Journal of Applied Analysis and Computation
Volume 10, Number 2, April 2020, 569–583
Website:http://jaac-online.com/
DOI:10.11948/20180314

# FUNCTIONAL EXPANSIONS FOR FINDING TRAVELING WAVE SOLUTIONS

Carmen Ionescu[1,†], Radu Constantinescu[1] and Mihail Stoicescu[1]

**Abstract** The paper proposes a generalized analytic approach which allows to find traveling wave solutions for some nonlinear PDEs. The solutions are expressed as functional expansions of the known solutions of an auxiliary equation. The proposed formalism integrates classical approaches as tanh method or $G'/G$ method, but it open the possibility of generating more complex solutions. A general class of second order PDEs is analyzed from the perspective of this formalism, and clear rules related to the balancing procedure are formulated. The KdV equation is used as a toy model to prove how the results obtained before through the $G'/G$ approach can be recovered and extended, in an unified and very natural way.

**Keywords** Traveling wave solutions, functional expansion, auxiliary equations, balancing procedure, KdV equation.

**MSC(2010)** 11.G, 12.G.

## 1. Introduction

Finding solutions for various partial differential equations (PDEs) is a fundamental issue and it presents a strong interest in many fields, where especially nonlinear phenomena are described. Sometimes, analytic methods for solving the nonlinear equations can be applied, otherwise numerical computations are considered, or, if neither analytic nor numerical approaches do not work, the simple decision on the equation's integrability is the maximal information which can be formulated. In this paper we will exclusively investigate analytic methods for finding solutions of nonlinear PDEs. Such solutions allow a better understanding of the physical phenomena and they bring researchers closer to the nature. There is not a clear procedure to solve nonlinear PDEs, but, during the time, many analytic methods for finding exact solutions were formulated. Cole and Hopf proposed a transformation method [10], Hirota used a bilinearization procedure [16], Ablowitz and Clarkson applied the inverse scattering formalism [2,7], etc. Adomian decomposition method is also very efficient in finding solutions for equations which do not respond to other solving methods [4,8,11]. A special situation is represented by the equations describing the constrained dynamical systems. Non-physical degrees of freedom has

[†]the corresponding author. Email address:crmnnsc@yahoo.com (C. Ionescu)
[1]Department of Physics, University of Craiova, 13 A.I.Cuza Street, 200585, Craiova, Romania

to be introduced in this case and specific procedures are required in order to get adequate reference frames where the dynamics is well defined [5,6].

An important class of the PDE's solutions is represented by the travelling waves. Despite not all equations accept such solutions, they are very important and there are many direct methods for finding them, as for example: the *tanh method* [21], the *tanh-coth method* [1,28], the *F-expansion method* [26], the *exp-function method* [15], the *elliptic function method* [20], imposing specific integrability conditions [14], the *extended trial equation method* [9], etc.

This paper will mainly refer to another interesting approach from finding traveling wave solutions, namely to the $G'/G$*-method* [27]. It supposes to look for solutions of an equation as an expansion in terms of the ratio $G'/G$, where $G$ represents a known solution of an auxiliary equation. Because it has proven to be effective, the method was generalized and improved [3,22,30]. However, it is not at all clear why the series are built in terms of this ratio. Why $G'/G$ and not $G'/G^2$ or any other expression can be considered? This is in fact the central aim of this work: to investigate how the $G'/G$-method can be further generalized and how all the mentioned methods for finding traveling wave solutions can be unified.

We will propose a new approach that is purely analytic. It will be called the *functional expansion* and it will be based on the very common approach supposing three main issues: *(i)* reduction of the PDE to an ODE; *(ii)* choice of an adequate auxiliary equation; *(iii)* choice of a specific expression of the solutions for the investigated ODE in terms of those of the auxiliary equation. The novelty of our approach is related to *(iii)* and it consists in looking for solutions in a general functional form. There were previous proposals and results which pointed out solutions having different form as the ratio $(G'/G)$, but all of them were related to a pre-established "basis", as, for example, $(G'/G, 1/G)$ [19,31], or $w(G)/G$ [18,25]. Our approach includes and extends all these attempts, presenting the advantage that it could solve equations for which the other approches do not work.

The first step, consisting in the reduction of the PDE to an ODE, is accomplished by introducing the wave variable. The auxiliary equation which will be attached to the resulting ODE is also an ODE but with well known solutions. There are many choices of auxiliary equations which were proposed in literature, starting with Riccati equation [29], till more complex, higher order linear or nonlinear equations [22]. Depending on the choice of the auxiliary equation, the solutions of the studied equation can be, them too, simpler or more complex.

The paper is structured as follows: in the next section, basic facts on the *functional expansion method* are presented; in section three we shall focus on a general form of second order differential equation which includes important examples as Korteweg de Vries, Dodd-Boulogh-Mikhailov, nonlinear Schrodinger equation, nonlinear Klein-Gordon equation, Burger type equation, Fisher's equation, etc. We will see how the proposed method can be applied to this general equation when rational functionals are considered in expansions and how general balancing rules can be established. In the fourth section the method is explicitly applied to the very simple case of the Korteweg de Vries equation. The solutions we are finding through our approach are compared with the solutions previously presented in literature. Final conclusions on the method will end the paper.

# 2. Functional expansions

To be more specific, let us consider that the dependent variable $u(x,t)$, defined in a $2D$ space $(x,t)$ satisfies the PDE:

$$F(u, u_t, u_x, u_{xx}, u_{tt}, \cdots) = 0. \tag{2.1}$$

Let us introduce the wave variable in the form:

$$\xi = x - Vt. \tag{2.2}$$

Here $V$ is a constant, identified as the wave velocity. With this transformation, the equation (2.1) becomes an ordinary differential equation:

$$\Delta(u, u', u'', \cdots) = 0, \tag{2.3}$$

where $u' = du(\xi)/d\xi$. We will look for a special class of analytical solutions of (2.3) which can be expressed as functions of the known solutions $G(\xi)$ of an auxiliary equation of the form:

$$\Theta(G, G', G'', \cdots) = 0. \tag{2.4}$$

## 2.1. Choice of the solution

Based on the main idea of the functional expansion method, we will consider now a very general choice for the solutions of the equation (2.3). This choice is of the form:

$$u(\xi) = \sum_{i=-m}^{m} P_i(G)\ H^i(G, G', G'', \cdots). \tag{2.5}$$

Here $P_i(G)$ are $2m+1$ functionals depending on $G(\xi)$ and that have to be determinated. $H(G, G', G'', \cdots)$ can be a very general expression containing $G(\xi)$ and its derivatives. Depending on the form of $P$ and $H$, one can generate very complex solutions. The choices covering almost all the approaches currently used in literature are $P$ as a rational expression, depending in fapt on $1/G$, and $H$ depending only on $G$ and $G'$. More strictly, $H$ is usually considered as a formal serie expansion at most linear in the two variables:

$$H(G, G') = h_0 + h_1 G + h_2 G',\ h_0,\ h_1, h_2 = const. \tag{2.6}$$

For example, if we will consider $h_0 = h_1 = 0$ and $h_2 = 1$, we get from (2.5) the expression:

$$u(\xi) = \sum_{i=-m}^{m} P_i(G) \left(G'\right)^i. \tag{2.7}$$

The generalized and improved $G\prime/G$ method [3], [22] corresponds to (2.10) with the choice:

$$P_i(G) = \frac{\pi_i}{G^i} \equiv \pi_i G^{-i};\ \pi_i = const.,\ i = \{-m, ..., 0, ...m\}. \tag{2.8}$$

The approach from [31] corresponds to (2.10) with $P_i = \pi_i G^{-i} + \pi_{i-1} G^{-i+1}$, while the $(w/g)$ method is recovered for $P = 1/G$ and $H = w(G)$, with an adequate choice for the auxiliary equation. The reprezentation used in [13] is also included in (2.5), but it does not accept the condensate form (2.10). It also imposes an $H(G, G')$ different from (2.6), namely:

$$H(G, G') = G' \sqrt{\sigma \left(1 + \frac{1}{\mu} \left(\frac{G'}{G}\right)^2\right)}. \tag{2.9}$$

For example, if we will consider $h_0 = h_1 = 0$ and $h_2 = 1$, we get from (2.5) the expression:

$$u(\xi) = \sum_{i=-m}^{m} P_i(G) \left(G'\right)^i. \tag{2.10}$$

The generalized and improved $G\prime/G$ method [3, 22] corresponds to (2.10) with the choice:

$$P_i(G) = \frac{\pi_i}{G^i} \equiv \pi_i G^{-i}; \ \pi_i = const., \ i = \{-m, ..., 0, ...m\}. \tag{2.11}$$

The approach from [31] corresponds to (2.10) with $P_i = \pi_i G^{-i} + \pi_{i-1} G^{-i+1}$, while the $(w/g)$ method is recovered for $P = 1/G$ and $H = w(G)$, with an adequate choice for the auxiliary equation. The reprezentation used in [?] is also included in (2.5), but it does not accept the condensate form (2.10). It also imposes an $H(G, G')$ different from (2.6), namely:

$$H(G, G') = G' \sqrt{\sigma \left(1 + \frac{1}{\mu} \left(\frac{G'}{G}\right)^2\right)}. \tag{2.12}$$

Coming back to the solutions of the form (2.10), the balancing procedure following the different powers in $G'$ leads to a system of ODE in the functionals $P_i(G)$. For this system we are looking to solutions given as expansions in $G$, more precisely as rationals with polynomial numerators and denominators. The choice that generalizes the previous expressions and that we will consider here is:

$$P_i(G) = \frac{\sum\limits_{\alpha=0}^{N_{i1}} \pi_{i\alpha} G^{\alpha}}{\sum\limits_{\beta=0}^{N_{i2}} \omega_{i\beta} G^{\beta}}. \tag{2.13}$$

The numbers $N_{i1}, N_{i2}$ can be integers representing the degree in $G$ of the numerator, respectively of the denominator. They define the degree of the functional $P_i$ as:

$$N(P_i) \equiv N_{i1} - N_{i2}. \tag{2.14}$$

The degree $N(P_i)$ has to be determined each time, and, to do that, a second balancing procedure has to be applied upon the equations in $P_i$ generated by the first balancing procedure.

It is important to note that, in almost all of the cases, the first balancing requirement leads to negative degrees for $P_i$. This explain why in the other methods evoked before, we find, as we already mentioned, a dependency of the form $P_i \equiv P_i(\frac{1}{G})$. This is also why a simpler representation as (2.13) can be considered for $P_i$. Defining $N_i \equiv |N(P_i)|$, we can choose the functionals $P_i$ as a sum of monomials:

$$P_i(G) = \sum_{\kappa=0}^{N_i} \pi_{i\kappa} G^{-\kappa}. \tag{2.15}$$

This is a generalization of the expression (2.11) used in the $(G'/G)$ approach, and a particular case of (2.13).

With (2.15), the solution (2.10) can be written down as:

$$u(\xi) = \sum_{i=-m}^{m} \left( \sum_{\kappa=0}^{N_i} \pi_{i\kappa} G^{-\kappa} \right) (G')^i. \tag{2.16}$$

In conclusion, the expansions of the type (2.5) are in fact the most general possible form of solutions and they includes almost all the choices used in various approaches to the direct finding of exact solutions of nonlinear differential equations. The effective use of (2.16) imposes the finding of the coefficients $\pi_{i\kappa}$,as well as of the two sumation limits. The last task is asking, as we will see, for two different balancing procedures: one following the powers of $G'$ and a second one following the powers of $G$.

Another important remark is that (2.10) generalizes the form of the solutions considered, without any supplementary explanation, in the $(G'/G)$-*method*. We will see that, by considering our expansion (2.10), the ratio $G'/G$ can appear in the most natural way.

## 2.2. Remarks on the auxiliary equation

The specific form of our generalized representations (2.5) also depends on the choice of the auxiliary equation (2.4). For example, when the *tanh method* is used, the auxiliary equation is chosen of Riccati-type, a first order ODE [21]. In this case, the functional $H(G, G', G'', ...)$ has the form (2.6), that is it contains at most linear terms. The value of the parameter $m$ depends on the model and it is established through a balancing procedure among the terms of higher order derivative, respectively of the higher nonlinearity. Similar approaches have been done considering other first order ODEs as auxiliary equations, as for example the equation (2.17) in [10], or the equation (2.18) in [16]:

$$G' = \frac{A}{G} + BG + CG^3, \tag{2.17}$$

$$G' = c_2 G^2 + c_4 G^4 + c_6 G^6. \tag{2.18}$$

If a *second order auxiliary equation* is considered, the second order derivative, $G''$ can be expressed in terms of $G$ and $G'$, so, again, $H = H(G, G')$. Examples of second order auxiliary equations are [2]:

$$G'' + \lambda G' + \mu G = 0, \tag{2.19}$$

$$AGG'' + B(G')^2 + CGG' + EG^2 = 0. \tag{2.20}$$

If we look for solutions of differential equations with order higher than two, auxiliary equations of higher orders could be considered. If, for example, we are dealing with an auxiliary equation of third order, the third order derivative can be in principle expressed in terms of the second and first orders, so it can be eliminated, and (2.5) stops at terms at maximum second order, $G''$.

# 3. The functional expansion method for a generalized second order differential equation

## 3.1. A generalized second order differential equation

Let us come back to the general algorithm of finding traveling wave solutions, to the moment when, introducing the wave variable, an ODE of the form (2.3) is generated. Specifically, we will consider here that this ODE has a form that belongs to a large class of second order ODEs:

$$A(u)u'' + B(u)u'^2 + C(u)u' + E(u) = 0. \tag{3.1}$$

Depending on the specific form of $A(u), B(u), C(u), E(u)$, the equation (3.1) includes a lot of interesting equations, investigated in various scientific domains, as, for example: the Dodd-Boulogh-Mikhailov equation describing fluid flows or QFT systems, the Buckmaster equation describing thin viscous fluid sheet flow, the nonlinear Schrodinger and Klein-Gordon equations, the Benjamin-Bona-Mahony equation, the Burger type or Hunter-Saxton equations, and many others.

If, in particular, all the coefficients in (3.1) are constants, it takes the form (2.20), an equation with already known solutions, which is sometime used as auxiliary equation. In fact, many of these equations can be exactly solved and they could be considered as auxiliary equations for other more complicated models with traveling wave solutions.

Our interest in the equation (3.1) will be related here on two objectives: to show how the functional expansion method can be effectively applied and to effectivelly compute through our approach the solutions for a particular case of (3.1), namely for the Korteweg de Vries equation. This last equation corresponds to the case when $A(u) = \delta = const$, $B(u) = 0$, $C(u) = 0$, and $E(u) = -Vu + k$. With these choices, its specific form is:

$$\delta u'' + \frac{1}{2}u^2 - Vu + k = 0. \tag{3.2}$$

The equation (3.2) will be investigated in details in the next section. Below, we will focus on the balancing procedures that our approach imposes. To do that, we will consider that all the coefficient functions from (3.1) have polynomial form. This supposition covers all the specific examples of equations we mentioned above, as included in (3.1). We will assume that the degrees of these polynomials, that is the highest power of $u(\xi)$ they are containing, are as follow:

$$Deg[A(u)] \equiv n_A; Deg[B(u)] = n_B; Deg[C(u)] = n_C; Deg[E(u)] = n_E. \tag{3.3}$$

## 3.2. The balancing procedure for the generalized equation

We come now back to (3.1) and we look for solutions in the form (2.10), where $G$ is a known solution of an auxiliary equation. The functionals $P_i(G)$ will be considered of the form (2.15) and the solutions of (3.1) will take the form (2.16). As we already mentioned, the main tasks consist in finding the two summation limits, that is the values for the parameters $m$ and $N_i$ appearing in (2.16). These tasks can be achieved following a combined balancing procedure, after $G'$ and, respectivelly, after $G$.

In the first step, we will analyse the polynomial degrees in $G'$ of each term appearing in (3.1), and we will balance the higher order derivative with the higher nonlinear term. We will use the notations $\dot{P}_i \equiv \frac{dP_i(G)}{dG}$ and $\ddot{P}_i \equiv \frac{d^2P_i(G)}{dG^2}$. The higher order derivative is represented by $A(u)u''$ and, taking into account (2.10), the term of the maximum degree in $G'$ is proportional with:

$$\ddot{P}_m P_m^{n_A} G'^{m(n_A+1)+2}. \tag{3.4}$$

The highest nonlinearity can be generated by any of the other terms from (3.1): $B(u)u'^2$, or $C(u)u'$, or $E(u)$. Using a similar evaluation reasoning as before, their degrees will be, respectively:

$$\dot{P}_m P_m^{n_A} G'^{m(n_B+2)+2}, \tag{3.5}$$

$$\dot{P}_m P_m^{n_C} G'^{m(n_C+1)+1}, \tag{3.6}$$

$$P_m^{n_E} G'^{mn_E}. \tag{3.7}$$

Depending on the value of these degrees, the following situations have to be considered:

1) If $n_E > n_B$ and $n_E > n_C$, then the higher nonlinear term is $E(u)$ and the balancing should be done between (3.4) and (3.7). It will give:

$$m(n_A+1)+2 = mn_E. \tag{3.8}$$

That is

$$m = \frac{2}{n_E - n_A - 1}. \tag{3.9}$$

We impose $m \in \mathbb{N}, n_E \in \mathbb{Z}, n_A \in \mathbb{Z}$. Then, $m$ can have the values $m = 2$ (for $n_E = n_A + 2$) or $m = 1$ (for $n_E = n_A + 3$). Independently if $B$ or $C$ vanish or not, the equation for $P_m$ with $m$ given by (3.9) will contain exactly the two terms which have to be balanced:

$$\ddot{P}_m + \alpha P_m^{\frac{m+2}{m}} = 0. \tag{3.10}$$

Here, the quantity $\alpha = \frac{e_{n_E}}{a_{n_A}}$ is known for a given equation.

**Remark.** The equation (3.10) accepts as solution:

$$P_m(G) = (-)^{\frac{m}{2}} m^{\frac{m}{2}} (m+1)^{\frac{m}{2}} a_{n_A}^{\frac{m}{2}} G^{-m} \tag{3.11}$$

It is something of the type $P_m \backsim G^{-m}$, form that corresponds to the usual choice in the $(G'/G)$ approach. It is very important to mention that (3.11) reffers to the maximal degree only, and, as we will see in the next section, it is not the most general solution of (3.10).

Step by step, we can solve all the equations for the functionals $\{P_k, k \in (m-1, ..., 0)\}$ and, implicitly, we can get the final solution of (3.1).

2) If $n_B > n_E$ and $n_B > n_C$ then $n_B = n_A - 1$, but $m$ can take any value $m = 1$ or $m = 2$. In this case, no solutions of the form $G'/G$ can be generated. It is for example, the case of the Hunter-Saxton equation which have the form:

$$(u - V)u'' + \frac{1}{2}u'^2 = 0. \tag{3.12}$$

For $m = 1$ this equation leads to a $P_1$ which satisfies the equation:

$$P_1 \ddot{P}_1 + \frac{1}{2} \dot{P}_1^2 = 0. \tag{3.13}$$

It does not admit solutions of the form $P_1 = a_{-1} G^{-1}$.

3) If the higher nonlinearity corresponds to $C(u)u'$, the balancing with $A(u)u''$ leads to:

$$m(n_A + 1) + 2 = m(n_C + 1) + 1.$$

We get:

$$m = \frac{1}{n_C - n_A},$$

that is

$$n_C = n_A + 1$$

or

$$n_C = n_A - 1.$$

This case creates also problems in finding solutions of the $G'/G$ type. To this case belong models as Fisher's and Chafee-Infante equations, particular cases of (3.1), for $B(u) = 0$.

**Conclusion.** Finding a solution for (3.1) in the form (2.10) supposes, as a first step, to evaluate the degrees in $G'$ of the terms appearing in the equation. The balancing conditions generate some differential and algebraic equations for the functionals $P_i$. To solve these equations, a new balancing procedure is required, following this time the degrees in $G$. The results we just mentioned allow making a clear distinction among the equations belonging to (3.1) type which can be solved with $G'/G$ method: equations of the form (3.1) with the dominant nonlinearity in $E(u)$. The other equations cannot be solved through $G'/G$ approach and they could admit more general solutions of the type (2.10).

# 4. The example of the KdV Equation

To prove that our approach generalizes other classical methods for solving nonlinear PDEs, we will show how it allows recovering for the simpler case of KdV, the solutions that can be generated through $G'/G$ method. As we will see, our approch also generates more general solutions.

## 4.1. Recovering the solutions given by the $G'/G$ method

We shall consider the Korteweg de Vries equation in the form:

$$u_t + u u_x + \delta u_{xxx} = 0.$$

By passing to the wave variable $\xi = x - Vt$, and by integrating once, we get the ODE:

$$\delta u''(\xi) + \frac{1}{2} u^2(\xi) - V u(\xi) + k = 0. \tag{4.1}$$

Here $\delta,\ k,\ V$ are constants which will be used as parameters. The balancing procedure between the terms $\delta u''(\xi)$ and $\frac{1}{2}u^2(\xi)$ leads to $m=2$, so the solutions (2.10) will have the form of the following expansion:

$$u(\xi)=\sum_{i=-2}^{2} P_i(G)(G')^i, \tag{4.2}$$

where the function $G(\xi)$ satisfy the auxiliary equation of the form:

$$G''+\lambda G'+\mu G=0. \tag{4.3}$$

With (4.2) and (4.3) in (4.1), by vanishing the coefficients of various powers of $G'$, we get the following system of equations in the functionals $\{P_k(G),\ k=-2,-1,0,1,2\}$:

$$2\delta\ddot{P}_2(G)+P_2^2(G)=0, \tag{4.4}$$

$$\delta\ddot{P}_1(G)-5\delta\dot{P}_2(G)+P_1(G)P_2(G)=0, \tag{4.5}$$

$$\begin{aligned}&\delta\ddot{P}_0(G)-3\delta\lambda\dot{P}_1(G)-5\delta\mu G\dot{P}_2(G)+2\delta(2\lambda^2-\mu)P_2(G)\\&+\frac{1}{2}P_1^2(G)+P_0(G)P_2(G)-VP_2(G)=0,\end{aligned} \tag{4.6}$$

$$\begin{aligned}&-\delta A\dot{P}_0(G)-3\delta\mu\dot{P}_1(G)G+\delta(\lambda^2-\mu)P_1(G)+6\lambda\mu GP_2(G)\\&+P_0(G)P_1(G)-VP_1(G)=0,\end{aligned} \tag{4.7}$$

$$\begin{aligned}&-\delta\mu G\dot{P}_0(G)+\frac{1}{2}P_0^2(G)-VP_0(G)+\delta\lambda\mu GP_1(G)+2\delta\mu^2G^2P_2(G)\\&+k+\delta\lambda\dot{P}_{-1}(G)+\delta\ddot{P}_{-2}(G)+P_1(G)P_{-1}(G)+P_2(G)P_{-2}(G)=0,\end{aligned} \tag{4.8}$$

$$P_{-2}(G)\left(\frac{1}{2}P_{-2}(G)+6\delta\mu^2G^2\right)=0, \tag{4.9}$$

$$10\delta\lambda\mu GP_{-2}(G)+2\delta\mu^2G^2P_{-1}(G)+P_{-1}(G)P_{-2}(G)=0, \tag{4.10}$$

$$\begin{aligned}&3\delta\mu G\dot{P}_{-2}(G)+3\delta\lambda\mu GP_{-1}(G)+(4\delta\lambda^2+2\delta\mu)P_{-2}(G)\\&+P_0(G)P_{-2}(G)+\frac{1}{2}P_{-1}^2(G)-VP_{-2}(G)=0,\end{aligned} \tag{4.11}$$

$$\begin{aligned}&3\delta\lambda\dot{P}_{-2}(G)+\delta\lambda^2P_{-1}(G)+\delta\mu(P_{-1}(G)+G\dot{P}_{-1}(G))\\&+P_0(G)P_{-1}(G)+P_1(G)P_{-2}(G)-VP_{-1}(G)=0.\end{aligned} \tag{4.12}$$

The system contains both differential and algebraic equations. The functionals $P_{-2},P_{-1},P_0,P_1,P_2$ have to be determinated and the solution will have the form:

$$u(\xi)=P_{-2}\cdot(G')^{-2}+P_{-1}\cdot(G')^{-1}+P_0(G)+P_1\cdot(G')+P_2\cdot(G')^2. \tag{4.13}$$

This is the solution given by the generalized and improved $(G'/G)$ method, if we will simply consider that $\{P_i, i=-2,-1,0,1,2\}$ are of the form (2.11). However, as we will see, a larger class of solutions is possible.

Let us prove for the moment that the choices (2.11) are compatible with the system. We note that $P_2$ and $P_1$ can be determined from (4.4) and, respectively, (4.5). These are differential equations which belong to the class of equations (3.1), so they obey the general results related to the balancing procedure given in the previous subsection. For the equation (4.4) the corresponding relation is (3.9) with $n_A = 0$, $n_E = 2$, so that $m = 2$.Following (2.15), the expression we are looking for $P_2$ will be:

$$P_2(G) = \sum_{\kappa=-n(P_2)}^{n(P_2)} \pi_{2\kappa} G^{-\kappa}. \tag{4.14}$$

Direct computations show that $n(P_2) = m = 2$, and (4.4) accepts as particular solution:

$$P_2(G) = \pi_{22} G^{-2} = -12\delta G^{-2}. \tag{4.15}$$

If we impose, for simplicity, $\mu = 0$ in the auxiliary equation, the balancing procedure for the equation (4.5) leads to the specific solution:

$$P_1(G) = \sum_{j=-1}^{1} \pi_{1j} G^{-j} = \pi_{11} G^{-1} = -12\delta\lambda G^{-1}. \tag{4.16}$$

The compatibility condition among the remaining equations from the system, (4.6)-(4.12) leads to the following expressions:

$$P_0 = V - \delta\lambda^2 = const, \tag{4.17}$$

$$P_{-2}(G) = P_{-1}(G) = 0, \tag{4.18}$$

$$k = \frac{1}{2}(V^2 - \delta^2\lambda^4). \tag{4.19}$$

Coming back with all these results in (4.13), we arrive to a KdV solution of the form:

$$u(\xi) = P_0(G) + P_1(G) \cdot (G') + P_2(G) \cdot (G')^2. \tag{4.20}$$

So, we recover the general solution of the KdV equation (4.1) which, practically, corresponds to the standard $(G'/G)$ approach:

$$u(\xi) = V - \delta\lambda^2 - 12\delta\lambda \frac{G'}{G} - 12\delta \left(\frac{G'}{G}\right)^2. \tag{4.21}$$

The previous expression, generated without any initial requirement on the form of the solution, contains practically all the particular solutions listed in literature, when the $G'/G$ method is applied to the KdV model.

Let us mention that in our case, when $\mu = 0$ and $\Delta \equiv \lambda^2 > 0$, the auxiliary equation (4.3) has the solution:

$$G(\xi) = e^{-(\lambda/2)\xi} \left(A_1 ch\frac{\lambda}{2}\xi + A_2 sh\frac{\lambda}{2}\xi\right) \tag{4.22}$$

with $A_1, A_2$ arbitrary constants. Using (4.22), the KdV solution (4.21) takes the form:

$$\begin{aligned} u(\xi) =& V - \delta\lambda^2 - 6\delta\lambda \frac{A_1 sh\frac{\lambda}{2}\xi + A_2 ch\frac{\lambda}{2}\xi}{A_1 ch\frac{\lambda}{2}\xi + A_2 sh\frac{\lambda}{2}\xi} \\ & - 6\delta\lambda(A_2 - A_1)^2 \left(\frac{ch\frac{\lambda}{2}\xi - sh\frac{\lambda}{2}\xi}{A_1 ch\frac{\lambda}{2}\xi + A_2 sh\frac{\lambda}{2}\xi}\right)^2. \end{aligned} \tag{4.23}$$

## 4.2. More than the $(G'/G)$ solutions for the KdV equation

In the previous subsection, it was illustrated how the *functional expansion method* we are proposing allows to recover the solution (4.21), the most general one mentioned in literature as being obtained using the standard $G'/G$ method. The natural question arising is if the new method could give more that the usual methods. We already mentioned in subsection 3.2 specific examples of equations for which the $(G'/G)$ method fails. How these equations can be solved through our method will be shown in a forthcoming paper. Here we will restrict ourselves to the KdV equation, and we will show that, even for this simple equation, the functional expansion method gives more general solutions that (4.21). The key issue for proving that is to consider the most general solution of the system (4.4)-(4.12). For example, for recovering the results brought by the standard $(G'/G)$ method, it was enough behind to consider the functional $P_2(G)$ in the form (4.15). We already mentioned that this form represents a particular solution of the equation (4.4), a most general one having the form:

$$P_2(G) = \frac{\pi_2}{\omega_2 G^2 + \omega_1 G + \omega_0},\ \omega_i = const. \tag{4.24}$$

It belongs to (2.13) with $N_{21} = 0, N_{22} = 2$. A simple check leads to the following relations among the parameters: $\pi_2 = -12\omega_2\delta$; $(\omega_1)^2 = 4\omega_2\omega_0$.

We can now use (4.24) for calculating the other functionals from the system (4.5)-(4.12), and to get then the KdV solution. To compare the new solution with (4.23), we will stay in the same case, considering $\mu = 0$ in the auxiliary equation, and $P_{-2} = P_{-1} = 0$, as solutions for (4.9)-(4.12). The KdV solution will have the form (4.20), and the compatibility of (4.5)-(4.8) will ask for $\lambda = \{-2, 1\}$. We are again in the case $\Delta = \lambda^2 > 0$, so we can consider for $G$ the solution (4.22). Direct computations leads to:

$$\begin{aligned}
P_1 &= -\frac{24\delta\omega_2}{2\omega_2 G + \omega_1},\\
P_0 &= V - \delta\lambda^2,\\
k &= \frac{1}{2}P_0^2 + \delta P_0 \lambda^2.
\end{aligned}$$

The KdV solution (4.21) will be now replaced by the most general one:

$$u(\xi) = V - \delta\lambda^2 - \frac{24\delta\omega_2 G'}{2\omega_2 G + \omega_1} - 12\delta\frac{(G')^2}{\omega_2 G^2 + \omega_1 G + \omega_0}. \tag{4.25}$$

By replacing here $G(\xi)$ given by (4.22), we come to the KdV solution:

$$\begin{aligned}
u(\xi) =& V - \delta\lambda^2 - 6\delta\lambda\frac{(A_2 - A_1)\left(ch\frac{\lambda}{2}\xi - sh\frac{\lambda}{2}\xi\right)}{\left(A_1 ch\frac{\lambda}{2}\xi + A_2 sh\frac{\lambda}{2}\xi\right) + \frac{\omega_1}{2\omega_2}e^{(\lambda/2)\xi}}\\
&- \frac{3\delta\lambda^2 e^{-\lambda\xi}(A_2 - A_1)^2\left(ch\frac{\lambda}{2}\xi - sh\frac{\lambda}{2}\xi\right)^2}{\omega_2 e^{-\lambda\xi}\left(A_1 ch\frac{\lambda}{2}\xi + A_2 sh\frac{\lambda}{2}\xi\right)^2 + \omega_1 e^{-(\lambda/2)\xi}\left(A_1 ch\frac{\lambda}{2}\xi + A_2 sh\frac{\lambda}{2}\xi\right) + \omega_0}
\end{aligned} \tag{4.26}$$

It is most general than (4.23) and it is different from the solutions listed in all the other papers dealing with KdV. See for example [22], where the improved and extended $(G'/G)$ method is used, or [17], where the $(G'/G, 1/G)$- expansion is used. The functional expansion brings a larger class of solutions.

# 5. Conclusions

In this article, a generalized approach to the direct finding of the traveling wave solutions of the nonlinear differential equations has been considered. It consists in looking for solutions in the form of the functional expansions (2.10), in terms of the solutions $G(\xi)$ of an auxiliary equation. The approach includes all the classical methods which have been previously used and it creates premises for generating more complex solutions. The algorithm contains the standard steps: (i) transforming the PDE into an ODE through the wave variable; (ii) choosing an adequate auxiliary equation; (iii) representing the solution of the ODE as an expansion in terms of the solutions of the auxiliary equation. The novelty we brought is related to the third step. The procedure is purely analytic and it is very simple to be applied for any type of equation admitting traveling wave solutions. It can be, for example, applied to equations describing the dynamics in tokamaks [23,24], or even to more complicated equations arrising from field theories [12].

In the present paper we considered the example of a generalized class of second order PDEs, containing many equations with important practical applications. An extensive study on the balancing procedure, limiting the form of the possible solutions, has been done on this equation. As a specific example, the KdV equation was considered in details and, particularly, the solutions generated through the $G'/G$ method have been recovered and extended. Here are few remarks summarizing the results.

**Remark 1.** In general, our approach can generate a larger class of solutions as $(G'/G)$ method is doing. They can arise when more complicated or higher order auxiliary equations are considered.

**Remark 2.** In our approach, two successive balancing procedures have been applied:

(i) A first balance between the higher derivative and the higher nonlinear terms was done for determining the value of $m$ in (2.10). It generated an ODE system in the functionals $P_{-m}(G), ..., P_0(G), ..., P_m(G)$.

(ii) The functionals $P_q(G)$ are expressed, them too, as expansions of the form (2.15) and, for determining their specific form, a second balancing procedure is required.

**Remark 3.** By considering simple models, as KdV, we concluded that, in this case, the functionals $P_i(G)$ are rational functions of the form:

$$P_q(G) = \frac{\pi_q}{\sum_{k=0}^{q} \omega_k G^k}, \tag{5.1}$$

where $q = -m, ..., -1, 0, 1, ..., m$. This result shows that, even for the simple KdV model, the functional expansion method brings a reacher class of solutions as the previous methods. Moreover, the new method we proposed allows to recover, in a very natural way, the solutions of the form $G'/G$. A pre-supposition that the solutions should have this form is not necessary. For other models or using other auxiliary equations, more complicated functionals might be compatible, and more general solutions, of the form $G'^p G^{-q}$ might be possible. Let us just recall that, the balancing evaluation of the Hunter-Saxton equation (3.12) leeds to a $P_1$ given by (3.13), and it seems that the solution $u(\xi)$ has in this case a dominant term of the form $G' G^{-2/3}$.

In conclusion, we proposed in this paper a method for finding the exact traveling wave solutions of various nonlinear equations which: (*i*) generalizes all the previous approaches; (*ii*) offers a richer class of solutions for the simpler equation as KdV, and (*iii*) gives the line on how the solutions should look for more complicated equations. Deeply investigations of this issue will be the challenge for future work.

# Acknowledgements

Part of this research has been done in the frame of the ICTP-SEENET-MTP Project NT03, "Cosmology - Classical and Quantum Challenges", as well as in the H2020 Project "Dynamics", 2017-RISE-777911.

# References

[1] M. A. Abdelkawy, A. H. Bhrawy, E. Zerrad and A. Biswas, *Application of tanh method to complex coupled nonlinear evolution equations*, Acta Physica Polonica A, 2016, 129(3), 278–283.

[2] M. J. Ablowitz and P.A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press, Cambridge, 1991.

[3] M. A. Akbar, N. H. Ali and E. M. E. Zayed, *A generalized and improved-expansion method for nonlinear evolution equations*, Mathematical Problems in Engineering, 2012. DOI:10.1155/2012/459879.

[4] A. F. Aljohani, R. Rach and E. El-Zahar, A.M. Wazwaz and A. Ebaid, *Solution of the hyperbolic Kepler equation by Adomian's asymptotic decomposition method*, Romanian Reports in Physics, 2018, 70(2), 112–126.

[5] A. Babalean, R. Constantinescu and C. Ionescu, *Non-minimal BRST terms for Yang-Mills theory*, Journal of Physics A: Mathematical and General, 1998, 31 (43), 8653–8659.

[6] A. Babalean, R. Constantinescu and C. Ionescu, *The gauge fixing problem in the sp(3) BRST canonical formalism*, Journal of Physics A: Mathematical and General, 1999, 32(16), 3005–3012.

[7] C. N. Babalic and A. S. Carstea, *Coupled Ablowitz–Ladik equations with branched dispersion*, Journal of Physics A: Mathematical and Theoretical, 2017, 50(41), Article Number: 415201.

[8] S. Bhalekar and J. Patade, *An analytical solution of Fisher's equation using decomposition method*, American Journal of Computational and Applied Mathematics, 2016. DOI:10.5923/j.ajcam.20160603.01.

[9] R. Cimpoiasu and A. S Pauna, *Complementary wave solutions for the long-short wave resonance model via the extended trial equation method and the generalized Kudryashov method*, Open Physics, 2018, 16(1), 419–426.

[10] J. D. Cole, *On a quasi-linear parabolic equation occurring in aerodynamics*, Quarterly of Applied Mathematics, 1951, 9(3), 225–236.

[11] R. Constantinescu and C. Ionescu, *Hot quark-gluon plasma and Chapline-Manton model*, Romanian Journal of Physics, 2011, 56(1–2), 53–61.

[12] R. Constantinescu and C. Ionescu, *The Yang-Mills fields—from the gauge theory to the mechanical model*, Central European Journal of Physics, 2009, 7(4), 711–720.

[13] S. Guo and Y. Zhou, *The extended (G'/G)-expansion method and its applications to the Whitham–Broer–Kaup–Like equations and coupled Hirota–Satsuma KdV equations*, Applied Mathematics and Computation, 2010, 215(9), 3214–3221.

[14] T. Harko and M. K. Mak, *Exact travelling wave solutions of non-linear reaction-convection-diffusion equations—An Abel equation based approach*, Journal of Mathematical Physics, 2015. https://doi.org/10.1063/1.4935299.

[15] J. He and X. Wu, *Exp-function method for nonlinear wave equations*, Chaos Solitons Fractals, 2006, 30(3), 700–708.

[16] R. Hirota, *Exact solution of the Korteweg—de Vries equation for multiple collisions of solitons*, Physical Review Letters, 1971, 27(18), 1192–1194.

[17] Md. Azmol Huda, Md. Samsuzzoha and M. Ali Akbar, *Searching soliton solutions to the Burger Huxley and the Klein Gordon equations*, Global Journal of Advanced Research (Scholary Peer Review Publishing System), 2019, 6(2), 67–77.

[18] W. Li, H. Chen and G. Zhang, *The $(\omega/g)$-expansion method and its application to Vakhnenko equation*, Chinese Physics B, 2009, 18(2), 400–404.

[19] L. Li and M. Wang, *The G'/G-expansion method and travelling wave solutions for a higher-order nonlinear Schrödinger equation*, Applied Mathematics Computational, 2009, 208(2), 440–445.

[20] S. Liu, Z. Fu, S. Liu and Q. Zhao, *Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations*, Physics Letters A, 2001, 289(1–2), 69–74.

[21] W. Malfliet, *Solitary wave solutions of nonlinear wave equations, American Journal of Physics*, 1992, 60, 650–654.

[22] H. Naher and F. A. Abdullah, *Further extension of the generalized and improved (G'/G)-expansion method for nonlinear evolution equation*, Journal of the Association of Arab Universities for Basic and Applied Sciences, 2016, 19, 52–58.

[23] M. Negrea, I. Petrisor and D. Constantinescu, *Aspects of the Diffusion of Electrons and Ions in Tokamak Plasma*, Romanian Journal of Physics, 2010, 55 (9–10), 1013–1023.

[24] I. Petrisor, *Some statistical features of particle dynamics in tokamak plasma*, Romanian Journal of Physics, 2016, 61(1–2), 217–234.

[25] A. R. Shehata and Safaa Abu-Amra, *Traveling wave solutions for some nonlinear partial differential equations by using modified (w/g)-expansion method*, European Journal of Mathematical Sciences, 2018, 4(2), 35–58.

[26] M. Wang and X. Li, *Application of F-expansion to periodic wave solutions for a new Hamiltonian amplitude equation*, Chaos, Solitons and Fractals, 2005, 24, 1257–1268.

[27] M. Wang, X. Li and J. Zhang, *The (G'/G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics*, Phys. Lett. A, 2008, 372, 417–423.

[28] A. M. Wazwaz, *The tanh-coth method for solitons and kink solutions for nonlinear parabolic equations*, Applied Mathematics and Computation, 2007, 188(2), 1467–1475.

[29] Z. Yan and H. Zhang, *New explicit solitary wave solutions and periodic wave solutions for Whitham–Broer–Kaup equation in shallow water*, Physics Letters A, 2001, 285(5–6), 355–362.

[30] J. Zhang, F. Jiang and X. Zhao, *An improved (G'/G)-expansion method for solving nonlinear evolution equations*, International Journal of Computer Mathematics, 2010, 87(8), 1716–1725.

[31] S. Zhang, *A generalized auxiliary equation method and its application to the (2+1)-dimensional KdV equations*, Applied Mathematics and Computation, 2007, 188(1), 1–6.