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INDICES DEFECT THEORY OF SINGULAR HAHN-STURM-LIOUVILLE OPERATORS

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Abstract In this article, we extend the results concerning the deficiency index problem to singular Hahn-Sturm-Liouville difference operators. We establish some criteria under which the singular Hahn-Sturm-Liouville equation is of limit-point case at infinity.

Keywords Hahn-Sturm-Liouville operator, symmetric operator, limit-point case, indices defect.

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1. Introduction

The deficiency index problem is one of the most important problems in spectral theory of differential operators. The problem on the defect index of the second-order differential equation

$$(\tau y)(x) := -\left(p\left(x\right)y'\left(x\right)\right)' + v\left(x\right)y\left(x\right) = \lambda y(x),\ x \in [a, b),$$

was first studied by Hermann Weyl [31]. In [31], Weyl shows that there may be close connections between the deficiency index problem and the problem of describing the spectrum of the minimal operator associated with a formal ordinary selfadjoint differential operator. Weyl was followed by Titchmarsh [30] and others mathematicians ([7,9–11,20,23,26,28,30,32]). The defect index d of τ is equal to the number of linearly independent square integrable solutions of equation $\tau y = \lambda y$ for each $\lambda \in \mathbb{C}$, $\operatorname{Im}\lambda \neq 0$. The deficiency index problem is the problem of determining d in terms of $p(x)$ and $v(x)$. Since the number of linearly independent self-adjoint boundary conditions required is given by the deficiency index, we study the deficiency index of this operator if we investigate the spectra of selfadjoint extensions of an operator. It is well known that a symmetric operator has a self-adjoint extension if and only if its defect indices are equal. So it is very important to determine the defect indices of the operator in the study of self-adjoint extensions of a minimal operator (see [2, 3, 7, 9, 26, 30]).

On the other hand, Hahn [12, 13] introduced the Hahn difference operator in 1949 in order to generalize two well-known difference operators, the quantum $q-$difference operator (see [17]) and the forward difference operator (see [18, 19]).

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The Hahn difference operator is defined by ($\omega > 0$, $q \in (0,1)$).

$$D_{\omega,q}f(x) = \frac{f(\omega + qx) - f(x)}{\omega + (q-1)x}.$$

By using this operator, researchers can study the construction of families of orthogonal polynomials and approximation problems (see, for example, [4, 8, 21-22, 27]). Recently, new developments of the theory and applications of Hahn difference operator were made. Hamza et al. [15] studied the theory of linear Hahn difference equations. The authors also study the existence and uniqueness of solution for the initial value problems for Hahn difference equations. In [16], the authors investigated Leibniz's rule and Fubini's theorem associated with Hahn difference operator. In [29], the author study the nonlocal boundary value problem for nonlinear Hahn difference equation. Annaby et al. [6] study the regular Hahn-Sturm-Liouville problem

$$-\frac{1}{q}D_{-\frac{\omega}{q},\frac{1}{q}}D_{\omega,q}y(x) + p(x)y(x) = \lambda y(x),$$
$$a_1 y(\omega_0) + a_2 D_{-\frac{\omega}{q},\frac{1}{q}}y(\omega_0) = 0,$$
$$b_1 y(b) + b_2 D_{-\frac{\omega}{q},\frac{1}{q}}y(b) = 0,$$

where $0 < \omega_0 \leq x < \infty$, $\alpha \in \mathbb{C}$, $a_i, b_i \in \mathbb{R} := (-\infty, \infty)$, $i = 1,2$, and $p(.)$ is a real-valued continuous function at ω_0 defined on $[\omega_0, b]$. They discussed the formulation of the self-adjoint operator and the properties of the eigenvalues and the eigenfunctions. Furthermore, they construct the Green's function and give an eigenfunction expansion theorem.

There is well known that the classical Sturm-Liouville eigenvalue problem is readily formulated as a constrained variational principle, namely as the isoperimetric problem, and many general properties of the eigenvalues can be derived using the variational principle. In this context, Malinowska and Torres [24] studied the Hahn quantum variational calculus. Necessary and sufficient optimality conditions for the basic, isoperimetric, and Hahn quantum Lagrange problems, are given. They also proved the validity of Leitmann's direct method for the Hahn quantum variational calculus. Later, in [25], the authors developed the variational Hahn calculus. They investigated problems of the calculus of variations using Hahn's difference operator and the Jackson-Nörlund integral.

To the best of our knowledge, there exists no work on the indices defect theory of the singular Hahn difference equations of the classical Sturm-Liouville type. Motivated by the discussion above, the defect index of the singular Hahn-Sturm-Liouville equations is studied in this paper. These results can be useful for the study of spectrum of Hahn-Sturm-Liouville operators. Hence, our study could fill an important gap in the spectral theory of the singular Hahn difference equations of the classical Sturm-Liouville type.

2. Preliminaries

Now, we will give some knowledge about Hahn difference operators [5-6, 12-13]. Throughout the paper, we let $q \in (0,1)$ and $\omega > 0$.

Define $\omega_0 := \omega / (1-q)$ and let I be a real interval containing ω_0.

Definition 2.1 ([12, 13]). Let $f : I \to \mathbb{R}$ be a function. The Hahn difference operator is defined by

$$D_{\omega,q}f(x) = \begin{cases} \frac{f(\omega+qx)-f(x)}{\omega+(q-1)x}, & x \neq \omega_0, \\ f'(\omega_0), & x = \omega_0, \end{cases}$$

provided that f is differentiable at ω_0. In this case, we call $D_{\omega,q}f$, the $\omega, q-$derivative of f.

Remark 2.1. The Hahn difference operator unifies two well known operators. When $q \to 1$, we get the forward difference operator, which is defined by

$$\Delta_\omega f(x) := \frac{f(\omega+x) - f(x)}{(\omega+x) - x}, \ x \in \mathbb{R}.$$

When $\omega \to 0$, we get the Jackson $q-$ difference operator, which is defined by

$$D_q f(x) := \frac{f(qx) - f(x)}{(qx) - x}, x \neq 0.$$

Furthermore, under appropriate conditions, we have

$$\lim_{\substack{q\to 1 \\ \omega \to 0}} D_{\omega,q} f(x) = f'(x).$$

In what follows, we present some important properties of the $\omega, q-$ derivative.

Theorem 2.1 ([5]). *Let $f, g : I \to \mathbb{R}$ be $\omega, q-$differentiable at $x \in I$ and $h(x) := \omega + qx$, then we have for all $x \in I$:*

$$\begin{aligned} D_{\omega,q}(af+bg)(x) &= aD_{\omega,q}f(x) + bD_{\omega,q}g(x), a, b \in I, \\ D_{\omega,q}(fg)(x) &= D_{\omega,q}(f(x))g(x) + f(\omega+xq)D_{\omega,q}g(x), \\ D_{\omega,q}\left(\frac{f}{g}\right)(x) &= \frac{D_{\omega,q}(f(x))g(x) - f(x)D_{\omega,q}g(x)}{g(x)g(\omega+xq)}, \\ D_{\omega,q}f\left(h^{-1}(x)\right) &= D_{-\omega q^{-1},q^{-1}}f(x). \end{aligned}$$

The concept of the $\omega, q-$integral of the function f can be defined as follows.

Definition 2.2 (Jackson-Nörlund Integral [5]). Let $f : I \to \mathbb{R}$ be a function and $a, b, \omega_0 \in I$. We define $\omega, q-$integral of the function f from a to b by

$$\int_a^b f(x)\, d_{\omega,q}(x) := \int_{\omega_0}^b f(x)\, d_{\omega,q}(x) - \int_{\omega_0}^a f(x)\, d_{\omega,q}(x),$$

where

$$\int_{\omega_0}^x f(t)\, d_{\omega,q}(t) := ((1-q)x - \omega) \sum_{n=0}^{\infty} q^n f\left(\omega \frac{1-q^n}{1-q} + xq^n\right), \ x \in I$$

provided that the series converges at $x = a$ and $x = b$. In this case, f is called $\omega, q-$integrable on $[a, b]$.

Similarly, one can define the $\omega, q-$integration for a function f over (ω_0, ∞) by

$$\int_{\omega_0}^{\infty} f(x)\, d_{\omega,q}(x) := \lim_{b\to\infty} \int_{\omega_0}^{b} f(x)\, d_{\omega,q}(x).$$

The following properties of $\omega, q-$integration can be found in [5].

Theorem 2.2 ([5]). *Let $f, g : I \to \mathbb{R}$ be $\omega, q-$integrable on I, $a, b, c \in I$, $a < c < b$ and $\alpha, \beta \in \mathbb{R}$. Then the following formulas hold:*

$$\int_a^b \{\alpha f(x) + \beta g(x)\}\, d_{\omega,q}(x) = \alpha \int_a^b f(x)\, d_{\omega,q}(x) + \beta \int_a^b g(x)\, d_{\omega,q}(x),$$

$$\int_a^a f(x)\, d_{\omega,q}(x) = 0,$$

$$\int_a^b f(x)\, d_{\omega,q}(x) = \int_a^c f(x)\, d_{\omega,q}(x) + \int_c^b f(x)\, d_{\omega,q}(x),$$

$$\int_a^b f(x)\, d_{\omega,q}(x) = -\int_b^a f(x)\, d_{\omega,q}(x).$$

Next, we present the $\omega, q-$integration by parts.

Lemma 2.1 ([5]). *Let $f, g : I \to \mathbb{R}$ be $\omega, q-$ integrable on I, $a, b \in I$, and $a < b$. Then the following formula holds:*

$$\int_a^b f(x)\, D_{\omega,q} g(x)\, d_{\omega,q}(x) + \int_a^b g(\omega + qx)\, D_{\omega,q} f(x)\, d_{\omega,q}(x)$$
$$= f(b)\, g(b) - f(a)\, g(a).$$

The next result is the fundamental theorem of Hahn calculus.

Theorem 2.3 ([5]). *Let $f : I \to \mathbb{R}$ be continuous at ω_0. Define*

$$F(x) := \int_{\omega_0}^{x} f(t)\, d_{\omega,q}(t),\ x \in I.$$

Then F is continuous at ω_0. Moreover, $D_{\omega,q}F(x)$ exists for every $x \in I$ and $D_{\omega,q}F(x) = f(x)$. Conversely,

$$\int_a^b D_{\omega,q} F(x)\, d_{\omega,q}(x) = f(b) - f(a).$$

Let $L^2_{\omega,q}((\omega_0, \infty), r)$ be the space of all complex-valued functions defined on $[\omega_0, \infty)$ such that

$$\|f\| := \left(\int_{\omega_0}^{\infty} |f(x)|^2\, r(x)\, d_{\omega,q} x \right)^{1/2} < +\infty,$$

where r is real-valued functions defined on $[\omega_0, \infty)$ and $r(x) > 0$ for all $x \in [\omega_0, \infty)$. The space $L^2_{\omega,q}((\omega_0, \infty), r)$ is a separable Hilbert space with the inner product

$$(f, g) := \int_{\omega_0}^{\infty} f(x)\, \overline{g(x)}\, r(x)\, d_{\omega,q} x,\ \ f, g \in L^2_{\omega,q}((\omega_0, \infty), r)$$

(see [5]).

The $\omega, q-$Wronskian of $y(.)$, $z(.)$ is defined to be

$$W_{\omega,q}(y, z)(x) := y(x)\, D_{\omega,q} z(x) - z(x)\, D_{\omega,q} y(x),\ \ x \in [\omega_0, \infty). \tag{2.1}$$

3. Main Results

Consider the symmetric Hahn-Sturm-Liouville expression $\mathcal{L}$ determined by

$$\mathcal{L}y := \frac{1}{r(x)}\left[-q^{-1}D_{-\omega q^{-1},q^{-1}}\left(p(x)D_{\omega,q}y(x)\right) + v(x)y(x)\right],\ x \in (\omega_0, \infty), \quad (3.1)$$

where λ is a complex parameter, p, v, w are real-valued continuous functions at ω_0 defined on $[\omega_0, \infty)$, and $p(x) > 0,\ r(x) > 0,\ x \in [\omega_0, \infty)$.

The singular Hahn-Sturm-Liouville equation

$$\mathcal{L}y = \lambda y \text{ on } (\omega_0, \infty), \quad (3.2)$$

where λ is a complex parameter, may be classified at the singular point ∞ as either limit-point (i.e., the deficiency indices of the operator $\mathcal{L}$ are equal to $(1,1)$) or limit-circle (i.e., the deficiency indices of the operator $\mathcal{L}$ are equal to $(2,2)$) according to whether the equation (3.2) has either at most one linearly independent solution in $L^2_{\omega,q}((\omega_0,\infty), r)$ or two independent solutions in $L^2_{\omega,q}((\omega_0,\infty), r)$. The deficiency index d of Hahn-Sturm-Liouville expression can be interpreted as the number of linearly independent solutions of the equation (3.2) which lie in $L^2_{\omega,q}((\omega_0,\infty), r)$ ([2,3,7,9,26,30]).

The equation (3.2) is called limit-point type at infinity if there is a solution y of the equation $\mathcal{L}y = 0$ which is not in $L^2_{\omega,q}((\omega_0,\infty), r)$, i.e.,

$$\int_{\omega_0}^{\infty} |y(x)|^2 r(x)\, d_{\omega,q}x = +\infty,$$

otherwise, i.e., if all solutions of the equation (3.2) are in $L^2_{\omega,q}((\omega_0,\infty), r)$, the equation (3.2) is called limit-circle type at infinity ([2,3,7,9,26,30]).

The maximal and minimal operators associated with a symmetric Hahn-Sturm-Liouville expression $\mathcal{L}$ in the Hilbert space $L^2_{\omega,q}((\omega_0,\infty), r)$ are defined as follows:

Definition 3.1. The linear set $D_{\max}$ consisting of all vectors $y \in L^2_{\omega,q}((\omega_0,\infty), r)$ such that y and $pD_{\omega,q}y$ are continuous functions at ω_0 defined on $[\omega_0,\infty)$ and $\mathcal{L}y \in L^2_{\omega,q}((\omega_0,\infty), r)$. We define the *maximal operator* $\mathcal{L}_{\max}$ on $D_{\max}$ by the equality $\mathcal{L}_{\max}y = \mathcal{L}y$. Let $\mathcal{D}_{\min}$ be the linear set of all vectors $y \in \mathcal{D}_{\max}$ satisfying the conditions

$$y(\omega_0) = (pD_{-\omega q^{-1},q^{-1}}y)(\omega_0) = 0,\ \ [y,z](\infty) = 0,$$

for arbitrary $z \in \mathcal{D}_{\max}$. The operator $\mathcal{L}_{\min}$, that is the restriction of the operator $\mathcal{L}_{\max}$ to $\mathcal{D}_{\min}$ is called the *minimal operator* and the equalities $\mathcal{L}_{\max} = \mathcal{L}^*_{\min}$ holds. Further $\mathcal{L}_{\min}$ is closed symmetric operator with deficiency indices (d,d), where $d = 1$ or $d = 2$ ([9,26]).

Lemma 3.1 ($\omega, q-$Green's formula)**.** *For every $y, z \in D_{\max}$, we have*

$$\int_{\omega_0}^{b} (\mathcal{L}y)(x)\overline{z(x)}d_{\omega,q}x - \int_{\omega_0}^{b} y(x)\overline{(\mathcal{L}z)(x)}d_{\omega,q}x$$
$$= [y,z](b) - [y,z](\omega_0),\ \ b \in (\omega_0,\infty), \quad (3.3)$$

where

$$[y,z](x) := p(x)\{y(x)\overline{D_{-\omega q^{-1},q^{-1}}z(x)} - D_{-\omega q^{-1},q^{-1}}y(x)\overline{z(x)}\}.$$

Proof. See [5]. □

Remark 3.1. Lemma 3.1 shows that for all $y,z\in\mathcal{D}_{\max}$,

$$[y,z](\infty):=\lim_{\xi\to\infty}[y,z](q^{-\xi})$$

exists and is finite.

Now, we will give some criteria under which the Hahn-Sturm-Liouville equation (3.2) is of limit-point case at infinity.

Theorem 3.1. *Assume there exists a point $\eta\in(\omega_0,\infty)$ such that*

$$\frac{v^2}{r}\in L^1_{\omega,q}((\eta,\infty),r), \tag{3.4}$$

and

$$r\notin L^1_{\omega,q}((\eta,\infty),r). \tag{3.5}$$

Then the deficiency indices of the symmetric operator $\mathcal{L}_{\min}$ are equal to $(1,1)$, i.e., equation (3.2) is in the limit-point case at infinity.

Proof. To obtain a contradiction, assume the deficiency indices of the symmetric operator $\mathcal{L}_{\min}$ are equal to $(2,2)$. Let φ and χ be linearly independent solutions of the equation $\mathcal{L}y=0$ for which

$$W_{\omega,q}(\varphi,\chi)=\varphi(pD_{\omega,q}\chi)-\chi(pD_{\omega,q}\varphi)=1. \tag{3.6}$$

$\omega,q-$integrating $D_{-\omega q^{-1},q^{-1}}(p(x)D_{\omega,q}\varphi(x))=qv(x)\varphi(x)$ from η to $h(x)$, we get

$$p(x)D_{\omega,q}\varphi(x)=p(h^{-1}(\eta)D_{\omega,q}\varphi\left(h^{-1}(\eta)\right)+q\int_\eta^{h(x)}v(x)\varphi(x)d_{\omega,q}x. \tag{3.7}$$

It follows from Cauchy-Schwarz inequality that

$$\left(\int_\eta^x v(x)\varphi(x)d_{\omega,q}x\right)^2\le\left(\int_\eta^x\frac{v^2(x)}{r(x)}d_{\omega,q}x\right)\left(\int_\eta^x\varphi^2(x)r(x)d_{\omega,q}x\right). \tag{3.8}$$

By virtue of (3.4), (3.5) and (3.7), we conclude that $pD_{\omega,q}\varphi$ is bounded on (η,∞). Likewise, $pD_{\omega,q}\chi$ is bounded on (η,∞). Hence, by (3.6), we have

$$C_1(|\varphi|+|\chi|)\sqrt{r}\ge\sqrt{r}, \tag{3.9}$$

where C_1 is a positive constant. Squaring (3.9), we get

$$C_2(|\varphi|^2+|\chi|^2)r\ge r, \tag{3.10}$$

where C_2 is a positive constant. Since $\varphi,\chi\in L^2_{\omega,q}((\eta,\infty),r)$ and $r\notin L^1_{\omega,q}((\eta,\infty),r)$, the inequality (3.10) contradicts our assumption. □

Theorem 3.2. *Let $p(x)>0$, $r(x)>0$ and $M(.)$ be a positive function defined on (ω_0,∞), M and $D_{\omega,q}M$ are continuous functions at ω_0. Suppose that for some $\eta\in(\omega_0,\infty)$, $r\notin L^1_{\omega,q}((\eta,\infty),r)$, the following three conditions are satisfied:*
(1)

$$\int_\eta^\infty\frac{\sqrt{r(x)}d_{\omega,q}x}{\sqrt{p(x)M(\omega+qx)}}=+\infty, \tag{3.11}$$

(2) there is a positive constant C such that

$$\left|\frac{\sqrt{p(x)}D_{\omega,q}M(x)}{M(x)\sqrt{r\left(x\right)}\sqrt{M\left(\omega+qx\right)}}\right|<C,\ \ x\in[\eta,\infty), \tag{3.12}$$

(3) there is a positive constant K such that

$$v(x)\geq -Kr\left(x\right)M(x),\ \ x\in[\eta,\infty). \tag{3.13}$$

Then the deficiency indices of the symmetric operator $\mathcal{L}_{\min}$ are equal to $(1,1)$, i.e., equation (3.2) is in the limit-point case at infinity.

Proof. Conversely, suppose that the deficiency indices of the symmetric operator $\mathcal{L}_{\min}$ are equal to $(2,2)$. Let φ and χ be linearly independent solutions of the equation $\mathcal{L}y=0$ for which

$$W_{\omega,q}\left(\varphi,\chi\right)=\varphi\left(pD_{\omega,q}\chi\right)-\chi\left(pD_{\omega,q}\varphi\right)=1. \tag{3.14}$$

If we multiply this identity by $\frac{\sqrt{r}}{\sqrt{pM}}$, we obtain

$$\begin{aligned}&\left[\sqrt{\frac{p(x)}{M(x)}}D_{\omega,q}\chi\left(x\right)\right]\left(\varphi\left(x\right)\sqrt{r\left(x\right)}\right)\\&-\left[\sqrt{\frac{p(x)}{M(x)}}D_{\omega,q}\varphi\left(x\right)\right]\left(\chi\left(x\right)\sqrt{r\left(x\right)}\right)\\=&\frac{\sqrt{r\left(x\right)}}{\sqrt{p(x)M(x)}},\ \ x\in[\eta,\infty).\end{aligned} \tag{3.15}$$

If we integrate both sides of (3.15), we get

$$\begin{aligned}\int_{\eta}^{\infty}\frac{\sqrt{r\left(x\right)}}{\sqrt{p(x)M(\omega+qx)}}d_{\omega,q}x=&\int_{\eta}^{\infty}\sqrt{\frac{p(x)r\left(x\right)}{M(\omega+qx)}}\varphi(x)D_{\omega,q}\chi(x)d_{\omega,q}x\\&-\int_{\eta}^{\infty}\sqrt{\frac{p(x)r\left(x\right)}{M(\omega+qx)}}\chi(x)D_{\omega,q}\varphi(x)d_{\omega,q}x.\end{aligned} \tag{3.16}$$

By the Cauchy-Schwarz inequality, we conclude that

$$\begin{aligned}&\left|\int_{\eta}^{\infty}\frac{\sqrt{r\left(x\right)}}{\sqrt{p(x)M(\omega+qx)}}d_{\omega,q}x\right|\\=&\left|\int_{\eta}^{\infty}\sqrt{\frac{p(x)r(x)}{M(\omega+qx)}}\varphi(x)D_{\omega,q}\chi(x)d_{\omega,q}x-\int_{\eta}^{\infty}\sqrt{\frac{p(x)r(x}{M(\omega+qx)}}\chi(x)D_{\omega,q}\varphi(x)d_{\omega,q}x\right|\\\leq&\left\|\varphi\right\|_{L^2_{\omega,q}((\eta,\infty),r)}\left(\int_{\eta}^{\infty}\frac{p(x)}{M(\omega+qx)}\left(D_{\omega,q}\chi(x)\right)^2d_{\omega,q}x\right)^{1/2}\\&+\left\|\chi\right\|_{L^2_{\omega,q}((\eta,\infty),r)}\left(\int_{\eta}^{\infty}\frac{p(x)}{M(\omega+qx)}\left(D_{\omega,q}\varphi(x)\right)^2d_{\omega,q}x\right)^{1/2}.\end{aligned} \tag{3.17}$$

Suppose y is a solution of equation $\mathcal{L}y=0$. Then

$$\frac{y\left[-q^{-1}D_{-\omega q^{-1},q^{-1}}\left(pD_{\omega,q}y\right)\right]}{M}=\frac{-vy^2}{M}.$$

Integrating both sides, we conclude that

$$-\int_{\eta}^{q^{-\xi}}\frac{vy^2}{M}d_{\omega,q}x=\int_{\eta}^{q^{-\xi}}\frac{y\left[-q^{-1}D_{-\omega q^{-1},q^{-1}}\left(pD_{\omega,q}y\right)\right]}{M}d_{\omega,q}x,\ \xi\in\mathbb{N}.$$

Using $\omega,q-$integration by parts, we obtain

$$\begin{aligned}
&-q\int_{\eta}^{q^{-\xi}}\frac{vy^2}{M}d_{\omega,q}x\\
&=\frac{y}{M}pD_{\omega,q}y\mid_{h^{-1}(q^{-\xi})}^{h^{-1}(\eta)}+\int_{\eta}^{q^{-\xi}}\left(pD_{\omega,q}y\right)D_{\omega,q}\left(\frac{y}{M}\right)d_{\omega,q}x\\
&=\frac{y}{M}pD_{\omega,q}y\mid_{h^{-1}(q^{-\xi})}^{h^{-1}(\eta)}+\int_{\eta}^{q^{-\xi}}\left(pD_{\omega,q}y\right)\left(\frac{MD_{\omega,q}y-yD_{\omega,q}M}{M(x)M\left(\omega+qx\right)}\right)d_{\omega,q}x\\
&=\frac{y}{M}pD_{\omega,q}y\mid_{h^{-1}(q^{-\xi})}^{h^{-1}(\eta)}+\int_{\eta}^{q^{-\xi}}\frac{p}{M(\omega+qx)}\left(D_{\omega,q}y\right)^2d_{\omega,q}x-\int_{\eta}^{q^{-\xi}}\frac{pyD_{\omega,q}yD_{\omega,q}M}{M(x)M(\omega+qx)}d_{\omega,q}x.
\end{aligned}$$

By (3.13), we have

$$-\int_{\eta}^{q^{-\xi}}\frac{vy^2}{M}d_{\omega,q}x<K\int_{\eta}^{q^{-\xi}}y^2rd_{\omega,q}x<K\int_{\eta}^{\infty}y^2rd_{\omega,q}x.$$

Hence, there is a constant K_1 such that

$$\begin{aligned}
K_1>&-(\frac{y}{M}pD_{\omega,q}y)h^{-1}(q^{-\xi})\\
&+\int_{\eta}^{q^{-\xi}}\frac{p}{M\left(\omega+qx\right)}\left(D_{\omega,q}y\right)^2d_{\omega,q}x-\int_{\eta}^{q^{-\xi}}\frac{pyD_{\omega,q}yD_{\omega,q}M}{M(x)M\left(\omega+qx\right)}d_{\omega,q}x.
\end{aligned}\tag{3.18}$$

Let

$$H\left(\xi\right)=\int_{\eta}^{q^{-\xi}}\frac{p}{M\left(\omega+qx\right)}\left(D_{\omega,q}y\right)^2d_{\omega,q}x.$$

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\left|\int_{\eta}^{q^{-\xi}}\frac{pyD_{\omega,q}yD_{\omega,q}M}{M(x)M\left(\omega+qx\right)}d_{\omega,q}x\right|^2\\
&=\left|\int_{\eta}^{q^{-\xi}}\left(\frac{pr}{M\left(\omega+qx\right)}\right)^{1/2}\frac{1}{M}D_{\omega,q}M\left(\frac{p}{M\left(\omega+qx\right)r}\right)^{1/2}yD_{\omega,q}yd_{\omega,q}x\right|^2\\
&\leq K_2^2\left(\int_{\eta}^{q^{-\xi}}\left(\frac{pr}{M\left(\omega+qx\right)}\right)^{1/2}yD_{\omega,q}yd_{\omega,q}x\right)^2\\
&\leq K_2^2H\left(\xi\right)\int_{\eta}^{q^{-\xi}}y^2rd_{\omega,q}x<K_3^2H\left(\xi\right),
\end{aligned}$$

where $K_2 > 0$ is a certain constant, and

$$K_3 = K_2 \left(\int_\eta^\infty y^2 r d_{\omega,q} x \right)^{1/2}.$$

By (3.18), we obtain

$$K_1 > -(\frac{y}{M} p D_{\omega,q} y) h^{-1}(q^{-\xi}) + H(\xi) - K_3 \sqrt{H(\xi)}.$$

If $H(\xi) \to \infty$ as $\xi \to \infty$, then for all large ξ, $(\frac{y}{M} p D_{\omega,q} y) h^{-1}(q^{-\xi}) > 0$. Then y and $D_{\omega,q} y$ have the same sign for all large ξ, which contradicts $y \in L^2_{\omega,q}((\omega_0, \infty), r)$. Thus

$$H(\infty) = \int_\eta^\infty \frac{p}{M(\omega + qx)} (D_{\omega,q} y)^2 d_{\omega,q} x < +\infty. \tag{3.19}$$

By the condition (3.11), the left hand-side of (3.16) is infinite. By virtue of (3.17) and (3.19), the right hand-side of (3.16) is finite, a contradiction. So, the theorem is proved. □

Now, we will give special cases of this theorem.

In the case $M(x) \equiv 1$ and $r(x) \equiv 1$ the following corollary results.

Corollary 3.1. *For sufficiently large x, let $v(x) > -K$, where K is a positive constant, and*

$$\int_{\omega_0}^\infty p(x)^{-1/2} d_{\omega,q} x = +\infty,$$

then the deficiency indices of the symmetric operator $\mathcal{L}_{\min}$ are equal to $(1,1)$, i.e., equation (3.2) is in the limit-point case at infinity.

Let us consider the minimal symmetric operator $T_{\min}$ generated by the $\omega, q-$Sturm-Liouville expression

$$Ty := -q^{-1} D_{-\omega q^{-1}, q^{-1}} D_{\omega,q} y(x) + v(x) y(x), \ x \in (\omega_0, \infty),$$

and

$$Ty = \lambda y \text{ on } (\omega_0, \infty), \tag{3.20}$$

where λ is a complex parameter.

Then we have a

Theorem 3.3. *For sufficiently large x, let*

$$v(x) > -Kx^2,$$

where K is a positive constant. Then the deficiency indices of the symmetric operator $T_{\min}$ are equal to $(1,1)$, i.e., equation (3.20) is in the limit-point case at infinity.

Proof. If we take $p(x) \equiv 1$, $r(x) \equiv 1$ and $M(x) = x^2$ in Theorem 3.2, we have the proof. □

Theorem 3.4. *If, for certain constants $c > 0$, $K > 0$, and $c < x_1 < x_2$, the inequality*

$$v(x_2) - v(x_1) > -K(x_2 - x_1) \tag{3.21}$$

holds, then the deficiency indices of the symmetric operator $T_{\min}$ are equal to $(1,1)$, i.e., equation (3.20) is in the limit-point case at infinity.

Proof. If we take $x_2 = x$ and keep x_1 fixed in (3.21), we get

$$v(x) > -K(x - x_1) + v(x_1).$$

So, for sufficiently large x, $v(x) > -K_1 x$, where $K_1 > 0$ is a constant. Applying $p(x) \equiv 1$, $r(x) \equiv 1$ and $M(x) = x$ in Theorem 3.2, we obtain the desired result. □

Theorem 3.5. *Let $p(x) \equiv 1$, $v \in L^2_{\omega,q}\left((\omega_0, \infty), \frac{1}{r}\right)$ and $r \notin L^1_{\omega,q}((\omega_0, \infty), r)$. Then the deficiency indices of the symmetric operator $\mathcal{L}_{\min}$ are equal to $(1,1)$, i.e., equation (3.2) is in the limit-point case at infinity.*

Proof. We will show that the equation

$$-q^{-1}D_{-\omega q^{-1},q^{-1}}D_{\omega,q}y(x) + v(x)y(x) = 0 \tag{3.22}$$

does not have two linearly independent solutions belonging to $L^2_{\omega,q}((\omega_0, \infty), r)$. Let y is a such solution, i.e. $y \in L^2_{\omega,q}((\omega_0, \infty), r)$. By the condition $v \in L^2_{\omega,q}\left((\omega_0, \infty), \frac{1}{r}\right)$, $D_{-\omega q^{-1},q^{-1}}D_{\omega,q}y(x) = qv(x)y(x)$ belongs to $L^1_{\omega,q}(\omega_0, \infty)$. Thus, we have

$$\begin{aligned}&\int_{\omega_0}^{h(q^{-\xi})} D_{-\omega q^{-1},q^{-1}}D_{\omega,q}y(x)d_{\omega,q}x \\ =&D_{-\omega q^{-1},q^{-1}}y(h(q^{-\xi})) - D_{-\omega q^{-1},q^{-1}}y(\omega_0).\end{aligned}$$

Since

$$D_{-\omega q^{-1},q^{-1}}y(h(q^{-\xi})) = D_{\omega,q}y(q^{-\xi})$$

and

$$D_{-\omega q^{-1},q^{-1}}y(\omega_0) = D_{\omega,q}y\left(h^{-1}(\omega_0)\right) = D_{\omega,q}y(\omega_0),$$

we get

$$\lim_{\xi\to\infty} D_{\omega,q}y(q^{-\xi}) = D_{\omega,q}y(\omega_0) + \int_{\omega_0}^{\infty} D_{-\omega q^{-1},q^{-1}}D_{\omega,q}y(x)d_{\omega,q}x. \tag{3.23}$$

Then the limit in the equality (3.23) exist. Therefore, the function $D_{\omega,q}y(q^{-\xi})$ is bounded as $\xi \to \infty$.

Now, let y_1 and y_2 be two linearly independent solutions of equation (3.22), then

$$W_{\omega,q}(y_1, y_2)(x) = y_1(x)D_{\omega,q}y_2(x) - y_2(x)D_{\omega,q}y_1(x) = c \neq 0.$$

If $y_1 \in L^2_{\omega,q}((\omega_0, \infty), r)$ and $y_2 \in L^2_{\omega,q}((\omega_0, \infty), r)$, then $D_{\omega,q}y_1$ and $D_{\omega,q}y_2$ are bounded. So, the function $y_1(x)D_{\omega,q}y_2(x) - y_2(x)D_{\omega,q}y_1(x) = c \neq 0$ also belongs to $L^2_{\omega,q}((\omega_0, \infty), r)$, which is impossible. The theorem is proved. □

Example 3.1. Consider the equation

$$-q^{-1}D_{-\omega q^{-1},q^{-1}}\left[p(x)D_{\omega,q}y(x)\right] + v(x)y(x) = \lambda y(x), \ x \in (\omega_0, \infty), \tag{3.24}$$

where $p(x) = \left\{x - \frac{\omega}{1-q}\right\}^{-2}$ and $v(x) = x^2$. We will show that the assumptions in Corollary 3.1 hold.

It is clear that $v(x) > -K$, where K is a positive constant.
Consider

$$\begin{aligned}\int_{\omega_0}^{\infty} p(x)^{-1/2} d_{\omega,q}x &= \int_{\omega_0}^{\infty} \left(x - \frac{\omega}{1-q}\right) d_{\omega,q}x \\ &= \lim_{b\to\infty} \int_{\omega_0}^{b} \left(x - \frac{\omega}{1-q}\right) d_{\omega,q}x \\ &= \lim_{b\to\infty} ((1-q)\,b - \omega) \sum_{n=0}^{\infty} q^{2n} \left(b - \frac{\omega}{1-q}\right) = +\infty.\end{aligned}$$

Thus, the assumptions of Corollary 3.1 hold, so we get that the equation (3.24) is in the limit-point case at infinity.

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