lambda_n \tau^{\alpha-1} (E_{\alpha, 2\alpha-2}(-\lambda_n \tau^\alpha) \\ &\quad + (2-\alpha) E_{\alpha, 2\alpha-1}(-\lambda_n \tau^\alpha))] \varphi_n(x). \end{aligned}$$

Therefore

$$\begin{aligned}
\left\| \frac{\partial}{\partial \tau} (\tau^{2-\alpha} \bar{v}_\tau(x, \tau)) \right\|_{L^2(\Omega)}^2 &= \sum_{n=1}^{\infty} |\lambda_n(\psi, \varphi_n)|^2 |\tau^{\alpha-1} (E_{\alpha, 2\alpha-2}(-\lambda_n \tau^\alpha) \\
&\quad + (2-\alpha) E_{\alpha, 2\alpha-1}(-\lambda_n \tau^\alpha))|^2 \\
&= \sum_{n=1}^{\infty} |\lambda_n^{\frac{1}{\alpha}}(\psi, \varphi_n)|^2 |\lambda_n^{\frac{\alpha-1}{\alpha}} \tau^{\alpha-1} (E_{\alpha, 2\alpha-2}(-\lambda_n \tau^\alpha) \\
&\quad + (2-\alpha) E_{\alpha, 2\alpha-1}(-\lambda_n \tau^\alpha))|^2 \\
&\leq C \sum_{n=1}^{\infty} |\lambda_n^{\frac{1}{\alpha}}(\psi, \varphi_n)|^2 \left| \frac{(\lambda_n \tau^\alpha)^{\frac{\alpha-1}{\alpha}}}{1 + \lambda_n \tau^\alpha} \right|^2 \\
&\leq C \|\psi\|_{D(L^{\frac{1}{\alpha}})}.
\end{aligned}$$

Then we know  $\frac{\partial}{\partial \tau} (\tau^{2-\alpha} \bar{v}_\tau(x, \tau)) \in L^1(0, T; L^2(\Omega))$ , i.e.  $\bar{v}_\tau(x, \tau) \in AC_{2-\alpha}([0, T]; L^2(\Omega))$ .

Next we prove the uniqueness of the weak solution to (3.3). Under the condition  $\psi = 0$ , we have to prove problem (3.3) has only a trivial solution. Since  $\{\varphi_n(x)\}_{n \in \mathbb{N}}$  is the orthonormal basis in  $L^2(\Omega)$ , denote  $\bar{v}_n(\tau) = (\bar{v}(\cdot, \tau), \varphi_n)$ , suppose  $\bar{v}(\cdot, \tau) \in H^2(\Omega) \cap H_0^1(\Omega)$ , then from (3.3), we know  $\bar{v}_n(\tau)$  satisfy (3.5) for  $\psi = 0$ . If  $\bar{v}_n \in C[0, T]$ , then we know  $I_{0+}^{2-\alpha} \bar{v}_n(0) = 0$ . Due to the uniqueness of the problem (3.5) instead of the initial condition by  $I_{0+}^{2-\alpha} \bar{v}_n(0) = 0$  in [16], we obtain that  $\bar{v}_n(\tau) = 0$ ,  $n = 1, 2, 3, \dots$ . Since  $\{\varphi_n\}_{n \in \mathbb{N}}$  is a complete orthonormal system in  $L^2(\Omega)$ , we have  $\bar{v}(x, \tau) = 0$  in  $\Omega \times (0, T)$ .  $\square$

Take a transformation on  $\tau$  as  $t = T - \tau$  and define  $v(x, t) = \bar{v}(x, \tau)$ , then we have

$$\begin{aligned}
D_{0+}^\alpha \bar{v}(x, \tau) &= \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{d\tau^2} \int_0^\tau \frac{\bar{v}(x, s)}{(\tau-s)^{\alpha-1}} ds \\
&= \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dt^2} \int_0^{T-t} \frac{v(x, T-s)}{(T-t-s)^{\alpha-1}} ds \\
&= \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dt^2} \int_t^T \frac{v(x, y)}{(y-t)^{\alpha-1}} dy \\
&= D_{T-}^\alpha v(x, t),
\end{aligned}$$

by a similar proof, we have  $D_{0+}^{\alpha-1} \bar{v}(x, \tau) = D_{T-}^{\alpha-1} v(x, t)$ , thus the function  $v(x, t)$  satisfies the following problem

$$\begin{cases} D_{T-}^\alpha v(x, t) + Lv(x, t) = 0, & x \in \Omega, 0 \leq t < T, \\ D_{T-}^{\alpha-1} v(x, t)|_{t=T} = \psi(x), & x \in \bar{\Omega}, \\ v(x, T) = 0, & x \in \bar{\Omega}, \\ v(x, t) = 0, & x \in \partial\Omega, 0 \leq t < T. \end{cases} \quad (3.8)$$

By Theorem 3.2, we can obtain the following theorem.

**Theorem 3.3.** *Let  $\psi \in L^2(\Omega)$ , then there exists a unique solution for problem (3.8) and the solution holds  $v(x, t) \in C([0, T]; L^2(\Omega)) \cap C([0, T]; H^2(\Omega) \cap H_0^1(\Omega))$  and we have an estimate*

$$\|v\|_{C([0, T]; L^2(\Omega))} \leq C \|\psi\|_{L^2(\Omega)}. \quad (3.9)$$

Moreover, if  $\psi \in D(L^{\frac{1}{\alpha}})$ , the solution  $v_t(x, t) \in AC_{2-\alpha}^T([0, T]; L^2(\Omega))$ .

#### 4. The uniqueness for the inverse source problem

Let  $t = T$  in (3.1), it is easy to know the source function  $f(x)$  satisfies the following first kind of Fredholm integral equation

$$\mathcal{A}f(x) = \int_{\Omega} f(\xi)k(x, \xi)d\xi = q(x), \quad (4.1)$$

where

$$\begin{aligned} k(x, \xi) &= \sum_{n=1}^{\infty} \nu_n(T)\varphi_n(\xi)\varphi_n(x), \\ \nu_n(T) &= \int_0^T g(T-\tau)\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)d\tau, \\ q(x) &= h(x) - u_1(x, T) - u_2(x, T), \end{aligned}$$

in which the functions  $u_1(x, t)$  and  $u_2(x, t)$  are defined by the

$$\begin{aligned} u_1(x, t) &= \sum_{n=1}^{\infty} (a, \varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n(x), \\ u_2(x, t) &= \sum_{n=1}^{\infty} (b, \varphi_n)tE_{\alpha,2}(-\lambda_n t^\alpha)\varphi_n(x). \end{aligned}$$

Let  $\mathcal{A}^*$  be the adjoint of  $\mathcal{A}$ , it is easy to know

$$\mathcal{A}^*q = \int_{\Omega} k(x, \xi)q(x)dx, \quad \xi \in \Omega.$$

Since  $\{\varphi_n\}_{n=1}^{\infty}$  are orthonormal in  $L^2(\Omega)$ , it is easy to verify

$$\mathcal{A}^*\mathcal{A}\varphi_n(\xi) = \nu_n^2(T)\varphi_n(\xi).$$

Hence, the singular values of  $\mathcal{A}$  are  $\sigma_n = |\nu_n(T)|$ . Define

$$\psi_n(x) = \begin{cases} \varphi_n(x), & \nu_n(T) \geq 0, \\ -\varphi_n(x), & \nu_n(T) < 0. \end{cases}$$

It is clear that  $\{\psi_n\}_{n=1}^{\infty}$  are orthonormal in  $L^2(\Omega)$ , we can verify

$$\begin{aligned} \mathcal{A}\varphi_n(\xi) &= \sigma_n\psi_n(x) = \nu_n(T)\varphi_n(x), \\ \mathcal{A}^*\psi_n(x) &= \sigma_n\varphi_n(\xi) = \nu_n(T)\psi_n(\xi). \end{aligned}$$

Therefore, the singular system of  $\mathcal{A}$  is  $(\sigma_n; \varphi_n, \psi_n)$ .

If for all  $n$ ,  $\nu_n(T) \neq 0$ , then the inverse source problem is unique. If there exists  $n$  such that  $\nu_n(T) = 0$ , then the inverse source problem is not unique. For this case, for any  $q \in R(\mathcal{A})$ , there exist infinitely many solutions for the integral equation (4.1) as

$$f(x) = \sum_{\nu_n \neq 0}^{\infty} (q, \varphi_n)/\nu_n(T)\varphi_n(x) + \sum_{\nu_n = 0} C_n\varphi_n(x), \text{ for any } C_n.$$

However the best-approximate solution in  $L^2(\Omega)$  is uniquely given by

$$f^\dagger(x) = \sum_{\nu_n \neq 0}^{\infty} (q, \varphi_n) / \nu_n(T) \varphi_n(x).$$

For a special case  $g(t) \equiv 1$ , by Lemma 2.1, we know  $\nu_n(T) = \frac{1 - E_{\alpha,1}(-\lambda_n T^\alpha)}{\lambda_n}$ . By the asymptotic behavior of Mittag-Leffler function at infinity (See (1.8.28) in [16]), for  $1 < \alpha < 2$ ,  $\beta \in \mathbb{R}$ ,  $\eta > 0$ , we have

$$E_{\alpha,\beta}(-\eta) = - \sum_{k=1}^N \frac{1}{\Gamma(\beta - \alpha k)(-\eta)^k} + O\left(\frac{1}{\eta^{N+1}}\right), \quad \eta \rightarrow \infty. \quad (4.2)$$

From which, we know that there exists  $L_0 > 0$  such that

$$E_{\alpha,1}(-\lambda_n T^\alpha) \leq \frac{1}{2\Gamma(1-\alpha)\lambda_n T^\alpha} < 0, \quad \lambda_n T^\alpha > L_0,$$

for  $1 < \alpha < 2$ , thus we know  $E_{\alpha,1}(-\lambda_n T^\alpha) = 1$  only if  $\lambda_n T^\alpha \leq L_0$ , which means that  $\nu_n(T) = 0$  only if  $\lambda_n T^\alpha \leq L_0$ . Since  $\lim_{n \rightarrow +\infty} \lambda_n = +\infty$ , there are only finite  $\lambda_n$  satisfying  $\lambda_n T^\alpha \leq L_0$ . Therefore, we can know there are only finite terms such that  $\nu_n(T) = 0$ . However, it remains unclear to give a condition for  $g$  such that  $\int_0^T g(T-\tau)\tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \tau^\alpha) d\tau \neq 0$  for all  $n$  for guaranteeing the uniqueness of the inverse source problem which is quite different from the case  $\alpha \in (0, 1)$  [28]. The main reason is the Mittag-Leffler functions  $E_{\alpha,1}(-t)$ ,  $E_{\alpha,\alpha}(-t)$  for  $\alpha \in (1, 2)$  have zero points over  $t > 0$ .

Under the conditions in Theorem 3.1 and by the proof of it, we know that  $\mathcal{A}f \in H^2(\Omega)$  for  $f \in L^2(\Omega)$ , thus the operator  $\mathcal{A}$  is a bounded linear operator from  $L^2(\Omega)$  into  $H^2(\Omega)$ . By the Sobolev embedding theorem, we can conclude that the space  $H^2(\Omega)$  is compactly embedded in  $L^2(\Omega)$ , thus  $\mathcal{A}$  is a compact operator from  $L^2(\Omega)$  into  $L^2(\Omega)$ . Further, we know that the inverse problem is ill-posed.

## 5. Tikhonov regularization method and gradient of functional

In order to overcome the ill-posedness of the inverse problem, we apply the Tikhonov regularization method to solve problem (4.1), that is to define the Tikhonov regularization functional

$$\begin{aligned} J(f) &= \frac{1}{2} \|\mathcal{A}f - h^\delta + u_1(x, T) + u_2(x, T)\|_{L^2(\Omega)}^2 + \frac{\mu}{2} \|f\|_{L^2(\Omega)}^2 \\ &= \frac{1}{2} \int_{\Omega} ((u_f(x, T) - h^\delta(x))^2 dx + \frac{\mu}{2} \int_{\Omega} f^2(x) dx, \end{aligned} \quad (5.1)$$

where  $\mu > 0$  is a regularization parameter and  $h^\delta \in L^2(\Omega)$  is a noisy data of  $h$  satisfying  $\|h^\delta - h\|_{L^2(\Omega)} \leq \delta$  and  $u_f(x, t)$  is the weak solution of direct problem (1.1). The first term express the error between the exact data and noisy data, and the second term denotes the penalty functional for stabilizing the numerical solution. Then the inverse source problem is transformed into a variational problem.

$$\min_{f \in L^2(\Omega)} J(f). \quad (5.2)$$

Since there is no guarantee the uniqueness of inverse source problem, we have to consider the best-approximate solution. By the standard discussion, it is not hard to know that the problem (5.2) there exists a unique minimizer called a Tikhonov regularized solution and the regularized solution converges to the best-approximate solution under a suitable choice of the regularization parameter  $\mu$ , refer to [7].

In this paper, we apply a conjugate gradient algorithm for finding the minimizer of functional (5.1). The key work is to find the gradient of the Tikhonov functional. By deducing a sensitivity problem and an adjoint problem, we can obtain the gradient of functional (5.1). We suppose that the solution  $u_f$  has the regularity in Theorem 3.1. Let the source term  $f(x)$  be perturbed by a small amount  $\delta f(x)$  in  $L^2(\Omega)$ , then the forward solution  $u_f$  has a small change  $w = u_{f+\delta f} - u_f$  which satisfies

**Sensitive problem:**

$$\begin{cases} \partial_{0+}^\alpha w(x, t) + Lw(x, t) = \delta f(x)g(t), & x \in \Omega, 0 < t \leq T, \\ w(x, 0) = 0, & x \in \bar{\Omega}, \\ \partial_t w(x, 0) = 0, & x \in \bar{\Omega}, \\ w(x, t) = 0, & x \in \partial\Omega, 0 < t \leq T. \end{cases} \quad (5.3)$$

From (5.1), we have

$$\begin{aligned} \delta J(f) &= J(f + \delta f) - J(f) \\ &= \int_{\Omega} (u_f(x, T) - h^\delta(x))w(x, T)dx + \mu \int_{\Omega} f\delta f dx \\ &\quad + \frac{1}{2}\|w(x, T)\|_{L^2(\Omega)}^2 + \frac{\mu}{2}\|\delta f\|_{L^2(\Omega)}^2. \end{aligned}$$

By Theorem 3.1, we have  $\|w(x, T)\|_{L^2(\Omega)} \leq \|\delta f(x)\|_{L^2(\Omega)}$ , then

$$\delta J(f) = \int_{\Omega} (u_f(x, T) - h^\delta)w(x, T)dx + \mu \int_{\Omega} f(x)\delta f(x)dx + o(\|\delta f(x)\|_{L^2(\Omega)}). \quad (5.4)$$

Let  $v(x, t)$  be an arbitrary function such that  $v \in C([0, T]; L^2(\Omega))$ ,  $v_t(x, t) \in AC_{2-\alpha}^T([0, T]; L^2(\Omega))$  and  $v(x, T) = 0$ ,  $w(x, t)$  is solution of (5.3), note that  $w(x, t) \in AC^2([0, T]; L^2(\Omega))$  from Theorem 3.1. Multiply  $v(x, t)$  on both sides of the first equation in (5.3) and integrating over  $\Omega \times (0, T)$ , by Lemma 2.5, we have

$$\begin{aligned} &\int_0^T \int_{\Omega} g(t)\delta f(x)v(x, t)dxdt = \int_{\Omega} \int_0^T \partial_{0+}^\alpha w(x, t)v(x, t)dt dx + \int_0^T \int_{\Omega} Lw(x, t)v(x, t)dxdt \\ &= \int_{\Omega} w(x, T)D_{T-}^{\alpha-1}v(x, T)dx + \int_{\Omega} \int_0^T w(x, t)(D_{T-}^\alpha v(x, t) + Lv(x, t))dt dx \\ &\quad - \sum_{i,j=1}^d \int_0^T \int_{\partial\Omega} (A_{ij}(x) \frac{\partial}{\partial x_i} w(x, t))v(x, t)n_j ds dt, \end{aligned} \quad (5.5)$$

where  $\vec{n} = (n_1, n_2, \dots, n_d)$  is the outside unit normal vector.

Suppose  $h_m^\delta \in D(L^{\frac{1}{\alpha}})$ , such that  $\|h_m^\delta - h^\delta\|_{L^2(\Omega)} \rightarrow 0$ . Let  $v_m(x, t)$  be the

solution of the following problem

$$\begin{cases} D_{T-}^{\alpha} v_m(x, t) + Lv_m(x, t) = 0, & x \in \Omega, 0 \leq t < T, \\ D_{T-}^{\alpha-1} v_m(x, t)|_{t=T} = u_f(x, T) - h_m^{\delta}(x), & x \in \bar{\Omega}, \\ v_m(x, T) = 0, & x \in \bar{\Omega}, \\ v_m(x, t) = 0, & x \in \partial\Omega, 0 \leq t < T. \end{cases} \quad (5.6)$$

By Theorem 3.3, we know  $v_m(x, t)$  holds the condition in Lemma 2.5. Taking  $v = v_m$  in (5.5), then we can apply the boundary conditions in (5.6) and (5.5) to obtain

$$\int_0^T \int_{\Omega} g(t) \delta f(x) v_m(x, t) dx dt = \int_{\Omega} (u_f(x, T) - h_m^{\delta}(x)) w(x, T) dx. \quad (5.7)$$

Let  $v$  be the weak solution for the following

**Adjoint problem:**

$$\begin{cases} D_{T-}^{\alpha} v(x, t) + Lv(x, t) = 0, & x \in \Omega, 0 \leq t < T, \\ D_{T-}^{\alpha-1} v(x, t)|_{t=T} = u_f(x, T) - h^{\delta}(x), & x \in \bar{\Omega}, \\ v(x, T) = 0, & x \in \bar{\Omega}, \\ v(x, t) = 0, & x \in \partial\Omega, 0 \leq t < T. \end{cases} \quad (5.8)$$

From (3.9) in Theorem 3.3, we have  $\|v_m - v\|_{C([0, T]; L^2(\Omega))} \leq C \|h_m^{\delta} - h^{\delta}\|_{L^2(\Omega)} \rightarrow 0, m \rightarrow \infty$ . Pass to limit as  $m \rightarrow \infty$  in (5.7), and noting that  $g(t) \in C^1[0, T]$ , then we have

$$\int_0^T \int_{\Omega} g(t) \delta f(x) v(x, t) dx dt = \int_{\Omega} (u_f(x, T) - h^{\delta}(x)) w(x, T) dx, \quad (5.9)$$

which implies that

$$\delta J(f) = \int_0^T \int_{\Omega} g(t) \delta f(x) v(x, t) dx dt + \mu \int_{\Omega} \delta f(x) f(x) dx + o(\|\delta f(x)\|_{L^2(\Omega)}).$$

Thus we conclude that

$$J'_f = \int_0^T g(t) v(x, t) dt + \mu f(x). \quad (5.10)$$

## 6. Conjugate gradient algorithm for solving the variational problem

We use a conjugate gradient algorithm to find the approximate minimizer of  $J(f)$ . Let  $f_k$  be the  $k$ th approximate solution to  $f(x)$ , we use the following iterative scheme

$$f_{k+1} = f_k + \beta_k d_k, \quad k = 0, 1, 2, \dots, \quad (6.1)$$

where  $\beta_k$  is a step size and  $d_k$  is a descent direction in the  $k$ th iteration with the form

$$d_k = -J'_{f_k} + \gamma_k d_{k-1}, \quad (6.2)$$

where  $\gamma_k$  is given by the Fletcher and Reeves method as

$$\gamma_k = \frac{\int_{\Omega} (J'_{f_k})^2 dx}{\int_{\Omega} (J'_{f_{k-1}})^2 dx}, \quad \gamma_0 = 0. \quad (6.3)$$

From (5.1), we have

$$J(f_k + \beta_k d_k) = \frac{1}{2} \int_{\Omega} (u_{f_k} + \beta_k w_k - h^{\delta}(x))^2 dx + \frac{\mu}{2} \int_{\Omega} (f_k + \beta_k d_k)^2 dx, \quad (6.4)$$

where  $w_k$  is a solution of the sensitive problem with  $\delta f = d_k$ , by

$$\frac{dJ}{d\beta_k} = \int_{\Omega} (u_{f_k}(x, T) - h^{\delta}(x) + \beta_k w_k) w_k dx + \mu \int_{\Omega} (f_k + \beta_k d_k) d_k dx = 0, \quad (6.5)$$

we can get  $\beta_k$  as

$$\beta_k = - \frac{\int_{\Omega} (u_{f_k}(x, T) - h^{\delta}(x)) w_k dx + \mu \int_{\Omega} f_k d_k dx}{\int_{\Omega} w_k^2 dx + \mu \int_{\Omega} d_k^2 dx}. \quad (6.6)$$

Next, we describe the process of the conjugate gradient algorithm:

- (1) Initialize  $f_0 = 0$  and  $k = 0$ ;
- (2) Solve the direct problem (1.1) with  $f = f_k$ , and obtain the residual  $r_k = u_{f_k} - h^{\delta}$ ;
- (3) Solve the adjoint problem (5.8) and determine the gradient  $J'_{f_k}$ ;
- (4) Calculate the conjugate coefficient  $\gamma_k$  by (6.3) and the descent direct  $d_k$  by (6.2);
- (5) Solve the sensitive problem with  $\delta f = d_k$  to obtain  $w_k$ ;
- (6) Calculate the step size  $\beta_k$  by (6.6);
- (7) Update the source term  $f_{k+1}$  by (6.1);
- (8) Increase  $k$  by one and go to step 2, repeat the process until a stopping condition is satisfied.

## 7. Numerical experiments

In this section, we present two examples in one-dimensional case and one example in two-dimensional case to demonstrate the effectiveness of the conjugate gradient algorithm. In numerical computations, we always set  $T = 1$ . The noisy is generated by adding a random perturbation, i.e

$$h^{\delta} = h + \epsilon h \cdot (2 \cdot \text{rand}(\text{size}(h)) - 1). \quad (7.1)$$

The corresponding noise level is calculated by  $\delta = \|h^{\delta} - h\|_{L^2(\Omega)}$ .

To test the accuracy of the reconstructed source item, we compute the approximate error denoted by

$$e_k = \|f_k(x) - f(x)\|_{L^2(\Omega)}, \quad (7.2)$$

where  $f_k(x)$  is the source term reconstructed at the  $k$ th iteration, and  $f(x)$  is the exact solution.

The residual  $E_k$  at the  $k$ th iteration is given by

$$E_k = \|u_{f_k}(x, T) - h^{\delta}\|_{L^2(\Omega)}. \quad (7.3)$$

In the iterative method, the key work is to find a well stopping principle. In this paper, we use the Morozov discrepancy principle, i.e. we choose  $k$  satisfying the following inequality

$$E_k \leq \tau\delta < E_{k-1}, \quad (7.4)$$

where  $\tau > 1$  is a constant.

We use a finite difference method developed in [30] to solve the direct problem and the sensitive problem in Examples 1-2 and using a finite element method in [18] to solve them in Example 3. For the adjoint problem, we solve problem (3.3) with  $\psi(x) = u_f(x, T) - h^\delta(x)$ , then get the solution of the adjoint problem by a relation  $v(x, t) = \bar{v}(x, \tau)$ . The detail is given in the following paragraphs.

Take an integration on both sides of the first equation in (3.3) for variable  $\tau$  from 0 to  $t$ , then we have

$$\begin{cases} D_{0+}^{\alpha-1}\bar{v}(x, t) + \int_0^t L\bar{v}(x, \tau)d\tau = u_f(x, T) - h^\delta(x), & x \in \Omega, 0 < t \leq T, \\ \bar{v}(x, 0) = 0, & x \in \bar{\Omega}, \\ \bar{v}(x, t) = 0, & x \in \partial\Omega, 0 < t \leq T. \end{cases} \quad (7.5)$$

For problem (7.5), the time-fractional derivative is approximated by using the scheme in [17], and the space derivative is approximated using the scheme in [34] for Examples 1-2 in one-dimensional case. To solve problem (7.5) in two dimensional case, we use the finite difference scheme in [17] to approach the time fractional derivative and employ a finite element method to discretize the resulted elliptic problem at each time step. The detail is given in the following.

Take the grid size for time is  $\Delta t = \frac{T}{N}$ . The grid points in the time interval  $[0, T]$  are labeled  $t_n = n\Delta t, n = 0, 1, \dots, N$ . The time-fractional derivative is approximated by

$$D_{0+}^{\alpha-1}\bar{v}(x, t_n) = (\Delta t)^{1-\alpha} \sum_{k=0}^n g_k^{\alpha-1} \bar{v}(x, t_{n-k}), \quad (7.6)$$

where  $g_k^{\alpha-1} = (-1)^k \binom{\alpha-1}{k}$ ,  $g_0^{\alpha-1} = 1$ .

Denote  $\bar{v}^n(x) = \bar{v}(x, t_n)$ ,  $G(x) = u(x, T) - h^\delta(x)$ . By the scheme (7.6), we can obtain the following Dirichlet problems for elliptic equations

$$\begin{cases} (\Delta t)^{1-\alpha} \bar{v}^1(x) + \frac{\Delta t}{2} L\bar{v}^1(x) = G(x), & x \in \Omega, \\ \bar{v}^1(x) = 0, & x \in \partial\Omega, \end{cases} \quad (7.7)$$

and

$$\begin{cases} (\Delta t)^{1-\alpha} \bar{v}^n(x) + \frac{\Delta t}{2} L\bar{v}^n(x) = G(x) - \Delta t \sum_{k=1}^{n-1} L\bar{v}^k(x) - (\Delta t)^{1-\alpha} \sum_{k=1}^{n-1} g_k^{\alpha-1} \bar{v}^{n-k}(x), & x \in \Omega, \\ \bar{v}^n(x) = 0, & x \in \partial\Omega, \end{cases} \quad (7.8)$$

for  $n = 2, \dots, N$ .

Let  $x_j, j = 1, 2, \dots, m$  be the mesh nodes located in  $\Omega$  and  $\psi_j$  be the corresponding finite element basis functions. Denote  $\bar{V}^n = (\bar{v}(x_1, t_n), \dots, \bar{v}(x_m, t_n))^T$ ,

$G = (G(x_1), \dots, G(x_m))^T$ , then by the standard finite element procedure, we can deduce the following linear equations for (7.7) and (7.8) as

$$[(\Delta t)^{1-\alpha} M + \frac{\Delta t}{2} K + \frac{\Delta t}{2} Q] \bar{V}^1 = MG, \quad (7.9)$$

and

$$[(\Delta t)^{1-\alpha} M + \frac{\Delta t}{2} K + \frac{\Delta t}{2} Q] \bar{V}^n = MG - \Delta t \sum_{k=1}^{n-1} (K+Q) \bar{V}^k - (\Delta t)^{1-\alpha} \sum_{k=1}^{n-1} g_k^{\alpha-1} M \bar{V}^{n-k}, \quad (7.10)$$

where  $M = ((\psi_i, \psi_j))_{m \times m}$ ,  $Q = ((c\psi_i, \psi_j))_{m \times m}$ ,  $K = (\sum_{k,l=1}^2 (a_{kl} \partial_{x_k} \psi_i, \partial_{x_l} \psi_j))_{m \times m}$  are mass matrix and stiff matrix in which  $(\cdot, \cdot)$  is the  $L^2$  inner product.

Take  $d = 1$ ,  $\Omega = (0, 1)$ ,  $L = -\partial_x(A(x)\partial_x) + c(x)$  in the following Examples 1-2.

**Example 1.** Suppose

$$f(x) = 10 \sin(3\pi x) e^{-x^5} + x^\alpha (1-x)^4$$

and the final data  $u(x, T)$  are obtained by solving the direct problem (1.1) with  $g(t) = e^t$  and initial value  $a(x) = x^2 \sin(\pi x)$ ,  $b(x) = x(1-x)$  and the diffusion coefficient  $A(x) = e^x$ , the zeroth order coefficient  $c(x) = \sin(x)$ .

**Example 2.** Suppose

$$f(x) = -|x - \frac{1}{2}| + \frac{1}{2}$$

and the final data  $u(x, T)$  are obtained by solving the direct problem (1.1) with  $g(t) = t^2$  and the initial value  $a(x) = x^6(1-x)\sin(\pi x)$ ,  $b(x) = x^2 \sin(\pi x)$  and the diffusion coefficients  $A(x) = x^2 + \sin(x)$ , the zeroth order coefficient  $c(x) = \sin(x)$ .

The inversion results for Examples 1-2 by using the discrepancy principle as stop principle for different noise levels in the cases of  $\alpha = 1.4$ ,  $\alpha = 1.8$  are shown in Figures 1-2 respectively by taking  $\mu = 0$ ,  $\tau = 1.01$ . From Figures 1-2, we can see that the proposed method is robust and stable when the measurement error is included.

In the following, we study the effectiveness of regularization parameter and iterative steps. In Figures 3-4, we illustrate the approximation error  $e_k$  and residuals  $E_k$  for Examples 1-2 with different regularization parameters for a fixed noise level  $\epsilon = 0.1$  in Example 1 and  $\epsilon = 0.01$  in Example 2. It is clearly seen that the small regularization parameter  $\mu$  yields faster convergence speeds and better numerical results than big  $\mu$ . Thus in the following, we take the regularization parameter as zero, that means we do not use the Tikhonov regularization. From Figures 3-4, the approximation error  $e_k$  will increase when the iteration steps become large which indicate the discrepancy principle plays a key role of regularization in this case. Therefore, in this paper we use the Morozov discrepancy principle to find a well-stopping step.

**Example 3.** Let  $d = 2$ ,  $\Omega = \{(x, y) : 0 \leq x^2 + y^2 \leq 1\}$ ,  $L = -\nabla(A(x, y)\nabla) + c(x, y)$ , we take

$$A(x, y) = \begin{pmatrix} x^2 + 3 & 1 + x + y \\ 1 + x + y & x^2 + 3 \end{pmatrix}$$

and  $c(x, y) = x^2y^2$ .

We take the exact spatial source function

$$f(x, y) = [0.3(1-3x)^2 e^{-9x^2-(3y+1)^2} - (0.2x-27x^2-(3y)^5) e^{-9x^2-9y^2} - 0.03e^{-(3x+1)^2-9y^2}] \chi_1,$$

and  $g(t) = t^2 + \sin(t)$ ,  $a(x, y) = xy\chi_2$ ,  $b(x, y) = [\sin(x) + \sin(y)]\chi_2$ , where

$$\chi_1 = \begin{cases} 1, & 0 \leq x^2 + y^2 \leq 0.7, \\ 0, & 0.7 < x^2 + y^2 \leq 1, \end{cases}$$

$$\chi_2 = \begin{cases} 1, & 0 \leq x^2 + y^2 \leq 0.9, \\ 0, & 0.9 < x^2 + y^2 \leq 1. \end{cases}$$

The final data  $u(x, y, T)$  are obtained by solving the direct problem (1.1). Figures 5-6 presents the exact source function and the regularized solutions by taking  $\mu = 0$ ,  $\tau = 1.01$  for  $\alpha = 1.4$ ,  $\alpha = 1.8$ ,  $\epsilon = 0.01$ . Figure 7 presents the absolute error between the exact source function and the numerical solutions for  $\alpha = 1.4, \alpha = 1.8$ , we can see that the numerical results match the exact ones quite well.

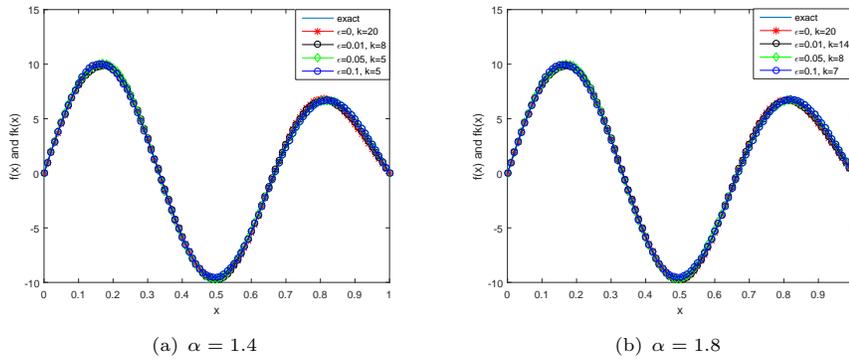


Figure 1. The numerical results for Example 1 with  $\mu = 0$ .

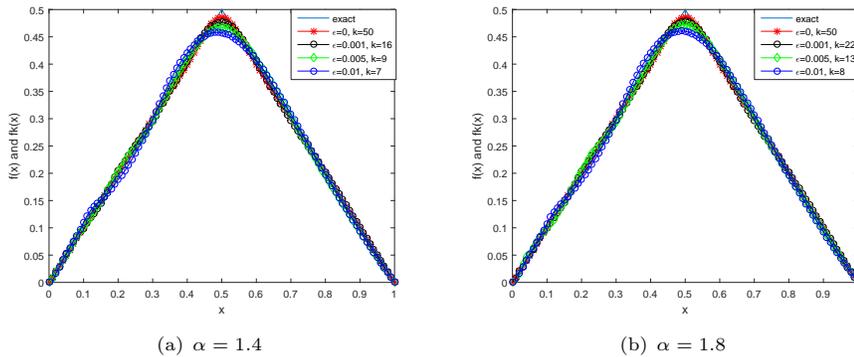


Figure 2. The numerical results for Example 2 with  $\mu = 0$ .

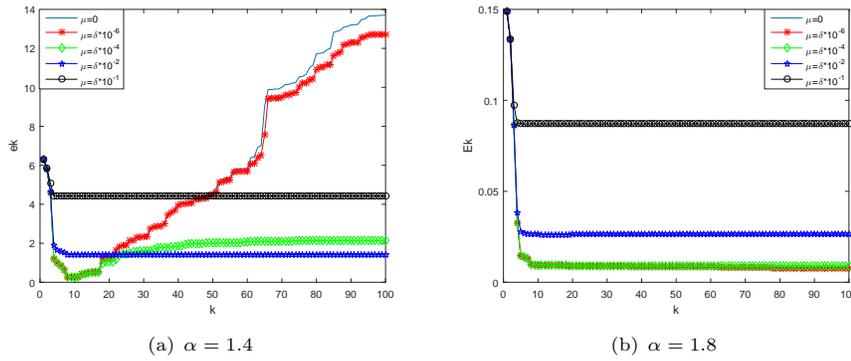


Figure 3. The errors  $e_k$  and  $E_k$  for Example 1 with different  $\mu$ .

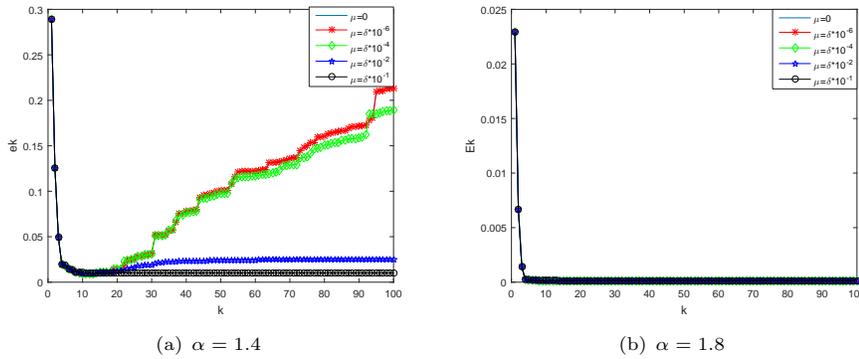


Figure 4. The errors  $e_k$  and  $E_k$  for Example 2 with different  $\mu$ .

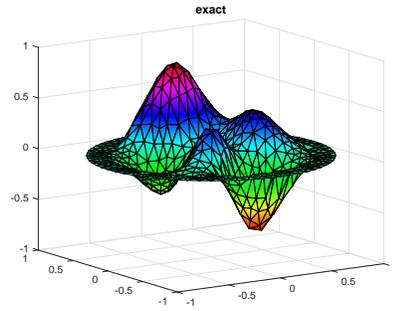
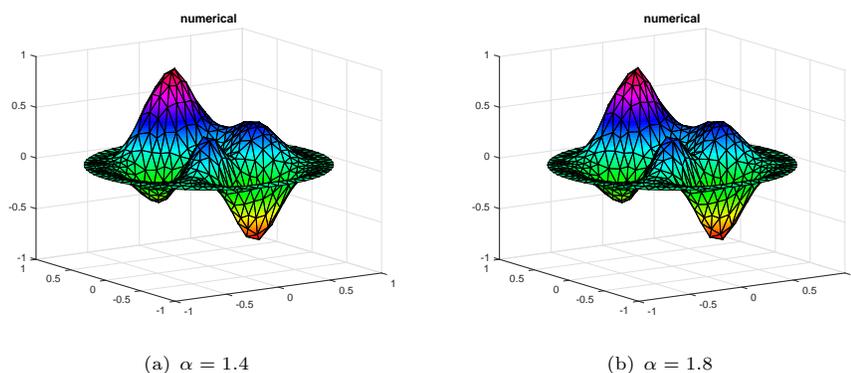


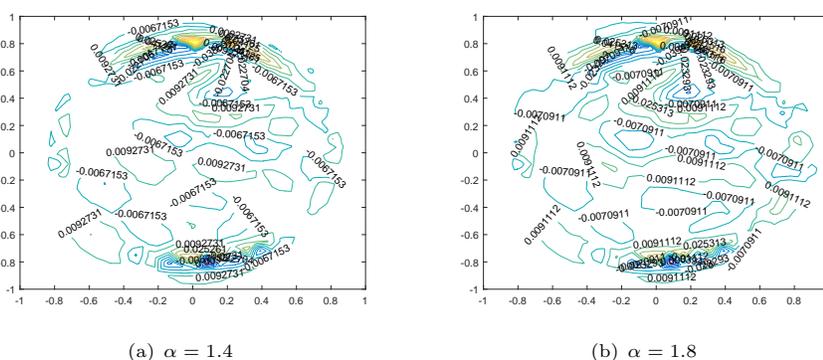
Figure 5. The exact solution for Example 3.

## 8. Conclusion

The inverse problem of determining the space-dependent source function in the time fractional diffusion wave equation is investigated. By the Fourier method, the regularity of direct problem as well as the existence and uniqueness of the adjoint problem are discussed. We give the conditions such that the uniqueness of the inverse source problem is satisfied. We use the Tikhonov regularization method to



**Figure 6.** The regularized solutions for Example 3.



**Figure 7.** Absolute errors for  $\alpha = 1.4$ ,  $\alpha = 1.8$  for Example 3.

overcome the ill-posedness, and provide a conjugate gradient algorithm to find an approximation to the minimizer of the Tikhonov regularization functional. From the computational results, it can be seen that the conjugate gradient method is effective and stable for solving the inverse source problem. Moreover we obtain an integration formula by parts for fractional derivatives which is useful for studying fractional diffusion wave equations.

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