Journal of Applied Analysis and Computation
Volume 10, Number 3, June 2020, 853–874
Website:http://jaac-online.com/
DOI:10.11948/20180312

A BLOCK-BY-BLOCK METHOD FOR THE IMPULSIVE FRACTIONAL ORDINARY DIFFERENTIAL EQUATIONS*

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Abstract In this paper, a block-by-block numerical method is constructed for the impulsive fractional ordinary differential equations (IFODEs). Firstly, the stability and convergence analysis of the scheme are established. Secondly, the numerical solution which converges to the exact solution with order $3+\gamma$ for $0<\gamma<1$ is proved, where γ is the order of the fractional derivative. Finally, a series of numerical examples are carried out to verify the correctness of the theoretical analysis.

Keywords Impulsive fractional ordinary differential equations, block-by-block method, stability analysis, convergence analysis.

MSC(2010) 65L12.

1. Introduction

Many physical processes, which exhibit abnormal diffusion process, non-exponential patterns, other non-local behaviors, can be described by fractional ODEs/PDEs [16]. Progress in the last two decades have demonstrated that many phenomena in various fields of science, mathematics, engineering, bioengineering, and economics can be more accurately described by using fractional derivatives [2,5].

The impulsive fractional ordinary differential equations have become important in recent years as mathematical models of phenomena in both the physical and the social sciences, such as physics, mechanics, viscoelasticity, electrochemistry, control, porous media, electromagnetic [7,14], etc. Recently, there are some papers [4,6,9,11,15] considering the existence of solutions to impulsive fractional differential equations. However, the numerical research of this type equations are not too much in the literature.

In this article, we will construct block-by-block method for the IFODEs. It is well known that the p-block-by-block approach leads to a system of p coupling

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*The authors were supported by National Natural Science Foundation of China (Grant numbers 11901135, 11961009, 11671166, U1530401), Foundation of Guizhou Science and Technology Department ([2017]1086, [2020]1Y015), The first author would like to acknowledge the financial support by the China Scholarship Council (201708525037).

unknowns $u^{pm+1}, u^{pm+2}, \cdots$, and u^{pm+p} at each block step $m+1$. Block-by-block method, proposed by Linz for a kind of nonlinear Volterra integral equations with nonsingular kernels [12], and then extended to initial value problems of fractional differential equations (FDEs) [10, 17]. It is a kind of linear multi-step methods for the integral equations [13, 19]. Our approach used in this paper is based on the idea of [1] for the fractional differential equations which without impulses.

The plan of this paper is as follows. In the next section, we describe the detailed construction of the block-by-block method for the impulsive fractional ordinary differential equation. In Section 3, An estimate for the local truncation errors is given. The convergence and stability analysis are obtained in Section 4. Finally numerical experiments are presented in Section 5 which support the theoretical error estimates. Some concluding remarks are given in the final section.

2. A block-by-block method

We consider the impulsive fractional ordinary differential equation as following:

$$\begin{cases} {}_0D_t^\gamma x(t) = f(t, x(t)), t \in J^{'} := J \backslash \{t_1, t_2, \cdots, t_{M-1}\}, J := [0, T], \\ \quad x(t_k^+) = x(t_k^-) + I_k(x(t_k^-)), k = 1, 2, \cdots, M-1, \\ \quad x(0) = x_0, \end{cases} \tag{2.1}$$

where $0 < \gamma < 1, x_0 \in R, f : J \times R \longrightarrow R$ is jointly continuous, $I_k : R \longrightarrow R$ and t_k satisfy $0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = T, x(t_k^-) = \lim\limits_{\epsilon \to 0^-} x(t_k + \epsilon)$ and $x(t_k^+) = \lim\limits_{\epsilon \to 0^+} x(t_k + \epsilon)$ represent the left and right limits of $x(t)$ at $t = t_k$. The operator ${}_0D_t^\gamma$ denotes the Caputo fractional derivative of order $\gamma (0 < \gamma < 1)$ with respect to t [16] and defined by

$${}_0D_t^\gamma x(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-\tau)^{-\gamma} x^{'}(\tau) d\tau, 0 < \gamma < 1, \tag{2.2}$$

where $\Gamma(\cdot)$ is the Gamma special function.

It has been proved [18] that the initial value problem (2.1) is equivalent to the following Volterra integral equation

$$x(t) = \begin{cases} x_0 + \dfrac{1}{\Gamma(\gamma)} \displaystyle\int_0^t (t-\tau)^{\gamma-1} f(\tau, x(\tau)) d\tau, t \in [0, t_1], \\ x_0 + \displaystyle\sum_{j=1}^{i} I_j(x(t_j^-)) + \dfrac{1}{\Gamma(\gamma)} \displaystyle\int_0^t (t-\tau)^{\gamma-1} f(\tau, x(\tau)) d\tau, \\ \qquad\qquad t \in (t_i, t_{i+1}], i = 1, 2, \cdots, M-2, \\ x_0 + \displaystyle\sum_{i=1}^{M-1} I_i(x(t_i^-)) + \dfrac{1}{\Gamma(\gamma)} \displaystyle\int_0^t (t-\tau)^{\gamma-1} f(\tau, x(\tau)) d\tau, t \in (t_{M-1}, T]. \end{cases} \tag{2.3}$$

For the sake of simplicity, we assume that the impulsive points are uniformly distributed in $[0, T]$, i.e., $h = \frac{T}{M}$. In order to construct a block-by-block scheme, we divide the interval $[t_i, t_{i+1}]$ into $2N$ equal sub-intervals with size $\Delta t = \frac{h}{2N}$ denoting

$t_k^i = t_k + i\Delta t = kh + i\Delta t, k = 0, 1, \cdots, M-1; i = 0, 1, \cdots, 2N$. The numerical solution of (2.3) at the point t_k^i is denoted by X_k^i, and set $f_k^i = f(t_k^i, X_k^i)$.

The idea is as follows. When $t \in (t_K, t_{K+1}], K = 1, 2, \cdots, M-1$, assuming that $X_K^j, j = 0, 1, \cdots, 2m$, are already known, then we will derive an approximation to $x(t_K^{2m+1})$ and $x(t_K^{2m+2})$. Firstly, we drive the $x(t_K^{2m+1})$ as following:

$$\begin{aligned}
x(t_K^{2m+1}) &= x_0 + \sum_{i=1}^{K} I_i(x(t_i^-)) + \frac{1}{\Gamma(\gamma)} \int_0^{t_K^{2m+1}} (t_K^{2m+1} - \tau)^{\gamma-1} f(\tau, x(\tau)) d\tau \\
&= x_0 + \sum_{i=1}^{K} I_i(x(t_i^-)) + \frac{1}{\Gamma(\gamma)} \int_0^{t_K} (t_K^{2m+1} - \tau)^{\gamma-1} f(\tau, x(\tau)) d\tau \\
&\quad + \frac{1}{\Gamma(\gamma)} \int_{t_K}^{t_K^{2m}} (t_K^{2m+1} - \tau)^{\gamma-1} f(\tau, x(\tau)) d\tau \\
&\quad + \frac{1}{\Gamma(\gamma)} \int_{t_K^{2m}}^{t_K^{2m+1}} (t_K^{2m+1} - \tau)^{\gamma-1} f(\tau, x(\tau)) d\tau \\
&\doteq x_0 + \sum_{i=1}^{K} I_i(x(t_i^-)) + R_1 + R_2 + R_3.
\end{aligned} \tag{2.4}$$

For R_1, we can directly obtain

$$\begin{aligned}
R_1 &= \frac{1}{\Gamma(\gamma)} \sum_{n=0}^{K-1} \sum_{l=0}^{N-1} \int_{t_n^{2l}}^{t_n^{2l+2}} (t_K^{2m+1} - \tau)^{\gamma-1} f(\tau, x(\tau)) d\tau \\
&\approx \frac{1}{\Gamma(\gamma)} \sum_{n=0}^{K-1} \sum_{l=0}^{N-1} \int_{t_n^{2l}}^{t_n^{2l+2}} (t_K^{2m+1} - \tau)^{\gamma-1} [\sum_{i=0}^{2} \psi_n^{i,l}(\tau) f_n^{2l+i}] d\tau \\
&= \sum_{n=0}^{K-1} \sum_{l=0}^{N-1} [W_{0,l,n}^{2m+1,K} f_n^{2l} + W_{1,l,n}^{2m+1,K} f_n^{2l+1} + W_{2,l,n}^{2m+1,K} f_n^{2l+2}],
\end{aligned} \tag{2.5}$$

where

$$W_{i,l,n}^{2m+1,K} = \frac{1}{\Gamma(\gamma)} \int_{t_n^{2l}}^{t_n^{2l+2}} (t_K^{2m+1} - \tau)^{\gamma-1} \psi_n^{i,l}(\tau) d\tau, i = 0, 1, 2, \tag{2.6}$$

with $\psi_n^{i,l}(t), i = 0, 1, 2$ being quadratic Lagrange polynomials associated with the points $t_n^{2l}, t_n^{2l+1}, t_n^{2l+2}$.

For R_2, we have

$$\begin{aligned}
R_2 &= \frac{1}{\Gamma(\gamma)} \sum_{l=0}^{m-1} \int_{t_K^{2l}}^{t_K^{2l+2}} (t_K^{2m+1} - \tau)^{\gamma-1} f(\tau, x(\tau)) d\tau \\
&\approx \frac{1}{\Gamma(\gamma)} \sum_{l=0}^{m-1} \int_{t_K^{2l}}^{t_K^{2l+2}} (t_K^{2m+1} - \tau)^{\gamma-1} \big[\sum_{i=0}^{2} \varphi_K^{i,l}(\tau) f_K^{2l+i}\big] d\tau \\
&= \sum_{l=0}^{m-1} [W_{0,l,K}^{2m+1,K} f_K^{2l} + W_{1,l,K}^{2m+1,K} f_K^{2l+1} + W_{2,l,K}^{2m+1,K} f_K^{2l+2}],
\end{aligned} \tag{2.7}$$

where $\varphi_K^{i,l}(t), i = 0,1,2; l = 0,1,\cdots,m-1$, are quadratic Lagrange polynomials associated with the points $t_K^{2l}, t_K^{2l+1}, t_K^{2l+2}$. And

$$W_{i,l,K}^{2m+1,K} = \frac{1}{\Gamma(\gamma)} \int_{t_K^{2l}}^{t_K^{2l+2}} (t_K^{2m+1} - \tau)^{\gamma-1} \varphi_K^{i,l}(\tau) d\tau, i = 0,1,2. \tag{2.8}$$

For R_3, we have

$$\begin{aligned} R_3 &\approx \frac{1}{\Gamma(\gamma)} \int_{t_K^{2m}}^{t_K^{2m+1}} (t_K^{2m+1} - \tau)^{\gamma-1} \Big[\sum_{i=0}^{2} \varphi_K^{i,m}(\tau) f_K^{2m+\frac{i}{2}} \Big] d\tau \\ &= \sum_{i=0}^{2} \varpi_{i,m,K}^{2m+1,K} f_K^{2m+\frac{i}{2}}, \end{aligned} \tag{2.9}$$

where $t_K^{2m+\frac{1}{2}} = t_K^{2m} + \frac{\Delta t}{2}, f_K^{2m+\frac{1}{2}} = f(t_K^{2m+\frac{1}{2}}, x(t_K^{2m+\frac{1}{2}}))$, and $\varphi_K^{i,m}(t), i = 0,1,2$, are quadratic interpolating functions, defined by the points $t_K^{2m}, t_K^{2m+\frac{1}{2}}, t_K^{2m+1}$. Here $\varpi_{i,m,K}^{1,K}, i = 0,1,2$, are defined

$$\varpi_{i,m,K}^{2m+1,K} = \frac{1}{\Gamma(\gamma)} \int_{t_K^{2m}}^{t_K^{2m+1}} (t_K^{2m+1} - \tau)^{\gamma-1} \varphi_K^{i,m}(\tau) d\tau, \ i = 0,1,2, \tag{2.10}$$

which can be exactly computed. The value of $f_K^{2m+\frac{1}{2}}$ is approximated by using the interpolation

$$f_K^{2m+\frac{1}{2}} \approx \frac{3}{8} f_K^{2m} + \frac{3}{4} f_K^{2m+1} - \frac{1}{8} f_K^{2m+2}. \tag{2.11}$$

Substituting (2.11) into (2.9), we obtain R_3

$$R_3 \approx W_{0,m,K}^{2m+1,K} f_K^{2m} + W_{1,m,K}^{2m+1,K} f_K^{2m+1} + W_{2,m,K}^{2m+1,K} f_K^{2m+2}, \tag{2.12}$$

where

$$\begin{aligned} &W_{0,m,K}^{2m+1,K} = \varpi_{0,m,K}^{2m+1,K} + \tfrac{3}{8} \varpi_{1,m,K}^{2m+1,K}, W_{1,m,K}^{2m+1,K} = \tfrac{3}{4} \varpi_{1,m,K}^{2m+1,K} + \varpi_{2,m,K}^{2m+1,K}, \\ &W_{2,m,K}^{2m+1,K} = -\tfrac{1}{8} \varpi_{1,m,K}^{2m+1,K}. \end{aligned}$$

Substituting (2.5), (2.7) and (2.12) into (2.4), we obtain

$$\begin{aligned} X_K^{2m+1} = x_0 &+ \sum_{i=1}^{K} I_i(x_{i-1}^{2N}) + \sum_{n=0}^{K-1} \sum_{l=0}^{N-1} \sum_{i=0}^{2} W_{i,l,n}^{2m+1,K} f_n^{2l+i} \\ &+ \sum_{l=0}^{m-1} \sum_{i=0}^{2} W_{i,l,K}^{2m+1,K} f_K^{2l+i} + \sum_{i=0}^{2} W_{i,m,K}^{2m+1,K} f_K^{2m+i}. \end{aligned} \tag{2.13}$$

Secondly, we compute the approximation of $x(t_K^{2m+2})$ as following:

$$\begin{aligned}
&x(t_K^{2m+2})\\
&= x_0 + \sum_{i=1}^{K} I_i(x(t_i^-)) + \frac{1}{\Gamma(\gamma)}\int_0^{t_K^{2m+2}} (t_K^{2m+2}-\tau)^{\gamma-1} f(\tau, x(\tau))d\tau\\
&= x_0 + \sum_{i=1}^{K} I_i(x(t_i^-)) + \frac{1}{\Gamma(\gamma)}\int_0^{t_K} (t_K^{2m+2}-\tau)^{\gamma-1} f(\tau, x(\tau))d\tau\\
&\quad + \frac{1}{\Gamma(\gamma)}\int_{t_K}^{t_K^{2m+2}} (t_K^{2m+2}-\tau)^{\gamma-1} f(\tau, x(\tau))d\tau\\
&\doteq x_0 + \sum_{i=1}^{K} I_i(x(t_i^-)) + R_4 + R_5. \qquad (2.14)
\end{aligned}$$

For R_4, R_5, by following the same calculation as for R_1 and R_2, we have

$$\begin{aligned}
R_4 &= \frac{1}{\Gamma(\gamma)} \sum_{n=0}^{K-1}\sum_{l=0}^{N-1} \int_{t_n^{2l}}^{t_n^{2l+2}} (t_K^{2m+2}-\tau)^{\gamma-1} f(\tau, x(\tau))d\tau\\
&\approx \frac{1}{\Gamma(\gamma)} \sum_{n=0}^{K-1}\sum_{l=0}^{N-1} \int_{t_n^{2l}}^{t_n^{2l+2}} (t_K^{2m+2}-\tau)^{\gamma-1} [\sum_{i=0}^{2} \psi_n^{i,l}(\tau) f_n^{2l+i}]d\tau\\
&= \sum_{n=0}^{K-1}\sum_{l=0}^{N-1} [W_{0,l,n}^{2m+2,K} f_n^{2l} + W_{1,l,n}^{2m+2,K} f_n^{2l+1} + W_{2,l,n}^{2m+2,K} f_n^{2l+2}], \qquad (2.15)\\
R_5 &= \frac{1}{\Gamma(\gamma)} \sum_{l=0}^{m} \int_{t_K^{2l}}^{t_K^{2l+2}} (t_K^{2m+2}-\tau)^{\gamma-1} f(\tau, x(\tau))d\tau\\
&\approx \frac{1}{\Gamma(\gamma)} \sum_{l=0}^{m} \int_{t_K^{2l}}^{t_K^{2l+2}} (t_K^{2m+2}-\tau)^{\gamma-1} \big[\sum_{i=0}^{2} \psi_K^{i,l}(\tau) f_K^{2l+i}\big]d\tau\\
&= \sum_{l=0}^{m} [W_{0,l,K}^{2m+2,K} f_K^{2l} + W_{1,l,K}^{2m+2,K} f_K^{2l+1} + W_{2,l,K}^{2m+2,K} f_K^{2l+2}], \qquad (2.16)
\end{aligned}$$

where

$$\begin{cases}
W_{i,l,n}^{2m+2,K} = \dfrac{1}{\Gamma(\gamma)} \displaystyle\int_{t_n^{2l}}^{t_n^{2l+2}} (t_K^{2m+2}-\tau)^{\gamma-1} \psi_n^{i,l}(\tau)d\tau,\\
\qquad\qquad i = 0,1,2;\ l = 0,1,\cdots,N-1;\ n = 0,1,\cdots,K-1,\\
W_{i,l,K}^{2m+2,K} = \dfrac{1}{\Gamma(\gamma)} \displaystyle\int_{t_K^{2l}}^{t_K^{2l+2}} (t_K^{2m+2}-\tau)^{\gamma-1} \psi_K^{i,l}(\tau)d\tau,\\
\qquad i = 0,1,2;\ l = 0,1,\cdots,m,
\end{cases} \qquad (2.17)$$

with $\psi_K^{i,l}(t)$ being quadratic Lagrange polynomials associated with the points t_K^{2l}, t_K^{2l+1}, t_K^{2l+2}.

Substituting (2.15), (2.16) into (2.14), we obtain

$$X_K^{2m+2} = x_0 + \sum_{i=1}^{K} I_i(x_{i-1}^{2N}) + \sum_{n=0}^{K-1}\sum_{l=0}^{N-1}\sum_{i=0}^{2} W_{i,l,n}^{2m+2,K} f_n^{2l+i} + \sum_{l=0}^{m}\sum_{i=0}^{2} W_{i,l,K}^{2m+2,K} f_K^{2l+i}. \tag{2.18}$$

To summarize, that is by combining (2.13) and (2.18), we arrive at the following overall scheme:

$$\begin{cases} X_K^{2m+1} = x_0 + \sum_{i=1}^{K} I_i(X_{i-1}^{2N}) + \sum_{n=0}^{K-1}\sum_{l=0}^{N-1}\sum_{i=0}^{2} W_{i,l,n}^{2m+1,K} f_n^{2l+i} \\ \qquad + \sum_{l=0}^{m-1}\sum_{i=0}^{2} W_{i,l,K}^{2m+1,K} f_K^{2l+i} + \sum_{i=0}^{2} W_{i,m,K}^{2m+1,K} f_K^{2m+i}, \\ X_K^{2m+2} = x_0 + \sum_{i=1}^{K} I_i(X_{i-1}^{2N}) + \sum_{n=0}^{K-1}\sum_{l=0}^{N-1}\sum_{i=0}^{2} W_{i,l,n}^{2m+2,K} f_n^{2l+i} \\ \qquad + \sum_{l=0}^{m}\sum_{i=0}^{2} W_{i,l,K}^{2m+2,K} f_K^{2l+i}, \\ K = 1, \cdots, M-1; m = 1, \cdots, N-1. \end{cases} \tag{2.19}$$

Remark 2.1. If $t \in (t_0, t_1]$, i.e., K=0, we assume $\sum_{l=i}^{j} S_l = 0$, with $j < i$, then the scheme (2.19) is also correct for $K = 0$, at this time the scheme (2.19) is the same as the scheme (2.8) of the [8].

In the next two sections, we will give a stability and convergence analysis for the (2.19). We start with an analysis for the local truncation error estimation.

3. Estimation of the truncation errors

We hereafter denote by C a generic constant which may not be the same at different occurrences, but independent of all discretization parameters.

Now we turn to obtain an estimate for the truncation errors of the scheme (2.19). We define the truncation error at the step n by

$$r_K^n(\Delta t) := x(t_K^n) - \tilde{x}_K^n, K = 0, 1, \cdots, M-1; n = 1, 2, \cdots, 2N, \tag{3.1}$$

where $\tilde{x}_K^n$ is an approximation to $x(t_K^n)$, evaluated by using the scheme (2.13), (2.18) with exact previous solutions, i.e.,

$$\begin{cases} \tilde{x}_K^{2m+1} =x_0 + \sum_{i=1}^{K} I_i(x(t_i^-)) + \sum_{n=0}^{K-1}\sum_{l=0}^{N-1}\sum_{i=0}^{2} W_{i,l,n}^{2m+1,K} f(t_n^{2l+i}, x(t_n^{2l+i})) \\ \qquad + \sum_{l=0}^{m-1}\sum_{i=0}^{2} W_{i,l,K}^{2m+1,K} f(t_K^{2l+i}, x(t_K^{2l+i})) + \sum_{i=0}^{2} W_{i,m,K}^{2m+1,K} f(t_K^{2m+i}, x(t_K^{2m+i})), \\ \tilde{x}_K^{2m+2} =x_0 + \sum_{i=1}^{K} I_i(x(t_i^-)) + \sum_{n=0}^{K-1}\sum_{l=0}^{N-1}\sum_{i=0}^{2} W_{i,l,n}^{2m+2,K} f(t_n^{2l+i}, x(t_n^{2l+i})) \\ \qquad + \sum_{l=1}^{m}\sum_{i=0}^{2} W_{i,l,K}^{2m+2,K} f(t_K^{2l+i}, x(t_K^{2l+i})), \\ \qquad K = 0,1,\cdots,M-1; m = 0,\cdots,N-1. \end{cases} \tag{3.2}$$

Then we have the following estimate for $r_K^n(\Delta t), K = 0,1,\cdots,M-1; n = 1,2,\cdots,2N$.

Lemma 3.1. *Let $r_K^n(\Delta t), K = 0,1,\cdots,M-1; n = 1,2,\cdots,2N$ being the truncation error defined in* (3.1). *If $f \in PC^4[0,T]$ and $0 < \gamma \le 1$, then it holds*

$$|r_K^n(\Delta t)| \le C\Delta t^{3+\gamma}. \tag{3.3}$$

Proof. When $K = 0$, the scheme (2.19) is equality with the numerical scheme of the [17]. The proof of (3.3) is the same as the Lemma 3.1 of the [17].

When $K \ge 1, n = 2m+1$, By comparing (2.4), (2.13), and (3.2), we have

$$\begin{aligned} & r_K^{2m+1}(\Delta t) = x(t_K^{2m+1}) - \tilde{x}_K^{2m+1} \\ = & x(t_K^{2m+1}) - \{x_0 + \sum_{i=1}^{K} I_i(x(t_i^-)) + \sum_{n=0}^{K-1}\sum_{l=0}^{N-1}\sum_{i=0}^{2} W_{i,l,n}^{2m+1,K} f(t_n^{2l+i}, x(t_n^{2l+i})) \\ & + \sum_{l=0}^{m-1}\sum_{i=0}^{2} W_{i,l,K}^{2m+1,K} f(t_K^{2l+i}, x(t_K^{2l+i})) + \sum_{i=0}^{2} W_{i,m,K}^{2m+1,K} f(t_K^{2m+i}, x(t_K^{2m+i}))\} \\ = & \frac{1}{\Gamma(\gamma)} \sum_{n=0}^{K-1}\sum_{l=0}^{N-1} \int_{t_n^{2l}}^{t_n^{2l+2}} (t_K^{2m+1} - \tau)^{\gamma-1} f(\tau, x(\tau)) d\tau \\ & + \frac{1}{\Gamma(\gamma)} \sum_{l=0}^{m-1} \int_{t_K^{2l}}^{t_K^{2l+2}} (t_K^{2m+1} - \tau)^{\gamma-1} f(\tau, x(\tau)) d\tau \\ & + \frac{1}{\Gamma(\gamma)} \int_{t_K^{2m}}^{t_K^{2m+1}} (t_K^{2m+1} - \tau)^{\gamma-1} f(\tau, x(\tau)) d\tau \\ & - \{\frac{1}{\Gamma(\gamma)} \sum_{n=0}^{K-1}\sum_{l=0}^{N-1} [\sum_{i=0}^{2} f(t_n^{2l+i}, x(t_n^{2l+i})) \int_{t_n^{2l}}^{t_n^{2l+2}} (t_K^{2m+1} - \tau)^{\gamma-1} \psi_n^{i,l}(\tau) d\tau] \\ & + \frac{1}{\Gamma(\gamma)} \sum_{l=0}^{m-1} [\sum_{i=0}^{2} f(t_K^{2l+i}, x(t_K^{2l+i})) \int_{t_K^{2l}}^{t_K^{2l+2}} (t_K^{2m+1} - \tau)^{\gamma-1} \varphi_K^{i,l}(\tau) d\tau] \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\gamma)}[f(t_K^{2m}, x(t_K^{2m}))\int_{t_K^{2m}}^{t_K^{2m+1}}(t_K^{2m+1}-\tau)^{\gamma-1}\varphi_K^{0,m}(\tau)d\tau \\
&+(\frac{3}{8}f(t_K^{2m}, x(t_K^{2m}))+\frac{3}{4}f(t_K^{2m+1}, x(t_K^{2m+1})) \\
&-\frac{1}{8}(t_K^{2m+2}, x(t_K^{2m+2})))\int_{t_K^{2m}}^{t_K^{2m+1}}(t_K^{2m+1}-\tau)^{\gamma-1}\varphi_K^{1,m}(\tau)d\tau \\
&+f(t_K^{2m+1}, x(t_K^{2m+1}))\int_{t_K^{2m}}^{t_K^{2m+1}}(t_K^{2m+1}-\tau)^{\gamma-1}\varphi_K^{2,m}(\tau)d\tau]\} \\
=&\frac{1}{\Gamma(\gamma)}\sum_{n=0}^{K-1}\sum_{l=0}^{N-1}\int_{t_n^{2l}}^{t_n^{2l+2}}(t_K^{2m+1}-\tau)^{\gamma-1}\{f(\tau,x(\tau))-\sum_{i=0}^{2}f(t_n^{2l+i}, x(t_n^{2l+i}))\psi_n^{i,l}(\tau)\}d\tau \\
&+\frac{1}{\Gamma(\gamma)}\sum_{l=0}^{m-1}\int_{t_K^{2l}}^{t_K^{2l+2}}(t_K^{2m+1}-\tau)^{\gamma-1}\{f(\tau,x(\tau))-\sum_{i=0}^{2}f(t_K^{2l+i}, x(t_K^{2l+i}))\varphi_K^{i,l}(\tau)\}d\tau \\
&+\frac{1}{\Gamma(\gamma)}\int_{t_K^{2m}}^{t_K^{2m+1}}(t_K^{2m+1}-\tau)^{\gamma-1}\{f(\tau,x(\tau))-\sum_{i=0}^{2}f(t_K^{2m+\frac{i}{2}}, x(t_K^{2m+\frac{i}{2}}))\varphi_K^{i,m}(\tau) \\
&+[f(t_K^{2m+\frac{1}{2}}, x(t_K^{2m+\frac{1}{2}}))-(\frac{3}{8}f(t_K^{2m}, x(t_K^{2m}))+\frac{3}{4}f(t_K^{2m+1}, x(t_K^{2m+1})) \\
&-\frac{1}{8}(t_K^{2m+2}, x(t_K^{2m+2})))]\varphi_K^{1,m}(\tau)\}d\tau \\
\doteq&\frac{1}{\Gamma(\gamma)}\sum_{n=0}^{K-1}\sum_{l=0}^{N-1}\int_{t_n^{2l}}^{t_n^{2l+2}}(t_K^{2m+1}-\tau)^{\gamma-1}A_0(\tau)d\tau \\
&+\frac{1}{\Gamma(\gamma)}\sum_{l=0}^{m-1}\int_{t_K^{2l}}^{t_K^{2l+2}}(t_K^{2m+1}-\tau)^{\gamma-1}A_1(\tau)d\tau \\
&+\frac{1}{\Gamma(\gamma)}\int_{t_K^{2m}}^{t_K^{2m+1}}(t_K^{2m+1}-\tau)^{\gamma-1}[A_2(\tau)+A_3(\tau)\varphi_K^{1,m}(\tau)]d\tau.
\end{aligned}
$$

By using Taylor theorem, it can be checked that for all $\tau \in [t_n^{2l}, t_n^{2l+2}]$, there exists $\xi_0(\tau) \in [t_n^{2l}, t_n^{2l+2}]$, such that

$$
A_0(\tau) = \frac{f^{(3)}(\xi_0(\tau), x(\xi_0(\tau)))}{3!}(\tau - t_n^{2l})(\tau - t_n^{2l+1})(\tau - t_n^{2l+2}). \tag{3.4}
$$

For all $\tau \in [t_K^{2l}, t_K^{2l+2}]$, there exists $\xi_1(\tau) \in [t_K^{2l}, t_K^{2l+2}]$, such as

$$
A_1(\tau) = \frac{f^{(3)}(\xi_1(\tau), x(\xi_1(\tau)))}{3!}(\tau - t_K^{2l})(\tau - t_K^{2l+1})(\tau - t_K^{2l+2}), \tag{3.5}
$$

and that for all $\tau \in [t_K^{2m}, t_K^{2m+1}]$, there exist $\xi_2(\tau), \xi_3(\tau) \in [t_K^{2m}, t_K^{2m+1}]$, such that

$$
\begin{aligned}
A_2(\tau) &= \frac{f^{(3)}(\xi_2(\tau), x(\xi_2(\tau)))}{3!}(\tau - t_K^{2m})(\tau - t_K^{2m+\frac{1}{2}})(\tau - t_K^{2m+1}), \\
A_3(\tau) &= \frac{1}{16}\Delta t^3 f^{(3)}(\xi_3(\tau), x(\xi_3(\tau))).
\end{aligned} \tag{3.6}
$$

Therefore, we have

$$
\begin{aligned}
& r_K^{2m+1}(\Delta t) \\
&= \frac{1}{\Gamma(\gamma)} \sum_{n=0}^{K-1} \sum_{l=0}^{N-1} \int_{t_n^{2l}}^{t_n^{2l+2}} (t_K^{2m+1}-\tau)^{\gamma-1} \frac{f^{(3)}(\xi_0(\tau), x(\xi_0(\tau)))}{3!} \prod_{i=0}^{2} (\tau - t_n^{2l+i}) d\tau \\
&\quad + \frac{1}{\Gamma(\gamma)} \sum_{l=0}^{m-1} \int_{t_K^{2l}}^{t_K^{2l+2}} (t_K^{2m+1}-\tau)^{\gamma-1} \frac{f^{(3)}(\xi_1(\tau), x(\xi_1(\tau)))}{3!} \prod_{i=0}^{2} (\tau - t_K^{2l+i}) d\tau \\
&\quad + \frac{1}{\Gamma(\gamma)} \int_{t_K^{2m}}^{t_K^{2m+1}} (t_K^{2m+1}-\tau)^{\gamma-1} \frac{f^{(3)}(\xi_2(\tau), x(\xi_2(\tau)))}{3!} \prod_{i=0}^{2} (\tau - t_K^{2m+\frac{i}{2}}) d\tau \\
&\quad + \frac{1}{\Gamma(\gamma)} \int_{t_K^{2m}}^{t_K^{2m+1}} (t_K^{2m+1}-\tau)^{\gamma-1} \frac{1}{16} \Delta t^3 f^{(3)}(\xi_3(\tau), x(\xi_3(\tau))) \varphi_K^{1,m}(\tau) d\tau \\
&\doteq S_1 + S_2 + S_3 + S_4.
\end{aligned} \tag{3.7}
$$

For S_1, we get

$$
\begin{aligned}
|S_1| \le & \Big| \frac{1}{\Gamma(\gamma)} \sum_{n=0}^{K-1} \sum_{l=0}^{N-1} \int_{t_n^{2l}}^{t_n^{2l+2}} (t_K^{2m+1}-\tau)^{\gamma-1} \frac{f^{(3)}(\xi_4, x(\xi_4))}{3!} \prod_{i=0}^{2} (\tau - t_n^{2l+i}) d\tau \Big| \\
& + \Big| \frac{1}{\Gamma(\gamma)} \sum_{n=0}^{K-1} \sum_{l=0}^{N-1} \int_{t_n^{2l}}^{t_n^{2l+2}} (t_K^{2m+1}-\tau)^{\gamma-1} \frac{f^{(3)}(\xi_0(\tau), x(\xi_0(\tau))) - f^{(3)}(\xi_4, x(\xi_4))}{3!} \\
& (\tau - t_n^{2l})(\tau - t_n^{2l+1})(\tau - t_n^{2l+2}) d\tau \Big| \doteq S_5 + S_6,
\end{aligned} \tag{3.8}
$$

where $\xi_4 = t_n^{2l+1}$. For S_5, it holds

$$
\begin{aligned}
|S_5| &\le \frac{M_1}{\Gamma(\gamma)} \sum_{n=0}^{K-1} \sum_{l=0}^{N-1} \Big| \int_{t_n^{2l}}^{t_n^{2l+2}} (t_K^{2m+1}-\tau)^{\gamma-1} \prod_{i=0}^{2} (\tau - t_n^{2l+i}) d\tau \Big| \\
&= \frac{M_1}{\Gamma(\gamma)} \sum_{n=0}^{K-1} \sum_{l=0}^{N-1} \Big| \int_{t_n^{2l}}^{t_n^{2l+1}} (t_K^{2m+1}-\tau)^{\gamma-1} \prod_{i=0}^{2} (\tau - t_n^{2l+i}) d\tau \\
&\quad + \int_{t_n^{2l+1}}^{t_n^{2l+2}} (t_K^{2m+1}-\tau)^{\gamma-1} \prod_{i=0}^{2} (\tau - t_n^{2l+i}) d\tau \Big| \\
&= \frac{M_1}{\Gamma(\gamma)} \sum_{n=0}^{K-1} \sum_{l=0}^{N-1} \Big| (t_K^{2m+1}-\tau_1^*)^{\gamma-1} \int_{t_n^{2l}}^{t_n^{2l+1}} \prod_{i=0}^{2} (\tau - t_n^{2l+i}) d\tau \\
&\quad + (t_K^{2m+1}-\tau_2^*)^{\gamma-1} \int_{t_n^{2l+1}}^{t_n^{2l+2}} \prod_{i=0}^{2} (\tau - t_n^{2l+i}) d\tau \Big| \\
&= \frac{M_1 \Delta t^4}{4\Gamma(\gamma)} \sum_{n=0}^{K-1} \sum_{l=0}^{N-1} \big| (t_K^{2m+1}-\tau_1^*)^{\gamma-1} - (t_K^{2m+1}-\tau_2^*)^{\gamma-1} \big| \\
&= \frac{M_1 \Delta t^4}{4\Gamma(\gamma)} \sum_{n=0}^{K-1} \sum_{l=0}^{N-1} \big| (\gamma-1)(t_K^{2m+1}-\tau_3^*)^{\gamma-2} (\tau_2^* - \tau_1^*) \big| \\
&\le \frac{M_1 \Delta t^4}{4\Gamma(\gamma)} |\gamma - 1| \sum_{n=0}^{K-1} \sum_{l=0}^{N-1} \Big| \int_{t_n^{2l}}^{t_n^{2l+2}} (t_K^{2m+1}-\tau)^{\gamma-2} d\tau \Big|
\end{aligned}
$$

$$\le \frac{M_1\Delta t^4}{4\Gamma(\gamma)}|\gamma-1|\sum_{n=0}^{K-1}|\int_{t_n^0}^{t_n^{2N}}(t_K^{2m+1}-\tau)^{\gamma-2}d\tau|$$

$$\le \frac{M_1\Delta t^4}{4\Gamma(\gamma)}\sum_{n=0}^{K-1}[|(t_K^{2m+1}-t_n^0)^{\gamma-1}|+|(t_K^{2m+1}-t_n^{2N})^{\gamma-1}|], \tag{3.9}$$

where $M_1=\sup_{t\in J'}|f^{(3)}(t,x(t))|, t_n^{2l}\le\tau_1^*\le t_n^{2l+1}\le\tau_2^*\le t_n^{2l+2}, \tau_1^*\le\tau_3^*\le\tau_2^*$.

The S_6, it can be bounded by

$$|S_6|\le \frac{M_2\Delta t}{\Gamma(\gamma)}\sum_{n=0}^{K-1}\sum_{l=0}^{N-1}\int_{t_n^{2l}}^{t_n^{2l+2}}(t_K^{2m+1}-\tau)^{\gamma-1}|\prod_{i=0}^{2}(\tau-t_n^{2l+i})|d\tau$$

$$\le \frac{M_2\Delta t^4}{\Gamma(\gamma)}\sum_{n=0}^{K-1}\sum_{l=0}^{N-1}|\int_{t_n^{2l}}^{t_n^{2l+2}}(t_K^{2m+1}-\tau)^{\gamma-1}d\tau|$$

$$= \frac{M_2\Delta t^4}{\Gamma(\gamma)}\sum_{n=0}^{K-1}|\int_{t_n^{0}}^{t_n^{2N}}(t_K^{2m+1}-\tau)^{\gamma-1}d\tau|$$

$$\le \frac{M_2\Delta t^4}{\Gamma(\gamma+1)}\sum_{n=0}^{K-1}[|(t_K^{2m+1}-t_n^0)^{\gamma}|+|(t_K^{2m+1}-t_n^{2N})^{\gamma}|], \tag{3.10}$$

where $M_2=\sup_{t\in J'}|f^{(4)}(t,x(t))|$. In the above derivation we have used the fact that

$$|\frac{f^{(3)}(\xi_0(\tau),x(\xi_0(\tau)))-f^{(3)}(\xi_4,x(\xi_4))}{3!}|\le M_2\Delta t,\ \text{for } \xi_4=t_n^{2l+1},\forall\tau\in[t_n^{2l},t_n^{2l+2}].$$

Bringing (3.9)-(3.10) into (3.8) gives

$$|S_1|\le \frac{M_1\Delta t^4}{4\Gamma(\gamma)}\sum_{n=0}^{K-1}[|(t_K^{2m+1}-t_n^0)^{\gamma-1}|+|(t_K^{2m+1}-t_n^{2N})^{\gamma-1}|]$$

$$+\frac{M_2\Delta t^4}{\Gamma(\gamma+1)}\sum_{n=0}^{K-1}[|(t_K^{2m+1}-t_n^0)^{\gamma}|+|(t_K^{2m+1}-t_n^{2N})^{\gamma}|]. \tag{3.11}$$

For S_2, we have

$$|S_2|\le\ |\frac{1}{\Gamma(\gamma)}\sum_{l=0}^{m-1}\int_{t_K^{2l}}^{t_K^{2l+2}}(t_K^{2m+1}-\tau)^{\gamma-1}\frac{f^{(3)}(\xi_5,x(\xi_5))}{3!}\prod_{i=0}^{2}(\tau-t_K^{2l+i})d\tau|$$

$$+|\frac{1}{\Gamma(\gamma)}\sum_{l=0}^{m-1}\int_{t_K^{2l}}^{t_K^{2l+2}}(t_K^{2m+1}-\tau)^{\gamma-1}\frac{f^{(3)}(\xi_1(\tau),x(\xi_1(\tau)))-f^{(3)}(\xi_5,x(\xi_5))}{3!}$$

$$(\tau-t_K^{2l})(\tau-t_K^{2l+1})(\tau-t_K^{2l+2})|\doteq S_7+S_8, \tag{3.12}$$

where $\xi_5=t_K^{2l+1}$.

The proof of S_7,S_8 are similar to the (3.9)-(3.10). Therefore, S_2 satisfies

$$|S_2|\le \frac{M_1\Delta t^4}{4\Gamma(\gamma)}[(t_K^{2m+1}-t_K^0)^{\gamma-1}+(t_K^{2m+1}-t_K^{2m})^{\gamma-1}]$$

$$+\frac{M_2\Delta t^4}{\Gamma(\gamma+1)}[(t_K^{2m+1}-t_K^0)^{\gamma}+(t_K^{2m+1}-t_K^{2m})^{\gamma}]. \tag{3.13}$$

For S_3, S_4, we have

$$
\begin{aligned}
&|S_3| \\
&\leq \frac{1}{\Gamma(\gamma)} \int_{t_K^{2m}}^{t_K^{2m+1}} (t_K^{2m+1} - \tau)^{\gamma-1} \left| \frac{f^{(3)}(\xi_2(\tau), x(\xi_2(\tau)))}{3!} \prod_{i=0}^{2} (\tau - t_K^{2m+\frac{i}{2}}) \right| d\tau \\
&\leq \frac{M_1 \Delta t^3}{\Gamma(\gamma)} \int_{t_K^{2m}}^{t_K^{2m+1}} (t_K^{2m+1} - \tau)^{\gamma-1} d\tau \leq \frac{M_1 \Delta t^4}{\Gamma(\gamma)} (t_K^{2m+1} - \tau_4^*)^{\gamma-1}, \qquad (3.14) \\
&|S_4| \\
&= \left| \frac{1}{\Gamma(\gamma)} \int_{t_K^{2m}}^{t_K^{2m+1}} (t_K^{2m+1} - \tau)^{\gamma-1} \frac{1}{16} \Delta t^3 f^{(3)}(\xi_3(\tau), x(\xi_3(\tau))) \varphi_K^{1,m}(\tau) d\tau \right| \\
&\leq \frac{M_1 \Delta t^3}{16\Gamma(\gamma)} \left| \int_{t_K^{2m}}^{t_K^{2m+1}} (t_K^{2m+1} - \tau)^{\gamma-1} \frac{-4(\tau - t_K^{2m})(\tau - t_K^{2m+1})}{\Delta t^2} d\tau \right| \\
&\leq \frac{M_1 \Delta t^3}{\Gamma(\gamma)} \int_{t_K^{2m}}^{t_K^{2m+1}} (t_K^{2m+1} - \tau)^{\gamma-1} d\tau \leq \frac{M_1 \Delta t^4}{\Gamma(\gamma)} (t_K^{2m+1} - \tau_4^*)^{\gamma-1}, \qquad (3.15)
\end{aligned}
$$

where $\tau_4^* \in (t_K^{2m}, t_K^{2m+1})$.

Combining (3.11), (3.13), (3.14) and (3.15), yields

$$
\begin{aligned}
&|r_K^{2m+1}(\Delta t)| \\
&\leq \frac{M_1 \Delta t^4}{4\Gamma(\gamma)} \sum_{n=0}^{K-1} [|(t_K^{2m+1} - t_n^0)^{\gamma-1}| + |(t_K^{2m+1} - t_n^{2N})^{\gamma-1}|] \\
&\quad + \frac{M_2 \Delta t^4}{\Gamma(\gamma+1)} \sum_{n=0}^{K-1} [|(t_K^{2m+1} - t_n^0)^{\gamma}| + |(t_K^{2m+1} - t_n^{2N})^{\gamma}|] \\
&\quad + \frac{M_1 \Delta t^4}{4\Gamma(\gamma)} [(t_K^{2m+1} - t_K^0)^{\gamma-1} + (t_K^{2m+1} - t_K^{2m})^{\gamma-1}] \\
&\quad + \frac{M_2 \Delta t^4}{\Gamma(\gamma+1)} [(t_K^{2m+1} - t_K^0)^{\gamma} + (t_K^{2m+1} - t_K^{2m})^{\gamma}] + \frac{2M_1 \Delta t^4}{\Gamma(\gamma)} (t_K^{2m+1} - \tau_4^*)^{\gamma-1} \\
&\leq \frac{M_1 \Delta t^4}{4\Gamma(\gamma)} \cdot 2K\Delta t^{\gamma-1} + \frac{M_2 \Delta t^4}{\Gamma(\gamma+1)} \cdot 2KT^{\gamma} + \frac{M_1 \Delta t^4}{4\Gamma(\gamma)} \cdot 2\Delta t^{\gamma-1} \\
&\quad + \frac{M_2 \Delta t^4}{\Gamma(\gamma+1)} [T^{\gamma} + \Delta t^{\gamma}] + \frac{2M_1 \Delta t^4}{\Gamma(\gamma)} \Delta t^{\gamma-1} \\
&\leq C\Delta t^{3+\gamma}, \qquad (3.16)
\end{aligned}
$$

where C only depends on M_1, M_2, γ, and T.

Similar to the truncation error at the previous steps, we get

$$|r_K^{2m+2}(\Delta t)| \leq C\Delta t^{3+\gamma},\ 0 < \gamma < 1.$$

The proof of Lemma 3.1 is complete. □

4. Convergence and stability analysis

In order to simplify the notations and without lose of generality, let us reformulate the scheme (2.19) by introducing the following coefficients:

$$
\begin{aligned}
&B^i_{n,0} = W^{i,K}_{0,0,n}, B^i_{n,2j+1} = W^{i,K}_{1,j,n}, i = 1,2,\cdots,2N; j = 0,1,\cdots,N-1,\\
&B^i_{n,2j+2} = W^{i,K}_{2,j,n} + W^{i,K}_{0,j+1,n}, j = 0,1,\cdots,N-2, \quad B^i_{n,2N} = W^{i,K}_{2,N-1,n}.\\
&C^0_K = \frac{W^{2m+1,K}_{0,0,K}}{\Delta t^\gamma},\ C^{2j+1}_K = \frac{W^{2m+1,K}_{1,j,K}}{\Delta t^\gamma}, j = 0,1,\cdots,m.\\
&C^{2j+2}_K = \frac{W^{2m+1,K}_{2,j,K} + W^{2m+1,K}_{0,j+1,K}}{\Delta t^\gamma},\ j = 0,1,\cdots,m-1,\ C^{2m+2}_K = \frac{W^{2m+1,K}_{2,m,K}}{\Delta t^\gamma}.\\
&E^0_K = \frac{W^{2m+2,K}_{0,0,K}}{\Delta t^\gamma},\ E^{2j+1}_K = \frac{W^{2m+2,K}_{1,j,K}}{\Delta t^\gamma}, j = 0,1,\cdots,m.\\
&E^{2j+2}_K = \frac{W^{2m+2,K}_{2,j,K} + W^{2m+2,K}_{0,j+1,K}}{\Delta t^\gamma},\ j = 0,1,\cdots,m-1,\ E^{2m+2}_K = \frac{W^{2m+2,K}_{2,m,K}}{\Delta t^\gamma}.
\end{aligned}
\tag{4.1}
$$

Then the numerical scheme (2.19) can be rewritten as follows:

$$
\begin{cases}
X^{2m+1}_K = x_0 + \sum\limits_{i=1}^{K} I_i(X^{2N}_{i-1}) + \sum\limits_{n=0}^{K-1}\sum\limits_{j=0}^{2N} B^{2m+1}_{n,j} f^j_n + \Delta t^\gamma \sum\limits_{j=0}^{2m+2} C^j_K f^j_K,\\
X^{2m+2}_K = x_0 + \sum\limits_{i=1}^{K} I_i(X^{2N}_{i-1}) + \sum\limits_{n=0}^{K-1}\sum\limits_{j=0}^{2N} B^{2m+2}_{n,j} f^j_n + \Delta t^\gamma \sum\limits_{j=0}^{2m+2} E^j_K f^j_K,\\
\qquad K = 0,1,\cdots,M-1; m = 0,1,\cdots,N-1.
\end{cases}
\tag{4.2}
$$

Lemma 4.1. *The coefficients $B^i_{n,j}, C^j_K$ and E^j_K, defined in* (4.1), *satisfy*

$$\sum_{n=0}^{K-1}\sum_{j=0}^{2N} |B^i_{n,j}| \le \frac{12KT^\gamma}{\Gamma(\gamma+1)}, i = 1,2,\cdots,2N, \tag{4.3}$$

$$|C^j_K| \le C(2m+2-j)^{\gamma-1},\ j = 0,1,\cdots,2m+2, \tag{4.4}$$

$$|E^j_K| \le C(2m+3-j)^{\gamma-1},\ j = 0,1,\cdots,2m+2. \tag{4.5}$$

Proof.

$$
\begin{aligned}
&\sum_{n=0}^{K-1}\sum_{j=0}^{2N} |B^i_{n,j}|\\
&= \sum_{n=0}^{K-1} [|W^{i,K}_{0,0,n}| + \sum_{j=0}^{N-1} |W^{i,K}_{1,j,n}| + \sum_{j=0}^{N-2} |W^{i,K}_{2,j,n} + W^{i,K}_{0,j+1,n}| + |W^{i,K}_{2,N-1,n}|]\\
&\le \sum_{n=0}^{K-1}\sum_{j=0}^{N-1} (|W^{i,K}_{0,j,n}| + |W^{i,K}_{1,j,n}| + |W^{i,K}_{2,j,n}|)\\
&= \frac{1}{\Gamma(\gamma)} \sum_{n=0}^{K-1}\sum_{j=0}^{N-1} (|\int_{t_n^{2j}}^{t_n^{2j+2}} (t^i_K - \tau)^{\gamma-1} \psi^{0,j}_n d\tau|
\end{aligned}
$$

$$
\begin{aligned}
&+|\int_{t_n^{2j}}^{t_n^{2j+2}}(t_K^i-\tau)^{\gamma-1}\psi_n^{1,j}d\tau|+|\int_{t_n^{2j}}^{t_n^{2j+2}}(t_K^i-\tau)^{\gamma-1}\psi_n^{2,j}d\tau|)\\
\leq&\frac{1}{\Gamma(\gamma)}\sum_{n=0}^{K-1}\sum_{j=0}^{N-1}\int_{t_n^{2j}}^{t_n^{2j+2}}(t_K^i-\tau)^{\gamma-1}(|\psi_n^{0,j}|+|\psi_n^{1,j}|+|\psi_n^{2,j}|)d\tau\\
\leq&\frac{6}{\Gamma(\gamma)}\sum_{n=0}^{K-1}\sum_{j=0}^{N-1}\int_{t_n^{2j}}^{t_n^{2j+2}}(t_K^i-\tau)^{\gamma-1}d\tau=\frac{6}{\Gamma(\gamma)}\sum_{n=0}^{K-1}\int_{t_n^{0}}^{t_n^{2N}}(t_K^i-\tau)^{\gamma-1}d\tau\\
\leq&\frac{6}{\Gamma(\gamma+1)}\sum_{n=0}^{K-1}[(t_K^i-t_n^0)^\gamma+(t_K^i-t_n^{2N})^\gamma]\leq\frac{12KT^\gamma}{\Gamma(\gamma+1)}.
\end{aligned}
$$

This proves (4.3). The proof of the coefficients C_K^j, E_K^j for (4.4) and (4.5) are similar as the Lemma 4.1 of the [1], we omitted them here. □

As for integer order differential equations, it is indicative to study the stability property of the scheme (4.2) with

$$
\begin{cases} f(t,x(t)):=\lambda x(t),\\ I_i(x)=\delta_i x, \end{cases} \tag{4.6}
$$

where λ is a real number, $\delta=max\{|\delta_1|,|\delta_2|,\cdots,|\delta_M|\}$.

Theorem 4.1. *Let $M_0=|x_0|$, then the scheme (4.2) with f, I_i given in (4.6) is stable with respect to the initial values under the condition*

$$
|\lambda\widetilde{B}_K^{2m+1}|\Delta t^\gamma<1, \tag{4.7}
$$

where

$$
|\widetilde{B}_K^{2m+1}|=\max\{|C_K^{2m+1}|+|C_K^{2m+2}|,|E_K^{2m+1}|+|E_K^{2m+2}|\}. \tag{4.8}
$$

That is, if (4.7) is satisfied, then

$$
|X_K^j|\leq CM_0,\ j=0,1,\cdots,2N; K=0,1,\cdots,M-1, \tag{4.9}
$$

where C only depends on $\delta,\lambda,K,\gamma,$ and T.

Proof. When $K=0$, plugging $f(t,x(t))=\lambda x(t)$ into (4.2) gives

$$
\begin{cases} X_0^{2m+1}=x_0+\lambda\Delta t^\gamma\sum\limits_{j=0}^{2m+2}C_0^jX_0^j,\\ X_0^{2m+2}=x_0+\lambda\Delta t^\gamma\sum\limits_{j=0}^{2m+2}E_0^jX_0^j,\ m=0,1,\cdots,N-1. \end{cases} \tag{4.10}
$$

Set

$$
|\widetilde{X}_0^{2i+1}|=|\widetilde{X}_0^{2i+2}|=\max\{|X_0^{2i+1}|,|X_0^{2i+2}|\},i=0,1,\cdots,m. \tag{4.11}
$$

According to Lemma 4.1, then (4.10) can be unified as

$$
\begin{aligned}
|\widetilde{X}_0^{2m+1}|\leq&|x_0|+C|\lambda|\Delta t^\gamma\sum_{j=0}^{2m}(2m+2-j)^{\gamma-1}|\widetilde{X}_0^j|\\
&+|\lambda|\Delta t^\gamma|\widetilde{B}_0^{2m+1}||\widetilde{X}_0^{2m+1}|.
\end{aligned} \tag{4.12}
$$

And (4.12) can be rewritten as

$$(1-|\lambda|\Delta t^{\gamma}|\widetilde{B}_0^{2m+1}|)|\widetilde{X}_0^{2m+1}| \le |x_0| + C|\lambda|\Delta t^{\gamma}\sum_{j=0}^{2m}(2m+2-j)^{\gamma-1}|\widetilde{X}_0^j|. \quad (4.13)$$

Under condition (4.7), (4.13) becomes

$$|\widetilde{X}_0^{2m+1}| \le C|x_0| + C|\lambda|\Delta t^{\gamma}\sum_{j=0}^{2m}(2m+2-j)^{\gamma-1}|\widetilde{X}_0^j|. \quad (4.14)$$

When $0<\gamma<1$, it holds $(2m+2-j)^{\gamma-1} \le (2m+1-j)^{\gamma-1}$, thus

$$|\widetilde{X}_0^{2m+1}| \le C|x_0| + C|\lambda|\Delta t^{\gamma}\sum_{j=0}^{2m}(2m+1-j)^{\gamma-1}|\widetilde{X}_0^j|. \quad (4.15)$$

Applying the discrete Gronwall Theorem 3.1 in [1] or Theorem 7.2 in [3] to (4.15) gives

$$|\widetilde{X}_0^{2m+1}| \le Cx_0E_{\gamma}(C|\lambda|\Gamma(\gamma)T^{\gamma}) \le CM_0, \quad (4.16)$$

where C depends on λ, γ, and T.

When $K \ge 1$, x_K^0 satisfy

$$X_K^0 = X_{K-1}^{2N} + I_K(X_{K-1}^{2N}) = X_{K-1}^{2N} + \delta_K X_{K-1}^{2N} = (1+\delta_K)X_{K-1}^{2N}. \quad (4.17)$$

As $K=1$, according to (4.17) and (4.16), we have

$$|X_1^0| = |(1+\delta_1)X_0^{2N}| \le CM_0. \quad (4.18)$$

Plugging (4.6) into (4.2) gives

$$\begin{cases} X_1^{2m+1} = x_0 + \delta_1 X_0^{2N} + \lambda\displaystyle\sum_{j=0}^{2N} B_{0,j}^{2m+1}X_0^j + \lambda\Delta t^{\gamma}\sum_{j=0}^{2m+2} C_1^j x_1^j, \\ X_1^{2m+2} = x_0 + \delta_1 X_0^{2N} + \lambda\displaystyle\sum_{j=0}^{2N} B_{0,j}^{2m+2}X_0^j + \lambda\Delta t^{\gamma}\sum_{j=0}^{2m+2} E_1^j X_1^j, \\ \qquad m = 0,1,\cdots,N-1. \end{cases} \quad (4.19)$$

Similar to (4.10), we obtain

$$\begin{aligned} |\widetilde{X}_1^{2m+1}| \le{}& C|M_0| + |\lambda|\sum_{j=0}^{2N}|\overline{B}_{0,j}^{2m+1}||\widetilde{X}_0^j| + C|\lambda|\Delta t^{\gamma}\sum_{j=0}^{2m}(2m+2-j)^{\gamma-1}|\widetilde{X}_1^j| \\ &+|\lambda|\Delta t^{\gamma}|\widetilde{B}_1^{2m+1}||\widetilde{X}_1^{2m+1}|, \end{aligned} \quad (4.20)$$

where

$$|\overline{B}_{0,j}^{2m+1}| = |\overline{B}_{0,j}^{2m+2}| = \max\{|B_{0,j}^{2m+1}|, |B_{0,j}^{2m+2}|\}.$$

By using (4.9) for $K = 0$, Lemma 4.1, we obtain

$$|\widetilde{X}_1^{2m+1}| \leq CM_0 + C|\lambda|\frac{12KT^\gamma}{\Gamma(\gamma+1)}M_0 + C|\lambda|\Delta t^\gamma \sum_{j=0}^{2m}(2m+2-j)^{\gamma-1}|\widetilde{X}_1^j| + |\lambda|\Delta t^\gamma|\widetilde{B}_1^{2m+1}||\widetilde{X}_1^{2m+1}|. \tag{4.21}$$

And (4.21) can be rewritten as flowing:

$$\begin{aligned}&(1-|\lambda|\Delta t^\gamma|\widetilde{B}_1^{2m+1}|)|\widetilde{X}_1^{2m+1}|\\ \leq\ &(1+|\lambda|\frac{12KT^\gamma}{\Gamma(\gamma+1)})CM_0 + C|\lambda|\Delta t^\gamma \sum_{j=0}^{2m}(2m+2-j)^{\gamma-1}|\widetilde{X}_1^j|.\end{aligned} \tag{4.22}$$

Under condition (4.7), (4.22) becomes

$$|\widetilde{X}_1^{2m+1}| \leq CM_0 + C|\lambda|\Delta t^\gamma \sum_{j=0}^{2m}(2m+2-j)^{\gamma-1}|\widetilde{X}_1^j|. \tag{4.23}$$

When $0 < \gamma < 1$, it holds $(2m+2-j)^{\gamma-1} \leq (2m+1-j)^{\gamma-1}$, thus

$$|\widetilde{X}_1^{2m+1}| \leq CM_0 + C|\lambda|\Delta t^\gamma \sum_{j=0}^{2m}(2m+1-j)^{\gamma-1}|\widetilde{X}_1^j|. \tag{4.24}$$

Applying the discrete Gronwall Theorem to (4.24) gives

$$|\widetilde{X}_1^{2m+1}| \leq CM_0 E_\gamma(C|\lambda|\Gamma(\gamma)T^\gamma) \leq CM_0, \tag{4.25}$$

where C depends on λ, γ, and T.

Combining the above estimate with (4.18),(4.25) yields

$$|X_1^j| \leq CM_0, j = 0, 1, \cdots, 2N. \tag{4.26}$$

We use mathematical induction method, If $l = K-1$, we have

$$|X_{K-1}^j| \leq CM_0,\ j = 0, 1, \cdots, 2N. \tag{4.27}$$

Next, we prove

$$|X_K^j| \leq CM_0,\ j = 0, 1, \cdots, 2N. \tag{4.28}$$

Plugging (4.6) into (4.2), we get

$$\begin{cases} X_K^{2m+1} = x_0 + \sum\limits_{i=1}^{K}\delta_i X_{i-1}^{2N} + \lambda\sum\limits_{n=0}^{K-1}\sum\limits_{j=0}^{2N}B_{n,j}^{2m+1}X_n^j + \lambda\Delta t^\gamma\sum\limits_{j=0}^{2m+2}C_K^jX_K^j,\\ X_K^{2m+2} = x_0 + \sum\limits_{i=1}^{K}\delta_i X_{i-1}^{2N} + \lambda\sum\limits_{n=0}^{K-1}\sum\limits_{j=0}^{2N}B_{n,j}^{2m+2}X_n^j + \lambda\Delta t^\gamma\sum\limits_{j=0}^{2m+2}E_K^jX_K^j,\\ \qquad m = 0, 1, \cdots, N-1.\end{cases} \tag{4.29}$$

Set

$$
\begin{aligned}
&|\widetilde{X}_K^{2i+1}| = |\widetilde{X}_K^{2i+2}| = \max\{|X_K^{2i+1}|, |X_K^{2i+2}|\}, i = 0, 1, \cdots, m, \\
&|\overline{B}_{n,j}^{2m+1}| = |\overline{B}_{n,j}^{2m+2}| = \max\{|B_{n,j}^{2m+1}|, |B_{n,j}^{2m+2}|\}.
\end{aligned} \tag{4.30}
$$

By using (4.27), we have

$$
\begin{aligned}
|\widetilde{X}_K^{2m+1}| \leq{} & CM_0 + CK\delta M_0 + |\lambda| \sum_{n=0}^{K-1} \sum_{j=0}^{2N} |\overline{B}_{n,j}^{2m+1}||\widetilde{X}_n^j| \\
& + C|\lambda|\Delta t^\gamma \sum_{j=0}^{2m} (2m+2-j)^{\gamma-1} |\widetilde{X}_K^j| + |\lambda|\Delta t^\gamma |\widetilde{B}_K^{2m+1}||\widetilde{X}_K^{2m+1}|.
\end{aligned} \tag{4.31}
$$

The proof is similar to the X_1^j, we get

$$
|\widetilde{X}_K^{2m+1}| \leq CM_0. \tag{4.32}
$$

According to (4.17), we obtain

$$
|X_K^0| = |X_{K-1}^{2N} + I_K(X_{K-1}^{2N})| = |(1+\delta_K)X_{K-1}^{2N}| \leq CM_0. \tag{4.33}
$$

Combining (4.32) with (4.33), then we obtain

$$
|X_K^j| \leq CM_0, j = 0, 1, \cdots, 2N. \tag{4.34}
$$

The proof of (4.9) is complete. □

In the error analysis, we consider the general $f(t,x)$ and $I_k(x)$ which satisfying the following Lipschitz condition with respect to the second variable: there exists a constant L_1, L_2, such that

$$
|I_k(x_1) - I_k(x_2)| \leq L_1|x_1 - x_2|, \ \forall x_1, x_2 \in R, \tag{4.35}
$$

$$
|f(t,x_1) - f(t,x_2)| \leq L_2|x_1 - x_2|, \ \forall x_1, x_2 \in R; t \in J'. \tag{4.36}
$$

Theorem 4.2. *Let x be the exact solution of (2.1), $\{X_K^j\}_{j=0}^{2N}, K = 0, 1, \cdots, M-1$ be the numerical solution of (4.2). If the time step size Δt satisfies*

$$
|\widetilde{B}_K^{2m+1}| L_2 \Delta t^\gamma < 1, \tag{4.37}
$$

where $|\widetilde{B}_K^{2m+1}|$ is define in (4.8). If $0 < \gamma \leq 1$, then the following error estimates hold:

$$
|x(t_K^j) - X_K^j| \leq C\Delta t^{3+\gamma}, \ j = 0, 1, \cdots, 2N; K = 0, 1, \cdots, M-1. \tag{4.38}
$$

where C only depends on f, γ, T, M, L_1 and L_2.

Proof. Let $e_l^j = x(t_l^j) - X_l^j, j = 0, 1, \cdots, 2N; l = 0, 1, \cdots, M-1$. As $l = 0$, it is readily seen that $e_0^0 = 0$, according to [8, 17], and $e_0^j, j \geq 1$, satisfy

$$
|e_0^j| \leq C\Delta t^{3+\gamma}. \tag{4.39}
$$

When $l \geq 1$, e_l^0 satisfy

$$\begin{aligned}|e_l^0| &= |x(t_l^0) - X_l^0| = |(x(t_{l-1}^{2N}) + I(x(t_{l-1}^{2N}))) - (X_{l-1}^{2N} + I(X_{l-1}^{2N}))| \\ &\leq |x(t_{l-1}^{2N}) - X_{l-1}^{2N}| + |I(x(t_{l-1}^{2N})) - I(X_{l-1}^{2N})| \\ &\leq |e_{l-1}^{2N}| + L_1|e_{l-1}^{2N}| = (1+L_1)|e_{l-1}^{2N}|. \end{aligned} \tag{4.40}$$

As $l = 1$, according to (4.40)

$$|e_1^0| \leq (1+L_1)|e_0^{2N}| \leq C(1+L_1)\Delta t^{3+\gamma} \leq C\Delta t^{3+\gamma}. \tag{4.41}$$

According to I_1, f satisfy (4.35) and (4.36), we get

$$\begin{cases} |e_1^{2m+1}| \leq L_1|e_0^{2N}| + L_2 \displaystyle\sum_{j=0}^{2N} |B_{0,j}^{2m+1}| \cdot |e_0^j| \\ \qquad\qquad +L_2\Delta t^\gamma \displaystyle\sum_{j=0}^{2m+2} |C_1^j| \cdot |e_1^j| + |r_1^{2m+1}(\Delta t)|, \\ |e_1^{2m+2}| \leq L_1|e_0^{2N}| + L_2 \displaystyle\sum_{j=0}^{2N} |B_{0,j}^{2m+2}| \cdot |e_0^j| \\ \qquad\qquad +L_2\Delta t^\gamma \displaystyle\sum_{j=0}^{2m+2} |E_1^j| \cdot |e_1^j| + |r_1^{2m+2}(\Delta t)|. \end{cases} \tag{4.42}$$

Let

$$\begin{aligned} &|\varepsilon_K^{2i+1}| = |\varepsilon_K^{2i+2}| = \max\{|e_K^{2i+1}|, |e_K^{2i+2}|\}, \\ &|r_K(\Delta t)| = \max\{|r_K^{2m+1}(\Delta t)|, |r_K^{2m+2}(\Delta t)|\}, K = 0, 1, \cdots, M-1. \end{aligned} \tag{4.43}$$

According to (4.8), (4.30) and (4.43), then (4.42) becomes

$$\begin{aligned}(1 - L_2\Delta t^\gamma |\widetilde{B}_1^{2m+1}|)|\varepsilon_1^{2m+1}| \leq \; & L_2 C\Delta t^\gamma \sum_{j=0}^{2m} (2m+2-j)^{\gamma-1}|\varepsilon_1^j| \\ & +L_1|\varepsilon_0^{2N}| + \sum_{j=0}^{2N} |\overline{B}_{0,j}^{2m+1}||\varepsilon_0^j| + |r_1(\Delta t)|. \end{aligned} \tag{4.44}$$

Under condition (4.37), and Lemma 3.1, Lemma 4.1, we get

$$|\varepsilon_1^{2m+1}| \leq L_2 C\Delta t^\gamma \sum_{j=0}^{2m} (2m+2-j)^{\gamma-1}|\varepsilon_1^j| + C\Delta t^{3+\gamma}. \tag{4.45}$$

When $0 < \gamma < 1$, it holds $(2m+2-j)^{\gamma-1} \leq (2m+1-j)^{\gamma-1}$, thus

$$|\varepsilon_1^{2m+1}| \leq L_2 C\Delta t^\gamma \sum_{j=0}^{2m} (2m+1-j)^{\gamma-1}|\varepsilon_1^j| + C\Delta t^{3+\gamma}. \tag{4.46}$$

Applying the discrete Gronwall Theorem to (4.46) gives

$$|\varepsilon_1^{2m+1}| \leq CE_\gamma(CL_2\Gamma(\gamma)T^\gamma)\Delta t^{3+\gamma} \leq C\Delta t^{3+\gamma}. \tag{4.47}$$

Combining the above estimate with (4.41) and (4.47) yields

$$|e_1^k| \leq C\Delta t^{3+\gamma},\ k > 0. \tag{4.48}$$

We use mathematical induction method, If $l = K - 1$, we obtain

$$|e_{K-1}^k| \leq C\Delta t^{3+\gamma},\ k > 0. \tag{4.49}$$

Next, we prove

$$|e_K^k| \leq C\Delta t^{3+\gamma},\ k > 0. \tag{4.50}$$

As $l = K$, according to (4.40), we have

$$|e_K^0| \leq (1 + L_1)|e_{K-1}^{2N}| \leq C(1 + L_1)\Delta t^{3+\gamma} \leq C\Delta t^{3+\gamma}. \tag{4.51}$$

For $e_K^j, j \geq 1$, according to I_i, f satisfy (4.35) and (4.36), we get

$$\begin{cases} |e_K^{2m+1}| \leq L_1 \displaystyle\sum_{i=1}^{K} |e_{i-1}^{2N}| + L_2 \sum_{n=0}^{K-1} \sum_{j=0}^{2N} |B_{n,j}^{2m+1}| \cdot |e_n^j| \\ \qquad\qquad +L_2 \Delta t^\gamma \displaystyle\sum_{j=0}^{2m+2} |C_K^j| \cdot |e_K^j| + |r_K^{2m+1}(\Delta t)|, \\ |e_K^{2m+2}| \leq L_1 \displaystyle\sum_{i=1}^{K} |e_{i-1}^{2N}| + L_2 \sum_{n=0}^{K-1} \sum_{j=0}^{2N} |B_{n,j}^{2m+2}| \cdot |e_n^j| \\ \qquad\qquad +L_2 \Delta t^\gamma \displaystyle\sum_{j=0}^{2m+2} |E_K^j| \cdot |e_K^j| + |r_K^{2m+2}(\Delta t)|. \end{cases} \tag{4.52}$$

The proof of (4.52) is similar as the (4.42), we have

$$|\varepsilon_K^{2m+1}| \leq C\Delta t^{3+\gamma}. \tag{4.53}$$

Combining the above estimate with (4.51) and (4.53) yields

$$|e_K^k| \leq C\Delta t^{3+\gamma},\ k > 0. \tag{4.54}$$

The proof is then complete. □

5. Numerical examples

We carry out in this section a series of numerical experiments and present some results to confirm our theoretical statements. Our main purpose is to check the convergence behavior of the numerical solution with respect to the step size Δt.

Example 5.1. We consider the problem (2.1) with $I_k(x) = 1, k = 1, 2, 3$, and the following right hand side function $f(t, x(t))$ and the corresponding exact solution $x(t)$

$$f(t, x(t)) = \begin{cases} \mu_1 t^4 + t^{4+\gamma} - x(t), t \in (t_0, t_1], \\ \mu_1 t^4 + 1 + t^{4+\gamma} - x(t), t \in (t_1, t_2], \\ \mu_1 t^4 + 2 + t^{4+\gamma} - x(t), t \in (t_2, t_3], \\ \mu_1 t^4 + 3 + t^{4+\gamma} - x(t), t \in (t_3, T], \end{cases} \quad x(t) = \begin{cases} t^{4+\gamma}, t \in (t_0, t_1], \\ 1 + t^{4+\gamma}, t \in (t_1, t_2], \\ 2 + t^{4+\gamma}, t \in (t_2, t_3], \\ 3 + t^{4+\gamma}, t \in (t_3, T], \end{cases}$$

where $\mu_1 = \frac{\Gamma(5+\gamma)}{24}$, the function f is linear with respect to x.

All the results presented in this example which corresponding to the numerical solution captured at $T = 1$. Figure 1 show the comparison of the exact solution and the numerical solution with $\gamma = 0.5$. From Figure 1, we find the numerical solution are well coincident with the exact solution. In Tables 1 we list the maximum errors $\max\limits_{i,j} |x(t_i^j) - X_i^j|$ as a function of Δt for several γ. Also shown are the corresponding rates. From Table 1, it is observed that for all γ smaller than 1, the convergence rate is close to $3 + \gamma$. This is in a good agreement with the theoretical prediction.

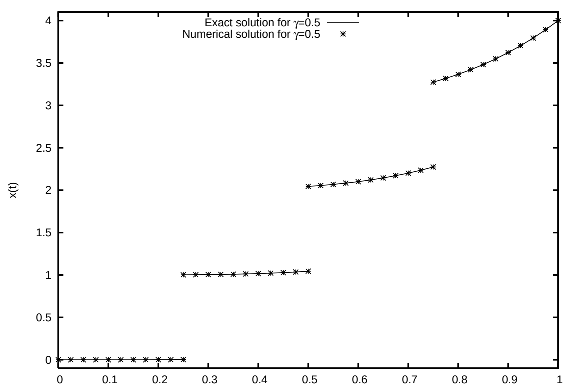

Figure 1. The exact solution and the numerical solution with $\gamma = 0.5$ for Example 5.1.

Table 1. Maximum errors and convergence order with $\gamma = 0.2, 0.5$ and 0.8 for Example 5.1.

Δt	$\gamma = 0.2$	Order	$\gamma = 0.5$	Order	$\gamma = 0.8$	Order
$\frac{1}{10}$	3.4541E-006	-	3.6569E-006	-	1.7258E-006	-
$\frac{1}{20}$	3.9595E-007	3.1249	3.3517E-007	3.4476	1.2614E-007	3.7741
$\frac{1}{40}$	4.4898E-008	3.1406	3.0359E-008	3.4647	9.2788E-009	3.7650
$\frac{1}{80}$	5.0541E-009	3.1511	2.7287E-009	3.4758	6.7948E-010	3.7714
$\frac{1}{160}$	5.6595E-010	3.1587	2.4400E-010	3.4832	4.9594E-011	3.7762
$\frac{1}{320}$	6.3119E-011	3.1645	2.1741E-011	3.4883	3.6104E-012	3.7799

Example 5.2. We consider the problem (2.1) with $I_k(x) = 0.8, k = 1, 2, 3$, and the following right hand side function $f(t, x(t))$

$$f(t,x(t)) = \begin{cases} \mu_2 + (1.5t^{0.5\gamma} - t^4)^3 - [x(t)]^{1.5}, t \in (t_0, t_1], \\ \mu_2 + [0.8 + (1.5t^{0.5\gamma} - t^4)^2]^{1.5} - [x(t)]^{1.5}, t \in (t_1, t_2], \\ \mu_2 + [1.6 + (1.5t^{0.5\gamma} - t^4)^2]^{1.5} - [x(t)]^{1.5}, t \in (t_2, t_3], \\ \mu_2 + [2.4 + (1.5t^{0.5\gamma} - t^4)^2]^{1.5} - [x(t)]^{1.5}, t \in (t_3, T], \end{cases}$$

where $\mu_2 = \frac{40320}{\Gamma(9-\gamma)} t^{8-\gamma} - 3\frac{\Gamma(5+0.5\gamma)}{\Gamma(5-0.5\gamma)} + \frac{9}{4}\Gamma(\gamma + 1)$, and the corresponding exact solution $x(t)$ is given

$$x(t) = \begin{cases} t^8 - 3t^{4+0.5\gamma} + 2.25t^{\gamma}, t \in (t_0, t_1], \\ 0.8 + t^8 - 3t^{4+0.5\gamma} + 2.25t^{\gamma}, t \in (t_1, t_2], \\ 1.6 + t^8 - 3t^{4+0.5\gamma} + 2.25t^{\gamma}, t \in (t_2, t_3], \\ 2.4 + t^8 - 3t^{4+0.5\gamma} + 2.25t^{\gamma}, t \in (t_3, T]. \end{cases}$$

Note that the function f is nonlinear with respect to x.

First we take $T = 1$. Figure 2 show the comparison of the exact solution and the numerical solution with $\gamma = 0.5$. From Figure 2, we find the numerical solution can be a good approximation of the exact solution. Tables 2 show the maximum errors and convergence orders as a function of the time step size for several $\gamma = 0.3, 0.5, 0.7$. Once again these results confirm that the convergence of the numerical solution is of order $3 + \gamma$ for $0 < \gamma < 1$.

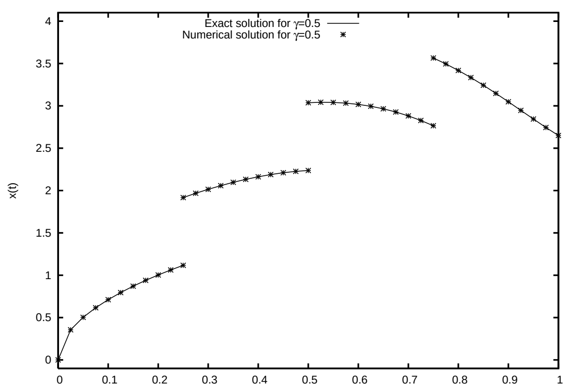

Figure 2. The exact solution and the numerical solution with $\gamma = 0.5$ for Example 5.2.

Table 2. Maximum errors and convergence order with $\gamma = 0.3, 0.5$ and 0.7 for Example 5.2.

Δt	$\gamma = 0.3$	Order	$\gamma = 0.5$	Order	$\gamma = 0.7$	Order
$\frac{1}{10}$	3.6542E-005	-	3.8928E-005	-	2.6572E-005	-
$\frac{1}{20}$	4.2330E-006	3.1093	3.7957E-006	3.3583	2.1901E-006	3.6008
$\frac{1}{40}$	4.7096E-007	3.1680	3.5688E-007	3.4108	1.7627E-007	3.6351
$\frac{1}{80}$	5.1135E-008	3.2032	3.2847E-008	3.4416	1.3995E-008	3.6548
$\frac{1}{160}$	5.4645E-009	3.2261	2.9834E-009	3.4607	1.1017E-009	3.6671
$\frac{1}{320}$	5.7739E-010	3.2421	2.6865E-010	3.4731	8.6235E-011	3.6753

6. Conclusion

In this work, we have developed and analyzed efficient numerical methods for the impulsive fractional differential equations. Firstly, the differential equation with initial value problem (2.1) transform the equivalent to the Volterra integral equation (2.3). Secondly, we construct a block-by-block method (2.19) for the integral equation (2.3), and then we obtain an estimate for the truncation errors of the numerical scheme. In Section 4, the convergence and the stability analysis are given. The numerical solution converges to the exact solution with order $3+\gamma$ for $0 < \gamma < 1$ is proved, where γ is the order of the Caputo fractional derivative. Finally, we carried out numerical tests confirmed the theoretical prediction. In the future, we will follow this idea to construct higher order schemes to impulsive fractional partial differential equations with stochastic or delay derivatives.

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