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EXISTENCE AND QUALITATIVE FEATURES OF ENTIRE SOLUTIONS FOR DELAYED REACTION DIFFUSION SYSTEM: THE MONOSTABLE CASE*

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Abstract The paper is concerned with the existence and qualitative features of entire solutions for delayed reaction diffusion monostable systems. Here the entire solutions mean solutions defined on the $(x,t) \in \mathbb{R}^{N+1}$. We first establish the comparison principles, construct appropriate upper and lower solutions and some upper estimates for the systems with quasimonotone nonlinearities. Then, some new types of entire solutions are constructed by mixing any finite number of traveling wave fronts with different speeds $c \geq c_*$ and propagation directions and a spatially independent solution, where $c_* > 0$ is the critical wave speed. Furthermore, various qualitative properties of entire solutions are investigated. In particularly, the relationship between the entire solution, the traveling wave fronts and a spatially independent solution are considered, respectively. At last, for the nonquasimonotone nonlinearity case, some new types of entire solutions are also investigated by introducing two auxiliary quasimonotone controlled systems and establishing some comparison theorems for Cauchy problems of the three systems.

Keywords Entire solutions, upper and lower solutions, traveling wave fronts, comparison principle, upper estimates.

MSC(2010) 35K57, 35R10, 35B40, 34K30, 58D25.

1. Introduction

The purpose of this paper is to consider the existence and qualitative features of entire solutions for a general delayed monostable reaction–diffusion system:

$$\begin{cases} u_t(x,t) = d_1 \Delta u(x,t) + h(u(x,t), v(x,t-\tau_1)), \\ v_t(x,t) = d_2 \Delta v(x,t) + g(u(x,t-\tau_2), v(x,t)), \end{cases} \tag{1.1}$$

where $(x,t) \in (\mathbb{R} \times \mathbb{R}^+)$, respectively, $d_i > 0$ $(i = 1,2)$ are diffusion coefficient, $\tau_i > 0$ $(i = 1,2)$ are time delays, the nonlinearities $h(u,v)$, $g(u,v) \in C^2(\mathbb{R}^2, \mathbb{R})$

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satisfy the following conditions,

(P1) $h,\, g \in C^2([\mathbf{0},\mathbf{K}],\mathbb{R})$, $h(\mathbf{0}) = g(\mathbf{0}) = 0$, $h(\mathbf{K}) = g(\mathbf{K}) = 0$, $h(u,v) > 0$, $g(u,v) > 0$ for $u \in (0,k_1)$ and $v \in (0,k_2)$, $h(u,v) \le \partial_1 h(\mathbf{0})u + \partial_2 h(\mathbf{0})v$ and $g(u,v) \le \partial_1 g(\mathbf{0})u + \partial_2 g(\mathbf{0})v$ for $(u,v) \in [\mathbf{0},\mathbf{K}]$, where $\mathbf{0} := (0,0)$, $\mathbf{K} := (k_1,k_2)$ and k_1, k_2 are positive constants.

(P2) $\partial_2 h(u,v) \ge 0$ and $\partial_1 g(u,v) \ge 0$ for $(u,v) \in [\mathbf{0},\mathbf{K}]$.

According to (P1), (1.1) only has two equilibria in the interval $[\mathbf{0},\mathbf{K}]$. It is clear that (P1) is a basic assumption to ensure that system (1.1) is monostable on $[\mathbf{0},\mathbf{K}]$. (P2) implies (1.1) is a quasimonotone system, otherwise, (1.1) is a nonquasimonotone system.

Such reaction–diffusion system (1.1) with kinds of special nonlinearities arise from many biological, chemical models (see [9, 18, 24]). And the spatial dynamics of special cases of (1.1) have been extensively studied. Some related works are illustrated in the sequel.

For example, when $h(u,v) = -\alpha u + h(v)$ and $g(u,v) = -\beta v + g(u)$, the system (1.1) reduces to

$$\begin{cases} u_t(x,t) = d_1\Delta u(x,t) - \alpha u(x,t) + h(v(x,t-\tau_1)), \\ v_t(x,t) = d_2\Delta v(x,t) - \beta v(x,t) + g(u(x,t-\tau_2)). \end{cases} \tag{1.2}$$

Wu and Hsu [39] studied the existence and qualitative features of entire solutions for the delayed monostable epidemic model arising from the spread of an epidemic by oral-faecal transmission of (1.2). Hsu and Yang [14] investigated the existence, monotonicity, uniqueness and asymptotic behavior of traveling waves for (1.2) without delays. The monostable and bistable traveling fronts for general monotone reaction–diffusion systems including (1.2) as a special case, were also considered in [32].

When $\tau_1 = d_2 = 0$, τ_2, $d_1 > 0$ and $h(v) = av$, then system (1.2) reduces to

$$\begin{cases} \partial_t u(x,t) = d_1\partial_{xx}u(x,t) - \alpha_1 u(x,t) + av(x,t), \\ \partial_t v(x,t) = -\alpha_2 v(x,t) + g(u(x,t-\tau_2)). \end{cases} \tag{1.3}$$

Thieme and Zhao [31] obtained the existence of spreading speed and minimal wave speed for the quasimonotone epidemic system (1.3). Xu and Zhao [44] proved the existence, uniqueness (up to translation) and globally exponential stability of bistable traveling wave fronts of (1.3) without delays. Moreover, Zhao and Wang [45] also considered the existence and nonexistence of monostable traveling wave fronts of (1.3) without delays. Furthermore, the qualitative behaviours of solutions for (1.3) with concave g and suitable boundary conditions was also treated in [1, 3, 6, 9] without delay. Assuming g is sigma-shaped, the authors in [7] also considered solutions of (1.3) with homogeneous Neumann boundary conditions without delay.

If $\tau_2 = d_1 = 0$, then the system (1.3) reduces to the ODE system

$$\begin{cases} \partial_t u(x,t) = -\alpha_1 u(x,t) + av(x,t), \\ \partial_t v(x,t) = -\alpha_2 v(x,t) + g(u(x,t)). \end{cases} \tag{1.4}$$

The system was proposed in [8] to model the 1973 cholera epidemic spread in the European Mediterranean regions. The system (1.4) is also used to model the spread of man-environment epidemics (see [4, 5]).

More recently, Hsu etc [15] extended (1.1) to a more general delayed system

$$\begin{cases} \partial_t u(x,t) = d_1 \partial_{xx} u(x,t) + h(u(x,t), u(x,t-\hat{\tau}_1), v(x,t-\tau_2)), \\ \partial_t v(x,t) = d_2 \partial_{xx} v(x,t) + g(v(x,t), u(x,t-\tau_1), v(x,t-\hat{\tau}_2)), \end{cases} \tag{1.5}$$

where $t \geq 0$, $x \in \mathbb{R}$, $\tau_1, \tau_2 \geq 0$ and $\hat{\tau}_1, \hat{\tau}_2 > 0$ are time delayed terms, $d_i > 0$ $(i = 1, 2)$ are diffusion coefficients and the nonlinearities $h(w, u, v),\ g(w, u, v) \in C^2(\mathbb{R}^3, \mathbb{R})$. They obtained the existence and stability of traveling wave fronts (i.e. monotone traveling waves) by using the monotone iteration scheme via an explicit construction of a pair of upper and lower solutions, the techniques of weighted energy method and comparison principle.

Mathematically, the important issue in population and epidemic dynamics is the interaction between traveling waves which can be described by a class of entire solutions that are defined in all space and time, and behave like a combination of traveling waves as $t \to -\infty$. From the dynamical points of view, the study of entire solutions can help us fully understand the transient dynamics and the structures of the global attractor. In particular, traveling waves are also entire solutions. From the viewpoint of biology, such entire solutions provide some new spread of epidemic and invasion ways of the species. For the study of the entire solution, we also refer to [2, 11–13, 16, 17, 19, 21–23, 25, 28, 30, 34–38, 40, 42, 43].

To the best of our knowledge, the entire solution of the general system (1.1) is not still considered. And it is easy to know that the system (1.2) in [39] is a special case of system (1.1). So the study of this article is meaningful.

Motivated by the above works, the aim of this paper is to construct some new types of entire solutions for (1.1) with quasimonotone or nonquasimonotone nonlinearities. We extend the arguments developed in [12, 13, 39, 42]. Based on the comparison theorem, upper and lower solutions, upper estimates and asymptotic behavior of the traveling wave fronts and a spatially independent solution at $x \to -\infty$, we prove the existence and qualitative features of entire solutions of (1.1).

On the other hand, if the condition (P2) does not hold, then (1.1) is a nonquasimonotone system. Thus, the comparison principle fails. To overcome the deficiency, encouraged by the works of [39, 42], we introduce two auxiliary quasimonotone delayed systems. Then, we establish a comparison theorem for the Cauchy problems of the three systems. The existence and qualitative features of entire solutions of (1.1) without the quasimonotone condition are still established by using the comparison theorem and considering a sequence of Cauchy problem of the three systems, where the combinations of any finite traveling wave fronts with different speeds $c \geq c_*$ and propagation directions and a spatially independent solution of the lower system are taken as the initial values.

This paper is organized as follows. In Section 2, we introduce some necessary notations and present the main results on the existence and qualitative features of entire solutions. Section 3 is devoted to proving that the existence and qualitative features of entire solutions in the quasimonotone nonlinearities case (i.e., Theorems 2.1–2.3). In Section 4, we prove the existence and qualitative features of entire solutions in the nonquasimonotone nonlinearities case (i.e., Theorem 2.4) by introduce two auxiliary quasimonotone delayed systems.

2. Preliminary and main results

A traveling wave solution of (1.1) is a special solution of the form

$$(u(x,t), v(x,t)) = (\phi_c(\xi), \psi_c(\xi)),\ \ \xi = x\cdot\nu + ct,\ \nu \in \mathbb{S}^{N-1},$$

where the velocity $c > 0$ is a constant corresponding to the wave speed and $\xi := x\cdot\nu + ct$ represents the moving coordinate. $(\phi_c(\xi), \psi_c(\xi)) \in C^2(\mathbb{R}, \mathbb{R}^2)$ is called the wave profile and satisfies

$$\begin{cases} c\phi_c'(\xi) = d_1\phi_c''(\xi) + h(\phi_c(\xi), \psi_c(\xi - c\tau_1)), \\ c\psi_c'(\xi) = d_2\psi_c''(\xi) + g(\phi_c(\xi - c\tau_2), \psi_c(\xi)). \end{cases} \tag{2.1}$$

Note that $\mathbb{S}^{N-1}$ stands for the unit sphere. A traveling wave solution is always considered as a development process from one equilibrium state to another, then we look for a solution of (2.1) which satisfies the asymptotic conditions:

$$\lim_{\xi\to-\infty}(\phi_c(\xi), \psi_c(\xi)) = (0,0),\ \ \lim_{\xi\to+\infty}(\phi_c(\xi), \psi_c(\xi)) = (k_1, k_2). \tag{2.2}$$

We say $(\phi_c(\xi),\ \psi_c(\xi))$ is a traveling wave front if $(\phi_c(\xi),\ \psi_c(\xi))$ is monotone and satisfies (2.2).

Then here are some marks. Two vectors $(u_1, \cdots, u_n) \le (v_1, \cdots, v_n)$ in $\mathbb{R}^N$ means $u_i \le v_i$ for $i = 1, \cdots, n$. An interval of $\mathbb{R}^N$ is defined according to this order. For convenience, denote by ∂_i the first differential operator with respect to the i-th variables, and by ∂_{ij} the second differential operator with respect to the i-th and j-th variables.

Throughout this article, in addition, (P1)–(P2) hold, we always assume that the nonlinearities $h(\cdot,\cdot)$ and $g(\cdot,\cdot)$ satisfy the following assumptions.

(P3) Assume $\alpha_1 < 0$, $\alpha_2 < 0$ and $\alpha_1\alpha_2 < \beta_1\beta_2$, where

$$\alpha_1 := \partial_u h(0,0),\ \ \alpha_2 := \partial_v g(0,0),$$
$$\beta_1 := \partial_v h(0,0),\ \ \beta_2 := \partial_u g(0,0).$$

(P4) $\partial_i h(u,v) \le \partial_i h(0,0),\ \ \partial_i g(u,v) \le \partial_i g(0,0)$ for any $(u,v) \in [\mathbf{0}, \mathbf{K}],\ \ i = 1,2$.

Clearly, (P3) can help us to investigate the characteristic roots of the linearized equations of (1.1) at the equilibria $\mathbf{0}$. And (P2) guarantees that $\beta_i \ge 0$, $i = 1,2$. According to the local analysis of (1.1) at the equilibria $\mathbf{0}$ and quasimonotonicity of h and g, we can construct a pair of upper and lower solutions for (1.1), then the existence result of the traveling wave fronts can be obtained by the framework established in [29]. In the rest of this paper, we may assume that (P1)–(P4) hold.

Motivated by the works of [39, 42], we will adopt the comparison principle, construct appropriate upper and lower solutions and some upper estimates to obtain some new types of entire solutions, which are mixing any finite number of traveling wave fronts with different speeds $c \ge c_*$ and directions and a spatially independent solution. Then obtain the various qualitative properties of entire solutions. Before to state our main results, we give the following definition and notation.

Definition 2.1. Let $k \in \mathbb{N}$ and η, $\eta_0 \in \mathbb{R}^k$, if the functions $\mathcal{Z}_\eta(x,t)$, $\partial_t\mathcal{Z}_\eta(x,t)$ and $\Delta\mathcal{Z}_\eta(x,t)$ converges uniformly in any compact set $\mathbb{G} \subset \mathbb{R}^{N+1}$ to $\mathcal{Z}_{\eta_0}(x,t)$, $\partial_t\mathcal{Z}_{\eta_0}(x,t)$ and $\Delta\mathcal{Z}_{\eta_0}(x,t)$ as $\eta \to \eta_0$, respectively, then we say that the functions $\mathcal{Z}_\eta(x,t) = (U_\eta(x,t), V_\eta(x,t))$ converge to $\mathcal{Z}_{\eta_0}(x,t) = (U_{\eta_0}(x,t), V_{\eta_0}(x,t))$ as $\eta \to \eta_0$ in the sense of topology τ.

Notation 2.1. For any $\gamma \in \mathbb{Z}^+$, $\nu_i \in \mathbb{S}^{N+1}$, $i = 1, \cdots, \gamma$ and $A, T \in \mathbb{R}$, denote the regions as follow:

$$\Theta^i_{A,T} := \{x \in \mathbb{R}^N | x \cdot \nu_i \geq A\} \times [T, +\infty),\ i = 1, \cdots, \gamma,\ \ \Theta^{\gamma+1}_{A,T} := \mathbb{R}^N \times [T, +\infty),$$
$$\widetilde{\Theta}^i_{A,T} := \{x \in \mathbb{R}^N | x \cdot \nu_i \leq A\} \times (-\infty, T]),\ i = 1, \cdots, \gamma,\ \ \widetilde{\Theta}^{\gamma+1}_{A,T} := \mathbb{R}^N \times (-\infty, T].$$

Now, we state our main results as follows.

Theorem 2.1. *Assume that* (P1)–(P4) *hold. For any* $\gamma \in \mathbb{N} \cup \{0\}$, $\nu_1, \cdots, \nu_\gamma \in \mathbb{S}^{N-1}$, $m_1, \cdots, m_{\gamma+1} \in \mathbb{R}$, $c_1, \cdots, c_\gamma \geq c_*$ *and* $\chi \in \{0, 1\}$ *with* $\gamma + \chi \geq 2$, *there exists an entire solution* $\mathcal{Z}_\eta(x,t) = (U_\eta(x,t), V_\eta(x,t))$ *of* (1.1) *such that*

$$\underline{\mathcal{Z}}(x,t) \leq \mathcal{Z}_\eta(x,t) \leq \mathbf{K} \ \text{for}\ (x,t) \in \mathbb{R}^{N+1}, \tag{2.3}$$

where $\eta := \eta_{\gamma,\chi} = (c_1, m_1, \nu_1 \cdots, c_\gamma, m_\gamma, \nu_\gamma, \chi m_{\gamma+1})$. *Furthermore, we have the following results:*

(i) *If* (P5) *holds, then* $\mathcal{Z}_\eta(x,t) \leq \mathcal{Z}^+(x,t)$ *for* $(x,t) \in \mathbb{R}^{N+1}$.

(ii) *If* $c_1, \cdots, c_\gamma > c_*$, *then* $\mathcal{Z}_\eta(x,t) \leq \overline{\mathcal{Z}}(x,t)$ *for* $(x,t) \in \mathbb{R}^{N+1}$.

Theorem 2.2. *Assume that* (P1)–(P4) *hold and the entire solution* $\mathcal{Z}_\eta(x,t)$ *of system* (1.1) *is derived in Theorem 2.1. Then the following statements hold.*

(i) *If* (P5) *holds or* $c_1, \cdots, c_\gamma > c_*$, *then* $\mathbf{0} < \mathcal{Z}_\eta(x,t) < \mathbf{K}$ *and* $\frac{\partial}{\partial t}\mathcal{Z}_\eta(x,t) > \mathbf{0}$ *for any* $(x,t) \in \mathbb{R}^{N+1}$.

(ii) $\lim\limits_{t\to+\infty} \sup\limits_{x\in\mathbb{R}^N} \|\mathcal{Z}_{\eta_{\gamma,1}}(x,t) - \mathbf{K}\| = 0$, $\lim\limits_{t\to+\infty} \sup\limits_{\|x\|\leq A} \|\mathcal{Z}_{\eta_{\gamma,0}}(x,t) - \mathbf{K}\| = 0$ *for any* $A \in \mathbb{R}_+$.
If (P5) *holds or* $c_1, \cdots, c_\gamma > c_*$, *then*
$$\lim_{t\to-\infty} \sup_{\|x\|\leq A} \|\mathcal{Z}_\eta(x,t)\| = 0$$
for any $A \in \mathbb{R}_+$.

(iii) *For any* $(x,t) \in \mathbb{R}^{N+1}$, $\mathcal{Z}_\eta(x,t)$ *is increasing with respect to* m_i, $i = 1, \cdots, \gamma+1$.

(iv) $\mathcal{Z}_\eta(x,t) \to \mathbf{K}$ *in the sense of topology* τ *as* $m_i \to +\infty$ *and uniformly on* $(x,t) \in \Theta^i_{A,T}$ *for* $A, T \in \mathbb{R}$, $i = 1, \cdots, \gamma+1$.

Theorem 2.3. *Assume that* (P1)–(P4) *hold and* $\mathcal{Z}_\eta(x,t)$ *is the entire solution of system* (1.1) *stated in Theorem 2.1, for any* $\nu \in \mathbb{S}^{N+1}$ *and* $c \geq c_*$, *then the following statements hold.*

(i) *If* (P5) *holds and there exists* $i_0 \in \{1, \cdots, \gamma\}$ *such that* $c\nu \cdot \nu_i < c_i$ *for any* $i \in \{1, \cdots, \gamma\} \backslash \{i_0\}$, *then it holds*
(a) *for* $c\nu \cdot \nu_{i_0} = c_{i_0}$, $\mathcal{Z}_\eta(x - ct\nu, t) \to \Phi_{c_{i_0}}(x \cdot \nu_{i_0} + m_{i_0})$ *as* $t \to -\infty$ *in the sense of topology* τ;
(b) *for* $c\nu \cdot \nu_{i_0} < c_{i_0}$, $\mathcal{Z}_\eta(x - ct\nu, t) \to \mathbf{0}$ *as* $t \to -\infty$ *in the sense of topology* τ;
(c) *for* $c\nu \cdot \nu_{i_0} > c_{i_0}$, $\mathcal{Z}_\eta(x - ct\nu, t) \to \mathbf{K}$ *as* $t \to -\infty$ *in the sense of topology* τ.

(ii) *If* (P5) *holds and there exists* $i_0 \in \{1, \cdots, \gamma\}$ *such that* $c\nu \cdot \nu_i > c_i$ *for any* $i \in \{1, \cdots, \gamma\} \backslash \{i_0\}$, *we have*
(a) *for* $c\nu \cdot \nu_{i_0} = c_{i_0}$, $\mathcal{Z}_\eta(x - ct\nu, t) \to \Phi_{c_{i_0}}(x \cdot \nu_{i_0} + m_{i_0})$ *as* $t \to +\infty$ *in the sense of topology* τ;
(b) *for* $c\nu \cdot \nu_{i_0} > c_{i_0}$, $\mathcal{Z}_\eta(x - ct\nu, t) \to \mathbf{0}$ *as* $t \to +\infty$ *in the sense of topology* τ;
(c) *for* $c\nu \cdot \nu_{i_0} < c_{i_0}$, $\mathcal{Z}_\eta(x - ct\nu, t) \to \mathbf{K}$ *as* $t \to +\infty$ *in the sense of topology* τ.

(iii) *For any* $c_1, \cdots, c_\gamma > c_*$ *and* $x \in \mathbb{R}^N$,

$$\mathcal{Z}_{\eta_{\gamma,1}}(x,t) \sim \Gamma(t+m_{\gamma+1}) \sim (1,b_*)e^{\lambda_*(t+m_{\gamma+1})} \text{ and } \mathcal{Z}_{\eta_{\gamma,0}}(x,t) = O(e^{\Lambda t}),$$

as $t \to -\infty$, *where* $\Lambda := \min_{i=1,\cdots,\gamma}\{c_i\lambda_1(c_i)\}$.

(iv) *For any* $A, T \in \mathbb{R}$,

$$\mathcal{Z}_{\eta_{\gamma,1}}(x,t) \to \mathcal{Z}_{\eta_{\gamma,0}}(x,t)$$

as $m_{\gamma+1} \to -\infty$ *in the sense of topology* τ, *and uniformly on* $(x,t) \in \widetilde{\Theta}^i_{A,T}$.

(v) *If* $c_{i_0} > c_*$, *for some* $i_0 \in \{1, \cdots, \gamma\}$, *then for any* $A, T \in \mathbb{R}$,

$$\mathcal{Z}_\eta(x,t) \to \mathcal{Z}_{c_i,m_i,i\in\{1,\cdots,\gamma\}\setminus\{i_0\};m_{\gamma+1}}(x,t)$$

as $m_{i_0} \to -\infty$ *in the sense of topology* τ, *and uniformly on* $(x,t) \in \widetilde{\Theta}^i_{A,T}$.

Theorem 2.4. *Assume that* (P1) *and* (P6) *hold. For any* $l \in \mathbb{N}\cup\{0\}$, $\zeta_1, \cdots, \zeta_{l+1} \in \mathbb{R}$, $\nu_1, \cdots, \nu_l \in \mathbb{S}^{N-1}$, $c_1, \cdots, c_l \geq c_*$ *and* $\chi \in \{0,1\}$ *with* $l + \chi \geq 1$, *there exists an entire solution* $\mathbf{Z}_\zeta(x,t) = (U_\zeta(x,t), V_\zeta(x,t))$ *of system* (1.1) *such that* $\mathbf{Z}_\zeta(x,t) > 0$ *for* $(x,t) \in \mathbb{R}^{N+1}$ *and*

$$\underline{\mathbf{Z}}(x,t) \leq \mathbf{Z}_\zeta(x,t) \leq \mathbf{K}^+ \text{ for } (x,t) \in \mathbb{R}^{N+1}, \tag{2.4}$$

where $\zeta := \zeta_{l,\chi} = (c_1, n_1, \nu_1, \cdots, c_l, n_l, \nu_l; \chi n_{l+1})$.

Furthermore, the following results hold,

(i) *If* (P7) *holds, then*

$$\mathbf{Z}_\zeta(x,t) \leq \mathbf{Z}^+(x,t) \text{ for } (x,t) \in \mathbb{R}^{N+1}. \tag{2.5}$$

(ii) *If* $c_1, \cdots, c_l > c_*$, *then*

$$\mathbf{Z}_\zeta(x,t) \leq \overline{\mathbf{Z}}(x,t) \text{ for } (x,t) \in \mathbb{R}^{N+1}. \tag{2.6}$$

(iii) *If* (P7) *holds or* $c_1, \cdots, c_l > c_*$, *then*

$$\lim_{t\to-\infty} \sup_{\|x\|\leq N} |\mathbf{Z}_\zeta(x,t)| = 0 \text{ for any } N \in \mathbb{R}_+.$$

(iv) $\lim_{t\to+\infty} \inf \inf_{x\in\mathbb{R}^N} \mathbf{Z}_{\zeta_{l,1}}(x,t) \geq \mathbf{K}^-$ *and* $\lim_{t\to+\infty} \inf \inf_{\|x\|\leq N} \mathbf{Z}_{\zeta_{l,0}}(x,t) \geq \mathbf{K}^-$ *for any* $N \in \mathbb{R}_+$.

(v) *If* $c_1, \cdots, c_l > c_*$, *then for every* $x \in \mathbb{R}^N$, *as* $t \to -\infty$,

$$\mathbf{Z}_{\zeta_{l,1}}(x,t) \sim (1,b_*)e^{\lambda_*(t+n_{l+1})} \text{ and } \mathbf{Z}_{\zeta_{l,0}}(x,t) = O(e^{\Lambda t}),$$

where $\Lambda := \min_{i=1,\cdots,l}\{c_i\lambda_1(c_i)\}$.

Remark 2.1. The convexity assumptions, special symbols, definitions and assumptions in the process of proving these main results, such as (P5), (P6), (P7) etc, will be given in the latter.

3. Entire solutions for quasi-monotone system

In this section, we consider the entire solutions of (1.1) with monstable quasi-monotone nonlinearity. At first, we give some preliminaries for the existence and asymptotic behavior of traveling wave fronts and spatially independent solutions of (1.1). Then, we consider the initial value problem of (1.1) and establish some comparison theorems and the upper estimate. Lastly, we prove the existence and investigate some qualitative features of the entire solutions of (1.1).

3.1. Traveling wave fronts and spatially independent solutions

In this subsection, we will first investigate the characteristic roots of the linearized equations for the profile equations (2.1) at the equilibria $\mathbf{0}$. From (2.1) and the notations in (P1), the linearized equations with respect to the trivial equilibrium $\mathbf{0}$ can be represented by

$$\begin{cases} cu_1'(\xi) = d_1u_1''(\xi) + \alpha_1u_1(\xi) + \beta_1u_2(\xi - c\tau_1), \\ cu_2'(\xi) = d_2u_2''(\xi) + \alpha_2u_2(\xi) + \beta_2u_1(\xi - c\tau_2). \end{cases} \tag{3.1}$$

From (3.1), one can see that characteristic equation at $\mathbf{0}$ has the form

$$\Delta_1(c,\lambda) \triangleq \Big(d_1\lambda^2 - c\lambda + \alpha_1\Big)\Big(d_2\lambda^2 - c\lambda + \alpha_2\Big) - \beta_1\beta_2e^{-c\lambda\tau} \text{ for } \tau = \tau_1 + \tau_2.$$

For convenience, we denote

$$f_i(c,\lambda) := d_i\lambda^2 - c\lambda + \alpha_i \text{ for } i = 1, 2$$

and

$$\begin{aligned} \lambda_1^{\pm} &:= \frac{c \pm \sqrt{c^2 - 4d_1\alpha_1}}{2d_1}, \quad & \lambda_2^{\pm} &:= \frac{c \pm \sqrt{c^2 - 4d_2\alpha_2}}{2d_2}, \\ \lambda_m^c &:= \min\{\lambda_1^+, \lambda_2^+\}, & \lambda_M^c &:= \max\{\lambda_1^+, \lambda_2^+\}. \end{aligned}$$

Similar to Lemmas 2.2 in [15], we easily obtain the following conclusions.

Lemma 3.1. *Assume that* (P1) *and* (P3) *hold. There exists a positive constant* c_* *such that*

(i) *if* $c \geq c_*$, *then* $\Delta_1(c,\lambda) = 0$ *has two positive real roots* $\lambda_1 := \lambda_1(c)$ *and* $\lambda_2 := \lambda_2(c)$ *with* $0 < \lambda_1 \leq \lambda_2 < \lambda_m^c$. *Moreover, if* $c > c_*$, *then* $\lambda_1 < \lambda_2$ *and* $\Delta_1(c,\lambda) > 0$ *in* $\lambda \in (\lambda_1, \lambda_2)$ *and if* $c = c_*$, *then* $\lambda_0 := \lambda_1(c_*) = \lambda_2(c_*)$.

(ii) *if* $\varepsilon > 0$ *and small enough, we have* $\Delta_1(c, \lambda_1 + \varepsilon) > 0$ *and* $f_i(c, \lambda_1 + \varepsilon) < 0$ *for* $i = 1, 2$.

Moreover, by using the theory in [26], the existence of traveling wave fronts of (2.1) can be reduced to look for a pair of suitable upper and lower solutions of (2.1). Inspiration from the upper and lower solutions of papers [15] and [39], the lower solution we choose coming from in [15], in order to get the upper estimate of traveling wave solutions, we choose the upper solution by means of the method in [39]. For completeness, we recall the definition for upper and lower solutions of system (2.1).

Definition 3.1. A continuous function $\bar{\Phi}(\xi) = (\bar{\phi}_1(\xi), \bar{\phi}_2(\xi))$ is called an upper solution of (2.1) if it is differentiable except at countable points and satisfies the following inequality

$$\begin{cases} c\bar{\phi}_1'(\xi) \geq d_1\bar{\phi}_1''(\xi) + h(\bar{\phi}_1(\xi), \bar{\phi}_2(\xi - c\tau_1)), \\ c\bar{\phi}_2'(\xi) \geq d_2\bar{\phi}_2''(\xi) + g(\bar{\phi}_1(\xi - c\tau_2), \bar{\phi}_2(\xi)). \end{cases} \tag{3.2}$$

The lower solution $\underline{\Phi}(\xi) = (\underline{\phi}_1(\xi), \underline{\phi}_2(\xi))$ of (2.1) can be similarly defined by only reversing the inequalities, i.e.,

$$\begin{cases} c\underline{\phi}_1'(\xi) \leq d_1\underline{\phi}_1''(\xi) + h(\underline{\phi}_1(\xi), \underline{\phi}_2(\xi - c\tau_1)), \\ c\underline{\phi}_2'(\xi) \leq d_2\underline{\phi}_2''(\xi) + g(\underline{\phi}_1(\xi - c\tau_2), \underline{\phi}_2(\xi)). \end{cases} \tag{3.3}$$

Let

$$\mathrm{C}_{[\mathbf{0},\mathbf{K}]}(\mathbb{R}, \mathbb{R}^2) = \Big\{\Psi : \Psi(\xi) \in \mathrm{C}(\mathbb{R}, \mathbb{R}^2),\ \ \mathbf{0} \leq \Psi(\xi) \leq \mathbf{K} \text{ for all } \xi \in \mathbb{R}.\Big\}$$

Assume that (2.1) has an upper solution $\bar{\Phi}(\xi) \in \mathrm{C}_{[\mathbf{0},\mathbf{K}]}(\mathbb{R}, \mathbb{R}^2)$ and a lower solution $\underline{\Phi}(\xi) \in \mathrm{C}_{[\mathbf{0},\mathbf{K}]}(\mathbb{R}, \mathbb{R}^2)$ satisfying the following conditions.

(H1) $\mathbf{0} \leq \underline{\Phi}(\xi) \leq \bar{\Phi}(\xi) \leq \mathbf{K}$.

(H2) $\sup\limits_{y \leq \xi} \underline{\Phi}(y) \leq \bar{\Phi}(\xi)$ for any $\xi \in \mathbb{R}$.

Similar to the frameworks of [26], we can verify the existence of traveling wave fronts as follows.

Proposition 3.1. *Assume that* (P1)–(P4) *hold. If* (2.1) *has an upper solution* $\bar{\Phi}(\xi)$ *and a lower solution* $\underline{\Phi}(\xi)$ *satisfying* (H1)–(H2), *in addition,* $h(\hat{\Phi}) \neq 0$, $g(\hat{\Phi}) \neq 0$ *for* $\hat{\Phi} \in (\mathbf{0}, \inf_{\xi\in\mathbb{R}} \bar{\Phi}(\xi)] \cup [\sup_{\xi\in\mathbb{R}} \underline{\Phi}(\xi), \mathbf{K})$, *then* (2.1) *has a monotone solution* $\Phi(\xi) = (\phi_1(\xi), \phi_2(\xi))$ *satisfying* (2.2), *which is a traveling wave front of* (1.1).

According to Proposition 3.1, to obtain the existence result for (2.1), we only need to find a pair of upper and lower solutions of (2.1) satisfying the conditions of proposition 3.1. The construction and the proof process of upper and lower solutions are from [39, Lemma 2.2] and [15, Lemma 3.3]. In order to prove the upper and lower solutions, we first recall two lemmas [14, Lemmas 3.1 & 3.2], both proofs are straightforward and skipped.

Lemma 3.2 (Lemma 3.1, [14]). *If* $\Phi(\xi)$ *and* $\Psi(\xi)$ *satisfy the inequality* (3.2) ((3.3), *resp.*) *for* ξ *in an open interval* $\mathbb{I} \subseteq \mathbb{R}$. *Then the function* $\psi(\xi)$ *defined by*

$$\psi(\xi) := \min\{\Phi(\xi), \Psi(\xi)\}\ (\psi(\xi) := \max\{\Phi(\xi), \Psi(\xi)\}, \text{resp.})$$

also satisfies the inequality (3.2) ((3.3), *resp.*) *for* $\xi \in \mathbb{I}$.

Lemma 3.3 (Lemma 3.2, [14]). *Let* $A = (a_{ij})$ *be a two by two matrix such that* $a_{ii} < 0, i = 1, 2$ *and* $a_{ij} > 0$ *for* $i \neq j$. *Then the system of the following inequalities*

$$a_{11}x_1 + a_{12}x_2 < 0\ (> 0, resp.) \quad and \quad a_{21}x_1 + a_{22}x_2 < 0\ (> 0, resp.)$$

have a solution (x_1, x_2) *with* $x_i > 0, i = 1, 2$ *if and only if* $\det A > 0\ (< 0, resp.)$.

In the following, we use the similar methods in [14,15,39] to construct a pair of upper and lower solutions of (2.1), respectively.

Now, we construct the upper solution of (2.1). Let us define the functions

$$\bar{\phi}_1(\xi) = e^{\lambda_1 \xi} \text{ and } \bar{\phi}_2(\xi) = b(c)e^{\lambda_1 \xi},$$

where $\bar{\xi}_1 := \ln k_1/\lambda_1$ and $\bar{\xi}_2 := (\ln k_2 - \ln b(c))/\lambda_1$. Let $b(c) := \beta_2 e^{-c\lambda_1\tau_2}/(c\lambda_1 - d_2\lambda_1^2 - \alpha_2) = (c\lambda_1 - d_1\lambda_1^2 - \alpha_1)/\beta_1 e^{-c\lambda_1\tau_1}$ and $\bar{\xi} := \min\{\bar{\xi}_1, \bar{\xi}_2\} > 0$.

Lemma 3.4. *Assume that* (P1), (P3) *hold. If* $c > c_*$, *then* $\bar{\phi}(\xi) = (\bar{\phi}_1(\xi), \bar{\phi}_2(\xi))$ *satisfies* (3.2) *for* $\xi < \bar{\xi}$.

Proof. The result can be proved by elementary computations. For the details, see part A of the Appendix.

Obviously, the equilibrium (k_1, k_2) is also an upper solution of (2.1). Let us define $\bar{\Psi}(\xi) = (\bar{\psi}_1(\xi), \bar{\psi}_2(\xi))$ by

$$\bar{\psi}_i(\xi) = \min\{k_i, \bar{\phi}_i(\xi)\} = \begin{cases} \bar{\phi}_i(\xi), & \text{for } \xi \le \bar{\xi}_i, \\ k_i, & \text{for } \xi \ge \bar{\xi}_i. \end{cases} \tag{3.4}$$

By Lemmas 3.2 and 3.4, $\bar{\Psi}(\xi)$ satisfies (3.2) for $\xi \in \mathbb{R}$ except at finitely many points, i.e. $\bar{\Psi}(\xi)$ is an upper solution of (2.1). This completes the proof. □

Next, let us start building the lower solution of (2.1). Defining the function as follows:

$$\underline{\phi}_i(\xi) := \kappa_i e^{\lambda_1 \xi}(1 - \delta_i e^{\varepsilon \xi}), \ \ i = 1, 2,$$

where $\varepsilon > 0$, $\kappa_i > 0$ and $\delta_i > 1$ are constants and will be determined later.

Lemma 3.5. *Assume that* (P1) *and* (P3) *hold. If* $c > c_*$, *then there exist positive constants* κ_i, ε *and* $\delta_i > 1$, $i = 1, 2$, $\underline{\xi} \le 0$, *such that* $\underline{\phi}(\xi) = (\underline{\phi}_1(\xi), \underline{\phi}_2(\xi))$ *satisfies* (3.3) *for* $\xi \le \underline{\xi}$.

Proof. The result can be proved by Lemmas 3.1 and 3.3. For the tedious details, see part B of the Appendix.

Obviously, the equilibrium $\mathbf{0}$ is also a lower solution of (2.1). Let us define $\underline{\Psi}(\xi) = (\underline{\psi}_1(\xi), \underline{\psi}_2(\xi))$ by

$$\underline{\psi}_i(\xi) = \max\{0, \underline{\phi}_i(\xi)\} = \begin{cases} \underline{\phi}_i(\xi), & \text{for } \xi \le \underline{\xi}_i, \\ 0, & \text{for } \xi \ge \underline{\xi}_i. \end{cases} \tag{3.5}$$

Note that $0 \le \underline{\psi}_i \le k_i$ for $i = 1, 2$. By Lemmas 3.2 and 3.5, $\underline{\Psi}(\xi)$ also satisfies (3.3) for all $\xi \in \mathbb{R}$ except at finitely many points, i.e. $\underline{\Psi}(\xi)$ (i.e., (3.5)) is a lower solution of (2.1). This completes the proof. □

Inspired by the framework of [39, Theorem 2.3], we can obtain the following existence and asymptotic behavior of traveling wave fronts of (1.1).

Theorem 3.1. *Let* (P1)–(P3). *For each* $c \ge c_*$, *system* (1.1) *has a traveling wave front* $\Phi(\xi) := (\phi(\xi), \psi(\xi))$, $\xi = x \cdot \nu + ct$, *which satisfies* $\Phi(-\infty) = \mathbf{0}$, $\Phi(+\infty) = \mathbf{K}$ *and* $\Phi(\xi) > \mathbf{0}$ *for all* $\xi \in \mathbb{R}$. *Moreover, if* $c > c_*$, *then*

$$\lim_{\xi \to -\infty} \Phi(\xi) e^{-\lambda_1(c)\xi} = (1, b(c)) \ \textit{ and } \ \Phi(\xi) \le (1, b(c)) e^{\lambda_1(c)\xi} \ \textit{ for all } \xi \in \mathbb{R}. \tag{3.6}$$

Then, we consider the solution $\Gamma = (\Gamma_1, \Gamma_2)$ of the following delayed differential system

$$\begin{cases} \Gamma_1'(t) = h(\Gamma_1(t), \Gamma_2(t-\tau_1)), \\ \Gamma_2'(t) = g(\Gamma_1(t-\tau_2), \Gamma_2(t)), \end{cases} \tag{3.7}$$

subject to the boundary conditions

$$\Gamma(-\infty) = \mathbf{0} \ \text{ and } \ \Gamma(+\infty) = \mathbf{K}, \tag{3.8}$$

which is called spatially independent heteroclinic solution of (1.1). Obviously, the characteristic function for (3.7) with respect to $(0,0)$ is given by

$$\Delta_2(c,\lambda) := (\lambda - \alpha_1)(\lambda - \alpha_2) - \beta_1\beta_2 e^{-\lambda(\tau_1+\tau_2)}.$$

Employing the framework of [39, Lemma 2.4] on careful local analysis near the equilibria $\mathbf{0}$ of (3.7), we can obtain that $\Delta_2(c,\lambda) = 0$ has a unique real root $\lambda_* > 0$. And similar to [39, Theorem 2.5], we have the following result.

Theorem 3.2. *Assume that* (P1)–(P3) *hold. There exists a solution* $\Gamma = (\Gamma_1, \Gamma_2)$ *of* (3.7) *and* (3.8) *such that*

$$\Gamma(t) > 0, \ \lim_{t\to-\infty} \Gamma(t)e^{-\lambda_* t} = (1, b_*) \ \text{ and } \Gamma(t) \le (1, b_*)e^{\lambda_* t} \ \text{ for all } t \in \mathbb{R},$$

where $b_* = \beta_2 e^{-\lambda_*\tau_2}/(\lambda_* - \alpha_2)$.

3.2. Initial value problem

In the subsection, we shall give the well posedness of the initial value problem of (1.1) and establish some comparison theorems of (1.1). A priori estimate of solutions of (1.1) is also established. For convenience, some notations and definitions are introduced.

Notation 3.1. (1) Let $X := BUC(\mathbb{R}^N, \mathbb{R}^2)$ be the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}^N$ into $\mathbb{R}^2$ with the supremum norm $\|\cdot\|_X$. Moreover, we denote

$$X^+ := \{\varphi = (\varphi_1, \varphi_2) \in X : \varphi_i(x) \ge 0, \, x \in \mathbb{R}^N, \, i = 1, 2\}.$$

(2) Let $\tau := \max\{\tau_1, \tau_2\}$ and $\mathcal{C} := C([-\tau, 0], X)$ be the Banach space of continuous functions from $[-\tau, 0]$ into X with the supremum norm. Moreover, we denote $\mathcal{C}^+ := \{\phi \in C : \phi(s) \in X^+, \, s \in [-\tau, 0]\}$.

(3) Denote the spaces

$$X_{[\mathbf{0},\mathbf{K}]} := \{\varphi \in X : \varphi(x) \in [\mathbf{0}, \mathbf{K}], \, x \in \mathbb{R}^N\}, \tag{3.9}$$

$$\mathcal{C}_{[\mathbf{0},\mathbf{K}]} := \{\phi \in \mathcal{C} : \phi(x,s) \in [\mathbf{0}, \mathbf{K}], \, x \in \mathbb{R}^N, \, s \in [-\tau, 0]\}. \tag{3.10}$$

(4) As usual, we identity an element $\phi \in \mathcal{C}$ as a function from $\mathbb{R} \times [-\tau, 0]$ into $\mathbb{R}^2$ defined by $\phi(x,s) = \phi(s)(x)$.

According to the above notations, it is verified to see that X^+ and $\mathcal{C}^+$ are closed cones of X and $\mathcal{C}$, respectively.

For any continuous function $Z : [-\tau, b) \to X$, $b > 0$, we define $Z_t \in \mathcal{C}$, $t \in [0, b)$ by $Z_t(s) = Z(t+s)$, $s \in [-\tau, 0]$. Then $t \to Z_t$ is a continuous function from $[0, b)$ to $\mathcal{C}$. Let us define $F = (F_1, \mathrm{F}_2) : \mathcal{C}_{[\mathbf{0},\mathbf{K}]} \to X$ by

$$F_1(\phi_1, \phi_2)(x, 0) = L_1\phi_1(x, 0) + h(\phi_1(x, 0), \phi_2(x, -\tau_1)),$$
$$F_2(\phi_1, \phi_2)(x, 0) = L_2\phi_2(x, 0) + g(\phi_1(x, -\tau_2), \phi_1(x, 0)),$$

where $L_1 := \max_{(u,v)\in[\mathbf{0},\mathbf{K}]} |\partial_1 h(u, v)|$ and $L_2 := \max_{(u,v)\in[\mathbf{0},\mathbf{K}]} |\partial_2 g(u, v)|$. It is clear to see that $F : \mathcal{C}_{[\mathbf{0},\mathbf{K}]} \to X$ is globally Lipschitz continuous.

We consider the initial value problem of (1.1) with the following initial data

$$u(x, s) = \phi_1(x, s), \quad v(x, s) = \phi_2(x, s), \quad x \in \mathbb{R}^N, \ s \in [-\tau, 0], \tag{3.11}$$

where $(\phi_1, \phi_2) \in \mathcal{C}$. Let $T(t) = (T_1(t), T_2(t))$ $(t > 0)$ be a family of linear operators on X defined by

$$T(t)[\varphi] := (T_1(t)[\varphi], T_2(t)[\varphi]), \quad \varphi = (\varphi_1, \varphi_2) \in X,$$

where

$$T_i(x, t) := \frac{1}{(4\pi d_i t)^{N/2}} \exp\Big\{ - \frac{\| x \|^2}{4d_i t} - L_i t\Big\}, \ \ i = 1, 2. \tag{3.12}$$

It is clear that $T(t)$ is a linear semigroup on X and $T(t)X^+ \subset X^+$.

Definition 3.2. A continuous function $Z = (u, v) : [-\tau, b) \to X_{[\mathbf{0},\mathbf{K}]}$, $b > 0$ is called an upper solution (or a lower solution) of (1.1) on $[0, b)$ if

$$Z(t) \geq \ (\text{or } \leq) T(t - r)Z(r) + \int_r^t T(t - \varrho)F(Z_\varrho)d\varrho$$

for any $0 \leq \varrho < t < b$. If Z is both an upper and a lower solution on $[0, b)$, then it is said to be a mild solution of (1.1).

Remark 3.1. Assume that $Z = (u, v) : \mathbb{R} \times [-\tau, b) \to \mathrm{X}_{[\mathbf{0},\mathbf{K}]}$ with $b > 0$ and Z is C in $x \in \mathbb{R}^N$, *and* C^1 in $t \in [0, b)$ and satisfies the following inequality

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} \geq (\leq) d_1\Delta u(x, t) + h(u(x, t), v(x, t - \tau_1)), \\ \dfrac{\partial v(x,t)}{\partial t} \geq (\leq) d_2\Delta v(x, t) + g(u(x, t - \tau_2), v(x, t)), \end{cases} \quad x \in \mathbb{R}^N, \ 0 < t < b.$$

Then by the positivity of $T(t) : X^+ \to X^+$, it follows that Definition 3.2 holds, and hence Z is an upper solution (or a lower solution) of (1.1) on $[0, b)$.

By using the theory of abstract functional differential equations ([27], Corollary 5), it is easy to prove the following result (the similar result can be found in [21,35, 39,42] and some references cited therein).

Lemma 3.6. *Assume that* (P1) *and* (P2) *hold. Consider the problem of system* (1.1) *with the initial data* (3.11)*.*

(1) *For any* $\phi \in \mathcal{C}_{[\mathbf{0},\mathbf{K}]}$, (1.1) *has a unique solution* $Z(x, t; \phi)$ *on* $(x, t) \in \mathbb{R}^N \times [0, +\infty)$ *with* $\mathbf{0} \leq Z(x, t; \phi) \leq \mathbf{K}$. *Moreover,* $Z(x, t; \phi)$ *is a classical solution of* (1.1) *for* $(x, t) \in \mathbb{R}^N \times (0, +\infty)$.

(2) *Letting $Z^+(x,t)$ and $Z^-(x,t)$ be a pair of upper and lower solutions of* (1.1) *on* $[0,+\infty)$ *with* $Z^+(x,s) \geq Z^-(x,s)$ *for* $x \in \mathbb{R}^N$ *and* $s \in [-\tau, 0]$, *it holds* $\mathbf{0} \leq Z^-(x,t) \leq Z^+(x,t) \leq \mathbf{K}$ *for* $(x,t) \in \mathbb{R}^N \times [0,+\infty)$.

According to the standard parabolic estimates (see Friedman [10]), similar to the proof of Proposition 4.3 in [35] and Lemma 2.10 in [39]. We have the following important results.

Lemma 3.7. *Suppose $Z(x,t) = (u(x,t), v(x,t))$ is a solution of* (1.1) *with initial value $\phi = (\phi_1, \phi_2) \in \mathcal{C}_{[\mathbf{0},\mathbf{K}]}$, then there exists a positive constant $M > 0$ independent of ϕ, such that for any $\eta > 0$, $x \in \mathbb{R}^N$ and $t > 3(\tau + 1)$,*

$$\begin{aligned} &\| \partial_t Z(x,t) \|, \ \| \partial_{tx_i} Z(x,t) \|, \ \| \partial_{tt} Z(x,t) \|, \ \| \partial_{x_i} Z(x,t) \|, \ \| \partial_{x_i t} Z(x,t) \| \leq M, \\ &\| \partial_{x_i x_j} Z(x,t) \|, \ \| \partial_{x_i^2 t} Z(x,t) \|, \ \| \partial_{x_i^2 x_j} Z(x,t) \| \leq M, \ \ \forall i,j = 1, \cdots, N. \end{aligned} \tag{3.13}$$

Lemma 3.8. *Assume that* (P1)–(P3) *hold. Let $Z^+ := (u^+, v^+) \in C(\mathbb{R}^N \times [-\tau, +\infty), [0,+\infty)^2)$ and $Z^- := (u^-, v^-) \in C(\mathbb{R}^N \times [-\tau, +\infty), (-\infty, k_1] \times (-\infty, k_2])$ be such that $Z^+(x,s) \geq Z^-(x,s)$ for $x \in \mathbb{R}^N$, $s \in [-\tau, 0]$, if*

$$\begin{cases} u_t^+(x,t) \geq d_1 \Delta u^+(x,t) + \alpha_1 u^+(x,t) + \beta_1 v^+(x, t - \tau_1), \\ v_t^+(x,t) \geq d_2 \Delta v^+(x,t) + \alpha_2 v^+(x,t) + \beta_2 u^+(x, t - \tau_2), \end{cases} \tag{3.14}$$

and

$$\begin{cases} u_t^-(x,t) \leq d_1 \Delta u^-(x,t) + \alpha_1 u^-(x,t) + \beta_1 v^-(x, t - \tau_1), \\ v_t^-(x,t) \leq d_2 \Delta v^-(x,t) + \alpha_2 v^-(x,t) + \beta_2 u^-(x, t - \tau_2), \end{cases} \tag{3.15}$$

for $x \in \mathbb{R}^N$ and $t > 0$, then $Z^+(x,t) \geq Z^-(x,t)$ for all $x \in \mathbb{R}^N$ and $t \geq 0$.

To obtain another comparison theorem which will be used to construct upper estimates, we need the concavity assumption of the functions g and h, technically.

(P5) Assume that $(u_i, v_i) \in [\mathbf{0}, \mathbf{K}]$, for $i = 1, \cdots m \in \mathbb{Z}^+$, then

$$L_1 \min \big\{k_1, \sum_{i=1}^m u_i\big\} + h\big(\min\{k_1, \sum_{i=1}^m u_i\}, \min\{k_2, \sum_{i=1}^m v_i\}\big) \leq \sum_{i=1}^m \big\{L_1 u_i + h(u_i, v_i)\big\}$$

and

$$L_2 \min \big\{k_2, \sum_{i=1}^m v_i\big\} + g\big(\min\{k_1, \sum_{i=1}^m u_i\}, \min\{k_2, \sum_{i=1}^m v_i\}\big) \leq \sum_{i=1}^m \big\{L_2 v_i + g(u_i, v_i)\big\}.$$

It is easy to see that (P5) is a general condition, but also holds for various models, see [20, 39, 42] for some special cases.

Lemma 3.9. *Assume that* (P1)–(P5) *hold. Let Z_i^0 $(i = 1, \cdots, m)$, $Z^0 \in \mathcal{C}_{[\mathbf{0},\mathbf{K}]}$ be $m + 1$ given functions with*

$$Z^0(x,s) \leq \min \Big\{ \mathbf{K}, \ \sum_{i=1}^m Z_i^0(x,s) \Big\}, \quad x \in \mathbb{R}^N, \ s \in [-\tau, 0], \ \ where \ m \in \mathbb{Z}^+.$$

Let Z_i $(i = 1, \cdots, m)$ *and* Z *be the solutions of the Cauchy problems of* (1.1) *with the initial values*

$$Z_i(x, s) = Z_i^0(x, s), \;\; x \in \mathbb{R}^N, \;\; s \in [-\tau, 0]$$

and

$$Z(x, s) = Z^0(x, s), \;\; x \in \mathbb{R}^N, \;\; s \in [-\tau, 0],$$

respectively. Then

$$0 \le Z(x, t) \le \min\Big\{\mathbf{K}, \; \sum_{i=1}^{m} Z_i(x, t)\Big\} \;\; \textit{for all } x \in \mathbb{R}^N \textit{ and } t \ge 0.$$

Proof. Set $W(x, t) = (W_1(x, t), W_2(x, t)) := \min\Big\{\mathbf{K}, \sum_{i=1}^{m} Z_i(x, t)\Big\}$, then $Z(x, s) \le W(x, s)$ for $x \in \mathbb{R}^N$ and $s \in [-\tau, 0]$. By Lemma 3.6, it need to show that $W(t)(\cdot) = W(\cdot, t) \in C([-\tau, +\infty), X_{[\mathbf{0},\mathbf{K}]})$ is an upper solution of (1.1), i.e.,

$$T(t - r)[W(r)](x) + \int_r^t T(t - \varrho)[F(W_\varrho)](x)d\varrho \le W(t)(x) \tag{3.16}$$

for $0 \le r < t < +\infty$. Since $\beta_1 \ge 0, \;\; \beta_2 \ge 0$, it holds

$$T(t - r)[W(r)](x) + \int_r^t T(t - \varrho)[F(W_\varrho)](x)d\varrho \le \mathbf{K} \tag{3.17}$$

for $0 \le r < t < +\infty$. We only need to show that

$$T(t - r)[W(r)](x) + \int_r^t T(t - \varrho)[F(W_\varrho)](x)d\varrho \le \sum_{i}^{m} Z_i(t)(x) \tag{3.18}$$

for $0 \le r < t < +\infty$, where $Z_i(t)(\cdot) = Z_i(\cdot, t) \in C([-\tau, +\infty), X_{[\mathbf{0},\mathbf{K}]})$. By (P5), we have

$$\begin{aligned}
&F_1(W_\varrho)(x) = L_1 W_1(x, \varrho) + h(W_1(x, \varrho), W_2(x, \varrho - \tau_1)) \\
&= L_1 \min\Big\{k_1, \sum_{i=1}^{m} u_i(x, \varrho)\Big\} + h\Big(\min\Big\{k_1, \sum_{i=1}^{m} u_i(x, \varrho)\Big\}, \min\Big\{k_2, \sum_{i=1}^{m} v_i(x, \varrho - \tau_1)\Big\}\Big) \\
&\le \sum_{i=1}^{m} \Big\{L_1 u_i(x, \varrho) + h(u_i(x, \varrho), v_i(x, \varrho - \tau_1))\Big\} \\
&= \sum_{i=1}^{m} F_1((Z_i)_\varrho)(x).
\end{aligned}$$

Similarly, we obtain

$$F_2(W_\varrho)(x) = L_2 W_2(x, \varrho) + g(W_1(x, \varrho - \tau_2), W_2(x, \varrho)) \le \sum_{i=1}^{m} F_2((Z_i)_\varrho)(x).$$

And we easily see that

$$T(t - r)[Z_i(r)](x) + \int_r^t T(t - \varrho)[F((Z_i)_\varrho)](x)d\varrho = Z_i(t)(x), \;\; i = 1, 2, \cdots, m$$

for $0 \le r < t < +\infty$, then

$$
\begin{aligned}
& T(t-r)[W(r)](x) + \int_r^t T(t-\varrho)[F(W_\varrho)](x)d\varrho \\
\le & \sum_{i=1}^m \Big\{ T(t-r)[Z_i(r)](x) + \int_r^t T(t-\varrho)[F((Z_i)\varrho)](x)d\varrho \Big\} \\
= & \sum_{i=1}^m Z_i(t)(x),
\end{aligned}
$$

for $0 \le r < t < +\infty$, which implies that (3.18) holds. Therefore, (3.16) is established, i.e., $W(x,t)$ is an upper solution of (1.1). □

3.3. Existence of entire solutions in the quasi-monotone case

In this subsection, we are ready to prove the existence result of entire solutions by using the properties of previous subsections to obtain some appropriate upper estimates for solutions of (1.1) which are inspired by the works [39, 42].

For any $n \in \mathbb{N}$, $\gamma \in \mathbb{N} \cup \{0\}$, $\nu_1, \cdots, \nu_\gamma \in \mathbb{S}^{N-1}$, $m_1, \cdots, m_{\gamma+1} \in \mathbb{R}$, $c_1, \cdots, c_\gamma \ge c_*$ and $\chi \in \{0, 1\}$ with $\gamma + \chi \ge 2$, we denote

$$
\begin{aligned}
z^n(x,s) &:= \max\Big\{ \max_{i=1,\cdots,\gamma} \Phi_{c_i}(x\cdot\nu_i + c_i s + m_i),\ \chi\Gamma(s+m_{\gamma+1}) \Big\}, \\
\underline{\mathcal{Z}}(x,t) &:= \max\Big\{ \max_{i=1,\cdots,\gamma} \Phi_{c_i}(x\cdot\nu_i + c_i t + m_i),\ \chi\Gamma(t+m_{\gamma+1}) \Big\},
\end{aligned}
$$

where $x \in \mathbb{R}^N$, $s \in [-n-\tau, -n]$ and $t > -n$. Let $\mathcal{Z}^n(x,t) := (u^n(x,t), v^n(x,t))$ be the unique solution of the initial value problem

$$
\begin{cases}
u_t = d_1\Delta u(x,t) + h(u(x,t), v(x,t-\tau_1)), \\
v_t = d_2\Delta v(x,t) + g(u(x,t-\tau_2), v(x,t)), \\
(u(x,s), v(x,s)) = z^n(x,s),
\end{cases}
\tag{3.19}
$$

for $x \in \mathbb{R}^N$, $s \in [-n-\tau, -n]$ and $t > -n$. From Lemma 3.6, we have

$$
\underline{\mathcal{Z}}(x,t) \le \mathcal{Z}^n(x,t) \le \mathbf{K} \ \text{ for } \ x \in \mathbb{R}^N,\ t \ge -n-\tau.
$$

The following results provide the appropriate upper estimates of $\mathcal{Z}^n(x,t)$.

Lemma 3.10. *Assume that* (P1)–(P5) *hold. The unique solution* $\mathcal{Z}^n(x,t)$ *of* (3.19) *satisfies*

$$
\mathcal{Z}^n(x,t) \le \mathcal{Z}^+(x,t) := \min\{\mathbf{K}, \mathcal{Q}(x,t)\} \ \textit{for}\ x \in \mathbb{R}^N,\ t \ge -n-\tau,
$$

where $\mathcal{Q}(x,t) := \sum_{i=1}^\gamma \Phi_{c_i}(x\cdot\nu_i + c_i t + m_i) + \chi\Gamma(t+m_{\gamma+1})$.

Proof. According to Lemma 3.9, it is easy to see that the conclusion holds. □

Lemma 3.11. *Assume that* (P1)–(P4) *hold. If* $c_1, \cdots, c_\gamma > c_*$, *then the unique solution* $\mathcal{Z}^n(x,t)$ *of* (3.19) *satisfies*

$$
\mathcal{Z}^n(x,t) \le \overline{\mathcal{Z}}(x,t) := \min\{\mathbf{K},\ \mathcal{Q}_1(x,t),\ \mathcal{Q}_2(x,t)\} \ \textit{for}\ x \in \mathbb{R}^N,\ t \ge -n-\tau,
$$

where

$$\mathcal{Q}_1(x,t) := \min_{1\le i_0\le\gamma} \Big\{ \Phi_{c_{i_0}}(x\cdot\nu_{i_0} + c_{i_0}t + m_{i_0}) + \chi(1,b_*)e^{\lambda_*(t+m_{\gamma+1})} + \sum_{1\le i\le\gamma, i\ne i_0} (1,b(c_i))e^{\lambda_1(c_i)(x\cdot\nu_i + c_it+m_i)} \Big\},$$

$$\mathcal{Q}_2(x,t) := \sum_{1\le i\le\gamma} (1,b(c_i))e^{\lambda_1(c_i)(x\cdot\nu_i + c_it+m_i)} + \chi\Gamma(t+m_{\gamma+1}).$$

Proof. For $x\in\mathbb{R}^N$ and $t\ge -n-\tau$, we know $\mathcal{Z}^n(x,t)\le\mathbf{K}$. Then, we only need to prove that $\mathcal{Z}^n(x,t)\le\mathcal{Q}_i(x,t)$ $(i=1,2)$ for $x\in\mathbb{R}^N$ and $t\ge -n-\tau$. We only prove $\mathcal{Z}^n(x,t)\le\mathcal{Q}_1(x,t)$, since $\mathcal{Z}^n(x,t)\le\mathcal{Q}_2(x,t)$ can be proved, similarly.

Given $i_0\in\{1,2,\cdots,\gamma\}$, set

$$w^n(x,t) = (w_1^n(x,t), w_2^n(x,t)) := \mathcal{Z}^n(x,t) - \Phi_{c_{i_0}}(x\cdot\nu_{i_0} + c_{i_0}t + m_{i_0}).$$

Then $\mathbf{0}\le w^n(x,t)\le\mathbf{K}$ for $x\in\mathbb{R}^N$ and $t\ge -n-\tau$. By (P1), it holds

$$\begin{cases} \partial_t w_1^n \le d_1\Delta w_1^n(x,t) + \alpha_1 w_1^n(x,t) + \beta_1 w_2^n(x,t-\tau_1), \\ \partial_t w_2^n \le d_2\Delta w_2^n(x,t) + \alpha_2 w_2^n(x,t) + \beta_2 w_1^n(x,t-\tau_2), \\ w^n(x,s) := \mathcal{Z}^n(x,s) - \Phi_{c_{i_0}}(x\cdot\nu_{i_0} + c_{i_0}s + h_{i_0}), \end{cases} \tag{3.20}$$

where $x\in\mathbb{R}^N,\ t>-n,\ s\in[-n-\tau,-n]$. Let

$$\begin{aligned} V(x,t) =& (V_1(x,t), V_2(x,t)) \\ :=& \sum_{1\le i\le\gamma, i\ne i_0} (1,b(c_i))e^{\lambda_1(c_i)(x\cdot\nu_i + c_it+m_i)} + \chi(1,b_*)e^{\lambda_*(t+m_{\gamma+1})}, \end{aligned}$$

where $b_* = \beta_2 e^{-\lambda_*\tau_2}/(\lambda_*-\alpha_2)$ and

$$b(c) = -\beta_2 e^{-c\lambda_1(c)\tau_2}/(d_2\lambda_1^2(c) - c\lambda_1(c) + \alpha_2).$$

It is easy to show that

$$\begin{cases} \partial_t V_1 = d_1\Delta V_1(x,t) + \alpha_1 V_1(x,t) + \beta_1 V_2(x,t-\tau_1), \\ \partial_t V_2 = d_2\Delta V_2(x,t) + \alpha_2 V_2(x,t) + \beta_2 V_1(x,t-\tau_2), \end{cases}$$

where $x\in\mathbb{R}^N,\ t>-n$. According to Theorems 3.1 and 3.2, we have

$$\begin{aligned} w^n(x,s) =& \mathcal{Z}^n(x,s) - \Phi_{c_{i_0}}(x\cdot\nu_{i_0} + c_{i_0}s + h_{i_0}) \\ \le& \sum_{1\le i\le\gamma, i\ne i_0} \Phi_{c_i}(x\cdot\nu_i + c_is + m_i) + \chi\Gamma(s+m_{\gamma+1}) \\ \le& \sum_{1\le i\le\gamma, i\ne i_0} (1,b(c_i))e^{\lambda_1(c_i)(x\cdot\nu_i + c_is+m_i)} + \chi(1,b_*)e^{\lambda_*(s+m_{\gamma+1})} \\ =& V(x,s) \end{aligned}$$

for $x\in\mathbb{R}^N,\ s\in[-n-\tau,-n]$. According to Lemma 3.8, it holds $w^n(x,t)\le V(x,t)$ for all $x\in\mathbb{R}^N,\ t\ge -n-\tau$, i.e.,

$$\mathcal{Z}^n(x,t) \le \Phi_{c_{i_0}}(x\cdot\nu_{i_0} + c_{i_0}t + m_{i_0}) + \chi(1,b_*)e^{\lambda_*(t+m_{\gamma+1})}$$

$$+\sum_{1\le i\le\gamma, i\ne i_0}(1,b(c_i))e^{\lambda_1(c_i)(x\cdot\nu_i+c_it+m_i)}.$$

By the arbitrariness of i_0, we can get $\mathcal{Z}^n(x,t)\le\mathcal{Q}_1(x,t)$ for all $x\in\mathbb{R}^N$ and $t\ge -n-\tau$. This completes the proof. □

3.3.1. Proof of Theorem 2.1

Proof. Let $F_1(\phi_1,\phi_{2\tau_1})(\xi)=L_1\phi_1(\xi)+h(\phi_1(\xi),\phi_2(\xi-c\tau_1))$ and $F_2(\phi_{1\tau_2},\phi_2)(\xi)=L_2\phi_2(\xi)+g(\phi_1(\xi-c\tau_2),\phi_2(\xi))$, then we have

$$\begin{cases}d_1\phi_1''(\xi)-c\phi_1'(\xi)-L_1\phi_1(\xi)+F_1(\phi_1,\phi_{2\tau_1})(\xi)=0,\\ d_2\psi_2''(\xi)-c\phi_2'(\xi)-L_2\phi_2(\xi)+F_2(\phi_{1\tau_2},\phi_2)(\xi)=0.\end{cases}\tag{3.21}$$

In view of $(\phi_1(\xi),\phi_2(\xi))\in[\mathbf{0},\mathbf{K}]$ and $\partial_2h(u,v)\ge0,\ \partial_1g(u,v)\ge0$ for $(u,v)\in[\mathbf{0},\mathbf{K}]$, we have

$$F_1(\phi_1,\phi_{2\tau_1})(\xi)\ge L_1\phi_1(\xi)+h(\phi_1(\xi),0)\ge0,$$
$$F_2(\phi_{1\tau_2},\phi_2)(\xi)\ge L_2\phi_2(\xi)+g(0,\phi_2(\xi))\ge0,$$

for $\xi\in\mathbb{R}$. By the theory of the ordinary differential equation, we can obtain

$$\begin{cases}\phi_1(\xi)=\frac{1}{\lambda_1^+-\lambda_1^-}[\int_{-\infty}^{\xi}e^{\lambda_1^-(\xi-s)}F_1(\phi_1,\phi_{2\tau_1})(s)ds+\int_{\xi}^{+\infty}e^{\lambda_1^+(\xi-s)}F_1(\phi_1,\phi_{2\tau_1})(s)ds],\\ \phi_2(\xi)=\frac{1}{\lambda_2^+-\lambda_2^-}[\int_{-\infty}^{\xi}e^{\lambda_2^-(\xi-s)}F_2(\phi_{1\tau_2},\phi_2)(s)ds+\int_{\xi}^{+\infty}e^{\lambda_2^+(\xi-s)}F_2(\phi_{1\tau_2},\phi_2)(s)ds].\end{cases}\tag{3.22}$$

It is easy to see that

$$\begin{aligned}0\le&\phi_1'(\xi)\\ =&\frac{1}{\lambda_1^+-\lambda_1^-}\Big\{\int_{-\infty}^{\xi}\lambda_1^-e^{\lambda_1^-(\xi-s)}F_1(\phi_1,\phi_{2\tau_1})(s)ds\\ &+\int_{\xi}^{+\infty}\lambda_1^+e^{\lambda_1^+(\xi-s)}F_1(\phi_1,\phi_{2\tau_1})(s)ds\Big\}\\ =&\frac{d_1}{\sqrt{c^2-4d_1\alpha_1}}\Big\{\int_{-\infty}^{\xi}\lambda_1^-e^{\lambda_1^-(\xi-s)}F_1(\phi_1,\phi_{2\tau_1})(s)ds\\ &+\int_{\xi}^{+\infty}\lambda_1^+e^{\lambda_1^+(\xi-s)}F_1(\phi_1,\phi_{2\tau_1})(s)ds\Big\}\\ \le&\frac{2d_1}{\sqrt{c^2-4d_1\alpha_1}}\Big\{L_1k_1+\max_{(u,v)\in[\mathbf{0},\mathbf{K}]}|h(u,v)|\Big\}\le M_0,\ \ \forall\xi\in\mathbb{R},\end{aligned}$$

where M_0 is a positive constant. Similarly, $0\le\phi_2'(\xi)\le M_0$ can be obtained. And it is easy to verify that there exists a constant $M>0$ such that

$$\sup_{x\in\mathbb{R}^N}|z_i^n(x+\ell,0)-z_i^n(x,0)|\le M\ell,\ i=1,2,\ \text{ for any }\ \ell>0.$$

We know that $\mathcal{Z}^n(x,t)=(u^n(x,t),v^n(x,t))$ is the unique solution of the initial problem of (3.19). By Lemma 3.6, we have

$$\underline{\mathcal{Z}}(x,t)\le\mathcal{Z}^n(x,t)\le\mathcal{Z}^{n+1}(x,t)\le\mathbf{K}\tag{3.23}$$

for all $x \in \mathbb{R}^N$ and $t \geq -n$. From a priori estimate of Lemma 3.7, there exists a positive constant C (independent of n) such that for any $\eta > 0$, $x \in \mathbb{R}^N$ and $t > -n + 3(\tau + 1)$,

$$\begin{aligned}&\| \partial_t \mathcal{Z}^n(x,t) \|, \ \| \partial_{tx_i} \mathcal{Z}^n(x,t) \|, \ \| \partial_{tt} \mathcal{Z}^n(x,t) \|, \ \| \partial_{x_i} \mathcal{Z}^n(x,t) \|, \ \| \partial_{x_i t} \mathcal{Z}^n(x,t) \|,\\ &\| \partial_{x_i x_j} \mathcal{Z}^n(x,t) \|, \ \| \partial_{x_i^2 t} \mathcal{Z}^n(x,t) \|, \ \| \partial_{x_i^2 x_j} \mathcal{Z}^n(x,t) \| \leq C, \ \ \forall i, j = 1, \cdots, N.\end{aligned}$$

Then there exist a function $\mathcal{Z}_\eta(x,t)$ satisfying $\mathbf{0} \leq \mathcal{Z}_\eta(x,t) \leq \mathbf{K}$ and a subsequence $\{\mathcal{Z}^{n_k}(x,t)\}_{k\in\mathbb{N}}$ of $\{\mathcal{Z}^n(x,t)\}_{n\in\mathbb{N}}$ (by a diagonal extraction process) such that $\{\mathcal{Z}^{n_k}(x,t)\}_{k\in\mathbb{N}}$, $\{\mathcal{Z}_t^{n_k}(x,t)\}_{k\in\mathbb{N}}$, $\{\Delta\mathcal{Z}^{n_k}(x,t)\}_{k\in\mathbb{N}}$ converge uniformly in any compact set $\mathbb{G} \subset \mathbb{R}^N$ to

$$\mathcal{Z}_\eta(x,t), \ \ \frac{\partial}{\partial t}\mathcal{Z}_\eta(x,t), \ \ \Delta\mathcal{Z}_\eta(x,t)$$

respectively. Since $\mathcal{Z}^n(x,t) \leq \mathcal{Z}^{n+1}(x,t)$ for $(x,t) \in \mathbb{R}^N \times [-n, +\infty)$, we have $\lim_{n\to\infty} \mathcal{Z}^n(x,t) = \mathcal{Z}_\eta(x,t)$. Obviously, $\mathcal{Z}_\eta(x,t)$ is an entire solution of (1.1). In particular, by (3.23), (2.3) holds. Moveover, by Lemmas 3.10 and 3.11, the assertions of (i) and (ii) hold. □

Remark 3.2. The entire solutions obtained in Theorem 2.1 contain the traveling wave fronts $\Phi(\xi)$ and the spatial independent solution $\Gamma(t)$ when $\gamma + \chi \geq 1$.

3.4. Qualitative properties of the entire solutions

In the previous subsection, some new types of entire solutions of (1.1) were constructed by considering a combination of any finite number of traveling waves with speeds $c \geq c_*$ and propagation directions and a spatial independent solution. In this subsection, we further investigate some qualitative properties of the entire solutions, i.e. we will prove Theorems 2.2 and 2.3.

3.4.1. Proof of Theorem 2.2.

Proof. (i) We prove $\mathcal{Z}_\eta(x,t) > \mathbf{0}$ for all $(x,t) \in \mathbb{R}^{N+1}$. Since

$$\mathcal{Z}^n(x,t) \geq \underline{\mathcal{Z}}(x,t) \geq \underline{\mathcal{Z}}(x,s) = z^n(x,s) = \mathcal{Z}^n(x,s)$$

for $x \in \mathbb{R}^N$, $s \in [-n-\tau, -n]$ and $t > -n$. Then, it follows from Lemma 3.6 that $\frac{\partial}{\partial t}\mathcal{Z}^n(x,t) \geq \mathbf{0}$ for $(x,t) \in \mathbb{R}^N \times [-n, +\infty)$. Therefore, $\frac{\partial}{\partial t}\mathcal{Z}_\eta(x,t) \geq \mathbf{0}$ for all $(x,t) \in \mathbb{R}^{N+1}$. First, we prove $\frac{\partial}{\partial t}U_\eta(x,t) > 0$ for all $(x,t) \in \mathbb{R}^{N+1}$. Note that

$$\begin{aligned}\frac{\partial^2}{\partial t^2}U_\eta =& d_1\Delta(U_\eta)_t + \partial_1 h(U_\eta, V_\eta(x, t-\tau_1))(U_\eta)_t(x,t)\\ &+ \partial_2 h(U_\eta, V_\eta(x,t-\tau_1))(V_\eta)_t(x, t-\tau_1)\\ \geq& d_1\Delta(U_\eta)_t + K_h(U_\eta)_t(x,t),\end{aligned}$$

where $x \in \mathbb{R}^N$ and $K_h := \min_{(u,v)\in[\mathbf{0},\mathbf{K}]} \partial_1 h(U_\eta, V_\eta(x, t-\tau_1)) < 0$. For any given $r \in \mathbb{R}$, it holds

$$(U_\eta)_t(x,t) \geq \int_{\mathbb{R}^N} \mathrm{K}_1(x-y, t-r)(U_\eta)_t(y,r)dy \geq 0, \ \ x \in \mathbb{R}^N, \ t > r, \tag{3.24}$$

where $\mathrm{K}_1(x,t) = \frac{1}{(4d_1\pi t)^{N/2}}\exp\{-\frac{\|x\|^2}{4d_1t} + K_h t\}$. Suppose to the contrary that there exists $(x_0,t_0) \in \mathbb{R}^{N+1}$ such that $(U_\eta)_t(x_0,t_0) = 0$, then $\int_{\mathbb{R}^N} \mathrm{K}_1(x_0-y,t_0-r)(U_\eta)_t(y,r)dy = 0$, which implies that $(U_\eta)_t(x_0,r) = 0$ for all $r \leq t_0$. Hence, $\lim_{t\to-\infty}(U_\eta)(x_0,t) = (U_\eta)(x_0,t_0)$. But following from (2.3) and the facts (i) and (ii) of the Theorem 2.1, it holds

$$\lim_{t\to-\infty}(U_\eta)(x_0,t) = 0, \ \ (U_\eta)(x_0,t_0) > 0.$$

This contradiction yields that $\frac{\partial}{\partial t}U_\eta(x,t) > 0$ for $(x,t) \in \mathbb{R}^{N+1}$. Similarly, we can show that $\frac{\partial}{\partial t}V_\eta(x,t) > 0$ for $(x,t) \in \mathbb{R}^{N+1}$. Therefore, we have $\frac{\partial}{\partial t}\mathcal{Z}_\eta(x,t) > \mathbf{0}$ for $(x,t) \in \mathbb{R}^{N+1}$.

Since $\frac{\partial}{\partial t}\mathcal{Z}_\eta(x,t) > \mathbf{0}$ for all $(x,t) \in \mathbb{R}^{N+1}$, we have $\mathcal{Z}_\eta(x,t) < \mathbf{K}$ for all $(x,t) \in \mathbb{R}^{N+1}$ directly.

(ii) When $\chi = 1$, then

$$\underline{\mathcal{Z}}(x,t) := \max\Big\{\max_{i=1,\cdots,\gamma}\Phi_{c_i}(x\cdot\nu_i + c_it + m_i), \Gamma(t+m_{\gamma+1})\Big\} \to \mathbf{K}$$

as $t \to +\infty$, $\forall x \in \mathbb{R}^N$, the expression (2.3) and the squeezing argument implies

$$\lim_{t\to+\infty}\sup_{x\in\mathbb{R}^N}\|\mathcal{Z}_{\eta_{\gamma,1}}(x,t) - \mathbf{K}\| = 0.$$

When $\chi = 0$, then

$$\underline{\mathcal{Z}}(x,t) := \max_{i=1,\cdots,\gamma}\Phi_{c_i}(x\cdot\nu_i + c_it + m_i) \to \mathbf{K}$$

is established as $t \to +\infty$, $\|x\| \leq A$. Similarly, it holds

$$\lim_{t\to+\infty}\sup_{\|x\|\leq A}\|\mathcal{Z}_{\eta_{\gamma,0}}(x,t) - \mathbf{K}\| = 0.$$

If (P5) holds, then

$$\mathcal{Z}^+(x,t) := \min\Big\{\mathbf{K}, \sum_{i=1}^{\gamma}\Phi_{c_i}(x\cdot\nu_i + c_it + m_i) + \chi\Gamma(t+m_{\gamma+1})\Big\} \to \mathbf{0}$$

as $t \to -\infty$, $\|x\| \leq A$ for $\chi = 0$ or $\chi = 1$. According to the (i) of the Theorem 2.1 and the squeezing argument, we have

$$\lim_{t\to-\infty}\sup_{\|x\|\leq A}\|\mathcal{Z}_\eta(x,t)\| = 0.$$

(iii) The assertion is clearly established, we omit the details of the proof.

(iv) Since

$$\underline{\mathcal{Z}}(x,t) := \max\Big\{\max_{i=1,\cdots,\gamma}\Phi_{c_i}(x\cdot\nu_i + c_it + m_i), \chi\Gamma(t+m_{\gamma+1})\Big\} \to \mathbf{K}$$

as $m_i \to +\infty$, when $x\cdot\nu_i + c_it \geq A_0$ (A_0 is an arbitrary constant). Since $(x,t) \in \Theta^i_{A,T}$ for $A, T \in \mathbb{R}$, then $x\cdot\nu_i + c_it \geq A + c_iT := A_0$. Therefore, whenever $\chi = 0$ or $\chi = 1$, the expression (2.3) and the squeezing argument imply $\mathcal{Z}_\eta(x,t) \to \mathbf{K}$. □

3.4.2. Proof of Theorem 2.3.

Proof. (i) We only prove statement (a), since the others can be proved similarly. According to (i) of Theorem 2.1, it holds

$$\begin{aligned}\mathcal{Z}_\eta(x-ct\nu,t) &\geq \underline{\mathcal{Z}}(x-ct\nu,t)\\ &\geq \max_{i=1,\cdots,\gamma}\Phi_{c_i}((x-ct\nu)\nu_i+c_it+m_i)\\ &:= \Phi_{c_{i_0}}(x\cdot\nu_{i_0}+(c_{i_0}-c\nu\cdot\nu_{i_0})t+m_{i_0}).\end{aligned}$$

Thus,

$$\begin{aligned}\mathbf{0} &\leq \mathcal{Z}_\eta(x-ct\nu,t)-\Phi_{c_{i_0}}((x-ct\nu)\nu_{i_0}+c_{i_0}t+m_{i_0})\\ &= \mathcal{Z}_\eta(x-ct\nu,t)-\Phi_{c_{i_0}}(x\cdot\nu_{i_0}+(c_{i_0}-c\nu\cdot\nu_{i_0})t+m_{i_0})\\ &\leq \sum_{1\leq i\leq\gamma, i\neq i_0}\Phi_{c_i}(x\cdot\nu_i+(c_i-c\nu\cdot\nu_i)t+m_i)+\chi\Gamma(t+m_{\gamma+1})\\ &\to \mathbf{0} \quad \text{as} \quad t\to-\infty,\end{aligned}$$

from the statement of (i), it's easy to know $\mathcal{Z}_\eta(x-ct\nu,t)\to\Phi_{c_{i_0}}(x\cdot\nu_{i_0}+m_{i_0})$ locally in x as $t\to-\infty$, and by Lemma 3.7, we know that the convergence also takes place in τ, then the statement of (a) is established. Similarly, we can prove (b)–(c) of (i) and the assertion of (ii).

(iii) When $c_1,\cdots,c_\gamma>c_*$, the assertion of part (ii) of Theorem 2.1 implies

$$\begin{aligned}\chi\Gamma(t+m_{\gamma+1}) &\leq \mathcal{Z}_{\eta_{\gamma,\chi}}(x,t)\\ &\leq \sum_{1\leq i\leq\gamma}(1,b(c_i))e^{\lambda_1(c_i)(x\cdot\nu_i+c_it+m_i)}+\chi(1,b_*)e^{\lambda_*(t+m_{\gamma+1})}. \quad (3.25)\end{aligned}$$

According to $\lim_{t\to-\infty}\Gamma(t)e^{-\lambda_*t}=(1,b_*)$, in order to prove the first part of this statement, it suffices to prove that $c\lambda_1(c)>\lambda_*$. Suppose to the contrary that there exists $c_0>c_*$ such that $c_0\lambda_1(c_0)\leq\lambda_*$. Note that $c_0\lambda_1(c_0)-\alpha_2-d_2\lambda_1^2(c_0)>0$. Then we have

$$\begin{aligned}0 &\geq c_0\lambda_1(c_0)-\lambda_*\\ &= d_1\lambda_1^2(c_0)+\alpha_1+\frac{\beta_1\beta_2e^{-c_0\lambda_1(c_0)(\tau_1+\tau_2)}}{c_0\lambda_1(c_0)-\alpha_2-d_2\lambda_1^2(c_0)}-\lambda_*\\ &> -(\lambda_*-\alpha_1)+\frac{\beta_1\beta_2e^{-\lambda_*(\tau_1+\tau_2)}}{\lambda_*-\alpha_2}\\ &= 0.\end{aligned}$$

This contradiction shows that $c\lambda_1(c)>\lambda_*$ for any $c>c_*$. Since $\lim_{t\to-\infty}\Gamma(t)e^{-\lambda_*t}=(1,b_*)$ and $c\lambda_1(c)>\lambda_*$ for any $c>c_*$, this statement obviously holds.

(iv) Recall that $\mathcal{Z}^n(x,t)$ is the unique solution of the initial value problem (3.19). For $\chi=1$, let us give simple marks $z^n(x,s)$ by $z^n_{\eta_{\gamma,1}}(x,s)$ and $\mathcal{Z}^n(x,t)$ by $\mathcal{Z}^n_{\eta_{\gamma,1}}(x,t)$, respectively. Similarly, for $\chi=0$, we denote $z^n(x,s)$ and $\mathcal{Z}^n(x,t)$ by $z^n_{\eta_{\gamma,0}}(x,s)$ and $\mathcal{Z}^n_{\eta_{\gamma,0}}(x,t)$, respectively. Let

$$W^n(x,t)=(W_1^n(x,t),W_2^n(x,t)):=\mathcal{Z}^n_{\eta_{\gamma,1}}(x,t)-\mathcal{Z}^n_{\eta_{\gamma,0}}(x,t)$$

for all $(x,t) \in \mathbb{R}^N \times [-n,+\infty)$, we easily see that $\mathbf{0} \le W^n(x,t) \le \mathbf{K}$ for all $(x,t) \in \mathbb{R}^N \times [-n,+\infty)$. By (P1), we have

$$\begin{aligned}\partial_t W_1^n =& d_1 \Delta W_1^n(x,t) + h(W_1^n(x,t), W_2^n(x,t-\tau_1)) \\ \le& d_1 \Delta W_1^n(x,t) + \alpha_1 W_1^n(x,t) + \beta_1 W_2^n(x,t-\tau_1)\end{aligned}$$

and

$$\begin{aligned}\partial_t W_2^n =& d_2 \Delta W_2^n(x,t) + g(W_1^n(x,t-\tau_2), W_2^n(x,t)) \\ \le& d_2 \Delta W_2^n(x,t) + \alpha_2 W_2^n(x,t) + \beta_2 W_1^n(x,t-\tau_2).\end{aligned}$$

Define the function

$$\widehat{W}(x,t) = (\widehat{W}_1(x,t), \widehat{W}_2(x,t)) := (1,b_*)e^{\lambda_*(t+m_{\gamma+1})},\ (x,t) \in \mathbb{R}^{N+1}.$$

By Theorem 3.2, we have

$$W^n(x,s) = \mathcal{Z}^n_{\eta_{\gamma,1}}(x,s) - \mathcal{Z}^n_{\eta_{\gamma,0}}(x,s) \le \Gamma(s+h) \le (1,b_*)e^{\lambda_*(s+h_{\gamma+1})} = \widehat{W}(x,s)$$

for $x \in \mathbb{R}^N$ and $s \in [-n-\tau,-n]$. Moreover, $\widehat{W}(x,t)$ satisfies the linear system

$$\begin{cases}\partial_t \widehat{W}_1 = d_1 \Delta \widehat{W}_1(x,t) + \alpha_1 \widehat{W}_1(x,t) + \beta_1 \widehat{W}_2(x,t-\tau_1), \\ \partial_t \widehat{W}_2 = d_2 \Delta \widehat{W}_2(x,t) + \alpha_2 \widehat{W}_2(x,t) + \beta_2 \widehat{W}_1(x,t-\tau_2).\end{cases}$$

Then, it follows from Lemma 3.8 that

$$0 \le W^n(x,t) = \mathcal{Z}^n_{\eta_{\gamma,1}}(x,t) - \mathcal{Z}^n_{\eta_{\gamma,0}}(x,t) \le \widehat{W}(x,t) = (1,b_*)e^{\lambda_*(t+m_{\gamma+1})}$$

for all $(x,t) \in \mathbb{R}^N \times [-n,+\infty)$. Since $\lim_{n\to+\infty} \mathcal{Z}^n_{\eta_{\gamma,\chi}}(x,t) = \lim_{n\to+\infty} W^n(x,t) = \mathcal{Z}_{\eta_{\gamma,\chi}}(x,t)$, it holds

$$0 \le \mathcal{Z}_{\eta_{\gamma,1}}(x,t) - \mathcal{Z}_{\eta_{\gamma,0}}(x,t) \le (1,b_*)e^{\lambda_*(t+m_{\gamma+1})} \text{ for all } (x,t) \in \mathbb{R}^{N+1},$$

which implies that $\mathcal{Z}_{\eta_{\gamma,1}}(x,t) \to \mathcal{Z}_{\eta_{\gamma,0}}(x,t)$ as $m_{\gamma+1} \to -\infty$ uniformly on $(x,t) \in \widetilde{\Theta}^{\gamma+1}_{A,T}$ for any $A, T \in \mathbb{R}$, for any subsequence $m^\ell_{\gamma+1}$ with $m^\ell_{\gamma+1} \to -\infty$ as $\ell \to +\infty$, the functions $\mathcal{Z}_{\eta^\ell_{\gamma,1}}(x,t)$ ($\eta^\ell_{\gamma,1} := (c_1, m_1, \nu_1, \cdots, c_\gamma, m_\gamma, \nu_\gamma, m^\ell_{\gamma+1})$) converge to a solution of (1.1) (up to extraction of some subsequence) in the sense of topology τ, which is $\mathcal{Z}_{\eta_{\gamma,0}}(x,t)$. The limit does not depend on the sequence $\eta^\ell_{\gamma,1}$, thus, all of the functions $\mathcal{Z}_{\eta_{\gamma,1}}(x,t)$ converge to $\mathcal{Z}_{\eta_{\gamma,0}}(x,t)$ in the sense of topology τ as $m_{\gamma+1} \to -\infty$, and this part follows.

The proof of part (v) is similar to that of part (iv). Thus, we omit it here. □

Remark 3.3. The convergence in the first statement of part (i) and (ii) of Theorem 2.3 means that only some fronts, i.e., those with small speeds, can be viewed as $t \to -\infty$, the others are being hidden. However, to see some of the traveling wave fronts as $t \to +\infty$, we need more restrictive conditions. A similar phenomenon has been observed by Hamel and Nadirashvili [13] for the Fisher-KPP equation, Wu and Hsu [39] and Wu and Ruan [42].

4. Entire solutions: non-quasimonotone case

In this section, we consider the entire solutions of (1.1) with nonquasimonotone nonlinearity. It is easy to known that the comparison principle is not applicable for nonquasimonotone systems, similar statements and results can be seen in [39, 42]. First, we introduce two auxiliary reaction-diffusion systems with quasi-monotone nonlinearities and establish a comparison theorem for the Cauchy problems of the three systems. Then, we prove the existence and qualitative properties of entire solutions using the comparison theorem.

Throughout this section, in addition to (P1), we further make the following assumptions.

(P6) There exist $\mathbf{K}^{\pm} = (k_1^{\pm}, k_2^{\pm}) > 0$ such that $h^{\pm},\ g^{\pm} \in C^2([\mathbf{0}, \mathbf{K}^{+}, \mathbb{R}])$ satisfy

- (i) $\partial_1 h^{\pm}, \partial_2 g^{\pm} \in C([\mathbf{0}, \mathbf{K}^{+}], \mathbb{R})$ and $h^{\pm}(\mathbf{K}^{\pm}) = g^{\pm}(\mathbf{K}^{\pm}) = 0$, $h^{\pm}(u,v) > 0$, $g^{\pm}(u,v) > 0$ for $(u,v) \in (\mathbf{0}, \mathbf{K}^{\pm})$, $\partial_i h^{\pm}(0,0) = \partial_i h(0,0)$ and $\partial_i g^{\pm}(0,0) = \partial_i g(0,0)$, $i = 1,2$;
- (ii) $\partial_2 h^{\pm}(u,v) \geq 0$ and $\partial_1 g^{\pm}(u,v) \geq 0$ for $(u,v) \in [\mathbf{0}, \mathbf{K}^{+}]$;
- (iii) $h^{-}(u,v) \leq h(u,v) \leq h^{+}(u,v) \leq \alpha_1 u + \beta_1 v$ and $g^{-}(u,v) \leq g(u,v) \leq g^{+}(u,v) \leq \alpha_2 v + \beta_2 u$ for $(u,v) \in [\mathbf{0}, \mathbf{K}^{+}]$;
- (iv) There exist positive constants $L_h,\ L_g > 0$ such that
$h^{\pm}(u, v_1) - h^{\pm}(u, v_2) \leq L_h \max\{0, v_1 - v_2\},\ \forall (u, v_1),\ (u, v_2) \in [\mathbf{0}, \mathbf{K}^{+}]$,
$g^{\pm}(u_1, v) - g^{\pm}(u_2, v) \leq L_g \max\{0, u_1 - u_2\},\ \forall (u_1, v),\ (u_2, v) \in [\mathbf{0}, \mathbf{K}^{+}]$.

(P7) Assume $(u_i, v_i) \in [\mathbf{0}, \mathbf{K}^{+}]$ for $i = 1, \cdots m \in \mathbb{Z}^{+}$, then

$$
\begin{aligned}
&L \min\big\{k_1^{+}, \sum_{i=1}^{m} u_i\big\} + h^{+}\big(\min\{k_1^{+}, \sum_{i=1}^{m} u_i\}, \min\{k_2^{+}, \sum_{i=1}^{m} v_i\}\big) \\
&\leq \sum_{i=1}^{m} \big\{L u_i + h^{+}(u_i, v_i)\big\}, \\
&L \min\big\{k_2^{+}, \sum_{i=1}^{m} v_i\big\} + g^{+}\big(\min\{k_1^{+}, \sum_{i=1}^{m} u_i\}, \min\{k_2^{+}, \sum_{i=1}^{m} v_i\}\big) \\
&\leq \sum_{i=1}^{m} \big\{L v_i + g^{+}(u_i, v_i)\big\}.
\end{aligned}
$$

It is easy to see that $h^{\pm} = h$, $g^{\pm} = g$ and $\mathbf{K}^{\pm} = \mathbf{K}$ if $\partial_2 h(u,v) \geq 0$ and $\partial_1 g(u,v) \geq 0$ for $(u,v) \in [\mathbf{0}, \mathbf{K}]$.

Similar to Lemma 3.6, it is easy to verify that for any $\phi \in \mathcal{C}_{[\mathbf{0},\mathbf{K}^{+}]}$, (1.1) has a unique solution $w(x,t;\phi)$ on $[0,+\infty)$ with $\mathbf{0} \leq w(x,t;\phi) \leq \mathbf{K}^{+}$ for all $x \in \mathbb{R}^N$ and $t \geq 0$. Moreover, $w(x,t;\phi)$ is a classical solution on $(\tau,+\infty)$. Here and in what follows, $X_{[\mathbf{0},\mathbf{K}^{\pm}]}$ and $\mathcal{C}_{[\mathbf{0},\mathbf{K}^{\pm}]}$ are defined as (3.9) and (3.10) by replacing $[\mathbf{0}, \mathbf{K}]$ with $[\mathbf{0}, \mathbf{K}^{\pm}]$.

According to (P6), we consider the following two auxiliary monotone delayed systems:

$$
\begin{cases}
u_t(x,t) = d_1 \Delta u(x,t) + h^{+}(u(x,t), v(x,t-\tau_1)), \\
v_t(x,t) = d_2 \Delta v(x,t) + g^{+}(u(x,t-\tau_2), v(x,t))
\end{cases}
\tag{4.1}
$$

and

$$
\begin{cases}
u_t(x,t) = d_1 \Delta u(x,t) + h^{-}(u(x,t), v(x,t-\tau_1)), \\
v_t(x,t) = d_2 \Delta v(x,t) + g^{-}(u(x,t-\tau_2), v(x,t)),
\end{cases}
\tag{4.2}
$$

where $(x,t) \in \mathbb{R}^{N+1}$. To obtain the entire solutions of (1.1) with the nonquasimonotone case, we need establish a comparison theorem for solutions of (1.1), (4.1) and (4.2).

It is easy to see that $\Delta_1(c,\lambda) = 0$ is also the characteristic equation of (4.1) and (4.2) with respect to the equilibrium $(0,0)$, since $\partial_i h^{\pm}(0,0) = \partial_i h(0,0)$ and $\partial_i g^{\pm}(0,0) = \partial_i g(0,0)$. By Theorems 3.1 and 3.2, we have the following results.

Proposition 4.1. *Assume that* (P1)–(P3) *and* (P6) *hold. For any* $c \geq c_*$ *and* $\nu \in \mathbb{R}^N$, (4.1) *and* (4.2) *have traveling wave fronts* $\Phi^+(\xi) = (\phi^+(\xi), \psi^+(\xi))$ *and* $\Phi^-(\xi) = (\phi^-(\xi), \psi^-(\xi))$ *with* $\xi = x \cdot \nu + ct$, *respectively, which satisfy* $\Phi^{\pm}(\cdot) > \mathbf{0}$, $\Phi^{\pm}(-\infty) = \mathbf{0}$ *and* $\Phi^{\pm}(+\infty) = \mathbf{K}^{\pm}$. *Moreover, if* $c > c_*$, *then*

$$\lim_{\xi \to -\infty} \Phi^{\pm}(\xi) e^{-\lambda_1(c)\xi} = (1, b(c)), \ \ \Phi^{\pm}(\xi) \leq (1, b(c)) e^{\lambda_1(c)\xi} \ \ \textit{for all } \xi \in \mathbb{R}.$$

Here, $c_*, \ \lambda_1(c), \ b(c)$ *are given as in Sec. 3.*

Proposition 4.2. *Assume that* (P1) *and* (P6) *hold. There exist solutions* $\Gamma^{\pm}(t) : \mathbb{R} \to [\mathbf{0}, \mathbf{K}^+]$ *of the following delayed differential system,*

$$\begin{cases} \Gamma_1'(t) = h^{\pm}(\Gamma_1(t), \Gamma_2(t-\tau_1)), \\ \Gamma_2'(t) = g^{\pm}(\Gamma_1(t-\tau_2), \Gamma_2(t)), \end{cases} \quad t \in \mathbb{R}, \tag{4.3}$$

which satisfy $\Gamma^{\pm}(-\infty) = \mathbf{0}$ *and* $\Gamma^{\pm}(+\infty) = \mathbf{K}^{\pm}$, $\Gamma^{\pm}(t) > \mathbf{0}$ *and*

$$\lim_{t \to -\infty} \Gamma^{\pm}(t) e^{-\lambda_* t} = (1, b_*), \ \ \Gamma^{\pm}(t) \leq (1, b_*) e^{\lambda_* t} \ \ \textit{for} \ \ t \in \mathbb{R}.$$

We first introduce some notations and definitions. Let us denote

$$L := \max_{(u,v) \in [\mathbf{0}, \mathbf{K}^+]} \max \Big\{ |\partial_1 h^+(u,v)|, |\partial_2 g^+(u,v)|, |\partial_1 h(u,v)|, |\partial_2 g(u,v)| \Big\},$$

and $\mathcal{F}(\phi) = (\mathcal{F}_1(\phi), \mathcal{F}_2(\phi)) : \mathcal{C}_{[\mathbf{0},\mathbf{K}^+]} \to X$ by

$$\begin{aligned} \mathcal{F}_1(\phi)(x) &:= L\phi_1(x,0) + h(\phi_1(x,0), \phi_2(x,-\tau_1)), \\ \mathcal{F}_2(\phi)(x) &:= L\phi_2(x,0) + g(\phi_1(x,-\tau_2), \phi_2(x,0)), \end{aligned}$$

for $\phi = (\phi_1, \phi_2) \in \mathcal{C}_{[\mathbf{0},\mathbf{K}^+]}$. Similarly, we define $F^{\pm}(\phi) := (F_1^{\pm}(\phi), F_2^{\pm}(\phi))$ by replacing (h,g) with $(h^{\pm}, g^{\pm})$, respectively. It is clear that $F^{\pm}(\cdot,\cdot)$ are nondecreasing in $\mathcal{C}_{[\mathbf{0},\mathbf{K}^+]}$ and

$$F^-(\phi_1,\phi_2) \leq \mathcal{F}(\phi_1,\phi_2) \leq F^+(\phi_1,\phi_2) \ \ \text{for} \ \ (\phi_1,\phi_2) \in \mathcal{C}_{[\mathbf{0},\mathbf{K}^+]}.$$

Furthermore, we denote the linear operate $\mathcal{T}(t) = diag(\mathcal{T}_1(t), \mathcal{T}_2(t))$ by $\mathcal{T}_i(0) = I$ and

$$\mathcal{T}_i(t)[\psi](x) := \int_{\mathbb{R}^N} \mathcal{T}_i(x-y,t)\psi(y)dy \ \ \text{for} \ \ x \in \mathbb{R}^N, \, t > 0, \, \psi \in BUC(\mathbb{R}^N, \mathbb{R}),$$

where $\mathcal{T}_i(x,t) := \frac{1}{(4\pi d_i t)^{N/2}} \exp\Big\{ -\frac{\|x\|^2}{4d_i t} - Lt \Big\}, \ \ i = 1,2.$

The following comparison principle plays an important role in the proof of our main result for the nonquasimonotone delay system.

Lemma 4.1. *Assume that* (P1) *and* (P6) *hold. Given any* $r \in \mathbb{R}$, *let* $w = (u, v)$, $w^{\pm} = (u^{\pm}, v^{\pm}) \in C([r-\tau, +\infty), X_{[\mathbf{0},\mathbf{K}^+]})$ *be such that*

$$w^-(t)(x) \le \mathcal{T}(t-r)[w^-(r)](x) + \int_r^t \mathcal{T}(t-\varrho)[F^-(w^-_\varrho)](x)d\varrho, \tag{4.4}$$

$$w(t)(x) = \mathcal{T}(t-r)[w(r)](x) + \int_r^t \mathcal{T}(t-\varrho)[\mathcal{F}(w_\varrho)](x)d\varrho, \tag{4.5}$$

$$w^+(t)(x) \ge \mathcal{T}(t-r)[w^+(r)](x) + \int_r^t \mathcal{T}(t-\varrho)[F^+(w^+_\varrho)](x)d\varrho, \tag{4.6}$$

for all $x \in \mathbb{R}^N$, $t > r$ *and* $w^-(x,s) \le w(x,s) \le w^+(x,s)$ *for* $x \in \mathbb{R}^N$, $s \in [r-\tau, r]$. *Then,*

$$w^-(x,t) \le w(x,t) \le w^+(x,t) \;\; \textit{for all} \;\; x \in \mathbb{R}^N, \;\; t \ge r.$$

Proof. It is clear that $\mathbf{0} \le w(x,t), w^{\pm}(x,t) \le \mathbf{K}^+$ for all $x \in \mathbb{R}^N$ and $t \ge \tau$. We only prove $w(x,t) \le w^+(x,t)$ for all $x \in \mathbb{R}^N$ and $t \ge r$, since the other case can be proved similarly. Let

$$X(x,t) = (X_1(x,t), X_2(x,t)) := w(x,t) - w^+(x,t) \;\; \text{for} \;\; x \in \mathbb{R}^N, \;\; t \ge r-\tau.$$

Note that $w(x,s) \le w^+(x,s)$ for $x \in \mathbb{R}^N$, $s \in [r-\tau, r]$. Since $h^+(u,v)$ is nondecreasing with respect to v and $g^+(u,v)$ is nondecreasing with respect to u for $(u,v) \in [\mathbf{0}, \mathbf{K}^+]$ and it is easy to know that $L + \partial_1 h(u,v) \ge 0$ and $L + \partial_2 g(u,v) \ge 0$, $(u,v) \in [\mathbf{0}, \mathbf{K}^+]$, for any $x \in \mathbb{R}^N$ and $t \in [r, r+\tau_m]$, where $\tau_m = \min\{\tau_1, \tau_2\} > 0$, it holds

$$\begin{aligned}
&\mathcal{F}_1(w_t)(x) - F_1^+(w_t^+)(x)\\
\le& F_1^+((w_t)(x) - F_1^+(w_t^+)(x)\\
=& LX_1(x,t) + h^+(u(x,t), v(x,t-\tau_1)) - h^+(u^+(x,t), v^+(x,t-\tau_1))\\
\le& LX_1(x,t) + h^+(u(x,t), v^+(x,t-\tau_1)) - h^+(u^+(x,t), v^+(x,t-\tau_1))\\
=& LX_1(x,t) + \partial_1 h^+(\theta u(x,t) + (1-\theta)u^+(x,t), v^+(x,t-\tau_1))X_1(x,t)\\
=& [L + \partial_1 h^+(\theta u(x,t) + (1-\theta)u^+(x,t), v^+(x,t-\tau_1))]X_1(x,t)\\
\le& 2L\max\{0, X_1(x,t)\},
\end{aligned} \tag{4.7}$$

where $\theta \in (0,1)$. Let $[B]_+ := \max\{B, 0\}$ for $B \in \mathbb{R}$. It then follows from (4.5)–(4.7) that

$$\begin{aligned}
X_1(t)(x) =& \mathcal{T}_1(t-r)[X_1(r)](x) + \int_r^t \mathcal{T}_1(t-\varrho)[\mathcal{F}_1(w_\varrho) - F_1^+(w_\varrho^+)](x)d\varrho\\
\le& 2L\int_r^t \int_{\mathbb{R}^N} \mathcal{T}_1(x-y, t-\varrho)[X_1(y,\varrho)]_+ dy d\varrho \;\forall x \in \mathbb{R}^N, \;\; t \in [r, r+\tau_m],
\end{aligned}$$

which implies that

$$[X_1(x,t)]_+ \le 2L\int_r^t \int_{\mathbb{R}^N} \mathcal{T}_1(y,\varrho)[X_1(x-y, t-\varrho)]_+ dy d\varrho.$$

Moreover, for any $\lambda > 0$, we set

$$\widetilde{X}_{i,\lambda}(t) := \sup_{x\in\mathbb{R}^N} [X_i(x,t)]_+ e^{-\lambda t} \quad \text{and} \quad \widetilde{X}_{i,\lambda} := \sup_{t\in[r-\tau,+\infty)} \widetilde{X}_{i,\lambda}(t) \;\; \text{for} \; i = 1,2.$$

Note that $\mathbf{0} \le (\widetilde{X}_{1,\lambda}, \widetilde{X}_{2,\lambda}) \le 2\mathbf{K}^+$. Then for $t \in [r, r+\tau_m]$, we obtain

$$\begin{aligned}\widetilde{X}_{1,\lambda}(t) &\le 2L \int_r^t \int_{\mathbb{R}^N} \mathcal{T}_1(y,\varrho)\widetilde{X}_{1,\lambda}(t-\varrho)e^{-\lambda\varrho}dyd\varrho \\ &\le 4Lk_1^+ \int_0^{\tau_m} e^{-(\lambda+L)\varrho}d\varrho,\end{aligned}$$

then, we have

$$\widetilde{X}_{1,\lambda} \le 4Lk_1^+[1-e^{-(\lambda+L)\tau_m}]/(\lambda+L) \to 0 \ \text{ as } \ \lambda \to +\infty.$$

Hence, $\widetilde{X}_{1,\lambda} \le 0$ for sufficiently large λ, which implies that $X_1(x,t) \le 0$ for $x \in \mathbb{R}^N$ and $t \in [r, r+\tau_m]$. Similarly, we can prove $X_2(x,t) \le 0$ for $x \in \mathbb{R}^N$ and $t \in [r, r+\tau_m]$. Repeating the same produce to each of the intervals $t \in [r+n\tau_m, r+(n+1)\tau_m]$, $n = 1, 2, \cdots$, there holds $(X_1(x,t), X_2(x,t)) \le (0,0)$. Then we have $w(x,t) \le w^+(x,t)$ for all $x \in \mathbb{R}^N,\ t \ge r-\tau$. This completes the proof. □

For the sake of convenience, we give some notations.

$$\begin{aligned}\mathbf{Z}(x,t) &:= \max\Big\{\max_{i=1,\cdots,l} \Phi_{c_i}^-(x\cdot\nu_i + c_i t + n_i), \chi\Gamma^-(t+n_{l+1})\Big\},\\ \mathbf{W}^+(x,t) &:= \sum_{i=1}^{l} \Phi_{c_i}^+(x\cdot\nu_i + c_i t + n_i) + \chi\Gamma^+(t+n_{l+1}),\\ \overline{\mathbf{W}}(x,t) &:= \sum_{i=1}^{l}(1, b(c_i))e^{\lambda_i(c_i)(x\cdot\nu_i+c_i t+n_i)} + \chi(1,b_*)e^{\lambda_*(t+n_{l+1})},\\ \mathbf{Z}^+(x,t) &:= \min\{\mathbf{K}^+, \mathbf{W}^+(x,t)\},\\ \overline{\mathbf{Z}}(x,t) &:= \min\{\mathbf{K}^+, \overline{\mathbf{W}}(x,t)\}.\end{aligned}$$

In the following, we give the detailed proof of the main results of this section.

4.0.3. Proof of Theorem 2.4

Proof. For $n \in \mathbb{Z}$, we denote

$$\phi^{n,-}(x,s) := \max\Big\{\max_{i=1,\cdots,l} \Phi_{c_i}^-(x\cdot\nu_i + c_i s + n_i),\ \chi\Gamma^-(s+n_{l+1})\Big\},$$

where $x \in \mathbb{R}^N$, $s \in [-n-\tau, -n]$. Let $\mathbf{Z}^n(x,t) = (U^n(x,t), V^n(x,t))$ be the unique mild solution of the initial value problem of (1.1) with initial condition

$$\mathbf{Z}^n(x,s) = \phi^{n,-}(x,s),\ \ x \in \mathbb{R}^N \ \text{ and } \ s \in [-n-\tau, -n].$$

It is clear that $\underline{\mathbf{Z}}(x,s) = \phi^{n,-}(x,s) = \mathbf{Z}^n(x,s) \le \mathbf{K}^+$ for $x \in \mathbb{R}^N$ and $s \in [-n-\tau, -n]$. Since $F^-(\cdot,\cdot)$ is nondecreasing in $\mathcal{C}_{[\mathbf{0},\mathbf{K}^+]}$, then it holds

$$\begin{aligned}\underline{\mathbf{Z}}(t)(x) &\le \mathcal{T}(t+n)[\underline{\mathbf{Z}}(-n)](x) + \int_{-n}^{t} \mathcal{T}(t-\varrho)[F^-(\underline{\mathbf{Z}}_\varrho)](x)d\varrho,\\ \mathbf{Z}^n(t)(x) &= \mathcal{T}(t+n)[\mathbf{Z}^n(-n)](x) + \int_{-n}^{t} \mathcal{T}(t-\varrho)[\mathcal{F}(\mathbf{Z}^n_\varrho)](x)d\varrho,\end{aligned}$$

$$\mathbf{K}^+ = \mathcal{T}(t+n)\mathbf{K}^+ + \int_{-n}^{t} \mathcal{T}(t-\varrho)[F^+(\mathbf{K}^+)](x)d\varrho,$$

for any $x \in \mathbb{R}^N$ and $t > -n$. It follows from Lemma 4.1 that

$$\underline{\mathbf{Z}}(x,t) \le \mathbf{Z}^n(x,t) \le \mathbf{K}^+ \text{ for } x \in \mathbb{R}^N, \ \ t > -n.$$

Now, we prove the following claim.

Claim. If $c_1, \cdots, c_l > c_*$, then

$$\mathbf{Z}^n(x,t) \le \overline{\mathbf{Z}}(x,t) \text{ for } x \in \mathbb{R}^N,\, t > -n, \tag{4.8}$$

and if (P7) holds, then

$$\mathbf{Z}^n(x,t) \le \mathbf{Z}^+(x,t) \text{ for } x \in \mathbb{R}^N,\, t > -n. \tag{4.9}$$

First, we prove (4.8). According to Theorems 3.1 and 3.2, if $c_1, \cdots, c_l > c_*$, then, $\phi^{n,-}(x,s) = \mathbf{Z}^n(x,s) \le \overline{\mathbf{Z}}(x,s)$ for $x \in \mathbb{R}^N$ and $s \in [-n-\tau, -n]$. By Lemma 4.1, it suffices to show that

$$\mathcal{T}(t+n)[\overline{\mathbf{Z}}(-n)](x) + \int_{-n}^{t} \mathcal{T}(t-\varrho)[F^+(\overline{\mathbf{Z}}_\varrho)](x)d\varrho \le \overline{\mathbf{Z}}(t)(x) \tag{4.10}$$

for $x \in \mathbb{R}^N$ and $t > -n$. Since $F^+(\cdot,\cdot)$ is nondecreasing in $\mathcal{C}_{[\mathbf{0},\mathbf{K}^+]}$, it is easy to see that

$$\mathcal{T}(t+n)[\overline{\mathbf{Z}}(-n)](x) + \int_{-n}^{t} \mathcal{T}(t-\varrho)[F^+(\overline{\mathbf{Z}}_\varrho)](x)d\varrho \le \mathbf{K}^+ \tag{4.11}$$

for $x \in \mathbb{R}^N$ and $t > -n$. For any $\phi = (\phi_1, \phi_2) \in \mathcal{C}_{[\mathbf{0},\mathbf{K}^+]}$, define $P = (P_1, P_2)$ by

$$P_1(\phi_1,\phi_2)(x,0) := (L+\alpha_1)\phi_1(x,0) + \beta_1\phi_2(x,-\tau_1),$$
$$P_2(\phi_1,\phi_2)(x,0) := (L+\alpha_2)\phi_2(x,0) + \beta_2\phi_1(x,-\tau_2).$$

Then, it is easy to see that $\overline{\mathbf{W}}(t)(\cdot) = \overline{\mathbf{W}}(t,\cdot)$ satisfies the integral equation

$$\overline{\mathbf{W}}(t)(x) = \mathcal{T}(t+n)[\overline{\mathbf{W}}(-n)](x) + \int_{-n}^{t} \mathcal{T}(t-\varrho)[Q(\overline{\mathbf{W}}_\varrho)](x)d\varrho. \tag{4.12}$$

Denote $\overline{\mathbf{Z}} = (\overline{U}, \overline{V})$ and $\overline{\mathbf{W}} = (\overline{\mathbf{W}}_1, \overline{\mathbf{W}}_2)$. By (P6), we have

$$\begin{aligned}
&F^+(\overline{\mathbf{Z}}_\varrho)(x)\\
=&\Big(L\overline{U}(x,\varrho) + h^+(\overline{U}(x,\varrho), \overline{V}(x,\varrho-\tau_1)), L\overline{V}(x,\varrho) + g^+(\overline{U}(x,\varrho-\tau_2), \overline{V}(x,\varrho))\Big)\\
\le&\Big((L+\alpha_1)\overline{U}(x,\varrho) + \beta_1\overline{V}(x,\varrho-\tau_1), (L+\alpha_2)\overline{V}(x,\varrho) + \beta_2\overline{U}(x,\varrho-\tau_2)\Big)\\
\le&\Big((L+\alpha_1)\overline{\mathbf{W}}_1(x,\varrho) + \beta_1\overline{\mathbf{W}}_2(x,\varrho-\tau_1), (L+\alpha_2)\overline{\mathbf{W}}_2(x,\varrho) + \beta_2\overline{\mathbf{W}}_1(x,\varrho-\tau_2)\Big)\\
=&P(\overline{\mathbf{W}}_\varrho)(x).
\end{aligned}$$

Then, it follows from (4.12) that

$$\begin{aligned}&\mathcal{T}(t+n)[\overline{\mathbf{Z}}(-n)](x)+\int_{-n}^{t}\mathcal{T}(t-\varrho)[F^{+}(\overline{\mathbf{Z}}_{\varrho})](x)d\varrho\\ \leq&\mathcal{T}(t+n)[\overline{\mathbf{W}}(-n)](x)+\int_{-n}^{t}\mathcal{T}(t-\varrho)[Q(\overline{\mathbf{W}}_{\varrho})](x)d\varrho\\ =&\overline{\mathbf{W}}(t)(x).\end{aligned} \tag{4.13}$$

Combing (4.12) and (4.13), (4.10) holds and (4.8) follows from Lemma 4.1.

Next, we prove (4.9). We only show that

$$\mathcal{T}(t+n)[\mathbf{Z}^{+}(-n)](x)+\int_{-n}^{t}\mathcal{T}(t-\varrho)[F^{+}(\mathbf{Z}^{+}_{\varrho})](x)d\varrho\leq\mathbf{Z}^{+}(t)(x) \tag{4.14}$$

for $x\in\mathbb{R}^N$ and $t>-n$. Since $F^{+}(\cdot,\cdot)$ is nondecreasing in $\mathcal{C}_{[\mathbf{0},\mathbf{K}^{+}]}$, it is easy to verify that

$$\mathcal{T}(t+n)[\mathbf{Z}^{+}(-n)](x)+\int_{-n}^{t}\mathcal{T}(t-\varrho)[F^{+}(\mathbf{Z}^{+}_{\varrho})](x)d\varrho\leq\mathbf{K}^{+}. \tag{4.15}$$

Moreover, by (P7) and a similar method as in the proof of Lemma 3.9, we have

$$\mathcal{T}(t+n)[\mathbf{Z}^{+}(-n)](x)+\int_{-n}^{t}\mathcal{T}(t-\varrho)[F^{+}(\mathbf{Z}^{+}_{\varrho})](x)d\varrho\leq\mathbf{W}^{+}(t)(x). \tag{4.16}$$

Combing (4.15) and (4.16), (4.14) holds and (4.9) follows from Lemma 4.1.

Moreover, $\mathbf{Z}^n(x,t)$ satisfies the regular estimates as in Lemma 3.7, that is, there exists a positive constant M, independent of n, such that for any $x\in\mathbb{R}^N$ and $t>-n+1$,

$$\begin{aligned}&\parallel\partial_t\mathbf{Z}^n(x,t)\parallel,\ \parallel\partial_{tx_i}\mathbf{Z}^n(x,t)\parallel,\ \parallel\partial_{t^2}\mathbf{Z}^n(x,t)\parallel,\ \parallel\partial_{x_i}\mathbf{Z}^n(x,t)\parallel,\ \parallel\partial_{x_it}\mathbf{Z}^n(x,t)\parallel,\\ &\parallel\partial_{x_ix_j}\mathbf{Z}^n(x,t)\parallel,\ \parallel\partial_{x_i^2t}\mathbf{Z}^n(x,t)\parallel,\ \parallel\partial_{x_i^2x_j}\mathbf{Z}^n(x,t)\parallel\leq M,\ \ \forall i,j=1,\cdots,N.\end{aligned}$$

By using the diagonal extraction process, there exists a subsequence $\{\mathbf{Z}^{n_k}(x,t)\}_{k\in\mathbb{N}}$ of $\{\mathbf{Z}^{n}(x,t)\}_{n\in\mathbb{N}}$ such that $\{\mathbf{Z}^{n_k}(x,t)\}_{k\in\mathbb{N}}$ converges to a function $\mathbf{Z}_\zeta(x,t)$ in the sense of topology τ. Clearly, $\mathbf{Z}_\zeta(x,t)$ is an solution of (1.1). By virtue of (4.8) and (4.9), it is easy to see that the assertions for part (i) and (ii). Note that $c\lambda_1(c)>\lambda_*$. The assertion of parts (iii)–(v) follow from (2.4)–(2.6). □

Appendix

A. Proof of Lemma 3.4.

For convenience, let us denote

$$\begin{aligned}\mathcal{H}_1(\psi_1,\psi_2)(\xi)&:=d_1\psi_1''(\xi)-c\psi_1'(\xi)+h(\psi_1(\xi),\psi_2(\xi-c\tau_1)),\\ \mathcal{H}_2(\psi_1,\psi_2)(\xi)&:=d_2\psi_2''(\xi)-c\psi_2'(\xi)+g(\psi_1(\xi-c\tau_2),\psi_2(\xi)).\end{aligned}$$

Then $\Psi(\xi):=(\psi_1(\xi),\psi_2(\xi))$ is an upper solution (or lower solution) of (2.1) if $\Psi'(\xi+)\leq\Psi'(\xi-)$ (or $\Psi'(\xi+)\geq\Psi'(\xi-)$) for $\xi\in\mathbb{R}$, and there exist $\xi_1,\cdots,\xi_n\in\mathbb{R}$ such that

$$\mathcal{H}_i(\psi_1,\psi_2)(\xi)\leq 0\ \ (\text{or}\,\mathcal{H}_i(\psi_1,\psi_2)(\xi)\geq 0)\ \ \text{for}\ \ \xi\in\mathbb{R}\setminus\{\xi_1,\cdots,\xi_n\},\ \ i=1,2.$$

It is easy to see that $\bar{\Psi}'(\xi+) \leq \bar{\Psi}'(\xi-)$ for $\xi \in \mathbb{R}$. Then, we only to show that $\mathcal{H}_i(\bar{\Psi})(\xi) \leq 0$ for $\xi \in \mathbb{R}\backslash\{\bar{\xi}_1, \bar{\xi}_2\}$, $i = 1, 2$. When $\xi > \bar{\xi}_1$, we easily check that

$$\mathcal{H}_1(\bar{\psi}_1, \bar{\psi}_2)(\xi) = h(k_1, \bar{\psi}_2(\xi - c\tau_1)) \leq h(k_1, k_2) = 0. \tag{4.17}$$

Using $h(u, v) \leq \alpha_1 u + \beta_1 v$ for $(u, v) \in [\mathbf{0}, \mathbf{K}]$, if $\xi < \bar{\xi}_1$, by directly computation, it holds

$$\begin{aligned}\mathcal{H}_1(\bar{\psi}_1, \bar{\psi}_2)(\xi) &= d_1\lambda_1^2 e^{\lambda_1\xi} - c\lambda_1 e^{\lambda_1\xi} + h(\bar{\phi}_1(\xi), \bar{\phi}_2(\xi - c\tau_1)) \\ &\leq d_1\lambda_1^2 e^{\lambda_1\xi} - c\lambda_1 e^{\lambda_1\xi} + \alpha_1 e^{\lambda_1\xi} + \beta_1 b(c) e^{\lambda_1(\xi - c\tau_1)} \\ &= e^{\lambda_1\xi}\Big\{d_1\lambda_1^2 - c\lambda_1 + \alpha_1 + \beta_1 b(c) e^{-c\lambda_1\tau_1}\Big\} \\ &= 0,\end{aligned} \tag{4.18}$$

since

$$b(c) = \frac{\beta_2 e^{-c\lambda_1\tau_2}}{c\lambda_1 - d_2\lambda_1^2 - \alpha_2} = \frac{c\lambda_1 - d_1\lambda_1^2 - \alpha_1}{\beta_1 e^{-\lambda_1 c\tau_1}}.$$

Similarly, if $\xi > \bar{\xi}_2$, it yields

$$\mathcal{H}_2(\bar{\psi}_1, \bar{\psi}_2)(\xi) = g(\bar{\psi}_1(\xi - c\tau_2), k_2) \leq g(k_1, k_2) = 0. \tag{4.19}$$

If $\xi < \bar{\xi}_2$, using $g(u, v) \leq \alpha_2 v + \beta_2 u$ for $(u, v) \in [\mathbf{0}, \mathbf{K}]$, we have

$$\begin{aligned}\mathcal{H}_2(\bar{\psi}_1, \bar{\psi}_2)(\xi) &= d_2\lambda_1^2 b(c) e^{\lambda_1\xi} - c\lambda_1 b(c) e^{\lambda_1\xi} + g(\bar{\psi}_1(\xi), \bar{\psi}_2(\xi - c\tau_2)) \\ &\leq d_2\lambda_1^2 b(c) e^{\lambda_1\xi} - c\lambda_1 b(c) e^{\lambda_1\xi} + \alpha_2 b(c) e^{\lambda_1\xi} + \beta_2 e^{\lambda_1(\xi - c\tau_2)} \\ &= e^{\lambda_1\xi}\Big\{b(c)\Big(d_2\lambda_1^2 - c\lambda_1 + \alpha_2\Big) + \beta_2 e^{-\lambda_1 c\tau_2}\Big\} = 0.\end{aligned} \tag{4.20}$$

From $b(c)$, we know that (4.20) is clearly established. □

B. Proof of Lemma 3.5.

By directly computation, we obtain

$$\begin{aligned}&\mathcal{H}_1(\underline{\phi}_1, \underline{\phi}_2) = d_1\underline{\phi}_1''(\xi) - c\underline{\phi}_1'(\xi) + h(\underline{\phi}_1(\xi), \underline{\phi}_2(\xi - c\tau_1)) \\ =&d_1(\kappa_1\lambda_1^2 e^{\lambda_1\xi} - \kappa_1(\lambda_1 + \varepsilon)^2\delta_1 e^{(\lambda_1+\varepsilon)\xi}) - c\kappa_1\lambda_1 e^{\lambda_1\xi} + c\kappa_1\delta_1(\lambda_1 + \varepsilon)e^{(\lambda_1+\varepsilon)\xi} \\ &+ \alpha_1\kappa_1 e^{\lambda_1\xi}(1 - \delta_1 e^{\varepsilon\xi}) + \beta_1\kappa_2 e^{\lambda_1(\xi - c\tau_1)}(1 - \delta_2 e^{\varepsilon(\xi - c\tau_1)}) \\ &+ \frac{1}{2}\Big\{h_{11}(u_1, v_1)\kappa_1^2 e^{2\lambda_1\xi}(1 - \delta_1 e^{\varepsilon\xi})^2 + h_{22}(u_3, v_3)\kappa_2^2 e^{2\lambda_1(\xi - c\tau_1)}(1 - \delta_2 e^{\varepsilon(\xi - c\tau_1)})^2 \\ &+ 2h_{12}(u_2, v_2)\kappa_1\kappa_2 e^{\lambda_1(2\xi - c\tau_1)}(1 - \delta_1 e^{\varepsilon\xi})(1 - \delta_2 e^{\varepsilon(\xi - c\tau_1)})\Big\} \\ =&e^{\lambda_1\xi}\Big(\kappa_1 f_1(c, \lambda_1) + \beta_1\kappa_2 e^{-\lambda_1 c\tau_1} - \kappa_1\delta_1 e^{\varepsilon\xi} f_1(c, \lambda_1 + \varepsilon) - \beta_1\kappa_2\delta_2 e^{\varepsilon\xi} e^{-(\lambda_1+\varepsilon)c\tau_1}\Big) \\ &+ \frac{1}{2}\Big\{h_{11}(u_1, v_1)\kappa_1^2 e^{2\lambda_1\xi}(1 - \delta_1 e^{\varepsilon\xi})^2 + h_{22}(u_3, v_3)\kappa_2^2 e^{2\lambda_1(\xi - c\tau_1)}(1 - \delta_2 e^{\varepsilon(\xi - c\tau_1)})^2 \\ &+ 2h_{12}(u_2, v_2)\kappa_1\kappa_2 e^{\lambda_1(2\xi - c\tau_1)}(1 - \delta_1 e^{\varepsilon\xi})(1 - \delta_2 e^{\varepsilon(\xi - c\tau_1)})\Big\}\end{aligned} \tag{4.21}$$

and

$$\mathcal{H}_2(\underline{\phi}_1, \underline{\phi}_2) = d_2\underline{\phi}_2''(\xi) - c\underline{\phi}_2'(\xi) + g(\underline{\phi}_1(\xi - c\tau_2), \underline{\phi}_2(\xi))$$

$$
\begin{aligned}
=&d_2\kappa_2\lambda_1^2e^{\lambda_1\xi}-d_2\kappa_2\delta_2(\lambda_1+\varepsilon)^2e^{(\lambda_1+\varepsilon)\xi}-c\kappa_2\lambda_1e^{\lambda_1\xi}+c\kappa_2\delta_2(\lambda_1+\varepsilon)e^{(\lambda_1+\varepsilon)\xi}\\
&+\alpha_2\kappa_2e^{\lambda_1\xi}(1-\delta_2e^{\varepsilon\xi})+\beta_2\kappa_1e^{\lambda_1(\xi-c\tau_2)}(1-\delta_1e^{\varepsilon(\xi-c\tau_2)})\\
&+\frac{1}{2}\Big\{g_{11}(u_4,v_4)\kappa_1^2e^{2\lambda_1(\xi-c\tau_2)}(1-\delta_1e^{\varepsilon(\xi-c\tau_2)})^2+g_{22}(u_6,v_6)\kappa_2^2e^{2\lambda_1\xi}(1-\delta_2e^{\varepsilon\xi})^2\\
&+2g_{12}(u_5,v_5)\kappa_1\kappa_2e^{\lambda_1(2\xi-c\tau_2)}(1-\delta_1e^{\varepsilon(\xi-c\tau_2)})(1-\delta_2e^{\varepsilon\xi})\Big\}\\
=&e^{\lambda_1\xi}\Big(\kappa_2f_2(c,\lambda_1)+\beta_2\kappa_1e^{-\lambda_1c\tau_2}-\kappa_2\delta_2e^{\varepsilon\xi}f_2(c,\lambda_1+\varepsilon)-\beta_2\kappa_1\delta_1e^{\varepsilon\xi}e^{-(\lambda_1+\varepsilon)c\tau_2}\Big)\\
&+\frac{1}{2}\Big\{g_{11}(u_4,v_4)\kappa_1^2e^{2\lambda_1(\xi-c\tau_2)}(1-\delta_1e^{\varepsilon(\xi-c\tau_2)})^2+g_{22}(u_6,v_6)\kappa_2^2e^{2\lambda_1\xi}(1-\delta_2e^{\varepsilon\xi})^2\\
&+2g_{12}(u_5,v_5)\kappa_1\kappa_2e^{\lambda_1(2\xi-c\tau_2)}(1-\delta_1e^{\varepsilon(\xi-c\tau_2)})(1-\delta_2e^{\varepsilon\xi})\Big\}, \qquad (4.22)
\end{aligned}
$$

where u_i lies between 0 and $\kappa_1e^{\lambda_1\xi}(1-\delta_1e^{\varepsilon\xi})$, v_i lies between 0 and $\kappa_2e^{\lambda_1(\xi-c\tau_1)}(1-\delta_2e^{\varepsilon(\xi-c\tau_1)})$ $(i=1,2,3)$ and u_i lies between 0 and $\kappa_1e^{\lambda_1(\xi-c\tau_2)}(1-\delta_1e^{\varepsilon(\xi-c\tau_2)})$ and v_i lies between 0 and $\kappa_2e^{\lambda_1\xi}(1-\delta_2e^{\varepsilon\xi})$ $(i=4,5,6)$. By Lemma 3.1, we have $\Delta_1(c,\lambda_1)=0$ and $f_i(c,\lambda_1)<0$ $(i=1,2)$ which imply that there exist $\kappa_i>0$ $(i=1,2)$ such that

$$
f_1(c,\lambda_1)\kappa_1+\beta_1\kappa_2e^{-\lambda_1c\tau_1}=0 \text{ and } f_2(c,\lambda_1)\kappa_2+\beta_2\kappa_1e^{-\lambda_1c\tau_2}=0. \qquad (4.23)
$$

By direct computation, we can take $\kappa_1=1$ and $\kappa_2=b(c)$. Moreover, by Lemma 3.1 again, we know $\Delta_1(c,\lambda_1+\varepsilon)>0$ and $f_i(c,\lambda_1+\varepsilon)<0$ $(i=1,2)$, where $\varepsilon>0$ and small enough. From Lemma 3.3, there exist $\zeta_i>0$ $(i=1,2)$ such that

$$
\begin{aligned}
m_1&:=f_1(\lambda_1+\varepsilon)\zeta_1+\beta_1e^{-(\lambda_1+\varepsilon)c\tau_1}\zeta_2<0,\\
m_2&:=f_2(\lambda_1+\varepsilon)\zeta_2+\beta_2e^{-(\lambda_1+\varepsilon)c\tau_2}\zeta_1<0. \qquad (4.24)
\end{aligned}
$$

Here (ζ_1,ζ_2) can be determined such that $\zeta_1>\kappa_1=1$ and $\zeta_2>\kappa_2=b(c)$. Fix (ζ_1,ζ_2) and set

$$
\delta_i:=\zeta_i/\kappa_i \text{ and } \underline{\xi}_i:=c\tau_i-(1/\varepsilon)\ln\delta_i,\ i=1,2.
$$

Note that $\delta_i>1$ and $\underline{\xi}_i\le 0$. Without loss of generality, we may assume $\delta_1<\delta_2$, $\underline{\xi}:=\max\{\underline{\xi}_1,\underline{\xi}_2\}=\underline{\xi}_1\le 0$. Then, if $\xi<\underline{\xi}$, we have

$$
\begin{aligned}
&0\le(1-\delta_2e^{\varepsilon\xi})\le 1,\\
&1-e^{\varepsilon c\tau_2}\le(1-\delta_1e^{\varepsilon\xi})\le 1,\\
&1-(\delta_2/\delta_1)e^{\varepsilon c(\tau_2-\tau_1)}\le(1-\delta_2e^{\varepsilon(\xi-c\tau_1)})\le 1,\\
&1-(\delta_2/\delta_1)e^{\varepsilon c(\tau_1-\tau_2)}\le(1-\delta_1e^{\varepsilon(\xi-c\tau_2)})\le 1.
\end{aligned}
$$

Obviously, $\delta_2/\delta_1=\zeta_2/(\zeta_1b(c))$ is invariant. Hence, there exists a constant $N>0$ (which is independent of $|(\zeta_1,\zeta_2|)$ such that

$$
|1-\delta_ie^{\varepsilon\xi}|,\ |1-\delta_1e^{\varepsilon(\xi-c\tau_2)}|,\ |1-\delta_2e^{\varepsilon(\xi-c\tau_1)}|\le N \text{ for } \xi\le\underline{\xi},\ i=1,2.
$$

Moreover, we have

$$
|\kappa_ie^{\lambda_1\xi}(1-\delta_ie^{\varepsilon\xi})|,\ |e^{\lambda_1(\xi-c\tau_2)}(1-\delta_1e^{\varepsilon(\xi-c\tau_2)})|,\ |b(c)e^{\lambda_1(\xi-c\tau_1)}(1-\delta_2e^{\varepsilon(\xi-c\tau_1)})|<N
$$

for $\xi \le \underline{\xi}$, $i = 1, 2$, where $\kappa_1 = 1$ and $\kappa_2 = b(c)$. Since $|u_i|, |v_i| \le N$, $i = 1, \cdots, 6$, all the second derivatives are bounded by a constant $M > 0$. Then for $\xi \le \underline{\xi}$, (4.21)–(4.24) imply

$$\begin{aligned}
&\mathcal{H}_1(\underline{\phi}_1, \underline{\phi}_2) = d_1\underline{\phi}_1'' - c\underline{\phi}_1'(\xi) + h(\underline{\phi}_1(\xi), \underline{\phi}_2(\xi - c\tau_1))\\
\ge& - e^{(\lambda_1+\varepsilon)\xi}m_1 - \frac{1}{2}\Big\{Me^{2\lambda_1\xi}(1-\delta_1 e^{\varepsilon\xi})^2 + Mb^2(c)e^{2\lambda_1(\xi-c\tau_1)}(1-\delta_2 e^{\varepsilon(\xi-c\tau_1)})^2\\
&+ 2Mb(c)e^{\lambda_1(2\xi-c\tau_1)}(1-\delta_1 e^{\varepsilon\xi})(1-\delta_2 e^{\varepsilon(\xi-c\tau_1)})\Big\}\\
\ge& - \frac{1}{2}\Big\{e^{(\lambda_1+\varepsilon)\xi}(2m_1 + MN^2 e^{(\lambda_1-\varepsilon)\xi}(1 + b^2(c)e^{-2\lambda_1 c\tau_1} + 2b(c)e^{-\lambda_1 c\tau_1}))\Big\}\\
=& - \frac{1}{2}\Big\{e^{(\lambda_1+\varepsilon)\xi}(2m_1 + MN^2 e^{(\lambda_1-\varepsilon)\xi}(1 + b(c)e^{-\lambda_1 c\tau_1})^2)\Big\}\\
>&0
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{H}_2(\underline{\phi}_1, \underline{\phi}_2) = d_2\underline{\phi}_2'' - c\underline{\phi}_2'(\xi) + g(\underline{\phi}_1(\xi - c\tau_2), \underline{\phi}_2(\xi))\\
\ge& - e^{(\lambda_1+\varepsilon)\xi}m_2 - \frac{1}{2}\Big\{Mb^2(c)e^{2\lambda_1\xi}(1-\delta_2 e^{\varepsilon\xi})^2 + Me^{2\lambda_1(\xi-c\tau_2)}(1-\delta_1 e^{\varepsilon(\xi-c\tau_2)})^2\\
&+ 2Mb(c)e^{\lambda_1(2\xi-c\tau_2)}(1-\delta_1 e^{\varepsilon(\xi-c\tau_2)})(1-\delta_2 e^{\varepsilon\xi})\Big\}\\
\ge& - \frac{1}{2}\Big\{e^{(\lambda_1+\varepsilon)\xi}(2m_1 + MN^2 e^{(\lambda_1-\varepsilon)\xi}(e^{-2\lambda_1 c\tau_2} + b^2(c) + 2b(c)e^{-\lambda_1 c\tau_1}))\Big\}\\
=& - \frac{1}{2}\Big(e^{(\lambda_1+\varepsilon)\xi}(2m_1 + MN^2 e^{(\lambda_1-\varepsilon)\xi}(b(c) + e^{-\lambda_1 c\tau_2})^2)\Big)\\
>&0
\end{aligned}$$

Therefore, $(\underline{\phi}_1(\xi), \underline{\phi}_2(\xi))$ satisfies (3.5) for $\xi \le \underline{\xi}$. □

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