Journal of Applied Analysis and Computation
Volume 9, Number 4, August 2019, 1527–1557
Website:http://jaac.ijournal.cn
DOI:10.11948/2156-907X.20180309

NEW PREDICTOR-CORRECTOR APPROACH FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS: ERROR ANALYSIS AND STABILITY*

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Abstract In this paper, the predictor-corrector approach is used to propose two algorithms for the numerical solution of linear and non-linear fractional differential equations (FDE). The fractional order derivative is taken to be in the sense of Caputo and its properties are used to transform FDE into a Volterra-type integral equation. Simpson's 3/8 rule is used to develop new numerical schemes to obtain the approximate solution of the integral equation associated with the given FDE. The error and stability analysis for the two methods are presented. The proposed methods are compared with the ones available in the literature. Numerical simulation is performed to demonstrate the validity and applicability of both the proposed techniques. As an application, the problem of dynamics of the new fractional order non-linear chaotic system introduced by Bhalekar and Daftardar-Gejji is investigated by means of the obtained numerical algorithms.

Keywords Predictor-corrector approach, fractional differential equation, Caputo derivative, Volterra integral equation.

MSC(2010) 26A33, 45D05, 65L07, 65L20.

1. Introduction

Fractional calculus is a branch of mathematical analysis which deals with integrals and derivatives of arbitrary (non-integer) order and their applications. However, the topic of fractional calculus did not attract much attention until recently, especially, concerning its applications. It might have been due to the reasons such as its intrinsic complexity [25], self-sufficiency of the classical calculus [3], lack of acceptable geometric or physical interpretation of fractional derivative [30, 32], multiple definitions for fractional derivatives [13], etc. However, the theoretical development and potential applications of fractional calculus in science and engineering helped to make it one of the hot topics for the researchers during the last few decades.

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*This project is supported by a research grant of the University of Tabriz.

Examples include Medical and biological sciences [38], psychology and social sciences [34], dynamical phenomena in physics [37], economy [11], electronics and control theory [1,33], etc. For more examples and details, we refer the reader to [6].

In mathematical modeling of the real world problems, the underlying dynamics of a system depends on its history (the stored information) [35]. This means that the future state of the system relies not only on its present state, but also upon the the preceding history of states as well. Since the integer order differential operator is a local operator, it cannot describe the hereditory behavior of such systems. On the other hand, the fractional order derivative, being nonlocal in its nature, is found to be a suitable tool to model nonlocal phenomena either in space or in time [18, 24]. Fractional order operators involved in the model cover the entire progress of the process under investigation. Thus such operators serve as an excellent instrument for the description of memory and hereditary properties of the systems [2, 5]. In fact, the nonlocal characteristic of fractional order operators played a key role in the popularity of fractional calculus [24, 26].

Mathematical modelling of real world systems in terms of fractional-order derivatives gives rise to a set of FDE, which cannot be solved analytically in many of the cases [8, 10]. However, analytic solutions for some FDE have been obtained by means of Adomians decomposition method, Mellin transform, Fourier transform and Laplace transform under certain circumstances, especially for homogeneous linear FDE with constant coefficients [23,29]. These solutions generally involve special functions such as Mittag-Leffler function, which are difficult to interpret [45]. In order to make such solutions plausible, high computational cost is required [23]. The non-availability or availability in terms of complicated mathematical functions of analytic solutions instigated researchers to develop numerical and approximate analytic methods for solving FDE. However, owing to difficulties in analysis of numerical methods for FDE, the scope of numerical methods for FDE is limited [8] and many researchers are working on this topic to develop new numerical algorithms for FDE.

Diethelm et al [20] suggested the predictor-corrector method for solving FDE. This method is a generalization of the classical one-step Adams-Bashforth-Moulton scheme, which is a well known technique for obtaining the numerical solution of first order equations. It was pointed out that the accuracy of predictor-corrector method can be improved with teh aid of the Richardson extrapolation, short memory principle and corresponding mixed numerical schemes. The detailed error analysis for this algorithm was given in [21]. Later, some researchers [12, 16, 17] focused on improving this method and solved some applied fractional order problems [15, 19, 27, 40]. Deng [17] introduced the new numerical approximation by combining the short memory principle and the predictor-corrector approach. Daftardar-Gejji et al [12] used the idea of iteration to modify Adams method and investigated dynamics of a fractional chaotic system by this new approach. Li et al [28] applied the Simpson's rule instead of trapezoidal quadrature formula to achieve higher order numerical algorithm for fractional differential equations. In a recent work, Yang et al [43] applied a new computational approach for solving local fractional wave equation based on the Gao-Yang-Kang version of the local fractional calculus. Motivated by the recent development on the topic, a new improved version of the predictor-corrector method possessing a better convergence has been proposed in this paper.

There are several definitions of fractional derivative in the literature. Among these definitions, Caputo and Riemann-Liouville type fractional derivatives gained much popularity [5,23]. In order to develop the fractional calculus without singular kernel of exponential function, the Yang-Srivastava-Machado fractional derivative was proposed in the articles [36,39]. Yang et al [42] introduced the concept of so called Yang-Gao-Machado-Baleanu fractional derivative to deal with the fractional calculus without singular kernel of sinc function. Cattani [9] studied Sinc-fractional operators on Shannon wavelet space. For the details on general fractional calculus operators involving special functions and variable-order fractional operators, see [41,44]. Fractional differential equations involving Riemann-Liouville fractional derivative need initial conditions in terms of the unknown function together with its Riemann-Liouville fractional derivative, which are unavailable for most of the practical applications and have no physical meaning. For this reason, the Riemann-Liouville fractional derivative is not always the automatic choice for real applications [14]. However, Fractional differential equations involving Caputo fractional derivative utilize classical initial conditions, which are accurately measurable and physically interpretable. Hence, we choose the Caputo fractional derivative in the present setting.

The rest of the paper is organized as follows. In Section 2, the detailed construction of the predictor-corrector scheme is described, and improved algorithm for this scheme is presented. In Section 3, the truncation error analysis of the proposed methods is derived through a series of Lemmas and Theorems. The stability of the numerical methods is proven in Section 4. To demonstrate the effectiveness of the proposed methods, we apply these methods to solve some numerical examples in Section 5. Fractional analogue of the new chaotic system introduced by Bhalekar and Daftardar-Gejji is also investigated in this section and relevant phase portraits are obtained by means of new improved numerical algorithm for different values of the order of fractional system. Finally, Some concluding remarks are given in Section 6.

2. The numerical method

We consider and investigate the the numerical solution for the following initial value problem:

$$\begin{cases} {}_0^C D_t^\alpha y(t) = f(t, y(t)), & 0 \le t \le T, \quad \alpha > 0, \\ y^{(k)}(x_0) = y_0^{(k)}, & k = 0, 1, \ldots, \lceil \alpha \rceil - 1, \end{cases} \tag{2.1}$$

where $\lceil \alpha \rceil$ is the first integer not less than α, ${}_0^C D_t^\alpha y(t)$ denotes Caputo fractional derivative defined by

$${}_0^C D_t^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{y^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad n-1 < \alpha \le n,$$

where $y^{(n)}(\tau)$ is the classical nth-order derivative of $y(\tau)$. Throughout the forthcoming analysis, it is assumed that $f(t, y(t))$ is a continuous function which satisfies a Lipschitz condition with respect to second argument, that is, $|f(t, y) - f(t, x)| \le L|y - x|$, which $L > 0$. Notice that continuity and Lipschitz conditions are sufficient to ensure existence of a unique solution to the problem (2.1) on the interval

$[0, T]$ [28]. Applying Laplace transform to both sides of (2.1), we get

$$s^{\alpha}Y(s) - \sum_{i=0}^{\lceil\alpha\rceil-1} s^{\alpha-i-1}y^{(i)}(0) = F(s, Y(s)),$$

which, on taking inverse Laplace transform, yields

$$y(t) = \sum_{i=0}^{\lceil\alpha\rceil-1} \frac{t^i}{k!}y^{(i)}(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau, y(\tau))d\tau. \tag{2.2}$$

Observe that the integral equation (2.2) is equivalent to the problem (2.1).

For computational convenience, let $F_j = f(t_j, y_j)$ and $F(t_j) = f(t_j, y(t_j))$, where y_j is the numerical approximation to $y(t_j)$.

2.1. Main algorithm

In this subsection, a new algorithm based on predictor-corrector scheme is designed for solving problem (2.1) by discretizing Eq. (2.2) with uniform grids $t_j = jh$, $(j = 0, 1, ..., N)$, $h = [\frac{T}{N}]$. Like the classic predictor-corrector method [20], the basic idea is to calculate approximations $y_j, j = 0, 1, .., k$ and then use these values to obtain the approximation y_{k+1} via Eq. (2.2). In order to construct the high order scheme, we use the Simpson's 3/8 rule with nodes t_j, taken with respect to the weight function $(t_{k+1} - \cdot)^{\alpha-1}$ to evaluate the integral in Eq. (2.2). In order to do so, we need nodes t_j, $(j = 0, 1, ..., k+1)$, $t_{j+\frac{1}{3}}$ and $t_{j+\frac{2}{3}}$, $(j = 0, 1, ..., k)$. New method for approximating $y_{k+\frac{1}{3}}$, $y_{k+\frac{2}{3}}$ and y_{k+1} will be developed in three steps.

Step 1. (An explicit algorithm for calculating $y_{k+\frac{1}{3}}$)
The discretized form of (2.2) to calculate $y\left(t_{k+\frac{1}{3}}\right)$ is

$$y\left(t_{k+\frac{1}{3}}\right) = \sum_{i=0}^{\lceil\alpha\rceil-1} \frac{t^i_{k+\frac{1}{3}}}{i!}y^{(i)}(0) + \frac{1}{\Gamma(\alpha)}\int_0^{t_{k+\frac{1}{3}}} (t_{k+\frac{1}{3}}-\tau)^{\alpha-1}F(\tau)d\tau. \tag{2.3}$$

The product rectangle formula is used to approximate the integral in (2.3) as follows.

$$\begin{aligned} I_{k+\frac{1}{3}} &= \int_0^{t_{k+\frac{1}{3}}} (t_{k+\frac{1}{3}}-\tau)^{\alpha-1}F(\tau)d\tau = \int_0^{t_{k+\frac{1}{3}}} (t_{k+\frac{1}{3}}-\tau)^{\alpha-1}\hat{F}(\tau)d\tau \\ &= \sum_{j=0}^{k-1}\left(\left[\int_{t_j}^{t_{j+\frac{1}{3}}} F(t_j) + \int_{t_{j+\frac{1}{3}}}^{t_{j+1}} F(t_{j+1})\right](t_{k+\frac{1}{3}}-\tau)^{\alpha-1}d\tau\right) \\ &\quad + \int_{t_k}^{t_{k+\frac{1}{3}}} (t_{k+\frac{1}{3}}-\tau)^{\alpha-1}F(t_k)d\tau. \end{aligned} \tag{2.4}$$

Now Eq. (2.4) simplifies to the following form:

$$\int_0^{t_{k+\frac{1}{3}}} (t_{k+\frac{1}{3}}-\tau)^{\alpha-1}F(\tau)d\tau = \sum_{j=0}^{k} e_{j,k+\frac{1}{3}}F(t_j), \tag{2.5}$$

where

$$e_{j,k+\frac{1}{3}} = \frac{h^{\alpha}}{\alpha}\begin{cases}(k+\frac{1}{3})^{\alpha}-k^{\alpha}, & j=0,\\ (k-j+1)^{\alpha}-(k-j)^{\alpha}, & 1\le j\le k.\end{cases} \tag{2.6}$$

In this way, $y_{k+\frac{1}{3}}$ can be determined by the following formula:

$$y_{k+\frac{1}{3}} = \sum_{i=0}^{\lceil\alpha\rceil-1}\frac{t^{i}_{k+\frac{1}{3}}}{i!}y^{(i)}(0)+\frac{1}{\Gamma(\alpha)}\sum_{j=0}^{k}e_{j,k+\frac{1}{3}}F_j. \tag{2.7}$$

Step 2. (An explicit algorithm for finding $y_{k+\frac{2}{3}}$)

The discretized form of (2.2) to calculate $y\left(t_{k+\frac{2}{3}}\right)$ is

$$y\left(t_{k+\frac{2}{3}}\right) = \sum_{i=0}^{\lceil\alpha\rceil-1}\frac{t^{i}_{k+\frac{2}{3}}}{i!}y^{(i)}(0)+\frac{1}{\Gamma(\alpha)}\int_0^{t_{k+\frac{2}{3}}}(t_{k+\frac{2}{3}}-\tau)^{\alpha-1}F(\tau)d\tau. \tag{2.8}$$

To determine $y_{k+\frac{2}{3}}$, the integral in (2.8) is approximated as follows:

$$\begin{aligned}I_{k+\frac{2}{3}} &= \int_0^{t_{k+\frac{2}{3}}}(t_{k+\frac{2}{3}}-\tau)^{\alpha-1}F(\tau)d\tau = \left[\int_0^{t_k}+\int_{t_k}^{t_{k+\frac{2}{3}}}\right](t_{k+\frac{2}{3}}-\tau)^{\alpha-1}\hat{F}(\tau)d\tau\\ &= \sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}(t_{k+\frac{2}{3}}-\tau)^{\alpha-1}\hat{F}_{j+1}(\tau)d\tau+\int_{t_k}^{t_{k+\frac{2}{3}}}(t_{k+\frac{2}{3}}-\tau)^{\alpha-1}\hat{F}_{k+1}(\tau)d\tau,\end{aligned} \tag{2.9}$$

where, in each interval, $\hat{F}_{j+1}(\tau)$ is the piecewise linear interpolation for $F(\tau)$ at the points t_j and $t_{j+\frac{1}{3}}$. For example, in the interval $[t_j, t_{j+1}]$, we have

$$\begin{aligned}\int_{t_j}^{t_{j+1}}(t_{k+\frac{2}{3}}-z)^{\alpha-1}\hat{g}(z)dz &= \int_{t_j}^{t_{j+1}}(t_{k+\frac{2}{3}}-z)^{\alpha-1}\frac{z-t_{j+\frac{1}{3}}}{t_j-t_{j+\frac{1}{3}}}g(t_j)dz\\ &\quad+\int_{t_j}^{t_{j+1}}(t_{k+\frac{2}{3}}-z)^{\alpha-1}\frac{z-t_j}{t_{j+\frac{1}{3}}-t_j}.g(t_{j+\frac{1}{3}})dz.\end{aligned}$$

After performing a series of calculations, we obtain

$$\int_0^{t_{k+\frac{2}{3}}}(t_{k+\frac{2}{3}}-z)^{\alpha-1}F(\tau)d\tau = \sum_{j=0}^{k}f_{j,k+\frac{2}{3}}F(t_j)+\sum_{j=0}^{k}h_{j,k+\frac{2}{3}}F(t_{j+\frac{1}{3}}), \tag{2.10}$$

where

$$f_{j,k+\frac{2}{3}} = \frac{h^{\alpha}}{\alpha(\alpha+1)}\begin{cases}\left(k-j+\frac{2}{3}\right)^{\alpha}(\alpha+3j-3k-1)\\ \quad+\left(k-j-\frac{1}{3}\right)^{\alpha}(2\alpha-3j+3k+1), & 0\le j\le k-1,\\ \left(\frac{2}{3}\right)^{\alpha}(\alpha-1), & j=k,\end{cases} \tag{2.11}$$

$$h_{j,k+\frac{2}{3}} = \frac{h^{\alpha}}{\alpha(\alpha+1)}\begin{cases}\left(k-j+\frac{2}{3}\right)^{\alpha}(-3j+3k+2)\\ \quad+\left(k-j-\frac{1}{3}\right)^{\alpha}(-3\alpha+3j-3k-2), & 0\le j\le k-1,\\ 2\left(\frac{2}{3}\right)^{\alpha}, & j=k.\end{cases} \tag{2.12}$$

Thus we get the following formula for determining $y_{k+\frac{2}{3}}$:

$$y_{k+\frac{2}{3}} = \sum_{i=0}^{\lceil\alpha\rceil-1} \frac{t_{k+\frac{2}{3}}^i}{i!} y^{(i)}(0) + \frac{1}{\Gamma(\alpha)} \left[\sum_{j=0}^{k} f_{j,k+\frac{2}{3}} F_j + \sum_{j=0}^{k} h_{j,k+\frac{2}{3}} F_{j+\frac{1}{3}} \right]. \tag{2.13}$$

Note that the values of y_j for $j = 0, 1, 2, \ldots, k$ in the above equation are known and the value of $y_{k+\frac{1}{3}}$ is calculated in the prior step.

Step 3. (A predictor-corrector algorithm to calculate y_{k+1})
The discretized form of (2.2) to find $y(t_{k+1})$ is

$$y(t_{k+1}) = \sum_{i=0}^{\lceil\alpha\rceil-1} \frac{t_{k+1}^i}{i!} y^{(i)}(0) + \frac{1}{\Gamma(\alpha)} \int_0^{t_{k+1}} (t_{k+1} - \tau)^{\alpha-1} F(\tau) d\tau. \tag{2.14}$$

In order to find y_{k+1}, the integral in (2.14) is approximated by the following procedure.

$$\begin{aligned} I_{k+1} &= \int_0^{t_{k+1}} (t_{k+1} - \tau)^{\alpha-1} F(\tau) d\tau = \int_0^{t_{k+1}} (t_{k+1} - \tau)^{\alpha-1} \hat{F}_k(\tau) d\tau \\ &= \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} (t_{k+1} - \tau)^{\alpha-1} \hat{F}_{j+1}(\tau) d\tau, \end{aligned} \tag{2.15}$$

where $\hat{F}_{j+1}(\tau)$ is the cubic interpolation for $F(\tau)$ at the nodes t_j, $t_{j+\frac{1}{3}}$, $t_{j+\frac{2}{3}}$ and t_{j+1}. After certain calculations, we get

$$I_{k+1} = \sum_{j=0}^{k+1} a_{j,k+1} F(t_j) + \sum_{j=0}^{k} b_{j,k+1} F(t_{j+\frac{1}{3}}) + \sum_{j=0}^{k} c_{j,k+1} F(t_{j+\frac{2}{3}}). \tag{2.16}$$

The weights of the above equation are given by

$$\begin{cases} a'_{0,k+1} = k^{\alpha+1}\Big[\alpha^2 + 5\alpha + 9\alpha k + 6 + 27k(k+1)\Big] + \frac{1}{2}\Big[k+1\Big]^{\alpha}\Big[2\alpha^3 \\ \qquad +\alpha^2(1-11k) + \alpha\big(36k^2 + 17k + 3\big) - 6k\big(9k^2 + 9k + 2\big)\Big], \\ a'_{j,k+1} = \Big[k-j\Big]^{\alpha+1}\Big[\alpha^2 + 5\alpha + 27j^2 + 27k^2 - 9j(\alpha + 6k + 3) + 9k(\alpha+3) \\ \qquad +6\Big] + \Big[k-j+2\Big]^{\alpha+1}\Big[\alpha^2 - 13\alpha + 27j^2 + 27k^2 - 9j(-\alpha + 6k + 9) \\ \qquad -9k(\alpha-9) + 60\Big] - \Big[k-j+1\Big]^{\alpha+1}\Big[11\alpha^2 + 55\alpha + 54j^2 \\ \qquad -108j(k+1) + 54k(k+2) + 120\Big], \qquad\qquad 1 \le j \le k, \\ a'_{k+1,k+1} = \alpha^2 - 4\alpha + 6, \end{cases} \tag{2.17}$$

$$\begin{aligned} b'_{j,k+1} = \frac{-9}{2}\Big[k-j\Big]^{\alpha+1}\Big[18j^2 + 18k^2 + (\alpha+2)(\alpha+3) - 4j(2\alpha + 9k + 6) + 8k \\ (\alpha+3)\Big] + 9\Big[k-j+1\Big]^{\alpha+1}\Big[9j^2 + 9k^2 + \alpha^2 + j(5\alpha - 18k - 3) + k(3 - 5\alpha)\Big], \end{aligned} \tag{2.18}$$

$$c'_{j,k+1} = 9\Big[k-j\Big]^{\alpha+1}\Big[9j^2+9k^2+(\alpha+2)(\alpha+3)-j(5(\alpha+3)+18k)+5k$$
$$(\alpha+3)\Big]-\frac{9}{2}\Big[k-j+1\Big]^{\alpha+1}\Big[\alpha^2+\alpha(8j-8k-3)+6(j-k)(3j-3k-2)\Big], \quad (2.19)$$

where $a_{j,k+1} = \frac{h^\alpha}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}a'_{j,k+1}$, $b_{j,k+1} = \frac{h^\alpha}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}b'_{j,k+1}$ and $c_{j,k+1} = \frac{h^\alpha}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}c'_{j,k+1}$. In this way, we obtain the following implicit formula for calculating y_{k+1}:

$$y_{k+1} = \sum_{i=0}^{\lceil\alpha\rceil-1}\frac{t_{k+1}^i}{i!}y^{(i)}(0)+\frac{1}{\Gamma(\alpha)}\Bigg[\sum_{j=0}^{k}a_{j,k+1}F_j+\sum_{j=0}^{k}b_{j,k+1}F_{j+\frac{1}{3}}$$
$$+\sum_{j=0}^{k}c_{j,k+1}F_{j+\frac{2}{3}}+a_{k+1,k+1}f(t_{k+1},y^P_{k+1})\Bigg]. \quad (2.20)$$

In the above equation, the values of $y_{k+\frac{1}{3}}$ and $y_{k+\frac{2}{3}}$ are found in Steps 1 and 2, respectively. Thus the rest of the problem is to design an explicit formula to find y^P_{k+1}. For that, we use the rectangle formula instead of the 3/8 Simpson's formula in Eq. (2.15) and get

$$\int_0^{t_{k+1}}(t_{k+1}-z)^{\alpha-1}F(\tau)d\tau = \int_0^{t_{k+1}}(t_{k+1}-\tau)^{\alpha-1}\hat{F}(\tau)d\tau$$
$$=\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}}(t_{k+1}-\tau)^{\alpha-1}F(t_k)d\tau = \sum_{j=0}^{k}d_{j,k+1}F(t_j), \quad (2.21)$$

where

$$d_{j,k+1} = \int_{t_j}^{t_{j+1}}(t_{k+1}-\tau)^{\alpha-1}d\tau = \frac{h^\alpha}{\alpha}\Big[(k-j+1)^\alpha-(k-j)^\alpha\Big], \quad 0\le j\le k. \quad (2.22)$$

In this way, y^P_{k+1} is determined by the following formula:

$$y^P_{k+1} = \sum_{i=0}^{\lceil\alpha\rceil-1}\frac{t_{k+1}^i}{i!}y^{(i)}(0)+\frac{1}{\Gamma(\alpha)}\sum_{j=0}^{k}d_{j,k+1}F_j. \quad (2.23)$$

Thus, assuming that the approximated values of $y_j, j = 0,1,..,k$ are already found, we can summarize the main algorithm based on predictor-corrector scheme for calculating y_{k+1} as follows.

1. Find the value of $y_{k+\frac{1}{3}}$ from Eq. (2.7).
2. Calculate the value of $y_{k+\frac{2}{3}}$ by substituting $y_{k+\frac{1}{3}}$ (from Step 1) in Eq. (2.13).
3. Obtain the value of y^P_{k+1} from Eq. (2.23).
4. Calculate the value of y_{k+1} by substituting the values of $y_{k+\frac{1}{3}}$, $y_{k+\frac{2}{3}}$, and y^P_{k+1} (from Steps 1, 2 and 3, respectively) in Eq. (2.20).

2.2. Improved algorithm

Here we improve the algorithms obtained in the last subsection by developing new predictor-corrector algorithms to approximate the values of $y(t)$ at the nodes $t_{k+\frac{1}{3}}$ and $t_{k+\frac{2}{3}}$ in two steps.

Step 1. (Design a predictor-corrector algorithm to calculate $y_{k+\frac{1}{3}}$)
We derive an implicit formula (corrector) for approximating $y_{k+\frac{1}{3}}$ by approximating Eq. (2.4) as follows.

$$\begin{aligned}
I_{k+\frac{1}{3}} &= \int_0^{t_{k+\frac{1}{3}}} (t_{k+\frac{1}{3}} - \tau)^{\alpha-1} F(\tau) d\tau \\
&= \int_0^{t_k} (t_{k+\frac{1}{3}} - \tau)^{\alpha-1} \tilde{F}_1(\tau) d\tau + \int_{t_k}^{t_{k+\frac{1}{3}}} (t_{k+\frac{1}{3}} - \tau)^{\alpha-1} \tilde{F}_2(\tau) d\tau \\
&= \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} (t_{k+\frac{1}{3}} - \tau)^{\alpha-1} \hat{F}_{j+1}(\tau) d\tau + \int_{t_k}^{t_{k+\frac{1}{3}}} (t_{k+\frac{1}{3}} - \tau)^{\alpha-1} \hat{F}_{k+1}(\tau) d\tau. \quad (2.24)
\end{aligned}$$

The first integral in the above equation is approximated by using the Simpson's 3/8 rule, while the second integral is approximated by using the trapezoidal quadrature formula. Evidently, $\hat{F}_{j+1}(\tau)$ $(j = 0, 1, \ldots, k-1)$ in Eq. (2.24) is cubic Lagrange interpolation of $F(\tau)$ at the nodes t_j, $t_{j+\frac{1}{3}}$, $t_{j+\frac{2}{3}}$ and t_{j+1}, and $\hat{F}_{k+1}(\tau)$ is the linear Lagrange interpolation of $F(\tau)$ at the nodes t_k and $t_{k+\frac{1}{3}}$. In consequence, we obtain

$$I_{k+\frac{1}{3}} = \sum_{j=0}^{k} II_{j,k+\frac{1}{3}} F(t_j) + \sum_{j=0}^{k} Il_{j,k+\frac{1}{3}} F(t_{j+\frac{1}{3}}) + \sum_{j=0}^{k-1} IM_{j,k+\frac{1}{3}} F(t_{j+\frac{2}{3}}), \quad (2.25)$$

with the weights given by

$$\begin{cases}
I'_{0,k+\frac{1}{3}} = \left[k - \frac{2}{3}\right]^{\alpha+1} \left[\alpha^2 - \alpha + 27k^2 + 9k(\alpha-1)\right] - \frac{1}{6}\left[k + \frac{1}{3}\right]^{\alpha} \Big[-6\alpha^3 \\
\qquad -23\alpha + \alpha^2(33k-25) + 3\alpha k(31-36k) + 18k\big(9(k-1)k+2\big)\Big], \\
I'_{j,k+\frac{1}{3}} = \left[-(j-k+\frac{2}{3})\right]^{\alpha+1} \Big[\alpha(\alpha-1) + 27j^2 - 9j(\alpha+6k-1) + 27k^2 \\
\qquad +9(\alpha-1)k\Big] - \left[k-j+\frac{1}{3}\right]^{\alpha+1} \Big[11\alpha(\alpha+5) + 54j^2 - 36j(3k+1) \\
\qquad +18k(3k+2) + 72\Big] + \left[k-j+\frac{4}{3}\right]^{\alpha+1} \Big[\alpha^2 + \alpha(9j-9k-7) \\
\qquad +9(j-k-1)(3j-3k-2)\Big], \qquad 1 \le j \le k-1, \\
I'_{k,k+\frac{1}{3}} = \frac{3^{-\alpha-1}}{2}\Big[-17\alpha^2 - 97\alpha + 2^{2\alpha+3}\big((\alpha-7)\alpha+18\big) - 144\Big],
\end{cases} \quad (2.26)$$

$$\begin{cases}
l'_{j,k+\frac{1}{3}} = \frac{3}{2}\left[k-j-\frac{2}{3}\right]^{\alpha+1}\Big[\alpha - 3\big(\alpha^2 + 8\alpha(k-j) + 18(j-k)^2\big) + 6\Big] \\
\qquad +3\left[k-j+\frac{1}{3}\right]^{\alpha+1}\Big[3\alpha^2 + 5\alpha(3j-3k+2) + 3(3j-3k+1) \\
\qquad (3j-3k+2)\Big], \qquad 0 \le j \le k-1, \\
l'_{k,k+\frac{1}{3}} = (\alpha+2)(\alpha+3)(3)^{-\alpha},
\end{cases} \quad (2.27)$$

$$M'_{j,k+\frac{1}{3}} = 3\left[k-j-\frac{2}{3}\right]^{\alpha+1}\Big[3\alpha^2 + 5\alpha(-3j+3k+1) + 9(j-k)(3j-3k-1)\Big]$$

$$-\frac{3}{2}\Big[k-j+\frac{2}{3}\Big]^{\alpha+1}\Big[3\alpha^2+\alpha(24j-24k+7)+18(j-k)(3j-3k+2)\Big], \quad (2.28)$$

where $I_{j,k+\frac{1}{3}}=\frac{h^\alpha}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}I'_{j,k+\frac{1}{3}}$, $l_{j,k+\frac{1}{3}}=\frac{h^\alpha}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}l'_{j,k+\frac{1}{3}}$ and $M_{j,k+\frac{1}{3}}=\frac{h^\alpha}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}M'_{j,k+\frac{1}{3}}$. In this way, we get the following implicit formula for finding $y_{k+\frac{1}{3}}$:

$$\begin{aligned}y_{k+\frac{1}{3}}=&\sum_{i=0}^{\lceil\alpha\rceil-1}\frac{t^i_{k+\frac{1}{3}}}{i!}y^{(i)}(0)+\frac{1}{\Gamma(\alpha)}\Big[\sum_{j=0}^{k}I_{j,k+\frac{1}{3}}F_j+\sum_{j=0}^{k-1}l_{j,k+\frac{1}{3}}F_{j+\frac{1}{3}}\\&+\sum_{j=0}^{k-1}M_{j,k+\frac{1}{3}}F_{j+\frac{2}{3}}+l_{k,k+\frac{1}{3}}f(t_{k+\frac{1}{3}},y^P_{k+\frac{1}{3}})\Big].\end{aligned} \quad (2.29)$$

Note that the product rectangle rule is applied to calculate $y_{\frac{1}{3}}$ as follows

$$\begin{aligned}\int_0^{t_{\frac{1}{3}}}(t_{\frac{1}{3}}-z)^{\alpha-1}\hat{g}(z)dz=&\int_0^{t_{\frac{1}{3}}}(t_{\frac{1}{3}}-z)^{\alpha-1}\frac{z-t_{\frac{1}{3}}}{0-t_{\frac{1}{3}}}g(0)dz\\&+\int_0^{t_{\frac{1}{3}}}(t_{\frac{1}{3}}-z)^{\alpha-1}\frac{z-0}{t_{\frac{1}{3}}-0}g(t_{\frac{1}{3}})dz.\end{aligned}$$

Hence, if $k=0$, we have

$$I'_{0,\frac{1}{3}}=\alpha(\alpha+2)(\alpha+3)(3)^{-\alpha},\quad l'_{0,\frac{1}{3}}=(\alpha+2)(\alpha+3)(3)^{-\alpha}. \quad (2.30)$$

We will follow the idea of subsection 2.1 to derive an explicit formula for determining the value of $y^P_{k+\frac{1}{3}}$. In Eq. (2.24), the product rectangle formula is utilized as follows:

$$\begin{aligned}I_{k+\frac{1}{3}}&=\int_0^{t_k}(t_{k+\frac{1}{3}}-\tau)^{\alpha-1}\tilde{F}(\tau)d\tau+\int_{t_k}^{t_{k+\frac{1}{3}}}(t_{k+\frac{1}{3}}-\tau)^{\alpha-1}\hat{F}(\tau)d\tau\\&=\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}(t_{k+\frac{1}{3}}-\tau)^{\alpha-1}F(t_j)d\tau+\int_{t_k}^{t_{k+\frac{1}{3}}}(t_{k+\frac{1}{3}}-\tau)^{\alpha-1}F(t_k)d\tau\\&=\sum_{j=0}^{k}N_{j,k+\frac{1}{3}}F(t_j),\end{aligned} \quad (2.31)$$

where

$$N_{j,k+\frac{1}{3}}=\frac{h^\alpha}{\alpha}\begin{cases}\Big(k-j+\frac{1}{3}\Big)^\alpha-\Big(k-j-\frac{2}{3}\Big)^\alpha, & 0\le j\le k-1,\\ 3^{-\alpha}, & j=k.\end{cases} \quad (2.32)$$

Hence we obtain the following formula for finding $y^P_{k+\frac{1}{3}}$:

$$y^P_{k+\frac{1}{3}}=\sum_{i=0}^{\lceil\alpha\rceil-1}\frac{t^i_{k+\frac{1}{3}}}{i!}y^{(i)}(0)+\frac{1}{\Gamma(\alpha)}\sum_{j=0}^{k}N_{j,k+\frac{1}{3}}F_j. \quad (2.33)$$

Thus a new predictor-corrector scheme for calculating $y_{k+\frac{1}{3}}$ is achieved in terms of (2.29) and (2.33).

Step 2. (Design a predictor-corrector algorithm to calculate $y_{k+\frac{2}{3}}$)
In this step, the ideas of the previous step will be employed to design a new predictor-corrector formula to calculate $y_{k+\frac{2}{3}}$. In order to derive an implicit formula (corrector) for finding $y_{k+\frac{2}{3}}$, Eq. (2.9) can be approximated as follows.

$$\begin{aligned} I_{k+\frac{2}{3}} &= \int_0^{t_{k+\frac{2}{3}}} (t_{k+\frac{2}{3}}-\tau)^{\alpha-1} F(\tau) d\tau \\ &= \int_0^{t_k} (t_{k+\frac{2}{3}}-\tau)^{\alpha-1} \tilde{F}_1(\tau) d\tau + \int_{t_k}^{t_{k+\frac{2}{3}}} (t_{k+\frac{2}{3}}-\tau)^{\alpha-1} \tilde{F}_2(\tau) d\tau \\ &= \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} (t_{k+\frac{2}{3}}-\tau)^{\alpha-1} \hat{F}_{j+1}(\tau) d\tau + \int_{t_k}^{t_{k+\frac{2}{3}}} (t_{k+\frac{2}{3}}-\tau)^{\alpha-1} \hat{F}_{k+1}(\tau) d\tau, \end{aligned} \tag{2.34}$$

where $\hat{F}_{j+1}(\tau)$ $j=0,1,\ldots,k-1$ is the same as given by Eq. (2.24). The Simpson's 1/3 formula is applied to approximate the second integral. Notice that $\hat{F}_{k+1}(\tau)$ is the piecewise quadratic interpolation of $F(\tau)$ at the nodes t_k, $t_{k+\frac{1}{3}}$ and $t_{k+\frac{2}{3}}$. After a series of calculations, we get

$$I_{k+\frac{2}{3}} = \sum_{j=0}^{k} p_{j,k+\frac{2}{3}} F(t_j) + \sum_{j=0}^{k} q_{j,k+\frac{2}{3}} F(t_{j+\frac{1}{3}}) + \sum_{j=0}^{k} r_{j,k+\frac{2}{3}} F(t_{j+\frac{2}{3}}), \tag{2.35}$$

with the weights given by

$$\begin{cases} p'_{0,k+\frac{2}{3}} = \frac{1}{6}\Big[k+\frac{2}{3}\Big]^{\alpha}\Big[2\alpha\ (3\alpha^2+7\alpha+2) - 162k^3 + 108\alpha k^2 - 3k(11\alpha^2 \\ \qquad +7\alpha-6)\Big]\Big[k-\frac{1}{3}\Big]^{\alpha+1}\Big[\alpha(\alpha+2)+27k^2+9k(\alpha+1)\Big], \\ p'_{j,k+\frac{2}{3}} = \Big[k-j+\frac{5}{3}\Big]^{\alpha+1}\Big[\alpha^2-10\alpha+27j^2-9j(-\alpha+6k+7)+27k^2 \\ \qquad -9k(\alpha-7)+36\Big] + \Big[-(j-k+\frac{1}{3})\Big]^{\alpha+1}\Big[\alpha(\alpha+2)+27j^2-9j \\ \qquad (\alpha+6k+1)+27k^2+9k(\alpha+1)\Big] - \Big[k-j+\frac{2}{3}\Big]^{\alpha+1}\Big[11\alpha(\alpha+5) \\ \qquad +54j^2-36j(3k+2)+18k(3k+4)+90\Big], \qquad 1\le j\le k-1, \\ p'_{k,k+\frac{2}{3}} = \Big[\frac{5}{3}\Big]^{\alpha+1}\Big[(\alpha-10)\alpha+36\Big] - \frac{1}{3}\Big[\frac{2}{3}\Big]^{\alpha+2}\Big[\alpha(5\alpha+28)+45\Big], \end{cases} \tag{2.36}$$

$$\begin{cases} q'_{j,k+\frac{2}{3}} = 3\Big[k-j+\frac{2}{3}\Big]^{\alpha+1}\Big[\alpha(3\alpha+5)+27j^2+j(15\alpha-54k+9)+27k^2 \\ \qquad -3k(5\alpha+3)\Big] - \frac{3}{2}\Big[k-j-\frac{1}{3}\Big]^{\alpha+1}\Big[\alpha(3\alpha+7)+54j^2 \\ \qquad -12j(2\alpha+9k+3)+54k^2+12k(2\alpha+3)\Big], \qquad 1\le j\le k-1, \\ q'_{k,k+\frac{2}{3}} = \alpha(\alpha+3)2\big(\frac{2}{3}\big)^{\alpha}, \end{cases} \tag{2.37}$$

$$\begin{cases} r'_{j,k+\frac{2}{3}} = 3\Big[-j+k-\frac{1}{3}\Big]^{\alpha+1}\Big[3\alpha^2+10\alpha+27j^2-3j(5\alpha+18k+9) \\ \qquad +27k^2+3k(5\alpha+9)+6\Big]-\frac{3}{2}\Big[-j+k+\frac{2}{3}\Big]^{\alpha+1}\Big[3\alpha^2-\alpha \\ \qquad +54j^2+24\alpha j-108jk+54k^2-24\alpha k-6\Big], \qquad 1\le j\le k-1, \\ r'_{k,k+\frac{2}{3}} = (2-\alpha)(\alpha+3)\big(\frac{2}{3}\big)^{\alpha}, \end{cases} \tag{2.38}$$

where $p_{j,k+\frac{2}{3}} = \frac{h^\alpha}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}p'_{j,k+\frac{2}{3}}$, $q_{j,k+\frac{2}{3}} = \frac{h^\alpha}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}q'_{j,k+\frac{2}{3}}$ and $r_{j,k+\frac{2}{3}} = \frac{h^\alpha}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}r'_{j,k+\frac{2}{3}}$. This leads to the following implicit formula for finding $y_{k+\frac{2}{3}}$:

$$y_{k+\frac{2}{3}} = \sum_{i=0}^{\lceil\alpha\rceil-1}\frac{t^i_{k+\frac{2}{3}}}{i!}y^{(i)}(0)+\frac{h^\alpha}{\Gamma(\alpha)}\Bigg[\sum_{j=0}^{k}p_{j,k+\frac{2}{3}}F_j+\sum_{j=0}^{k}q_{j,k+\frac{2}{3}}F_{j+\frac{1}{3}} + \sum_{j=0}^{k-1}r_{j,k+\frac{2}{3}}F_{j+\frac{2}{3}}+r_{k,k+\frac{2}{3}}f(t_{k+\frac{2}{3}},y^P_{k+\frac{2}{3}})\Bigg]. \tag{2.39}$$

Here we mention that the Simpson's rule at the nodes t_0, $t_{\frac{1}{3}}$ and $t_{\frac{2}{3}}$ is applied to calculate $y_{\frac{2}{3}}$. In particular, for $k=0$, we have

$$p'_{0,\frac{2}{3}} = \alpha^2(\alpha+3)\Big(\frac{2}{3}\Big)^{\alpha},\ q'_{0,\frac{2}{3}} = \alpha(\alpha+3)2\Big(\frac{2}{3}\Big)^{\alpha},\ r'_{0,\frac{2}{3}} = (2-\alpha)(\alpha+3)\Big(\frac{2}{3}\Big)^{\alpha}. \tag{2.40}$$

The integral in Eq. (2.34) is approximated by the product rectangle rule to obtain a predictor formula for $y^P_{\frac{2}{3}}$ given by

$$\begin{aligned} I_{k+\frac{2}{3}} &= \int_0^{t_k}(t_{k+\frac{2}{3}}-\tau)^{\alpha-1}\tilde{F}(\tau)d\tau+\int_{t_k}^{t_{k+\frac{2}{3}}}(t_{k+\frac{2}{3}}-\tau)^{\alpha-1}\hat{F}(\tau)d\tau \\ &= \sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}(t_{k+\frac{2}{3}}-\tau)^{\alpha-1}F(t_j)d\tau+\int_{t_k}^{t_{k+\frac{2}{3}}}(t_{k+\frac{2}{3}}-\tau)^{\alpha-1}F(t_k)d\tau \\ &= \sum_{j=0}^{k}v_{j,k+\frac{2}{3}}F(t_j), \end{aligned} \tag{2.41}$$

where

$$v_{j,k+\frac{2}{3}} = \frac{h^\alpha}{\alpha}\begin{cases}\Big(k-j+\frac{2}{3}\Big)^{\alpha}-\Big(k-j-\frac{1}{3}\Big)^{\alpha}, & 0\le j\le k-1,\\ \Big(\frac{2}{3}\Big)^{\alpha}, & j=k.\end{cases} \tag{2.42}$$

In this way, we obtain the following formula for finding $y^P_{k+\frac{2}{3}}$:

$$y^P_{k+\frac{2}{3}} = \sum_{i=0}^{\lceil\alpha\rceil-1}\frac{t^i_{k+\frac{2}{3}}}{i!}y^{(i)}(0)+\frac{1}{\Gamma(\alpha)}\sum_{j=0}^{k}v_{j,k+\frac{2}{3}}F_j. \tag{2.43}$$

In consequence, after obtaining the approximations for y_j for $y_j, j=0,1,..,k$, the improved version of the predictor-corrector scheme for finding y_{k+1} can be summarized as follows.

1. Obtain the value of $y^P_{k+\frac{1}{3}}$ from Eq. (2.33).
2. Calculate $y_{k+\frac{1}{3}}$ by substituting the value of $y^P_{k+\frac{1}{3}}$ obtained from Step 1 into Eq. (2.29).
3. Calculate the value of $y^P_{k+\frac{2}{3}}$ From Eq. (2.43).
4. Find $y_{k+\frac{2}{3}}$ by substituting the values of $y_{k+\frac{1}{3}}$ and $y^P_{k+\frac{2}{3}}$ found in Steps 2 and 3 into Eq. (2.39).
5. Approximate the value of y^P_{k+1} from Eq. (2.23).
6. Calculate the value of y_{k+1} by substituting $y_{k+\frac{1}{3}}$, $y_{k+\frac{2}{3}}$, and y^P_{k+1} found in the steps 2, 4 and 5respectively into Eq. (2.20).

Note that the steps 5 and 6 of the improved algorithm are the same as the last two steps of the main algorithm derived in section 2.1.

3. Error analysis

In this section, we present the error analysis for the numerical algorithms. For computational convenience, let $E_l = y(t_l) - y_l$ and $E^P_l = y(t_l) - y^P_l$. We emphasize that C denotes a fixed constant which has different values for different formulae in the forthcoming analysis.

3.1. Truncation error analysis for the main algorithm

We first discuss the errors of Simpson's 3/8 rule, trapezoidal quadrature formula and product rectangle rule used in the main predictor-corrector algorithm.

Lemma 3.1. *For the weights of the the main predictor-corrector algorithm the following inequalities hold:*

$$\sum_{j=0}^{k}\left|e_{j,k+\frac{1}{3}}\right| \le \tfrac{C_e}{\alpha}T^\alpha,\ \sum_{j=0}^{k}\left|f_{j,k+\frac{2}{3}}\right| \le \tfrac{C_f}{\alpha}T^\alpha,\ \sum_{j=0}^{k}\left|h_{j,k+\frac{2}{3}}\right| \le \tfrac{C_h}{\alpha}T^\alpha,$$
$$\sum_{j=0}^{k}|a_{j,k+1}| \le \tfrac{C_a^P}{\alpha}T^\alpha,\ \sum_{j=0}^{k}|b_{j,k+1}| \le \tfrac{C_b}{\alpha}T^\alpha,\ \ \sum_{j=0}^{k}|c_{j,k+1}| \le \tfrac{C_c}{\alpha}T^\alpha,$$
$$\sum_{j=0}^{k}|d_{j,k+1}| \le \tfrac{C_d}{\alpha}T^\alpha,$$

where the constants $c_* > 0$ and $c^P_* > 0$ are independent of all discretization parameters.

Proof. Observe that

$$\left|e_{0,k+\frac{1}{3}}\right| = \left|\int_{t_0}^{t_{\frac{1}{3}}}(t_{k+\frac{1}{3}}-\tau)^{\alpha-1}d\tau\right| = \frac{1}{\alpha}\left[(t_{k+\frac{1}{3}}-t_0)^\alpha - (t_{k+\frac{1}{3}}-t_{\frac{1}{3}})^\alpha\right]$$
$$=\frac{1}{\alpha}\left[(t_{k+\frac{1}{3}})^\alpha - (t_k)^\alpha\right] \le \frac{1}{\alpha}(t_{k+1})^\alpha = \frac{1}{\alpha}T^\alpha.$$

For $1 \le j \le k$, we have

$$\sum_{j=1}^{k}\left|e_{j,k+\frac{1}{3}}\right| = \sum_{j=1}^{k}\left[\left|\int_{t_{j-\frac{2}{3}}}^{t_j}(t_{k+\frac{1}{3}}-\tau)^{\alpha-1}d\tau + \int_{t_j}^{t_{j+\frac{1}{3}}}(t_{k+\frac{1}{3}}-\tau)^{\alpha-1}d\tau\right|\right]$$

$$=\frac{1}{\alpha}\sum_{j=1}^{k}\left[(t_{k+\frac{1}{3}}-t_{j-\frac{2}{3}})^{\alpha}-(t_{k+\frac{1}{3}}-t_{j})^{\alpha}\right]+\left[(t_{k+\frac{1}{3}}-t_{j})^{\alpha}-(t_{k+\frac{1}{3}}-t_{j+\frac{1}{3}})^{\alpha}\right],$$

$$\begin{aligned}\sum_{j=1}^{k}\left|e_{j,k+\frac{1}{3}}\right| &= \frac{1}{\alpha}\sum_{j=1}^{k}\left[(t_{k+\frac{1}{3}}-t_{j-\frac{2}{3}})^{\alpha}-(t_{k+\frac{1}{3}}-t_{j+\frac{1}{3}})^{\alpha}\right]=\frac{1}{\alpha}(t_{k+\frac{1}{3}}-t_{\frac{1}{3}})^{\alpha}\\ &\leq \frac{1}{\alpha}t_{k+1}^{\alpha}=\frac{1}{\alpha}T^{\alpha}.\end{aligned}$$

Thus $\sum\limits_{j=0}^{k}\left|e_{j,k+\frac{1}{3}}\right| \leq \frac{C_e}{\alpha}T^{\alpha}$. The proof of the inequality $\sum\limits_{j=0}^{k}\left|d_{j,k+\frac{1}{3}}\right| \leq \frac{C_d}{\alpha}T^{\alpha}$ is similar, so we omit it. We derive the estimate for $\sum\limits_{j=0}^{k}\left|f_{j,k+\frac{2}{3}}\right|$ for two cases.

Case 1 ($0 \leq j \leq k$).

$$\begin{aligned}\sum_{j=0}^{k-1}\left|f_{j,k+\frac{2}{3}}\right| &= \sum_{j=0}^{k-1}\left|\int_{t_j}^{t_{j+1}}(t_{k+\frac{2}{3}}-\tau)^{\alpha-1}\frac{\tau-t_{j+\frac{1}{3}}}{t_j-t_{j+\frac{1}{3}}}d\tau\right|\\ &\leq \sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}(t_{k+\frac{2}{3}}-\tau)^{\alpha-1}\left|\frac{\tau-t_{j+\frac{1}{3}}}{t_j-t_{j+\frac{1}{3}}}\right|d\tau.\end{aligned}$$

By the first integral mean value theorem, for $\tilde{\tau}_j \in [t_j, t_{j+1}]$, the above equation can be rewritten as follows:

$$\begin{aligned}\sum_{j=0}^{k-1}\left|f_{j,k+\frac{2}{3}}\right| &\leq \sum_{j=0}^{k-1}\left|\frac{\tilde{\tau}_j-t_{j+\frac{1}{3}}}{t_j-t_{j+\frac{1}{3}}}\right|\int_{t_j}^{t_{j+1}}(t_{k+\frac{2}{3}}-\tau)^{\alpha-1}d\tau\\ &\leq \left|\frac{h}{-\frac{1}{3}h}\right|\frac{1}{\alpha}\sum_{j=0}^{k-1}\left[(t_{k+\frac{2}{3}}-t_j)^{\alpha}-(t_{k+\frac{2}{3}}-t_{j+1})^{\alpha}\right]\\ &= \frac{3}{\alpha}\left[(t_{k+\frac{2}{3}}-t_0)^{\alpha}-(t_{k+\frac{2}{3}}-t_k)^{\alpha}\right]\\ &= \frac{3}{\alpha}\left[(t_{k+\frac{2}{3}})^{\alpha}-(t_{\frac{2}{3}})^{\alpha}\right]\leq\frac{3}{\alpha}t_{k+1}^{\alpha}=\frac{3}{\alpha}T^{\alpha}.\end{aligned}$$

Case 2 ($j = k$).

$$\begin{aligned}\left|f_{k,k+\frac{2}{3}}\right| &\leq \int_{t_k}^{t_{k+\frac{2}{3}}}(t_{k+\frac{2}{3}}-\tau)^{\alpha-1}\left|\frac{\tau-t_{k+\frac{1}{3}}}{t_k-t_{k+\frac{1}{3}}}\right|d\tau\leq\left|\frac{\tilde{\tau}-t_{k+\frac{1}{3}}}{t_k-t_{k+\frac{1}{3}}}\right|\int_{t_k}^{t_{k+\frac{2}{3}}}(t_{k+\frac{2}{3}}-\tau)^{\alpha-1}d\tau\\ &\leq \left|\frac{h}{-\frac{1}{3}h}\right|\frac{1}{\alpha}\left[(t_{k+\frac{2}{3}}-t_k)^{\alpha}-(t_{k+\frac{2}{3}}-t_{k+\frac{2}{3}})^{\alpha}\right].\end{aligned}$$

Therefore it follows that $\sum\limits_{j=0}^{k}\left|f_{j,k+\frac{2}{3}}\right| \leq \frac{C_f}{\alpha}T^{\alpha}$. The rest of the inequalities can be established in a similar manner. □

The errors of the compound Simpson's 3/8 formula given by Eq. (2.15) are presented in the following Lemma.

Lemma 3.2. *Let $F(\tau) \in C^4[0,T]$. Then*

$$\left| \int_0^{t_{k+1}} (t_{k+1}-\tau)^{\alpha-1} \left(F(\tau) - \hat{F}_k(\tau) \right) d\tau \right| \leq Ch^4. \tag{3.1}$$

Proof. By Taylor's theorem, for all $\tau \in [t_j, t_{j+1}]$, there exist $\xi_j(\tau) \in [t_j, t_{j+1}]$ such that

$$\begin{aligned} I &= \left| \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} (t_{k+1}-\tau)^{\alpha-1} \left(F(\tau) - \hat{F}_{j+1}(\tau) \right) d\tau \right| \\ &\leq \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \left| (t_{k+1}-\tau)^{\alpha-1} \frac{F^{(4)}(\xi_j(\tau))}{4!} (\tau - t_j)(\tau - t_{j+\frac{1}{3}})(\tau - t_{j+\frac{2}{3}})(\tau - t_{j+1}) \right| d\tau \\ &\leq \frac{M_4}{4!} \sum_{j=0}^{k} \left| (\tilde{\tau}_j - t_j)(\tilde{\tau}_j - t_{j+1/3})(\tilde{\tau}_j - t_{j+2/3})(\tilde{\tau}_j - t_{j+1}) \right| \int_{t_j}^{t_{j+1}} (t_{k+1}-\tau)^{\alpha-1} d\tau \\ &\leq h^4 \frac{M_4}{4!} \frac{1}{\alpha} \sum_{j=0}^{k} \left[(t_{k+1}-t_j)^{\alpha} - (t_{k+1}-t_{j+1})^{\alpha} \right] = \left(\frac{M_4}{4!} \frac{1}{\alpha} (t_{k+1})^{\alpha} \right) h^4 \end{aligned}$$

where $\tilde{\tau}_j \in [t_j, t_{j+1}]$ and $M_4 = \sup_{t\in[0,T]} \left| F^{(4)}(t) \right|$. □

The errors of the trapezoidal quadrature formula given by Eq. (2.9) are described in the following Lemma.

Lemma 3.3. *Let $F(\tau) \in C^2[0,T]$. Then*

$$\left| \int_0^{t_k} (t_{k+\frac{2}{3}}-\tau)^{\alpha-1} \left(F(\tau) - \hat{F}_k(\tau) \right) d\tau \right| \leq Ch^2, \tag{3.2}$$

$$\left| \int_{t_k}^{t_{k+\frac{2}{3}}} (t_{k+\frac{2}{3}}-\tau)^{\alpha-1} \left(F(\tau) - \hat{F}_k(\tau) \right) d\tau \right| \leq Ch^{2+\alpha}. \tag{3.3}$$

Proof. We do not provide the the proof of Eq. (3.2) as it is similar to that of Lemma 3.2. Eq. (3.3) can be proven as follows.

$$\begin{aligned} I &= \left| \int_{t_k}^{t_{k+\frac{2}{3}}} (t_{k+\frac{2}{3}}-\tau)^{\alpha-1} \frac{F^{(2)}(\xi_j(\tau))}{2!} (\tau - t_k)(\tau - t_{k+\frac{1}{3}}) d\tau \right| \\ &\leq \frac{M_2}{2!} \left| \int_{t_k}^{t_{k+\frac{2}{3}}} (t_{k+\frac{2}{3}}-\tau)^{\alpha-1} (\tau - t_k)(\tau - t_{k+\frac{1}{3}}) d\tau \right| \\ &= \left(\frac{M_2}{2!} \left| -\frac{2^{\alpha+1} 3^{-\alpha-2} (\alpha-2)}{\alpha\left(\alpha^2+3\alpha+2\right)} \right| \right) h^{\alpha+2}, \end{aligned}$$

where $M_2 = \sup_{t\in[0,T]} \left| F^{(2)}(t) \right|$. □

The errors of the rectangle quadrature formula described by Eq. (2.4) is given in the following Lemma.

Lemma 3.4. *Let $F(\tau) \in C^1[0,T]$. Then*

$$\left| \int_0^{t_{k+\frac{1}{3}}} (t_{k+\frac{1}{3}}-\tau)^{\alpha-1} \left(F(\tau) - \hat{F}(\tau) \right) d\tau \right| \leq Ch. \tag{3.4}$$

Proof. Note that

$$
\begin{aligned}
I =& \Bigg| \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+\frac{1}{3}}} (t_{k+\frac{1}{3}} - \tau)^{\alpha-1} \left(F(\tau) - \hat{F}(\tau) \right) d\tau \\
&+ \sum_{j=0}^{k-1} \int_{t_{j+\frac{1}{3}}}^{t_{j+1}} (t_{k+\frac{1}{3}} - \tau)^{\alpha-1} \left(F(\tau) - \hat{F}(\tau) \right) d\tau \\
&+ \int_{t_k}^{t_{k+\frac{1}{3}}} (t_{k+\frac{1}{3}} - \tau)^{\alpha-1} \left(F(\tau) - \hat{F}(\tau) \right) d\tau \Bigg| = |I_1 + I_2 + I_3| . \qquad (3.5)
\end{aligned}
$$

We estimate the above integrals separately. In order to estimate $|I_1|$, we use Taylor's theorem. Thus, for all $\tau \in [t_j, t_{j+\frac{1}{3}}]$, there exist $\xi_j(\tau) \in [t_j, t_{j+\frac{1}{3}}]$ such that

$$
\begin{aligned}
|I_1| \leq& \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+\frac{1}{3}}} (t_{k+\frac{1}{3}} - \tau)^{\alpha-1} \left| \frac{F^{(1)}(\xi_j(\tau))}{1!} (\tau - t_j) \right| d\tau \\
\leq& \sum_{j=0}^{k-1} |M_1(\bar{\tau}_j - t_j)| \int_{t_j}^{t_{j+\frac{1}{3}}} (t_{k+\frac{1}{3}} - \tau)^{\alpha-1} d\tau,
\end{aligned}
$$

where $\bar{\tau}_j \in [t_j, t_{j+\frac{1}{3}}]$ and $M_1 = \sup_{t\in[0,T]} \left| F^{(1)}(t) \right|$. Thus

$$
\begin{aligned}
|I_1| \leq& hM_1 \frac{1}{\alpha} \sum_{j=0}^{k-1} \left[(t_{k+\frac{1}{3}} - t_j)^{\alpha} - (t_{k+\frac{1}{3}} - t_{j+\frac{1}{3}})^{\alpha} \right] = hM_1 \frac{1}{\alpha} \left[(t_{k+\frac{1}{3}})^{\alpha} - (t_1)^{\alpha} \right] \\
\leq& hM_1 \frac{1}{\alpha} (t_{k+1})^{\alpha} = \left(M_1 \frac{1}{\alpha} T^{\alpha} \right) h = C_1 h. \qquad (3.6)
\end{aligned}
$$

In a similar manner, we can establish that

$$
|I_2| \leq= C_2 h. \qquad (3.7)
$$

The estimate for $|I_3|$ is also based on Taylor theorem. Thus, for all $\tau \in [t_j, t_{j+\frac{1}{3}}]$, there exist $\xi_j(\tau) \in [t_j, t_{j+\frac{1}{3}}]$ such that

$$
\begin{aligned}
|I_3| \leq& M_1 \left| \int_{t_k}^{t_{k+\frac{1}{3}}} (t_{k+\frac{1}{3}} - \tau)^{\alpha-1} (\tau - t_{k+1}) d\tau \right| \\
=& \left(M_1 \left| - \frac{3^{-\alpha-1}(3\alpha+2)}{\alpha(\alpha+1)} \right| \right) h^{\alpha+1} \leq C_3 h^{\alpha+1}. \qquad (3.8)
\end{aligned}
$$

Combining Eqs. (3.5–3.8), we find that

$$
I \leq |I_1| + |I_2| + |I_3| = C_1 h + C_2 h + C_3 h^{\alpha+1} \leq Ch.
$$

□

The errors of the rectangle quadrature formula given in Eq. (2.21) is presented in the following Lemma.

Lemma 3.5. *Let $F(\tau) \in C^1[0,T]$. Then*

$$
\left| \int_0^{t_{k+1}} (t_{k+1} - \tau)^{\alpha-1} \left(F(\tau) - \hat{F}(\tau) \right) d\tau \right| \leq Ch. \qquad (3.9)
$$

We omit the proof as it is similar to that of Lemma 3.4. The error analysis for the (2.23) in given in the following Lemma.

Lemma 3.6. *Assume that the solution y of the given initial value problem satisfies*

$$\left|\int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1} {}_0^C D_t^\alpha y(t)dt - \sum_{j=0}^{k} d_{j,k+1}\ {}_0^C D_t^\alpha y(t_j)\right| \leq Ch^{\delta_1}, \tag{3.10}$$

where $\delta_1 > 0$. Then, for $C_{1P}, C_{2P} > 0$, we have

$$\left|E_{k+1}^P\right| \leq C_{1P}h^{\delta_1} + C_{2P} \max_{0\leq j\leq k} |E_j|. \tag{3.11}$$

Proof. By the predictor formula (2.23) and Eq. (2.2), one can find that

$$\begin{aligned}
\left|E_{k+1}^P\right| &= \left|y(t_{k+1}) - y_{k+1}^P\right| \\
&= \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_{k+1}} (t_{k+1}-\tau)^{\alpha-1} F(\tau)d\tau - \sum_{j=0}^{k} d_{j,k+1}F_j \pm \sum_{j=0}^{k} d_{j,k+1}F(t_j)\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_{k+1}} (t_{k+1}-\tau)^{\alpha-1} F(\tau)d\tau - \sum_{j=0}^{k} d_{j,k+1}F(t_j)\right| + \sum_{j=0}^{k} |d_{j,k+1}|\,|F(t_j)-F_j|,
\end{aligned}$$

Using Lemma 3.1, Eq. (2.1) and the Lipschitz property, we have

$$\begin{aligned}
\left|E_{k+1}^P\right| \leq& \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1} {}_0^C D_t^\alpha y(t)dt - \sum_{j=0}^{k} d_{j,k+1}\ {}_0^C D_t^\alpha y(t_j)\right| \\
&+ \frac{L}{\Gamma(\alpha)} \sum_{j=0}^{k} d_{j,k+1} |y(t_j) - y_j| \\
\leq& \frac{1}{\Gamma(\alpha)} Ch^{\delta_1} + \frac{L}{\Gamma(\alpha)} \max_{0\leq j\leq k} |E_j| \sum_{j=0}^{k} |d_{j,k+1}| \leq C_{1P}h^{\delta_1} + C_{2P} \max_{0\leq j\leq k} |E_j|,
\end{aligned}$$

where $C_{1P} = \frac{C}{\Gamma(\alpha)}, \quad C_{2P} = \frac{LC_d T^\alpha}{\Gamma(\alpha+1)}$. □

Lemma 3.7. *Assume that the solution y of the given initial value problem satisfies*

$$\left|\int_0^{t_{k+\frac{1}{3}}} (t_{k+\frac{1}{3}}-t)^{\alpha-1} {}_0^C D_t^\alpha y(t)dt - \sum_{j=0}^{k} e_{j,k+\frac{1}{3}}\ {}_0^C D_t^\alpha y(t_j)\right| \leq Ch^{\delta_2}, \tag{3.12}$$

where $\delta_2 > 0$. Then, for $C_{1\frac{1}{3}}, C_{2\frac{1}{3}} > 0$, we have

$$\max_{0\leq j\leq k} \left|E_{j+\frac{1}{3}}\right| \leq C_{1\frac{1}{3}}h^{\delta_2} + C_{2\frac{1}{3}} \max_{0\leq j\leq k} |E_j|. \tag{3.13}$$

Proof. We complete the proof by means of mathematical induction. In view of the given initial conditions, the induction basis ($j = 0$) is evident. Assuming that (3.13) holds for $j = 0, 1, 2, \ldots, k-1$; it will be shown that it holds for $j = k$. From (2.7) and (2.3), it follows that

$$\left|E_{k+\frac{1}{3}}\right| \leq \frac{1}{\Gamma(\alpha)}\left[\left|\int_0^{t_{k+\frac{1}{3}}} (t_{k+\frac{1}{3}}-\tau)^{\alpha-1} F(\tau)d\tau - \sum_{j=0}^{k} e_{j,k+\frac{1}{3}} F(t_j)\right|\right.$$

$$+\left|\sum_{j=0}^{k} e_{j,k+\frac{1}{3}}(F(t_j)-F_j)\right|\Bigg],$$

We have, by the Lipschitz condition

$$\begin{aligned}\left|E_{k+\frac{1}{3}}\right| &\leq \frac{1}{\Gamma(\alpha)}\left[Ch^{\delta_2}+L\sum_{j=0}^{k}\left|e_{j,k+\frac{1}{3}}\right||y(t_j)-y_j|\right]\\ &\leq \frac{1}{\Gamma(\alpha)}\left[Ch^{\delta_2}+L\max_{0\leq j\leq k}|y(t_j)-y_j|\sum_{j=0}^{k}\left|e_{j,k+\frac{1}{3}}\right|\right]\\ &\leq C_{1\frac{1}{3}}h^{\delta_2}+C_{2\frac{1}{3}}\max_{0\leq j\leq k}|E_j|,\end{aligned}$$

which, by Lemma 3.1, implies that $C_{1\frac{1}{3}}=\frac{C}{\Gamma(\alpha)}, \quad C_{2\frac{1}{3}}=\frac{LC_eT^{\alpha}}{\Gamma(\alpha+1)}$. □

Lemma 3.8. *Assume that the condition (3.12) of Lemma 3.7 is satisfied and that the solution y of the given initial value problem satisfies*

$$\begin{aligned}\Bigg|\int_0^{t_{k+\frac{2}{3}}}(t_{k+\frac{2}{3}}-t)^{\alpha-1}\,{}_0^CD_t^{\alpha}y(t)dt-\sum_{j=0}^{k}f_{j,k+\frac{2}{3}}\,{}_0^CD_t^{\alpha}y(t_j)\\ -\sum_{j=0}^{k}h_{j,k+\frac{2}{3}}\,{}_0^CD_t^{\alpha}y(t_{j+\frac{1}{3}})\Bigg|\leq Ch^{\delta_3},\end{aligned}\tag{3.14}$$

where $\delta_3>0$. Then, for $C_{1\frac{2}{3}},C_{2\frac{2}{3}}>0$ we have

$$\max_{0\leq j\leq k}\left|E_{j+\frac{2}{3}}\right|\leq C_{1\frac{2}{3}}h^{\min\{\delta_2,\delta_3\}}+C_{2\frac{2}{3}}\max_{0\leq j\leq k}|E_j|.\tag{3.15}$$

Proof. By the formula for $y_{k+\frac{2}{3}}$, Eq. (2.8) and Lipschitz condition, we deduce that

$$\begin{aligned}\left|E_{k+\frac{2}{3}}\right|\leq&\frac{1}{\Gamma(\alpha)}\Bigg[\Bigg|\int_0^{t_{k+\frac{2}{3}}}(t_{k+\frac{2}{3}}-\tau)^{\alpha-1}F(\tau)d\tau-\sum_{j=0}^{k}f_{j,k+\frac{2}{3}}F(t_j)\\ &-\sum_{j=0}^{k}h_{j,k+\frac{2}{3}}F(t_{j+\frac{1}{3}})\Bigg|+L_1\sum_{j=0}^{k}\left|f_{j,k+\frac{2}{3}}\right||E_j|+L_2\sum_{j=0}^{k}\left|h_{j,k+\frac{2}{3}}\right|\left|E_{j+\frac{1}{3}}\right|\Bigg],\end{aligned}$$

$$\left|E_{k+\frac{2}{3}}\right|\leq\frac{1}{\Gamma(\alpha)}\left[Ch^{\delta_3}+L_1\max_{0\leq j\leq k}|E_j|\sum_{j=0}^{k}\left|f_{j,k+\frac{2}{3}}\right|+L_2\max_{0\leq j\leq k}\left|E_{j+\frac{1}{3}}\right|\sum_{j=0}^{k}\left|h_{j,k+\frac{2}{3}}\right|\right].$$

By applying the Lemmas 3.1 and 3.7, we obtain

$$\begin{aligned}\left|E_{k+\frac{2}{3}}\right|\leq&\frac{1}{\Gamma(\alpha)}\left[Ch^{\delta_3}+L_1\max_{0\leq j\leq k}|E_j|\left(\frac{C_f}{\alpha}T^{\alpha}\right)+L_2\left(C_{1\frac{1}{3}}h^{\delta_2}+C_{2\frac{1}{3}}\max_{0\leq j\leq k}|E_j|\right)\left(\frac{C_h}{\alpha}T^{\alpha}\right)\right]\\ =&C_1h^{\delta_2}+C_2h^{\delta_3}+C_3\max_{0\leq j\leq k}|E_j|,\end{aligned}$$

where

$$C_1 = \frac{L_2 C_{1\frac{1}{3}} C_h T^\alpha}{\Gamma(\alpha+1)}, \quad C_2 = \frac{CT^\alpha}{\Gamma(\alpha)}, \quad C_3 = \frac{L_1 C_f T^\alpha}{\Gamma(\alpha+1)} + \frac{L_2 C_{2\frac{1}{3}} C_h T^\alpha}{\Gamma(\alpha+1)}.$$

Choosing sufficiently large values of the constants completes the proof. □

In the following theorem, based on the error estimate of the preceding subsection, we present the truncation error analysis for the main predictor-corrector method described by (2.7), (2.13), (2.20) and (2.23).

Theorem 3.1. *Let the assumptions (3.10), (3.12) and (3.14) (of Lemmas 3.6, 3.7 and 3.8, respectively) hold and that the solution y of the initial value problem satisfies the inequality*

$$\Big| \int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1} {}_0^C D_t^\alpha y(t) dt - \sum_{j=0}^{k+1} a_{j,k+1} {}_0^C D_t^\alpha (y_j) - \sum_{j=0}^{k} b_{j,k+1} {}_0^C D_t^\alpha (y_{j+\frac{1}{3}}) - \sum_{j=0}^{k} c_{j,k+1} {}_0^C D_t^\alpha (y_{j+\frac{2}{3}}) \Big| \le Ch^{\delta_4}, \tag{3.16}$$

with $\delta_4 > 0$. Then, for the main algorithm, we have

$$\max_{0\le j\le N} |E_j| = O(h^q),$$

where $q = \min\{\alpha+\delta_1,\ \delta_2,\ \min\{\delta_2,\delta_3\},\ \delta_4\}$, and $N = \left[\frac{T}{h}\right]$.

Proof. The proof is based on mathematical induction. Assume that

$$\max_{0\le j\le k} |E_j| \le C_0\ h^q \tag{3.17}$$

holds for $j = 0, 1, 2, \ldots, k$ for some $k \le N-1$ and it will be shown that it holds true for $j = k+1$. In view of the given initial conditions, the induction basis ($j = 0$) is trivial. For y_{k+1}^P, it follows from Eqs. (2.14) and (2.20), and the Lipschitz property of $F(\tau)$ that

$$\begin{aligned}
|E_{k+1}| \le & \frac{1}{\Gamma(\alpha)} \Bigg[\Big| \int_0^{t_{k+1}} (t_{k+1}-\tau)^{\alpha-1} F(\tau) d\tau - \sum_{j=0}^{k+1} a_{j,k+1} F(t_j) - \sum_{j=0}^{k} b_{j,k+1} F(t_{j+\frac{1}{3}}) \\
& - \sum_{j=0}^{k} c_{j,k+1} F(t_{j+\frac{2}{3}}) \Big| + \sum_{j=0}^{k} |a_{j,k+1}|\, |F(t_j) - F_j| + \sum_{j=0}^{k} |b_{j,k+1}| \left| F(t_{j+\frac{1}{3}}) - F_{j+\frac{1}{3}} \right| \\
& + \sum_{j=0}^{k} |c_{j,k+1}| \left| F(t_{j+\frac{2}{3}}) - F_{j+\frac{2}{3}} \right| + |a_{k+1,k+1}|\, |f(t_{k+1}, y_{k+1}) - f(t_{k+1}, y_{k+1}^P)| \Bigg], \\
|E_{k+1}| \le & \frac{1}{\Gamma(\alpha)} \Bigg[Ch^{\delta_4} + L_1 \max_{0\le j\le k} |E_j| \sum_{j=0}^{k} |a_{j,k+1}| + L_2 \max_{0\le j\le k} \left| E_{j+\frac{1}{3}} \right| \sum_{j=0}^{k} |b_{j,k+1}| \\
& + L_3 \max_{0\le j\le k} \left| E_{j+\frac{2}{3}} \right| \sum_{j=0}^{k} |c_{j,k+1}| + L_4\, |a_{k+1,k+1}|\, \left| E_{k+1}^P \right| \Bigg].
\end{aligned}$$

Applying Lemmas 3.1, 3.6, 3.7 and 3.8 together with the induction hypothesis (3.17) and the value of $a_{k+1,k+1}$ given in (2.17), we obtain

$$
\begin{aligned}
|E_{k+1}| \leq & \frac{1}{\Gamma(\alpha)}\Bigg[Ch^{\delta_4} + L_1 \max_{0\leq j\leq k}|E_j|\left(\frac{C_a}{\alpha}T^{\alpha}\right) + L_2\left(C_{1\frac{1}{3}}h^{\delta_2} + C_{2\frac{1}{3}}\max_{0\leq j\leq k}|E_j|\right) \\
& \left(\frac{C_b}{\alpha}T^{\alpha}\right) + L_3\left(C_{1\frac{2}{3}}h^{\min\{\delta_2,\delta_3\}} + C_{2\frac{2}{3}}\max_{0\leq j\leq k}|E_j|\right)\left(\frac{C_c}{\alpha}T^{\alpha}\right) \\
& + L_4\left|\frac{\alpha^2-4\alpha+6}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}h^{\alpha}\right|\left(C_{1P}h^{\delta_1} + C_{2P}\max_{0\leq j\leq k}|E_j|\right)\Bigg] \\
= & C_1h^{\delta_1+\alpha} + C_2h^{\delta_2} + C_3h^{\min\{\delta_2,\delta_3\}} + C_4h^{\delta_4} + C_5h^q + C_6h^{q+\alpha}, \qquad (3.18)
\end{aligned}
$$

where

$$
\begin{aligned}
C_1 =& \frac{L_4(\alpha^2-4\alpha+6)C_{1P}}{\Gamma(\alpha+4)}, \quad C_2 = \frac{L_2C_{1\frac{1}{3}}C_bT^{\alpha}}{\Gamma(\alpha+1)}, \quad C_3 = \frac{L_3C_{1\frac{2}{3}}C_cT^{\alpha}}{\Gamma(\alpha+1)}, \\
C_4 =& \frac{C}{\Gamma(\alpha)}, \quad C_5 = \frac{L_1C_aT^{\alpha} + L_2C_{2\frac{1}{3}}C_bT^{\alpha} + L_3C_{2\frac{2}{3}}C_cT^{\alpha}}{\Gamma(\alpha+1)}, \\
C_6 =& \frac{L_4(\alpha^2-4\alpha+6)C_{2P}}{\Gamma(\alpha+4)}.
\end{aligned}
$$

In view of the relations $q = \min\{\alpha+\delta_1,\ \delta_2,\ \min\{\delta_2,\delta_3\},\ \delta_4\}$, and $q < q+\alpha$ and by choosing C sufficiently large, the above equation takes the form:

$$|E_{k+1}| \leq Ch^q.$$

□

3.2. Truncation error analysis of the improved algorithm

In this subsection, we presents some results concerning the error bound for the improved predictor-corrector approach.

Lemma 3.9. *For the weights of the the main predictor- corrector algorithm, the following inequalities hold:*

$$
\begin{aligned}
&\sum_{j=0}^{k}\left|II_{j,k+\frac{1}{3}}\right| \leq \tfrac{C_{II}}{\alpha}T^{\alpha},\ \sum_{j=0}^{k}\left|Il_{j,k+\frac{1}{3}}\right| \leq \tfrac{C_{Il}}{\alpha}T^{\alpha},\ \sum_{j=0}^{k-1}\left|IM_{j,k+\frac{1}{3}}\right| \leq \tfrac{C_{IM}}{\alpha}T^{\alpha}, \\
&\sum_{j=0}^{k}\left|N_{j,k+\frac{1}{3}}\right| \leq \tfrac{C_N}{\alpha}T^{\alpha},\ \sum_{j=0}^{k}\left|Ip_{j,k+\frac{2}{3}}\right| \leq \tfrac{C_{Ip}}{\alpha}T^{\alpha},\ \sum_{j=0}^{k}\left|Iq_{j,k+\frac{2}{3}}\right| \leq \tfrac{C_{Iq}}{\alpha}T^{\alpha}, \\
&\sum_{j=0}^{k}\left|Ir_{j,k+\frac{2}{3}}\right| \leq \tfrac{C_{Ir}}{\alpha}T^{\alpha},\ \sum_{j=0}^{k}\left|v_{j,k+\frac{1}{3}}\right| \leq \tfrac{C_v}{\alpha}T^{\alpha},
\end{aligned}
$$

where the constants $c_* > 0$ and $c_*^P > 0$ are independent of all discretization parameters.

We do not provide the proof of this lemma as it is similar to that of Lemma 3.1. The errors for the compound Simpson's 3/8 formulas (2.24) and (2.34) are given by the following Lemmas. We omit the proof for these Lemmas as those are similar to that of the Lemma 3.2.

Lemma 3.10. *Let $F(\tau) \in C^4[0,T]$. Then*

$$\left| \int_0^{t_k} (t_{k+\frac{1}{3}} - \tau)^{\alpha-1} \left(F(\tau) - \tilde{F}_1(\tau) \right) d\tau \right| \leq Ch^4. \tag{3.19}$$

Lemma 3.11. *Let $F(\tau) \in C^4[0,T]$. Then*

$$\left| \int_0^{t_k} (t_{k+\frac{2}{3}} - \tau)^{\alpha-1} \left(F(\tau) - \tilde{F}_1(\tau) \right) d\tau \right| \leq Ch^4. \tag{3.20}$$

The following lemma presents the error bound for the compound Simpson's 1/3 formula (2.34).

Lemma 3.12. *Let $F(\tau) \in C^3[0,T]$. Then*

$$\left| \int_{t_k}^{t_{k+\frac{2}{3}}} (t_{k+\frac{2}{3}} - \tau)^{\alpha-1} \left(F(\tau) - \tilde{F}_2(\tau) \right) d\tau \right| \leq Ch^{3+\alpha}. \tag{3.21}$$

Proof. By Taylor theorem, for all $\tau \in [t_k, t_{k+\frac{2}{3}}]$, there exist $\xi_j(\tau) \in [t_k, t_{k+\frac{2}{3}}]$ such that

$$\begin{aligned} I &\leq \frac{M_3}{3!} \left| \int_{t_k}^{t_{k+\frac{2}{3}}} (t_{k+\frac{2}{3}} - \tau)^{\alpha-1} (\tau - t_k)(\tau - t_{k+\frac{1}{3}})(\tau - t_{k+\frac{2}{3}}) d\tau \right| \\ &= \left(\frac{M_3}{3!} \left| \frac{2^{\alpha+2} 3^{-\alpha-3} (\alpha - 1)}{(\alpha+1)(\alpha+2)(\alpha+3)} \right| \right) h^{\alpha+3}, \quad M_3 = \sup_{t \in [0,T]} \left| F^{(3)}(t) \right|. \end{aligned}$$

□

The error for the trapezoidal quadrature formula (2.24) is described in the following Lemma, whose proof is omitted as it is similar to that of Eq. (3.3).

Lemma 3.13. *Let $F(\tau) \in C^2[0,T]$. Then*

$$\left| \int_{t_k}^{t_{k+\frac{1}{3}}} (t_{k+\frac{1}{3}} - \tau)^{\alpha-1} \left(F(\tau) - \tilde{F}_2(\tau) \right) d\tau \right| \leq Ch^{2+\alpha}. \tag{3.22}$$

The errors for the rectangle quadrature formulas (2.31) and (2.41) are given by the following Lemmas.

Lemma 3.14. *Let $F(\tau) \in C^1[0,T]$. Then*

$$\left| \int_0^{t_k} (t_{k+\frac{1}{3}} - \tau)^{\alpha-1} \left(F(\tau) - \tilde{F}(\tau) \right) d\tau \right| \leq Ch, \tag{3.23}$$

$$\left| \int_{t_k}^{t_{k+\frac{1}{3}}} (t_{k+\frac{1}{3}} - \tau)^{\alpha-1} \left(F(\tau) - \hat{F}(\tau) \right) d\tau \right| \leq Ch^{1+\alpha}. \tag{3.24}$$

Lemma 3.15. *Let $F(\tau) \in C^1[0,T]$. Then*

$$\left| \int_0^{t_k} (t_{k+\frac{2}{3}} - \tau)^{\alpha-1} \left(F(\tau) - \tilde{F}(\tau) \right) d\tau \right| \leq Ch, \tag{3.25}$$

$$\left| \int_{t_k}^{t_{k+\frac{2}{3}}} (t_{k+\frac{2}{3}} - \tau)^{\alpha-1} \left(F(\tau) - \hat{F}(\tau) \right) d\tau \right| \leq Ch^{1+\alpha}. \tag{3.26}$$

The proof for Lemmas 3.14 and 3.15 is similar to that Lemma 3.4. The following two Lemmas can be proven by employing the arguments for proof of Lemma (3.6).

Lemma 3.16. *Assume that the solution y of the initial value problem satisfies the inequality*

$$\left| \int_0^{t_{k+\frac{1}{3}}} (t_{k+\frac{1}{3}} - t)^{\alpha-1} {}_0^C D_t^\alpha y(t) dt - \sum_{j=0}^{k} N_{j,k+\frac{1}{3}} \, {}_0^C D_t^\alpha y(t_j) \right| \leq C h^{\delta_2}, \tag{3.27}$$

where $\delta_2 > 0$. Then, for $C_{1P\frac{1}{3}}, C_{2P\frac{1}{3}} > 0$, we have

$$\left| E^P_{k+\frac{1}{3}} \right| \leq C_{1P\frac{1}{3}} h^{\delta_2} + C_{2P\frac{1}{3}} \max_{0 \leq j \leq k} |E_j| . \tag{3.28}$$

Lemma 3.17. *Assume that the solution y of the initial value problem satisfies the inequality:*

$$\left| \int_0^{t_{k+\frac{2}{3}}} (t_{k+\frac{2}{3}} - t)^{\alpha-1} \, {}_0^C D_t^\alpha y(t) dt - \sum_{j=0}^{k} v_{j,k+\frac{2}{3}} \, {}_0^C D_t^\alpha y(t_j) \right| \leq C h^{\delta_3}, \tag{3.29}$$

where $\delta_3 > 0$. Then, for $C_{1P\frac{2}{3}}, C_{2P\frac{2}{3}} > 0$, we have

$$\left| E^P_{k+\frac{2}{3}} \right| \leq C_{1P\frac{2}{3}} h^{\delta_3} + C_{2P\frac{2}{3}} \max_{0 \leq j \leq k} |E_j| . \tag{3.30}$$

Lemma 3.18. *Let assumptions (3.27) and (3.29) (of Lemmas 3.16 and 3.17, respectively) be satisfied and that the solution y of the initial value problem is such that*

$$\begin{aligned} &\Bigg| \int_0^{t_{k+\frac{2}{3}}} (t_{k+\frac{2}{3}} - t)^{\alpha-1} \, {}_0^C D_t^\alpha y(t) dt - \sum_{j=0}^{k} I_{j,k+\frac{1}{3}} \, {}_0^C D_t^\alpha (y_j) \\ &\qquad - \sum_{j=0}^{k} l_{j,k+\frac{1}{3}} \, {}_0^C D_t^\alpha (y_{j+\frac{1}{3}}) - \sum_{j=0}^{k-1} M_{j,k+\frac{1}{3}} \, {}_0^C D_t^\alpha (y_{j+\frac{2}{3}}) \Bigg| \leq C h^{\delta_5}, \end{aligned} \tag{3.31}$$

$$\begin{aligned} &\int_0^{t_{k+\frac{2}{3}}} (t_{k+\frac{2}{3}} - t)^{\alpha-1} \, {}_0^C D_t^\alpha y(t) dt - \sum_{j=0}^{k} p_{j,k+\frac{2}{3}} \, {}_0^C D_t^\alpha y(t_j) \\ &\qquad - \sum_{j=0}^{k} q_{j,k+\frac{2}{3}} \, {}_0^C D_t^\alpha y(t_{j+\frac{1}{3}}) - \sum_{j=0}^{k-1} r_{j,k+\frac{2}{3}} \, {}_0^C D_t^\alpha y(t_{j+\frac{2}{3}}) \Bigg| \leq C h^{\delta_6}, \end{aligned} \tag{3.32}$$

where $\delta_5, \delta_6 > 0$. Moreover assume that

$$\max_{0 \leq j \leq k} |E_j| \leq C_0 \ h^q. \tag{3.33}$$

Then

$$\max_{0 \leq j \leq k} \left| E_{j+\frac{1}{3}} \right| \leq C_{1\frac{1}{3}} h^{p_1} + C_{2\frac{1}{3}} \max_{0 \leq j \leq k} |E_j| , \tag{3.34}$$

$$\max_{0 \leq j \leq k} \left| E_{j+\frac{2}{3}} \right| \leq C_{1\frac{2}{3}} h^{p_2} + C_{2\frac{2}{3}} \max_{0 \leq j \leq k} |E_j| , \tag{3.35}$$

for all $j = 0, 1, 2, \ldots, k$, where $p_1 = \min\{\alpha + \delta_2, \ \delta_5, \ p_2\}$ and $p_2 = \min\{\alpha + \delta_3, \ \delta_6, \ p_1\}$.

Proof. We make use of mathematical induction to establish this result. In view of the given initial condition, the induction basis ($j = 0$) is presupposed. Assume that (3.34) and (3.35) hold for $j = 0, 1, 2, ..., k-1$. Then it will be shown that the given inequities hold true for $j = k$. Let us first consider Eq. (3.34). By construction of $y_{k+\frac{1}{3}}$, Eq. (2.3) and the Lipschitz property of $F(\tau)$, we have

$$\begin{aligned}\left|E_{k+\frac{1}{3}}\right| \leq &\frac{1}{\Gamma(\alpha)}\Bigg[\Bigg|\int_0^{t_{k+\frac{1}{3}}}(t_{k+\frac{1}{3}}-\tau)^{\alpha-1}F(\tau)d\tau - \sum_{j=0}^{k} I_{j,k+\frac{1}{3}}F(t_j)\\ &- \sum_{j=0}^{k} l_{j,k+\frac{1}{3}}F(t_{j+\frac{1}{3}}) - \sum_{j=0}^{k-1} M_{j,k+\frac{1}{3}}F(t_{j+\frac{2}{3}})\Bigg| + L_1 \max_{0\leq j\leq k}|E_j|\sum_{j=0}^{k}\left|I_{j,k+\frac{1}{3}}\right|\\ &+ L_2 \max_{0\leq j\leq k-1}\left|E_{j+\frac{1}{3}}\right|\sum_{j=0}^{k-1}\left|l_{j,k+\frac{1}{3}}\right| + L_3 \max_{0\leq j\leq k-1}\left|E_{j+\frac{2}{3}}\right|\sum_{j=0}^{k-1}\left|M_{j,k+\frac{1}{3}}\right|\\ &+ L_4\left|l_{k,k+\frac{1}{3}}\right|\left|E^P_{k+\frac{1}{3}}\right|\Bigg].\end{aligned}$$

In view of Lemmas 3.9 and 3.16, the induction hypothesis, and the value of $l_{k,k+\frac{1}{3}}$ given by (2.27), the above inequality leads to

$$\begin{aligned}\left|E_{k+\frac{1}{3}}\right| \leq &\frac{1}{\Gamma(\alpha)}\Bigg[Ch^{\delta_5} + L_1C_0h^q\left(\frac{C_I}{\alpha}T^\alpha\right) + L_2\left(C_{1\frac{1}{3}}h^{p_1} + C_{2\frac{1}{3}}h^q\right)\left(\frac{C_l}{\alpha}T^\alpha\right)\\ &+L_3\left(C_{1\frac{2}{3}}h^{p_2}+C_{2\frac{2}{3}}h^q\right)\left(\frac{C_M}{\alpha}T^\alpha\right)+L_4\left|\frac{h^\alpha(3)^{-\alpha}}{\alpha(\alpha+1)}\right|\left(C_{1P\frac{1}{3}}h^{\delta_2}+C_{2P\frac{1}{3}}C_0h^q\right)\Bigg]\\ =&C_1h^{\delta_2+\alpha} + C_2h^{\delta_5} + C_3h^{p_1} + C_4h^{p_2} + C_5h^q + C_6h^{q+\alpha},\end{aligned}$$

where

$$\begin{aligned}C_1 =&\frac{L_4(3)^{-\alpha}C_{1P\frac{1}{3}}}{\Gamma(\alpha+2)},\ C_2 = \frac{C}{\Gamma(\alpha)},\ C_3 = \frac{L_2C_{1\frac{1}{3}}C_MT^\alpha}{\Gamma(\alpha+1)},\ C_4 = \frac{L_3C_{1\frac{2}{3}}C_lT^\alpha}{\Gamma(\alpha+1)},\\ C_5 =&\frac{L_1C_0C_IT^\alpha}{\Gamma(\alpha+1)} + \frac{L_2C_{2\frac{1}{3}}C_lT^\alpha}{\Gamma(\alpha+1)} + \frac{L_3C_{2\frac{2}{3}}C_MT^\alpha}{\Gamma(\alpha+1)},\ C_6 = \frac{L_4(3)^{-\alpha}C_{2P\frac{1}{3}}}{\Gamma(\alpha+2)}.\end{aligned}$$

By the relations $p_1 = \min\{\alpha+\delta_2,\ \delta_5,\ p_2\}$, and $q < q+\alpha$ together with sufficiently large values of $C_{1\frac{1}{3}}$ and $C_{2\frac{1}{3}}$, the above inequality can be written as $\left|E_{k+\frac{1}{3}}\right| \leq C_{1\frac{1}{3}}h^{p_1} + C_{2\frac{1}{3}}h^q$. Thus Eq. (3.34) holds true for $j = 0, 1, 2, \ldots, k$.
Next we prove the validity of Eq. (3.35). Observe that

$$\begin{aligned}\left|E_{k+\frac{2}{3}}\right| \leq &\frac{1}{\Gamma(\alpha)}\Bigg[\Bigg|\int_0^{t_{k+\frac{2}{3}}}(t_{k+\frac{2}{3}}-\tau)^{\alpha-1}F(\tau)d\tau - \sum_{j=0}^{k}Ip_{j,k+\frac{2}{3}}F(t_j) - \sum_{j=0}^{k}Iq_{j,k+\frac{2}{3}}F(t_{j+\frac{1}{3}})\\ &- \sum_{j=0}^{k}Ir_{j,k+\frac{2}{3}}F(t_{j+\frac{2}{3}})\Bigg| + L_1\max_{0\leq j\leq k}|E_j|\sum_{j=0}^{k}\left|p_{j,k+\frac{2}{3}}\right| + L_2\max_{0\leq j\leq k}\left|E_{j+\frac{1}{3}}\right|\\ &\sum_{j=0}^{k-1}\left|q_{j,k+\frac{2}{3}}\right| + L_3\max_{0\leq j\leq k-1}\left|E_{j+\frac{2}{3}}\right|\sum_{j=0}^{k-1}\left|r_{j,k+\frac{1}{3}}\right| + L_4\left|r_{k,k+\frac{2}{3}}\right|\left|E^P_{k+\frac{2}{3}}\right|\Bigg],\end{aligned}$$

which, on using Lemmas 3.9 and 3.17, Eq. (3.34), the induction hypothesis, and the value of $r_{k,k+\frac{1}{3}}$ from (2.38), leads to

$$\left|E_{k+\frac{2}{3}}\right| \leq \frac{1}{\Gamma(\alpha)}\Bigg[Ch^{\delta_6} + L_1C_0h^q\left(\frac{C_p}{\alpha}T^\alpha\right) + L_2\left(C_{1\frac{1}{3}}h^{p_1} + C_{2\frac{1}{3}}h^q\right)\left(\frac{C_q}{\alpha}T^\alpha\right)$$
$$+ L_3\left(C_{1\frac{2}{3}}h^{p_2} + C_{2\frac{2}{3}}h^q\right)\left(\frac{C_r}{\alpha}T^\alpha\right) + L_4\left|\frac{(2-\alpha)\left(\frac{2}{3}\right)^\alpha h^\alpha}{\alpha(\alpha+1)(\alpha+2)}\right|\Big(C_{1P\frac{2}{3}}h^{\delta_3}$$
$$+ C_{2P\frac{2}{3}}C_0h^q\Big)\Bigg] = C_1h^{\delta_3+\alpha} + C_2h^{\delta_6} + C_3h^{p_1} + C_4h^{p_2} + C_5h^q + C_6h^{q+\alpha},$$

where

$$C_1 = \frac{L_4(2-\alpha)\left(\frac{2}{3}\right)^\alpha C_{1P\frac{2}{3}}}{\Gamma(\alpha+3)}, C_2 = \frac{C}{\Gamma(\alpha)}, C_3 = \frac{L_2C_{1\frac{1}{3}}C_qT^\alpha}{\Gamma(\alpha+1)}, C_4 = +\frac{L_3C_{1\frac{2}{3}}C_{Ir}T^\alpha}{\Gamma(\alpha+1)},$$
$$C_5 = \frac{L_1C_0C_pT^\alpha}{\Gamma(\alpha+1)} + \frac{L_2C_{2\frac{1}{3}}C_qT^\alpha}{\Gamma(\alpha+1)} + \frac{L_3C_{2\frac{2}{3}}C_rT^\alpha}{\Gamma(\alpha+1)}, C_6 = \frac{L_4(2-\alpha)\left(\frac{2}{3}\right)^\alpha C_{2P\frac{2}{3}}}{\Gamma(\alpha+3)}.$$

In view of the relations $p_2 = \min\{\alpha+\delta_3,\ \delta_6,\ p_1\}$, $q < q+\alpha$ and fixing $C_{1\frac{2}{3}}$ and $C_{2\frac{2}{3}}$ to be sufficiently large, the above inequality takes the form: $\left|E_{k+\frac{2}{3}}\right| \leq C_{1\frac{2}{3}}h^{p_2} + C_{2\frac{2}{3}}h^q$. In consequence we deduce that Eq. (3.35) holds true for $j = 0, 1, 2, \ldots, k$. This completes the proof. □

Utilizing the error estimates obtained in the preceding subsection, we present the truncation error analysis for the improved predictor-corrector approaches described by (2.20), (2.23), (2.29), (2.33), (2.39) and (2.43).

Theorem 3.2. *Let the assumptions (3.16), (3.10), (3.27), (3.29), (3.31) and (3.32) (of Theorem 3.1 and Lemmas 3.6, 3.16, 3.17 and 3.18, respectively) be satisfied. Then, for the improved algorithm, we have*

$$\max_{0\leq j\leq N}|E_j| = O(h^q),$$

where $q = \min\{\alpha+\delta_1,\ p_1,\ p_2,\ \delta_4\}$, *and* $N = \left[\frac{T}{h}\right]$.

The proof is similar to that of Theorem 3.1, so it is omitted.

4. Stability analysis

An important characteristic of a numerical method applied to approximate the solution of a given initial value problem is its stability, that is, a small change in the initial data results in a small change in the computed solutions [4, 28]. In this section, we discuss the stability of the main algorithm and the improved algorithm of predictor-corrector method. For that, let us set $\tilde{E}_l = y_l - \tilde{y}_l$ and $\tilde{E}_l^P = y_l - \tilde{y}_l^P$.

4.1. Stability analysis of the main algorithm

Theorem 4.1. *Let* y_{k+1} *and* $\tilde{y}_{k+1}$ *denote numerical solutions given by (2.20) with the initial conditions* $y_0^{(i)}$ *and* $\tilde{y}_0^{(i)}$ *respectively. Then*

$$|y_{k+1} - \tilde{y}_{k+1}| = \left|\tilde{E}_{k+1}\right| \leq K\left\|y_0 - \tilde{y}_0\right\|_\infty, \tag{4.1}$$

for any k, that is, the main predictor-corrector scheme described by (2.7), (2.13), (2.20) and (2.23) is numerically stable.

Proof. We make use of mathematical induction to complete the proof. According to the given initial condition, the induction basis ($j = 0$) is presupposed. Suppose that Eq. (4.1) is true for $j = 0, 1, 2, ..., k$. Then we have to show that the inequality indeed holds for $j = k + 1$. We first consider the following formulae for y^p_{k+1}, $y_{k+\frac{1}{3}}$ and $y_{k+\frac{2}{3}}$ with the predictor step y^p_{k+1}:

$$\left|y^P_{k+1} - \tilde{y}^P_{k+1}\right| = \left|\tilde{E}^P_{k+1}\right| = \left|\sum_{i=0}^{\lceil\alpha\rceil-1} \frac{t^i_{k+1}}{i!} y^{(i)}(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k} d_{j,k+1} f(t_j, y_j)\right| - \left|\sum_{i=0}^{\lceil\alpha\rceil-1} \frac{t^i_{k+1}}{i!} \tilde{y}^{(i)}(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k} d_{j,k+1} f(t_j, \tilde{y}_j)\right|,$$

$$\begin{aligned}\left|\tilde{E}^P_{k+1}\right| &\leq \sum_{i=0}^{\lceil\alpha\rceil-1} \frac{t^i_{k+1}}{i!} \left\|\tilde{E}_0\right\|_\infty + \frac{L_1}{\Gamma(\alpha)} \sum_{j=0}^{k} \left|d_{j,k+1}\right| \left|\tilde{E}_j\right| \\ &\leq K_{1P} \left\|\tilde{E}_0\right\|_\infty + K_{2P} \max_{0\leq j\leq k} \left|\tilde{E}_j\right|. \end{aligned} \tag{4.2}$$

In a similar manner, from Eq. (2.7), we can obtain

$$\left|\tilde{E}_{k+\frac{1}{3}}\right| = \left|\tilde{E}_{k+\frac{1}{3}}\right| \leq K_{1\frac{1}{3}} \left\|\tilde{E}_0\right\|_\infty + K_{2\frac{1}{3}} \max_{0\leq j\leq k} \left|\tilde{E}_j\right|. \tag{4.3}$$

By Eqs. (2.13) and (4.3), Lemma 3.1 and the Lipschitz property of $F(\tau)$ for $y_{k+\frac{2}{3}}$, we get

$$\left|\tilde{E}_{k+\frac{2}{3}}\right| \leq \sum_{i=0}^{\lceil\alpha\rceil-1} \frac{t^i_{k+\frac{2}{3}}}{i!} \left\|\tilde{E}_0\right\|_\infty + \frac{L_1}{\Gamma(\alpha)} \sum_{j=0}^{k} \left|f_{j,k+\frac{2}{3}}\right| \left|\tilde{E}_j\right| + \frac{L_2}{\Gamma(\alpha)} \sum_{j=0}^{k} \left|h_{j,k+\frac{2}{3}}\right| \left|\tilde{E}_{j+\frac{1}{3}}\right|.$$

$$\begin{aligned}\left|\tilde{E}_{k+\frac{2}{3}}\right| \leq{}& \sum_{i=0}^{\lceil\alpha\rceil-1} \frac{t^i_{k+\frac{2}{3}}}{i!} \left\|\tilde{E}_0\right\|_\infty + \frac{L_1}{\Gamma(\alpha)} \sum_{j=0}^{k} \left|f_{j,k+\frac{2}{3}}\right| \max_{0\leq j\leq k} \left|\tilde{E}_j\right| + \frac{L_2}{\Gamma(\alpha)} \sum_{j=0}^{k} \left|h_{j,k+\frac{2}{3}}\right| \\ & \left(K_{1\frac{1}{3}} \left\|\tilde{E}_0\right\|_\infty + K_{2\frac{1}{3}} \max_{0\leq j\leq k} \left|\tilde{E}_j\right|\right) \leq K_{1\frac{2}{3}} \left\|\tilde{E}_0\right\|_\infty + K_{2\frac{2}{3}} \max_{0\leq j\leq k} \left|\tilde{E}_j\right|. \end{aligned} \tag{4.4}$$

Similarly, for y_{k+1}, we have

$$\begin{aligned}\left|\tilde{E}_{k+1}\right| \leq{}& \sum_{i=0}^{\lceil\alpha\rceil-1} \frac{t^i_{k+1}}{i!} \left\|\tilde{E}_0\right\|_\infty + \frac{L_1}{\Gamma(\alpha)} \sum_{j=0}^{k} \left|a_{j,k+1}\right| \max_{0\leq j\leq k} \left|\tilde{E}_j\right| + \frac{L_2}{\Gamma(\alpha)} \sum_{j=0}^{k} \left|b_{j,k+1}\right| \left|\tilde{E}_{j+\frac{1}{3}}\right| \\ & + \frac{L_3}{\Gamma(\alpha)} \sum_{j=0}^{k} \left|c_{j,k+1}\right| \left|\tilde{E}_{j+\frac{2}{3}}\right| + \frac{L_4}{\Gamma(\alpha)} \left|a_{k+1,k+1}\right| \left|\tilde{E}^P_{k+1}\right|, \end{aligned}$$

$$\left|\tilde{E}_{k+1}\right| \leq \sum_{i=0}^{\lceil\alpha\rceil-1} \frac{t^i_{k+1}}{i!} \left\|\tilde{E}_0\right\|_\infty + \frac{L_1}{\Gamma(\alpha)} \left(\frac{C_a}{\alpha} T^\alpha\right) \max_{0\leq j\leq k} \left|\tilde{E}_j\right|$$

$$+\frac{L_2}{\Gamma(\alpha)}\left(\frac{C_b}{\alpha}T^{\alpha}\right)\left(K_{1\frac{1}{3}}\left\|\tilde{E}_0\right\|_{\infty}+K_{2\frac{1}{3}}\max_{0\le j\le k}\left|\tilde{E}_j\right|\right)$$
$$+\frac{L_3}{\Gamma(\alpha)}\left(\frac{C_c}{\alpha}T^{\alpha}\right)\left(K_{1\frac{2}{3}}\left\|\tilde{E}_0\right\|_{\infty}+K_{2\frac{2}{3}}\max_{0\le j\le k}\left|\tilde{E}_j\right|\right)$$
$$+\frac{L_4}{\Gamma(\alpha)}\left|\frac{\alpha^2-4\alpha+6}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}h^{\alpha}\right|\left(K_{1P}\left\|\tilde{E}_0\right\|_{\infty}+K_{2P}\max_{0\le j\le k}\left|\tilde{E}_j\right|\right),$$

where we have used Lemma 3.1, Eqs. (4.2), (4.3) and (4.4). Simplifying the above equation and applying the induction hypothesis leads to the completion of the proof. □

4.2. Stability analysis of the improved algorithm

This subsection is devoted to the stability analysis of the improved algorithm.

Lemma 4.1. *Let $y_{k+\frac{1}{3}}$ and $\tilde{y}_{k+\frac{1}{3}}$ be the numerical solutions given by (2.29), and $y_{k+\frac{2}{3}}$ and $\tilde{y}_{k+\frac{2}{3}}$ represent numerical solutions given by (2.39), with the initial conditions $y_0^{(i)}$ and $\tilde{y}_0^{(i)}$, respectively. Then*

$$\max_{0\le j\le k}\left|\tilde{E}_{j+\frac{1}{3}}\right|\le K_{1\frac{1}{3}}\left\|\tilde{E}_0\right\|_{\infty}+K_{2\frac{1}{3}}\max_{0\le j\le k}\left|\tilde{E}_j\right|, \tag{4.5}$$
$$\max_{0\le j\le k}\left|\tilde{E}_{j+\frac{2}{3}}\right|\le K_{1\frac{2}{3}}\left\|\tilde{E}_0\right\|_{\infty}+K_{2\frac{2}{3}}\max_{0\le j\le k}\left|\tilde{E}_j\right|. \tag{4.6}$$

Proof. The principle of mathematical induction is the main tool of the proof. In view of the given initial condition, the induction basis is presupposed. Suppose that Eqs. (4.5) and (4.6) hold for $j=0,1,2,...,k-1$. Similar to (4.2), one can achieve the following inequalities:

$$\left|\tilde{E}^P_{k+\frac{1}{3}}\right|\le K_{1P\frac{1}{3}}\left\|\tilde{E}_0\right\|_{\infty}+K_{2P\frac{1}{3}}\max_{0\le j\le k}\left|\tilde{E}_j\right|, \tag{4.7}$$
$$\left|\tilde{E}^P_{k+\frac{2}{3}}\right|\le K_{1P\frac{2}{3}}\left\|\tilde{E}_0\right\|_{\infty}+K_{2P\frac{2}{3}}\max_{0\le j\le k}\left|\tilde{E}_j\right|. \tag{4.8}$$

Using Lemma 3.9, Eq. (4.7) and the induction hypothesis, one can prove Eq. (4.5) as follows.

$$\begin{aligned}\left|\tilde{E}_{k+\frac{1}{3}}\right|\le{}&\sum_{i=0}^{\lceil\alpha\rceil-1}\frac{t^i_{k+\frac{1}{3}}}{i!}\left\|\tilde{E}_0\right\|_{\infty}+\frac{L_1}{\Gamma(\alpha)}\sum_{j=0}^{k}\left|II_{j,k+\frac{1}{3}}\right|\max_{0\le j\le k}\left|\tilde{E}_j\right|+\frac{L_2}{\Gamma(\alpha)}\sum_{j=0}^{k-1}\left|Il_{j,k+\frac{1}{3}}\right|\\&\left(K_{1\frac{1}{3}}\left\|\tilde{E}_0\right\|_{\infty}+K_{2\frac{1}{3}}\max_{0\le j\le k-1}\left|\tilde{E}_j\right|\right)+\frac{L_3}{\Gamma(\alpha)}\sum_{j=0}^{k-1}\left|IM_{j,k+\frac{1}{3}}\right|\left(K_{1\frac{2}{3}}\left\|\tilde{E}_0\right\|_{\infty}\right.\\&\left.+K_{2\frac{2}{3}}\max_{0\le j\le k-1}\left|\tilde{E}_j\right|\right)+\frac{L_4}{\Gamma(\alpha)}\left|Il_{k,k+\frac{1}{3}}\right|\left(K_{1P\frac{1}{3}}\left\|\tilde{E}_0\right\|_{\infty}+K_{2P\frac{1}{3}}\max_{0\le j\le k}\left|\tilde{E}_j\right|\right).\end{aligned} \tag{4.9}$$

Next Eq. (4.6) can be proven by utilizing Lemma 3.9, Eqs. (4.6) and (4.9) and the induction hypothesise. □

Theorem 4.2. *Let y_{k+1} and $\tilde{y}_{k+1}$ denote numerical solutions given by (2.20) of the given problem with the initial conditions $y_0^{(i)}$ and $\tilde{y}_0^{(i)}$, respectively. Then*

$$|y_{k+1} - \tilde{y}_{k+1}| \leq K \left\| y_0 - \tilde{y}_0 \right\|_\infty , \tag{4.10}$$

for any k, that is, the improved predictor-corrector scheme described by (2.20), (2.23), (2.29), (2.33), (2.39) and (2.43) is numerically stable.

We do not provide the proof as it is similar to that of Theorem (4.1).

5. Numerical results and discussion

In this section, we illustrate and verify the obtained numerical schemes with the aid of examples.

Example 5.1. Consider the following initial value problem

$${}_0^C D_t^\alpha y(t) = \frac{t^{2-\alpha}}{\Gamma(3-\alpha)} - y(t) + t^2, \quad y(0) = 0. \tag{5.1}$$

The initial value problem (5.1) is solved by the main predictor-corrector scheme for $\alpha = 0.9$. The exact Solution of the initial value problem (5.1) is $y(t) = t^2$. By the main scheme of predictor-corrector formula, the numerical solutions are obtained and the comparison with the absolute errors of Ref. [31] is presented at different positions. From the results listed in Table. 1, the absolute errors for the presented main scheme (E_1) is found to be less than the one obtainted in [31] (E_2). In all given cases, one can note that the main predictor-corrector scheme is more accurate.

Table 1. The absolute errors of the present main scheme (E_1) and numerical method of [31] (E_2) for (5.1) with $\alpha = 0.9$.

	$t = 1.008$		$t = 5.008$		$t = 7.008$	
h	E_1	E_2	E_1	E_2	E_1	E_2
0.016	3.3322e-04	0.025858	0.0222	0.096058	0.0233	0.128336
0.008	1.8320e-04	0.020939	0.0110	0.088030	0.0125	0.120168
0.004	9.5437e-05	0.018472	0.0015	0.084016	0.0196	0.116084

Example 5.2. Consider the problem

$${}_0^C D_t^\alpha y(t) + y^4(t) = \frac{\Gamma(2\alpha+1)t^\alpha}{\Gamma(\alpha+1)} - \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + \left(t^{2\alpha} - t^2\right)^4, \quad y(0) = 0. \tag{5.2}$$

The initial value problem (5.2) is solved by the improved predictor-corrector scheme and the fractional predictor-corrector Adams method [22] for $\alpha = 0.75$, $h = 0.1$. Exact solution for the problem (5.2) is $y(t) = t^{2\alpha} - t^2$. Solutions are plotted in Fig. 1. All the three solutions coincide in this case and it is clear that both the methods are in very good agreement.

The absolute errors at different times for the fractional Adams method and the improved algorithm are given in Table 2, which show that the improved algorithm is more accurate than the fractional Adams method.

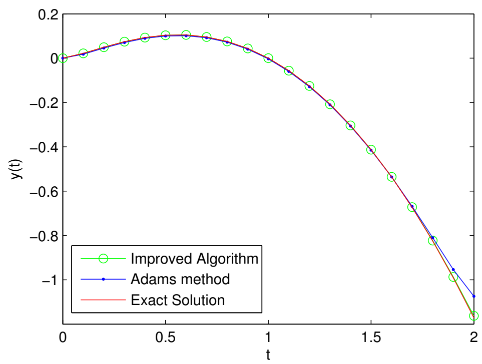


Figure 1. Solutions of (5.2) with $\alpha = 0.75$, $h = 0.1$.

Table 2. The absolute errors of the present improved algorithm and fractional Adams method for (5.2) with $\alpha = 0.75$, $h = 0.1$.

	$t = 0.5$	$t = 1$	$t = 1.5$	$t = 2$
Fractional Adams method	0.0036	0.0034	0.0019	0.0975
Improved algorithm	1.4824e-04	1.2566e-04	1.8503e-04	0.0088

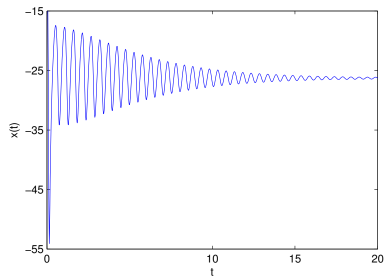
Example 5.3 (Bhalekar-Gejji System)**.** Let us consider a fractional order chaotic system due to Bhalekar and Daftardar-Gejji [7] given by

$$\begin{cases} D^{\alpha}x = \omega x - y^2, \\ D^{\alpha}y = \mu(z - y), \\ D^{\alpha}z = ay - bz + xy, \end{cases} \tag{5.3}$$

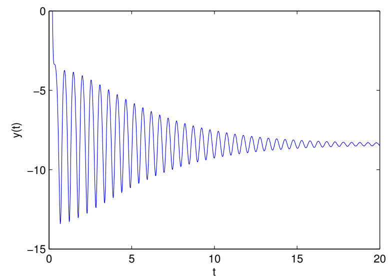
with $\omega = -2.667$, $\mu = 10$, $a = 27.3, b = 1$. The fractional order Bhalekar-Gejji System is solved numerically by improved predictor-corrector scheme. The phase portraits of system (5.3) are shown in Figs. 2–3 for $\alpha = 0.88, 0.89$, with initial conditions $(0, 10, 10)$ and the step size $h = 0.02$. One can observe that the given system shows stable orbits for $\alpha = 0.88$ and indicates chaotic behavior for $\alpha = 0.89$. The simulation results obtained for the Bhalekar-Gejji System are in agreement with the ones presented in Ref. [12].

6. Conclusions

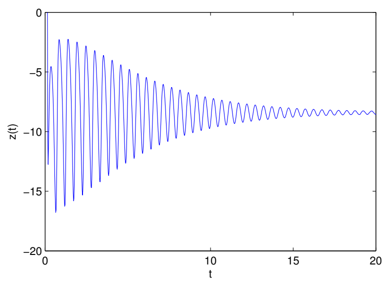
A new predictor-corrector method to solve fractional order non-linear differential equations is established. The possible improvements for the fractional predictor-corrector algorithm are also discussed. The local truncation errors and the stability analysis for the new schemes are derived. Validation tests presented in the paper have shown the applicability and efficiency of the proposed methods. It is shown that the results obtained by using the obtained schemes are in complete agreement with the exact solution and the results obtained by other methods [31] and [22]. The algorithm developed in this paper is applicable to fractional differential equations as well as to fractional order systems. As an application, the improved numerical algorithm is applied to show the chaotic behavior and dynamics of Bhalekar and Daftardar-Gejji fractional order nonlinear system.



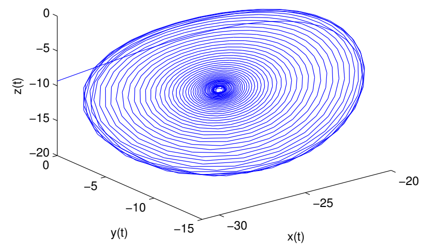
(a) $x(t)$ waveform



(b) $y(t)$ waveform

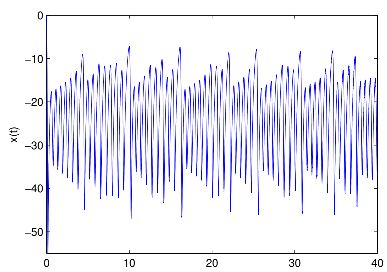


(c) $z(t)$ waveform

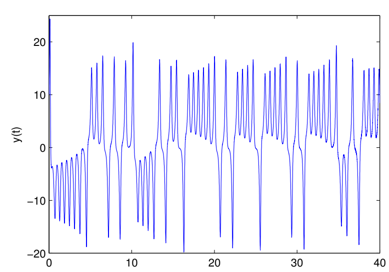


(d) 3-D view

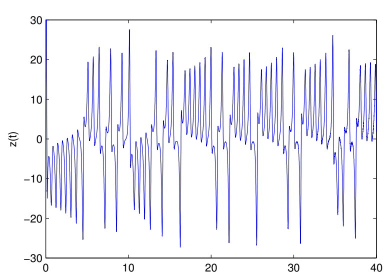
Figure 2. Waveform of the non-linear system (5.3) for $\alpha = 0.88$.



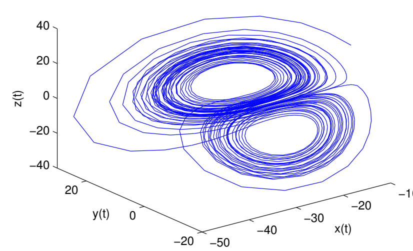
(a) $x(t)$ waveform



(b) $y(t)$ waveform



(c) $z(t)$ waveform



(d) 3-D view

Figure 3. Waveform of the non-linear system (5.3) for $\alpha = 0.89$.

Acknowledgements. The authors are grateful to the anonymous referees for their useful suggestions which improve the contents of this article.

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