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STEPANOV-LIKE PSEUDO ALMOST PERIODIC SOLUTIONS FOR IMPULSIVE PERTURBED PARTIAL STOCHASTIC DIFFERENTIAL EQUATIONS AND ITS OPTIMAL CONTROL*

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Abstract This paper is mainly concerned with the Stepanov-like pseudo almost periodicity to a class of impulsive perturbed partial stochastic differential equations. Firstly, we prove the existence of p-mean piecewise Stepanov-like pseudo almost periodic mild solutions for the impulsive stochastic dynamical system in a Hilbert space under non-Lipschitz conditions. The results are obtained by using the fixed point techniques with fractional power arguments. Then the existence of optimal pairs of system governed by impulsive partial stochastic differential equations is also obtained. Finally, an example is provided to illustrate the developed theory.

Keywords Impulsive stochastic perturbed partial differential equations, Stepanov-like pseudo almost periodic solutions, optimal controls, fractional power operators, fixed point.

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1. Introduction

The concept of Stepanov-like pseudo almost periodicity as a natural generalization of the concept of pseudo almost periodicity as well as the one of Stepanov-like almost periodicity. In recent years, there has been a significant development in the existence of Stepanov-like pseudo almost periodic solutions to deterministic abstract differential equations; see [3, 11, 14] and references therein. In the real world, stochastic perturbation is unavoidable and omnipresent. Therefore, the concept of Stepanov-like pseudo almost periodicity is of great importance in probability for investigating stochastic processes. Recently, the existence of Stepanov-like pseudo almost periodic or Stepanov-like almost periodic solutions to some stochastic differential equations have been considered in many publications such as [4, 5, 12, 31] and the references therein.

On the other hand, the impulsive effects exist widely in many evolution processes in which states are changed abruptly at certain moments of time, involving such fields as finance, economics, mechanics, electronics and telecommunications,

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etc. (see [21]). The existence and qualitative properties of piecewise almost periodic solutions, piecewise pseudo almost periodic solutions and piecewise weighted pseudo almost periodic solutions for impulsive deterministic differential equations have been considered by many authors; see [16, 22, 23, 25, 26]. However, besides impulse effects and delays, stochastic effects likewise exist in real systems. Several dynamical systems have variable structures subject to stochastic abrupt changes, which may result from abrupt phenomena such as sudden environment changes, changes in the interconnections of subsystems, stochastic failures and repairs of the components, etc. Therefore, impulsive stochastic differential equations describing these dynamical systems subject to both impulse and stochastic changes have attracted considerable attention [15, 19, 20, 27]. More recently, Yan and Lu [28] obtained the existence and exponential stability of piecewise pseudo almost periodic mild solutions for nonlinear impulsive stochastic differential equations by using a fixed point theorem.

In this article, we investigate the existence of p-mean piecewise Stepanov-like pseudo almost periodic mild solutions for the following impulsive stochastic perturbed partial differential equations such as

$$\begin{aligned} d[x(t)+g(t,B_1x(t))] &= A[x(t)+g(t,B_1x(t))]dt+f(t,B_2x(t))dt \\ &\quad +F(t,B_3x(t))dW(t), \quad t\in\mathbb{R}, t\neq t_i, i\in\mathbb{Z}, \end{aligned} \tag{1.1}$$

$$\Delta x(t_i)=x(t_i^+)-x(t_i^-)=I_i(x(t_i)), \quad i\in\mathbb{Z}, \tag{1.2}$$

where A is the infinitesimal generator of an uniformly exponentially stable analytic semigroup $(T(t))_{t\geq 0}$ on $L^p(\mathbb{P},\mathbb{H})$, and $B_j, j=1,2,3$, are arbitrary linear (possibly unbounded) operators on $L^p(\mathbb{P},\mathbb{H})$, $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega,\mathcal{F},\mathbb{P},\mathcal{F}_t)$, where $\mathcal{F}_t=\sigma\{W(s)-W(\tau); s,\tau\leq t\}$. g, f, F and I_i, t_i satisfy suitable conditions which will be established later. $x(t_i^+), x(t_i^-)$ represent the right-hand side and the left-hand side limits of $x(\cdot)$ at t_i, respectively.

The asymptotic properties of solutions to dynamical systems is one of the fundamental tasks of the analysis theory and finds its application in various fields; see [6, 8, 9, 16, 22, 23, 25, 26, 28]. However, many systems arising from realistic models can be described as partial stochastic differential equations with impulse and Stepanov-like pseudo almost periodic coefficients, and the systems deserve a study because it is a more general hybrid system, and that of can be more accurate description of the actual phenomenon in the real world. So it is natural to extend the concept of Stepanov-like pseudo almost periodic periodicity to dynamical systems represented by these impulsive systems in an infinite interval. Motivated by the above consideration, we study the problems, which are natural generalizations of the concepts for impulsive equations well known in the theory of infinite dimensional systems. This is one of our motivations.

The study of optimal control problems for differential control systems in practical engineering applications will be more important to the system stability and performance. For instance [13, 18, 24], and the authors [1, 29, 30] discussed the optimal control of partial stochastic differential systems with impulse under a finite interval. However, the optimal control of impulsive stochastic differential control systems with Stepanov-like pseudo almost periodic coefficients is an untreated topic. Therefore, it is interesting to study the control problems. This is another of our motivations.

This paper has three main contributions: (i) We introduce the new concept of p-mean piecewise Stepanov-like pseudo almost periodic mild solutions for impulsive stochastic systems, which is natural generalizations of the concept of pseudo almost periodicity for stochastic differential systems in abstract spaces. (ii) We study and obtain the existence of p-mean piecewise Stepanov-like pseudo almost periodic mild solutions to system (1.1)-(1.2) for the non-Lipschitz conditions cases by using the Krasnoselskii-Schaefer type fixed point theorem along with a new composition theorem, stochastic analysis, analytic semigroup and fractional powers of closed operators. Moreover, we consider the Lagrange problem of systems governed by impulsive partial stochastic differential equations with Stepanov-like pseudo almost coefficients and an existence result of optimal controls is presented. (iii) The known results appeared in [11, 12, 14, 16, 23, 25, 28, 31] are generalized to the impulsive stochastic control systems with Stepanov-like pseudo almost periodic coefficients settings.

The paper is organized as follows. In Section 2, we introduce some notations and necessary preliminaries. In Section 3, we give the existence of Stepanov-like pseudo almost periodic mild solutions for (1.1)-(1.2). In Section 4, we establish the existence result of optimal controls for a Lagrange problem. In Section 5, an example is given to illustrate our results. Finally, concluding remarks are given in Section 6.

2. Preliminaries

Throughout the paper, we assume that $(\mathbb{H}, \|\cdot\|), (\mathbb{K}, \|\cdot\|_{\mathbb{K}})$ are real separable Hilbert spaces and $(\Omega, \mathcal{F}, \mathbb{P})$ is supposed to be a filtered complete probability space. Let $L(\mathbb{K}, \mathbb{H})$ be the space of all linear bounded operators from $\mathbb{K}$ into $\mathbb{H}$, equipped with the usual operator norm $\|\cdot\|_{L(\mathbb{K},\mathbb{H})}$; in particular, this is simply denoted by $L(\mathbb{H})$ when $\mathbb{K} = \mathbb{H}$. Furthermore, $L_2^0(\mathbb{K}, \mathbb{H})$ denotes the space of all $\mathbb{Q}$-Hilbert-Schmidt operators from $\mathbb{K}$ to $\mathbb{H}$ with the norm $\| \psi \|^2_{L_2^0} = \mathrm{Tr}(\psi \mathbb{Q} \psi^*) < \infty$ for any $\psi \in L(\mathbb{K}, \mathbb{H})$. Let $\mathbb{T}$ be the set consisting of all real sequences $\{t_i\}_{i\in\mathbb{Z}}$ such that $\gamma = \inf_{i\in\mathbb{Z}}(t_{i+1} - t_i) > 0$, $\lim_{i\to\infty} t_i = \infty$ and $\lim_{i\to-\infty} t_i = -\infty$. For $\{t_i\}_{i\in\mathbb{Z}} \in \mathbb{T}$, let $PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ be the space consisting of all bounded piecewise continuous processes $f : \mathbb{R} \to L^p(\mathbb{P}, \mathbb{H})$ such that $f(\cdot)$ is continuous at t for any $t \notin \{t_i\}_{i\in\mathbb{Z}}$ and $f(t_i) = f(t_i^-)$ for all $i \in \mathbb{Z}$; let $PC(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{K}), L^p(\mathbb{P}, \mathbb{H}))$ be the space formed by all piecewise continuous processes $f : \mathbb{R} \times L^p(\mathbb{P}, \mathbb{K}) \to L^p(\mathbb{P}, \mathbb{H})$ such that for any $x \in L^p(\mathbb{P}, \mathbb{K})$, $f(\cdot, x) \in PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ and for any $t \in \mathbb{R}, f(t, \cdot)$ is continuous at $x \in L^p(\mathbb{P}, \mathbb{K})$.

Definition 2.1 ([5]). A function $f \in C(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ is said to be p-mean almost periodic if for each $\varepsilon > 0$, there exists an $l(\varepsilon) > 0$, such that every interval J of length $l(\varepsilon)$ contains a number τ with the property that $E \| f(t+\tau) - f(t) \|^p < \varepsilon$ for all $t \in R$. Denote by $AP(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ the set of such functions.

Definition 2.2 ([21]). A sequence $\{x_n\}$ is called p-mean almost periodic if for any $\varepsilon > 0$, there exists a relatively dense set of its ε-periods, i.e., there exists a natural number $l = l(\varepsilon)$, such that for $k \in \mathbb{Z}$, there is at least one number q in $[k, k+l]$, for which inequality $E \| x_{n+q} - x_n \|^p < \varepsilon$ holds for all $n \in \mathbb{N}$. Denote by $AP(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H})))$ the set of such sequences.

Define $l^\infty(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H})) = \{x : \mathbb{Z} \to L^p(\mathbb{P}, \mathbb{H}) : \| x \| = \sup_{n\in\mathbb{Z}}(E \| x(n) \|^p)^{1/p} <$

$\infty\}$, and

$$PAP_0(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H})) = \left\{ x \in l^\infty(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H})) : \lim_{n\to\infty} \frac{1}{2n} \sum_{j=-n}^{n} E \parallel x(j) \parallel^p = 0 \right\}.$$

Definition 2.3 ([28])**.** A sequence $\{x_n\}_{n\in\mathbb{Z}} \in l^\infty(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H}))$ is called p-mean pseudo almost periodic if $x_n = x_n^1 + x_n^2$, where $x_n^1 \in AP(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H})), x_n^2 \in PAP_0(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H}))$. Denote by $PAP(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H}))$ the set of such sequences.

Definition 2.4 (Compare with [21])**.** For $\{t_i\}_{i\in\mathbb{Z}} \in \mathbb{T}$, the function $f \in PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ is said to be p-mean piecewise almost periodic if the following conditions are fulfilled:

(i) $\{t_i^j = t_{i+j} - t_i\}, j \in \mathbb{Z}$, is equipotentially almost periodic, that is, for any $\varepsilon > 0$, there exists a relatively dense set Q_ε of $\mathbb{R}$ such that for each $\tau \in Q_\varepsilon$ there is an integer $\tilde{q} \in \mathbb{Z}$ such that $|t_{i+\tilde{q}} - t_i - \tau| < \varepsilon$ for all $i \in \mathbb{Z}$.

(ii) For any $\varepsilon > 0$, there exists a positive number $\tilde{\delta} = \tilde{\delta}(\varepsilon)$ such that if the points t' and t'' belong to a same interval of continuity of φ and $|t' - t''| < \tilde{\delta}$, then $E \parallel f(t') - f(t'') \parallel^p < \varepsilon$.

(iii) For every $\varepsilon > 0$, there exists a relatively dense set $\tilde{\Omega}(\varepsilon)$ in $\mathbb{R}$ such that if $\tau \in \tilde{\Omega}(\varepsilon)$, then

$$E \parallel f(t+\tau) - f(t) \parallel^p < \varepsilon$$

for all $t \in \mathbb{R}$ satisfying the condition $|t - t_i| > \varepsilon, i \in \mathbb{Z}$. The number τ is called ε-translation number of f.

We denote by $AP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ the collection of all the p-mean piecewise almost periodic functions. Obviously, the space $AP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ endowed with the sup norm defined by $\parallel f \parallel_\infty = \sup_{t\in\mathbb{R}}(E \parallel f(t) \parallel^p)^{1/p}$ for any $f \in AP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ is a Banach space. Let $UPC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ be the space of all stochastic functions $f \in PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ such that f satisfies the condition (ii) in Definition 2.4.

Definition 2.5 ([21])**.** The function $f \in PC(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{K}), L^p(\mathbb{P}, \mathbb{H}))$ is said to be p-mean piecewise almost periodic in $t \in \mathbb{R}$ uniform in $x \in L^p(\mathbb{P}, \mathbb{K})$ if for every compact subset $K \subseteq L^p(\mathbb{P}, \mathbb{K})$, $\{f(\cdot, x) : x \in K\}$ is uniformly bounded, and given $\varepsilon > 0$, there exists a relatively dense subset Ω_ε such that

$$E \parallel f(t+\tau, x) - f(t, x) \parallel^p < \varepsilon$$

for all $x \in K, \tau \in \Omega_\varepsilon$, and $t \in \mathbb{R}$ satisfying $|t - t_i| > \varepsilon$. Denote by $AP_T(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{K}), L^p(\mathbb{P}, \mathbb{H}))$ the set of all such functions.

Denote

$$PC_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H})) = \left\{ f \in PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H})) : \lim_{t\to\infty} E \parallel f(t) \parallel^p = 0 \right\},$$

$$PAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H})) = \left\{ f \in PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H})) : \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} E \parallel f(t) \parallel^p dt = 0 \right\},$$

$$
\begin{aligned}
&PAP_T^0(\mathbb{R}\times L^p(\mathbb{P},\mathbb{K}),L^p(\mathbb{P},\mathbb{H}))\\
&\quad=\Big\{f\in PC(\mathbb{R}\times L^p(\mathbb{P},\mathbb{K}),L^p(\mathbb{P},\mathbb{H})):\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}E\parallel f(t,x)\parallel^p dt=0\\
&\qquad\text{uniformly with respect to } x\in K,\\
&\qquad\text{where } K \text{ is an arbitrary compact subset of } L^p(\mathbb{P},\mathbb{K})\Big\}.
\end{aligned}
$$

Definition 2.6 ([28]). A function $f \in PC(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$ is said to be p-mean piecewise pseudo almost periodic if it can be decomposed as $f = h + \varphi$, where $h \in AP_T(L^p(\mathbb{P},\mathbb{H}))$ and $\varphi \in PAP_T^0(L^p(\mathbb{P},\mathbb{H}))$. Denote by $PAP_T(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$ the set of all such functions. $PAP_T(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$ is a Banach space with the sup norm $\|\cdot\|_\infty$.

Remark 2.1 ([28]). (i) $PAP_T^0(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$ is a translation invariant set of $PC(\mathbb{R}, L^p(\mathbb{P},\mathbb{H})))$. (ii) $PC_T^0(\mathbb{R}, L^p(\mathbb{P},\mathbb{H})) \subset PAP_T^0(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$.

Lemma 2.1 ([28]). *Let $\{f_n\}_{n\in\mathbb{N}} \subset PAP_T^0(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$ be a sequence of functions. If f_n converges uniformly to f, then $f \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$.*

Definition 2.7 ([28]). A function $f \in PC(\mathbb{R}\times L^p(\mathbb{P},\mathbb{K}), L^2(\mathbb{P},\mathbb{H}))$ is said to be p-mean piecewise pseudo almost periodic if it can be decomposed as $f = h+\varphi$, where $h \in AP_T(\mathbb{R}\times L^p(\mathbb{P},\mathbb{K}))$ and $\varphi \in PAP_T^0(L^p(\mathbb{R}\times L^p(\mathbb{P},\mathbb{K}))$. Denote by $PAP_T(\mathbb{R}\times L^p(\mathbb{P},\mathbb{K}), L^p(\mathbb{P},\mathbb{H}))$ the set of all such functions.

Lemma 2.2 ([28]). *Let $f \in PAP_T(\mathbb{R}\times L^p(\mathbb{P},\mathbb{K}), L^p(\mathbb{P},\mathbb{H}))$. Assume further that there exists a number $L_f > 0$ satisfying*

$$E \parallel f(t,x) - f(t,y) \parallel^p \leq L_f E \parallel x - y \parallel^p$$

for all $t \in \mathbb{R}, x, y \in L^p(\mathbb{P},\mathbb{K})$. If $\phi(\cdot) \in PAP_T(\mathbb{R}, L^p(\mathbb{P},\mathbb{K}))$ then $f(\cdot,\phi(\cdot)) \in PAP_T(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$.

Lemma 2.3 ([28]). *Assume the sequence of vector-valued functions $\{I_i\}_{i\in\mathbb{Z}}$ is pseudo almost periodic, and there exist $L_i > 0$ satisfying*

$$E \parallel I_i(x) - I_i(y) \parallel^p \leq L_i E \parallel x - y \parallel^p)$$

for all $x, y \in L^p(\mathbb{P},\mathbb{K})$, $i \in \mathbb{Z}$. If $\phi \in PAP_T(\mathbb{R}, L^p(\mathbb{P},\mathbb{H})) \cap UPC(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$ such that $\mathbb{R}(\phi) \subset L^p(\mathbb{P},\mathbb{K})$, then $I_i(\phi(t_i))$ is pseudo almost periodic.

Definition 2.8 ([4]). The Bochner transform $x^b(t,s), t \in \mathbb{R}, s \in [0,1]$, of a stochastic process $x : \mathbb{R} \to L^p(\mathbb{P},\mathbb{H})$ is defined by

$$x^b(t,s) := x(t+s).$$

Remark 2.2 ([4]). A stochastic process $\psi(t,s), t \in \mathbb{R}, s \in [0,1]$, is the Bochner transform of a certain stochastic process x,

$$\psi(t,s) = x^b(t,s),$$

if and only if

$$\psi(t+\tau, s-\tau) = \psi(s,t)$$

for all $t \in \mathbb{R}$, $s \in [0,1]$ and $\tau \in [s-1, s]$.

Definition 2.9 ([4]). The Bochner transform $F^b(t,s,\tilde{u}), t \in \mathbb{R}, s \in [0,1], \tilde{u} \in L^p(\mathbb{P},\mathbb{H})$, of a function $F : \mathbb{R} \times L^p(\mathbb{P},\mathbb{H}) \to L^p(\mathbb{P},\mathbb{H})$ is defined by

$$F^b(t,s,\tilde{u}) := F(t+s,\tilde{u})$$

for each $\tilde{u} \in L^p(\mathbb{P},\mathbb{H})$.

Definition 2.10 ([4]). The space $BS^p(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$ of all Stepanov bounded stochastic processes consists of all measurable stochastic processes $x : \mathbb{R} \to L^p(\mathbb{P},\mathbb{H})$ such that

$$x^b \in L^\infty(\mathbb{R}, L^p((0,1), L^p(\mathbb{P},\mathbb{H}))).$$

This is a Banach space with the norm

$$\| x \|_{S^p} = \| x^b \|_{L^\infty(\mathbb{R},L^p)} = \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} E \| x(\tau) \|^p \, d\tau \right)^{\frac{1}{p}}.$$

Definition 2.11. A stochastic process $f \in BS^p(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$ is said to be Stepanov-like p-mean piecewise pseudo almost periodic (or S^p-pseudo almost periodic) if it can be decomposed as $f = h + \varphi$, where $h^b \in AP_T(\mathbb{R}, L^p((0,1), L^p(\mathbb{P},\mathbb{H})))$ and $\varphi^b \in PAP_T^0(\mathbb{R}, L^p((0,1), L^p(\mathbb{P},\mathbb{H})))$. Denote the set of all such stochastically continuous processes by $PAPS_T^p(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$.

In other words, a stochastic process $f \in L^p_{\text{loc}}(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$ is said to be Stepanov-like p-mean piecewise pseudo almost periodic if its Bochner transform $f^b : \mathbb{R} \to L^p((0,1), L^p(\mathbb{P},\mathbb{H}))$ is p-mean piecewise pseudo almost periodic in the sense that there exist two functions $h, \varphi : \mathbb{R} \to L^p(\mathbb{P},\mathbb{H})$ such that $f = h + \varphi$, where $h^b \in AP_T(\mathbb{R}, L^p((0,1), L^p(\mathbb{P},\mathbb{H})))$ and $\varphi^b \in PAP_T^0(\mathbb{R}, L^p((0,1), L^p(\mathbb{P},\mathbb{H})))$ i.e.,

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \left(\int_t^{t+1} E \| \varphi(\tau) \|^p \, d\tau \right)^{\frac{1}{p}} dt = 0.$$

Obviously, the following inclusions hold:

$$AP_T(\mathbb{R}, L^p(\mathbb{P},\mathbb{H})) \subset PAP_T(\mathbb{R}, L^p(\mathbb{P},\mathbb{H})) \subset PAPS_T^p(\mathbb{R}, L^p(\mathbb{P},\mathbb{H})).$$

Definition 2.12. A stochastic process $f \in BS^p(\mathbb{R} \times L^p(\mathbb{P},\mathbb{K}), L^p(\mathbb{P},\mathbb{H}))$ is said to be Stepanov-like p-mean piecewise pseudo almost periodic (or S^p-pseudo almost periodic) if it can be decomposed as $f = h + \varphi$, where $h^b \in AP_T(\mathbb{R} \times L^p((0,1), L^p(\mathbb{P},\mathbb{K}), L^p(\mathbb{P},\mathbb{H})))$ and $\varphi^b \in PAP_T^0(\mathbb{R} \times L^p((0,1), L^p(\mathbb{P},\mathbb{K}), L^p(\mathbb{P},\mathbb{H})))$. Denote the set of all such stochastically continuous processes by $PAPS^p(\mathbb{R} \times L^p(\mathbb{P},\mathbb{K}), L^p(\mathbb{P},\mathbb{H}))$.

We need the following composition of Stepanov-like p-mean pseudo almost periodic processes.

Lemma 2.4. *Assume $f \in PAPS_T^p(\mathbb{R} \times L^p(\mathbb{P},\mathbb{K}), L^p(\mathbb{P},\mathbb{H}))$. Suppose that $f(t,x)$ satisfies*

$$E \| f(t,x) - f(t,y) \|^p \leq \Lambda(E \| x - y \|^p) \tag{2.1}$$

for all $t \in \mathbb{R}, x, y \in L^p(\mathbb{P},\mathbb{K})$, where Λ is a concave and continuous nondecreasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that $\Lambda(0) = 0, \Lambda(s) > 0$ for $s > 0$ and $\int_{0^+} \frac{ds}{\Lambda(s)} = +\infty$. Here, the symbol $\int_{0^+}$ stands for $\lim_{\varepsilon\to 0^+} \int_\varepsilon^{+\infty}$. If $\phi(\cdot) \in PAPS_T^p(\mathbb{R}, L^p(\mathbb{P},\mathbb{K}))$ then $f(\cdot,\phi(\cdot)) \in PAPS_T^p(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$.

Proof. Assume that $f^p = f_1^p + f_2^p, \phi^p = \phi_1^p + \phi_2^p$, where $f_1^p \in AP_T(\mathbb{R} \times L^p((0,1), L^p(\mathbb{P}, \mathbb{K}))), L^p(\mathbb{P}, \mathbb{H}))$, $f_2^p \in PAP_T^0(\mathbb{R} \times L^p((0,1), L^p(\mathbb{P}, \mathbb{K}), L^p(\mathbb{P}, \mathbb{H})))$, $\phi_1 \in AP_T(\mathbb{R}, L^p((0,1), L^p(\mathbb{P}, \mathbb{K})))$, and $\phi_2 \in PAP_T^0(\mathbb{R}, L^p((0,1), L^p(\mathbb{P}, \mathbb{K})))$. Consider the decomposition

$$f^p(t, \phi^p(t)) = f_1^p(t, \phi_1^p(t)) + [f^p(t, \phi^p(t)) - f^p(t, \phi_1^p(t))] + f_2^p(t, \phi_1^p(t)).$$

Since $f_1^p(\cdot, \phi_1^p(\cdot)) \in AP_T(\mathbb{R}, L^p((0,1), L^p(\mathbb{P}, \mathbb{H})))$, it remains to prove that both $[f^p(\cdot, \phi^p(\cdot)) - f^p(\cdot, \phi_1^p(\cdot))]$ and $f_2^p(\cdot, \phi_1^p(\cdot))$ belong to $PAP_T^0(\mathbb{R}, L^p((0,1), L^p(\mathbb{P}, \mathbb{H})))$. Indeed, using (2.1), it follows that

$$\begin{aligned}
&\frac{1}{2r} \int_{-r}^{r} \int_{t}^{t+1} E \parallel f^p(\tau, \phi^p(\tau)) - f^p(\tau, \phi_1^p(\tau)) \parallel^p d\tau dt \\
\leq &\frac{1}{2r} \int_{-r}^{r} \int_{t}^{t+1} \Lambda(E \parallel \phi^p(\tau) - \phi_1^p(\tau) \parallel^p) d\tau dt \\
= &\frac{1}{2r} \int_{-r}^{r} \int_{t}^{t+1} \Lambda(E \parallel \phi_2^p(\tau) \parallel^p) d\tau dt,
\end{aligned}$$

noting that Λ is concave, continuous and $\Lambda(0) = 0$, we deduce that

$$\begin{aligned}
&\frac{1}{2r} \int_{-r}^{r} \int_{t}^{t+1} \Lambda(E \parallel \phi_2^p(\tau) \parallel^p) d\tau dt \\
\leq &\Lambda\left(\frac{1}{2r} \int_{-r}^{r} \int_{t}^{t+1} E \parallel \phi_2^p(\tau) \parallel^p d\tau dt\right) \to 0 \quad \text{as } r \to \infty,
\end{aligned}$$

which implies that $[f(\cdot, \phi(\cdot)) - f(\cdot, \phi_1(\cdot))] \in PAP_T^0(\mathbb{R}, L^p(0,1; L^p(\mathbb{P}, \mathbb{H}))$.

Since $\phi_1^p(\mathbb{R})$ is relatively compact in $L^p(\mathbb{P}, \mathbb{K})$ and f_1^p is uniformly continuous on sets of the form $\mathbb{R} \times K$ where $K \subset L^p(\mathbb{P}, \mathbb{K})$ is compact subset, for $\varepsilon > 0$ there exists $\xi \in (0,1)$ such that

$$E \parallel f_1^p(t, z) - f_1^p(t, \tilde{z}) \parallel^p \leq \varepsilon, \ \ z, \tilde{z} \in \phi_1^p(\mathbb{R})$$

with $|z - \tilde{z}| < \xi$. Now, fix $z_1, \ldots, z_n \in \phi_1^p(\mathbb{R})$ such that $\phi_1^p(\mathbb{R}) \subset \bigcup_{j=1}^n B_\xi(z_j, L^p(0, \varsigma; L^p(\mathbb{P}, \mathbb{K}))$. Obviously, the sets $D_j = (\phi_1^p)^{-1}(B_\xi(z_j))$ form an open covering of $\mathbb{R}$, and therefore using the sets $B_1 = D_1, B_2 = D_2 \backslash D_1$ and $B_j = D_j \backslash \bigcup_{k=1}^{j-1} D_k$ one obtains a covering of $\mathbb{R}$ by disjoint open sets. For $t \in B_j, \phi_1^p(t) \in B_\xi(z_j)$,

$$\begin{aligned}
E \parallel f_2^p(t, \phi_1^p(t)) \parallel^p &\leq 3^{p-1} E \parallel f^p(t, \phi_1^p(t)) - f^p(t, z_j) \parallel^p \\
&\quad + 3^{p-1} E \parallel -f_1^p(t, \phi_1^p(t)) + f_1^p(t, z_j) \parallel^p \\
&\quad + 3^{p-1} E \parallel f_2^p(t, z_j) \parallel^p \\
&\leq 3^{p-1} \Lambda(E \parallel \phi_1^p(t) - z_j \parallel^p) + 3^{p-1}\varepsilon + 3^{p-1} E \parallel f_2^p(t, z_j) \parallel^p \\
&\leq 3^{p-1} \Lambda(\varepsilon) + 3^{p-1}\varepsilon + 3^{p-1} E \parallel f_2^p(t, z_j) \parallel^p .
\end{aligned}$$

Now using the previous inequality it follows that

$$\begin{aligned}
&\frac{1}{2r} \int_{-r}^{r} \int_{t}^{t+1} E \parallel f_2^p(t, \phi_1^p(\tau)) \parallel^p d\tau dt \\
= &\frac{1}{2r} \sum_{j=1}^{n} \int_{B_j \cap [-r,r]} \int_{t}^{t+1} E \parallel f_1^p(\tau, \phi_1^p(\tau)) \parallel^p d\tau dt
\end{aligned}$$

$$
\begin{aligned}
&\leq 3^{p-1}\frac{1}{2r}\sum_{j=1}^{n}\int_{B_j\cap[-r,r]}\int_t^{t+1} E\parallel f^p(\tau,\phi_1^p(\tau))-f^p(\tau,z_j)\parallel^p d\tau dt\\
&\quad+3^{p-1}\frac{1}{2r}\sum_{j=1}^{n}\int_{B_j\cap[-r,r]}\int_t^{t+1} E\parallel f_1^p(\tau,\phi_1^p(\tau))-f_1^p(\tau,z_j)\parallel^p d\tau dt\\
&\quad+3^{p-1}\frac{1}{2r}\sum_{j=1}^{n}\int_{B_j\cap[-r,r]}\int_t^{t+1} E\parallel f_2^p(\tau,z_j)\parallel^p d\tau dt\\
&\leq 3^{p-1}\frac{1}{2r}\int_{-r}^{r}\int_t^{t+1}[\Lambda(\varepsilon)+\varepsilon]d\tau dt\\
&\quad+3^{p-1}\sum_{j=1}^{n}\frac{1}{2r}\int_{-r}^{r}\int_t^{t+1} E\parallel f_2^p(\tau,z_j)\parallel^p d\tau dt.
\end{aligned}
$$

In view of the above it is clear that $f_2^p(\cdot,\phi_1^p(\cdot))$ belongs to $PAP_T^0(\mathbb{R},L^p(0,1;L^p(\mathbb{P},\mathbb{H}))$. □

Next, we introduce a useful compactness criterion on $PC(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$. Let $h:\mathbb{R}\to\mathbb{R}^+$ be a continuous function such that $h(t)\geq 1$ for all $t\in\mathbb{R}$ and $h(t)\to\infty$ as $|t|\to\infty$. Define

$$PC_h^0(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))=\left\{f\in PC(\mathbb{R},L^p(\mathbb{P},\mathbb{H})):\lim_{|t|\to\infty}\frac{E\parallel f(t)\parallel^p}{h(t)}=0\right\}$$

endowed with the norm $\parallel f\parallel_h=\sup_{t\in\mathbb{R}}\frac{E\parallel f(t)\parallel^p}{h(t)}$, it is a Banach space.

Lemma 2.5 ([28]). *A set $\tilde{B}\subseteq PC_h^0(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$ is relatively compact if and only if it verifies the following conditions:*

(i) $\lim\limits_{|t|\to\infty}\frac{E\|f(t)\|^p}{h(t)}=0$ *uniformly for $f\in\tilde{B}$.*

(ii) $\tilde{B}(t)=\{f(t):f\in\tilde{B}\}$ *is relatively compact in $L^p(\mathbb{P},\mathbb{H})$ for every $t\in\mathbb{R}$.*

(iii) The set $\tilde{B}$ is equicontinuous on each interval $(t_i,t_{i+1})(i\in Z)$.

Let $0\in\rho(A)$, then it is possible to define the fractional power A^α, for $0<\alpha\leq 1$, as a closed linear operator on its domain $D(A^\alpha)$. Furthermore, the subspace $D(A^\alpha)$ is dense in $\mathbb{H}$ and the expression $\parallel x\parallel_\alpha=\parallel A^\alpha x\parallel, x\in D(A^\alpha)$, defines a norm on $D(A^\alpha)$. Hereafter we denote by $\mathbb{H}_\alpha$ the Banach space $D(A^\alpha)$ with norm $\parallel x\parallel_\alpha$.

Lemma 2.6 ([17]). *Let $0<\alpha\leq\beta\leq 1$. Then the following properties hold:*

(a) $\mathbb{H}_\beta$ is a Banach space and $\mathbb{H}_\beta\hookrightarrow\mathbb{H}_\alpha$ is continuous.

(b) The function $s\to A^\beta T(s)$ is continuous in the uniform operator topology on $(0,\infty)$ and there exists $M_\beta>0$ such that $\parallel A^\beta T(t)\parallel\leq M_\beta e^{-\delta t}t^{-\beta}$ for each $t>0$.

(c) For each $x\in D(A^\beta)$ and $t\geq 0$, $A^\beta T(t)x=T(t)A^\beta x$.

(d) $A^{-\beta}$ is a bounded linear operator in $\mathbb{H}$ with $D(A^\beta)=Im(A^{-\beta})$.

Similar to [16], one has

Lemma 2.7. *Assume that* $f \in AP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$, *the sequence* $\{x_i\}_{i\in\mathbb{Z}} \in AP(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H}))$, *and* $\{t_i^j\}, j \in \mathbb{Z}$ *are equipotentially almost periodic. Then, for each* $\varepsilon > 0$, *there exist relatively dense sets* Ω_ε *of* $\mathbb{R}$ *and* Ω_ε *of* $\mathbb{Z}$ *such that*

(i) $E \parallel f(t+\tau) - f(t) \parallel^p < \varepsilon$ *for all* $t \in \mathbb{R}$, $|t - t_i| > \varepsilon$, $\tau \in \Omega_\varepsilon$ *and* $i \in \mathbb{Z}$.

(ii) $E \parallel x_{i+\tilde{q}} - x_i \parallel^p < \varepsilon$ *for all* $\tilde{q} \in \Omega_\varepsilon$ *and* $i \in \mathbb{Z}$.

(iii) $E \parallel x_i^{\tilde{q}} - \tau \parallel^p < \varepsilon$ *for all* $\tilde{q}, \tau \in \Omega_\varepsilon$ *and* $i \in \mathbb{Z}$.

Lemma 2.8 (Krasnoselskii-Schaefer type fixed point theorem [7])**.** *Let* Φ_1, Φ_2 *be two operators such that:*

(a) Φ_1 *is a contraction, and*

(b) Φ_2 *is completely continuous.*

Then either:

(i) the operator equation $x = \Phi_1 x + \Phi_2 x$ *has a solution, or*

(ii) the set $G = \{x \in \mathbb{H} : \lambda\Phi_1(\frac{x}{\lambda}) + \lambda\Phi_2 x = x\}$ *is unbounded for* $\lambda \in (0, 1)$.

3. Existence of Stepanov-like pseudo almost periodic mild solution

In this section, we investigate the existence of p-mean piecewise Stepanov-like pseudo almost periodic mild solution for system (1.1)-(1.2). To do this, we first introduce the notion of mild solution to system (1.1)-(1.2).

Definition 3.1. An $\mathcal{F}_t$ -progressively measurable process $\{x(t)\}_{t\in\mathbb{R}}$ is called a mild solution of system (1.1)-(1.2) if for any $t \in \mathbb{R}, t > \sigma, \sigma \neq t_i, i \in \mathbb{Z}$,

$$\begin{aligned} x(t) &= T(t-\sigma)[x(\sigma) + g(\sigma, B_1 x(\sigma))] - g(t, B_1 x(t)) \\ &\quad + \int_\sigma^t T(t-s) f(s, B_2 x(s)) ds + \int_\sigma^t T(t-s) F(s, B_3 x(s)) dW(s) \\ &\quad + \sum_{\sigma < t_i < t} T(t-t_i) I_i(x(t_i)). \end{aligned} \tag{3.1}$$

In order to obtain our main results, we assume that the operator $A^{-q} : \mathbb{H} \to \mathbb{H}_\alpha$ is compact for $0 \le \alpha < q < 1$. In addition, we make the following hypotheses:

(H1) A is the infinitesimal generator of a exponentially stable analytic semigroup $(T(t))_{t\ge 0}$ on $L^p(\mathbb{P}, \mathbb{H})$ such that for all $t \ge 0$, $\parallel T(t) \parallel \le Me^{-\delta t}$ with $M, \delta > 0$.

(H2) The operators $B_j : L^p(\mathbb{P}, \mathbb{H}_\alpha) \to L^p(\mathbb{P}, \mathbb{H})$ are bounded for $\alpha \in (0, 1)$, $j = 1, 2, 3$, and $\varpi_0 := \max_{j=1,2,3} \parallel B_j \parallel_{L(L^p(\mathbb{P},\mathbb{H}_\alpha), L^p(\mathbb{P},\mathbb{H}))}$.

(H3) The function $g \in PAP_T(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{H}), L^p(\mathbb{P}, \mathbb{H}_\beta))$, and there exist constants $\beta, L_g > 0$ such that $0 < \alpha \le \beta < 1$, and

$$E \parallel g(t_1, \psi_1) - g(t_2, \psi_2) \parallel_\beta^p \le L_g[|t_1 - t_2| + \parallel \psi_1 - \psi_2 \parallel^p], \\ t_1, t_2 \in \mathbb{R}, \quad \psi_1, \psi_2 \in L^p(\mathbb{P}, \mathbb{H}),$$

$$E \parallel g(t, \psi) \parallel_\beta^p \le L_g(\parallel \psi \parallel^p + 1), \quad t \in \mathbb{R},\ \psi \in L^p(\mathbb{P}, \mathbb{H}).$$

(H4) The functions $f \in PAPS^p_T(\mathbb{R}\times L^p(\mathbb{P},\mathbb{H}), L^p(\mathbb{P},\mathbb{H}))$, $F \in PAPS^p_T(\mathbb{R}\times L^p(\mathbb{P},\mathbb{H}), L^p(\mathbb{P}, L^0_2))$, and for each $t \in \mathbb{R}$, $\psi_1, \psi_2 \in L^p(\mathbb{P},\mathbb{H})$,

$$E \parallel f(t,\psi_1) - f(t,\psi_2) \parallel^p + E \parallel F(t,\psi_1) - F(t,\psi_2) \parallel^p_{L^0_2} \leq \Lambda(E \parallel \psi_1 - \psi_2 \parallel^p),$$

where Λ is a concave and continuous nondecreasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that $\Lambda(0) = 0$, $\Lambda(s) > 0$ for $s > 0$ and $\int_{0^+} \frac{ds}{\Lambda(s)} = +\infty$.

(H5) For any $\rho_1 > 0$, there exists a constant $\mu > 0$ and nondecreasing continuous function $\Theta : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $t \in \mathbb{R}$, and $\psi \in L^p(\mathbb{P},\mathbb{H})$ with $E \parallel x \parallel^p > \mu$,

$$\parallel f(t,\psi) \parallel^p_{S^p} + \parallel F(t,\psi) \parallel^p_{S^p} \leq \rho_1 \Theta(E \parallel \psi \parallel^p).$$

(H6) The functions $I_i \in PAP(\mathbb{Z}, L^p(\mathbb{P},\mathbb{H}))$, and there exist constants $c_i > 0$ such that

$$E \parallel I_i(x_1) - I_i(x_2) \parallel^p \leq c_i E \parallel x_1 - x_2 \parallel^p_\alpha,\ x_1, x_2 \in L^p(\mathbb{P},\mathbb{H}_\alpha),\ i \in \mathbb{Z},$$

$$E \parallel I_i(x) \parallel^p \leq c_i(E \parallel x \parallel^p_\alpha + 1),\ x \in L^p(\mathbb{P},\mathbb{H}_\alpha),\ i \in \mathbb{Z}.$$

Also, we need to introduce a few preliminary and important results.

Lemma 3.1. *If the assumptions (H1), (H2) and (H4) hold and if Υ is the function, for $x \in PAP_T(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}_\alpha))$, defined by*

$$\Upsilon(t) := \int_{-\infty}^t T(t-s) f(s, B_2 x(s)) ds \tag{3.2}$$

for each $t \in \mathbb{R}$, then $\Upsilon \in PAP_T(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}_\alpha))$.

Proof. Let $x \in PAP_T(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}_\alpha))$. Since $B_2 \in L(L^p(\mathbb{P},\mathbb{H}_\alpha), L^p(\mathbb{P},\mathbb{H}))$ then $B_2 x \in PAP_T(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$. Setting $f(t) = f(t, B_2 x(t))$ and using Lemma 2.4, it follows that $f \in PAPS^p_T(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$. Moreover, it follows that

$$E\left\| \int_{-\infty}^t T(t-s) f(s) ds \right\|^p_\alpha \leq M^p_\alpha E\left[\int_{-\infty}^t (t-s)^{-\alpha} e^{-\delta(t-s)} \parallel f(s) \parallel ds \right]^p,$$

and hence the function $s \to T(t-s)f(s)$ is integrable over $(-\infty, t)$ for each $t \in \mathbb{R}$.

Let $f = f_1 + f_2$, where $f^b_1 \in AP_T(\mathbb{R}, L^p((0,1), L^p(\mathbb{P},\mathbb{H})))$ and $f^b_2 \in PAP^0_T(\mathbb{R}, L^p((0,1), L^p(\mathbb{P},\mathbb{H})))$, such that

$$\Upsilon(t) = \int_{-\infty}^t T(t-s) f_1(s) ds + \int_{-\infty}^t T(t-s) f_2(s) ds =: \Upsilon_1(t) + \Upsilon_2(t).$$

Next we only need to verify $\Upsilon_1 \in AP_T(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}_\alpha))$ and $\Upsilon_2 \in PAP^0_T(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}_\alpha))$. Thus, the following verification procedure is divided into three steps.

Step 1. $\Upsilon_1 \in UPC(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}_\alpha))$.

Let $t', t'' \in (t_i, t_{i+1})$, $i \in \mathbb{Z}$, $t'' < t'$. By $\{T(t)\}_{t\geq 0}$ is an exponentially stable analytic semigroup, for any $\varepsilon > 0$, there exists $0 < \xi < (1 - \frac{p\alpha}{p-1})^{p-1} (\frac{\varepsilon}{2\tilde{f}_1})^{\frac{1}{p(\alpha-1)-1}}$ such that $0 < t' - t'' < \xi$, we have

$$\parallel T(t' - t'') - I \parallel^p \leq \frac{\tilde{\delta}_1 \varepsilon}{2\tilde{f}_1},$$

where $\tilde{f}_1 = 2^{p-1} M_\alpha^p \parallel f_1 \parallel_{S^p}^p$, $\tilde{\delta}_1 = [(\Gamma(1-\frac{p\alpha}{p-1})\delta^{\frac{p\alpha}{p-1}-1})^{p-1}\frac{e^\delta}{e^{-\delta}-1}]^{-1}$. Using Hölder's inequality, we have

$$
\begin{aligned}
& E \parallel \Upsilon_1(t') - \Upsilon_1(t'') \parallel_\alpha^p \\
\leq\ & 2^{p-1} E \left\| \int_{-\infty}^{t''} T(t''-s)[T(t'-t'')-I] f_1(s) ds \right\|_\alpha^p \\
& + 2^{p-1} E \left\| \int_{t''}^{t'} T(t'-s) f_1(s) ds \right\|_\alpha^p \\
\leq\ & 2^{p-1} M_\alpha^p \parallel T(t'-t'') - I \parallel^p \left(\int_{-\infty}^{t''} (t''-s)^{-\frac{p}{p-1}\alpha} e^{-\delta(t''-s)} ds \right)^{p-1} \\
& \times \left(\sum_{n=1}^{\infty} \int_{t''-n}^{t''-n+1} e^{-\delta(t''-s)} E \parallel f_1(s) \parallel^p ds \right) \\
& + 2^{p-1} M_\alpha^p \left(\int_{t''}^{t'} (t'-s)^{-\frac{p}{p-1}\alpha} e^{-\delta(t'-s)} \right)^{p-1} \\
& \times \left(\int_{t''}^{t'} e^{-\delta(t'-s)} E \parallel f_1(s) \parallel^p ds \right) \\
\leq\ & 2^{p-1} M_\alpha^p \parallel T(t'-t'') - I \parallel^p \left(\Gamma(1-\frac{p\alpha}{p-1}) \delta^{\frac{p\alpha}{p-1}-1} \right)^{p-1} \\
& \times \left[\sum_{n=1}^{\infty} e^{-\delta(n-1)} \int_{n-1}^{n} E \parallel f_1(t''-s) \parallel^p ds \right] \\
& + 2^{p-1} M_\alpha^p \left(\int_{t''}^{t'} (t'-s)^{-\frac{p}{p-1}\alpha} \right)^{p-1} \left(\int_{t''}^{t'} E \parallel f_1(s) \parallel^p ds \right) \\
\leq\ & 2^{p-1} M_\alpha^p \parallel T(t'-t'') - I \parallel^p \left(\Gamma(1-\frac{p\alpha}{p-1}) \delta^{\frac{p\alpha}{p-1}-1} \right)^{p-1} \frac{e^\delta}{e^\delta - 1} \parallel f_1 \parallel_{S^p}^p \\
& + 2^{p-1} M_\alpha^p \left(1 - \frac{p\alpha}{p-1} \right)^{1-p} \parallel f_1 \parallel_{S^p}^p (t'-t'')^{p(1-\alpha)-1} \\
<\ & 2^{p-1} M_\alpha^p \parallel \frac{\tilde{\delta}_1 \varepsilon}{2\tilde{f}_1} \left(\Gamma(1-\frac{p\alpha}{p-1}) \delta^{\frac{p\alpha}{p-1}-1} \right)^{p-1} \frac{e^\delta}{e^\delta - 1} \parallel f_1 \parallel_{S^p}^p \\
& + 2^{p-1} M_\alpha^p \left(1 - \frac{p\alpha}{p-1} \right)^{1-p} \parallel f_1 \parallel_{S^p}^p \left[\left(\frac{\varepsilon}{2\tilde{f}_1} \right)^{\frac{1}{p(1-\alpha)-1}} \right]^{p(1-\alpha)-1} \\
=\ & \varepsilon.
\end{aligned}
$$

Consequently, $\Upsilon_1 \in UPC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$.

Step 2. $\Upsilon_1 \in AP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$.

Consider for each $n = 1, 2, \ldots$, the integrals

$$\Upsilon_1^{(n)}(t) = \int_{t-n}^{t-n+1} T(t-s) f_1(s) ds$$

for each $t \in (t_i, t_{i+1})$, $i \in \mathbb{N}$. Set

$$\Upsilon_1^{(n)}(t) = \int_{t-n}^{t-n+1} T(t-s) f_1(s) ds = \int_{n-1}^{n} T(s) f_1(t-s) ds$$

for each $t \in (t_i, t_{i+1})$, $i \in \mathbb{N}$.

From assumption (H4) and Hölder's inequality, it follows that

$$\begin{aligned} E \parallel \Upsilon_1^{(n)}(t) \parallel_\alpha^p &\leq M_\alpha^p E\bigg[\int_{n-1}^{n} s^{-\alpha} e^{-\delta s} \parallel f_1(t-s) \parallel ds\bigg]^p \\ &\leq M_\alpha^p \bigg(\int_{n-1}^{n} s^{-\frac{p}{p-1}\alpha} e^{-\delta s} ds\bigg)^{p-1} \\ &\quad \times \bigg(\int_{n-1}^{n} e^{-\delta s} E \parallel f_1(t-s) \parallel^p ds\bigg) \\ &\leq M_\alpha^p (n-1)^{-p\alpha} \bigg(\int_{n-1}^{n} e^{-\delta s} ds\bigg)^{p-1} \\ &\quad \times e^{-\delta(n-1)} \bigg(\int_{n-1}^{n} E \parallel f_1(t-s) \parallel^p ds\bigg) \\ &\leq M_\alpha^p (n-1)^{-p\alpha} e^{-p\delta(n-1)} \parallel f_1 \parallel_{S^p}^p . \end{aligned}$$

Since the series

$$\sum_{n=1}^{\infty} \int_{n-1}^{n} (n-1)^{-p\alpha} e^{-p\delta(n-1)} < \infty,$$

we deduce from the well-known Weirstrass test that the series $\sum_{n=1}^{\infty} \Upsilon_1^{(n)}(t)$ is convergent in the sense of the norm $\parallel \cdot \parallel_{S^p}$ uniformly on $\mathbb{R}$. Now let $\Upsilon_1(t) = \sum_{n=1}^{\infty} \Upsilon_1^{(n)}(t)$. Observe that

$$\Upsilon_1(t) = \int_{-\infty}^{t} T(t-s) f_1(s) ds,$$

and hence $\Upsilon_1(t) \in PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$. Moreover, for any $t \in \mathbb{R}$, we have

$$E \parallel \Upsilon_1(t) \parallel_\alpha^p \leq \sum_{n=1}^{\infty} E \parallel \Upsilon_1^{(n)}(t) \parallel_\alpha^p \leq C_1(M_\alpha, p, \alpha, \delta) \parallel f_1 \parallel_{S^p}^p,$$

where $C_1(M_\alpha, p, \alpha, \delta)$ depends only on the fixed constants $M_\alpha, p, \alpha, \delta$.

Now we show that each $\Upsilon_1^{(n)} \in AP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$. Indeed, by $f_1^b \in AP_T(\mathbb{R}, L^p(0,1), L^p(\mathbb{P}, \mathbb{H})))$, given $\varepsilon > 0$, one can find $l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ contains at least s' with the property that

$$\int_{t}^{t+1} E \parallel f_1(s+s') - f_1(s) \parallel^p ds < \varepsilon \tag{3.3}$$

for all $t \in (t_i, t_{i+1})$, $i \in \mathbb{Z}$. On the other hand, using the inequality (3.3), exponential stable of $T(t)_{t\geq 0}$ and Höder's inequality, we obtain that

$$\begin{aligned} & E \parallel \Upsilon_1^{(n)}(t+s') - \Upsilon_1^{(n)}(t) \parallel_\alpha^p \\ \leq & E \bigg\| \int_{n-1}^{n} T(s)[f_1(t+s'-s) - f_1(t-s)] d\tau \bigg\|_\alpha^p \\ \leq & M_\alpha^p E \bigg(\int_{n-1}^{n} s^\alpha e^{-\delta s} \parallel f_1(t+s'-s) - f_1(t-s) \parallel ds\bigg)^p \end{aligned}$$

$$\leq M_\alpha^p\left(\int_{n-1}^{n} s^{-\frac{p-1}{p}\alpha}e^{-\delta s}ds\right)\times\left(\int_{n-1}^{n} e^{-\delta s}E\parallel f_1(t+s'-s)-f_1(t-s)\parallel^2 ds\right)$$
$$\leq M_\alpha^p(n-1)^{-p\alpha}e^{-p\delta(n-1)}\left(\int_{t-n}^{t-n+1} E\parallel f_1(s+s')-f_1(s)\parallel^p ds\right)$$
$$< M_\alpha^p(n-1)^{-p\alpha}e^{-p\delta(n-1)}\varepsilon.$$

Therefore, we deduce that $\Upsilon_1^{(n)}\in AP_T(\mathbb{R},L^p(\mathbb{P},\mathbb{H}_\alpha))$.

Step 3. $\Upsilon_2\in PAP_T^0(\mathbb{R},L^p(\mathbb{P},\mathbb{H}_\alpha))$.

We will prove that $\Upsilon_2^{(n)}\in PAP_T^0(\mathbb{R},L^p(\mathbb{P},\mathbb{H}_\alpha))$. It is obvious that $\Phi_2^{(n)}\in BC(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$, the left task is to show that

$$\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}E\parallel\Upsilon_2^{(n)}(t)\parallel^p dt=0$$

for each $t\in(t_i,t_{i+1})$, $i\in\mathbb{N}$, and $n=1,2,3,\ldots$. Then, by using the exponential stable of $T(t)_{t\geq 0}$ and Höder's inequality, it follows that

$$\begin{aligned}
E\parallel\Upsilon_2^{(n)}(t)\parallel_\alpha^p &\leq M_\alpha^p E\left[\int_{n-1}^{n} s^{-\alpha}e^{-\delta s}\parallel f_2(t-s)\parallel ds\right]^p\\
&\leq M_\alpha^p\left(\int_{n-1}^{n} s^{-\frac{p}{p-1}\alpha}e^{-\delta s}ds\right)^{p-1}\\
&\quad\times\left(\int_{n-1}^{n} e^{-\delta s}E\parallel f_2(t-s)\parallel^p ds\right)\\
&\leq M_\alpha^p(n-1)^{-p\alpha}\left(\int_{n-1}^{n} e^{-\delta s}ds\right)^{p-1}\\
&\quad\times e^{-\delta(n-1)}\left(\int_{n-1}^{n} E\parallel f_2(t-s)\parallel^p ds\right)\\
&\leq M_\alpha^p(n-1)^{-p\alpha}e^{-p\delta(n-1)}\left(\int_{n-1}^{n} E\parallel f_2(t-s)\parallel^p ds\right).
\end{aligned}$$

Then, for $r>0$, we see that

$$\frac{1}{2r}\int_{-r}^{r}E\parallel\Upsilon_2^{(n)}(t)\parallel_\alpha^p dt\leq M_\alpha^p(n-1)^{-p\alpha}e^{-p\delta(n-1)}\frac{1}{2r}\int_{-r}^{r}\int_{n-1}^{n}E\parallel f_2(t-s)\parallel^p ds.$$

Since $f_2^b\in PAP_T^0(\mathbb{R},L^p((0,1),L^p(\mathbb{P},\mathbb{H}_\alpha)))$, the above inequality leads to $\Upsilon_2^{(n)}\in PAP_T^0(\mathbb{R},L^p(\mathbb{P},\mathbb{H}_\alpha))$ for each $n=1,2,\ldots$. The above inequality leads also to

$$E\parallel\Upsilon_2^{(n)}(t)\parallel_\alpha^p\leq M_\alpha^p(n-1)^{-p\alpha}e^{-p\delta(n-1)}\parallel f_2\parallel_{S^p}.$$

Since the series

$$\sum_{n=1}^{\infty}\int_{n-1}^{n}(n-1)^{-p\alpha}e^{-p\delta(n-1)}<\infty,$$

we deduce from the well-known Weirstrass test that the series $\sum_{n=1}^{\infty}\Upsilon_2^{(n)}(t)$ is convergent in the sense of the norm $\|\cdot\|_{S^p}$ uniformly on $\mathbb{R}$. Furthermore,

$$\Upsilon_2(t)=\sum_{n=1}^{\infty}\Upsilon_2^{(n)}(t)$$

for each $(t_i, t_{i+1}), i \in \mathbb{N}$. Clearly, $\Upsilon_2(t) \in PC_T(R, L^p(\mathbb{P}, \mathbb{H}_\alpha))$. Applying $\Upsilon_2^{(n)} \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$ and the inequality

$$\frac{1}{2r}\int_{-r}^{r} E \parallel \Upsilon_2(t) \parallel_\alpha^p dt \leq \frac{1}{2r}\int_{-r}^{r} E\left\| \Upsilon_2(t) - \sum_{n=1}^{m} \Upsilon_2^{(n)}(t)\right\|_\alpha^p dt + \sum_{n=1}^{m}\frac{1}{2r}\int_{-r}^{r} E \parallel \Upsilon_2^{(n)}(t) \parallel_\alpha^p .$$

We deduce that the uniformly limit $\Upsilon_2(t) = \sum_{n=1}^{\infty} \Upsilon_2^{(n)}(t) \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$. $\square$

Lemma 3.2. *If the assumptions (H1), (H2), (H4) hold and if Ψ is the function, for $x \in PAP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$, defined by*

$$\Psi(t) := \int_{-\infty}^{t} T(t-s)F(s, B_3x(s))dW(s) \tag{3.4}$$

for each $t \in \mathbb{R}$, then $\Psi \in PAP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$.

Proof. Let $x \in PAP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$. Since $B_3 \in L(L^p(\mathbb{P}, \mathbb{H}_\alpha), L^p(\mathbb{P}, \mathbb{H}))$ then $B_3x \in PAP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$. Setting $F(t) = F(t, B_3x(t))$ and using Lemma 2.4, it follows that $F \in PAPS_T^p(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$. Moreover, it follows that

$$E\left\|\int_{-\infty}^{t} T(t-s)F(s)dW(s)\right\|_\alpha^p \leq M_\alpha C_p E\left[\int_{-\infty}^{t}(t-s)^{-2\alpha}e^{-2\delta(t-s)} \parallel F(s) \parallel_{L_2^0}^2 ds\right]^{p/2},$$

and hence the function $s \to T(t-s)F(s)$ is integrable over $(-\infty, t)$ for each $t \in \mathbb{R}$.

Let $F = F_1 + F_2$, where $F_1^b \in AP_T(\mathbb{R}, L^p((0,1), L^p(\mathbb{P}, \mathbb{H})))$ and $F_2^b \in PAP_T^0(\mathbb{R}, L^p((0,1), L^p(\mathbb{P}, \mathbb{H})))$, such that

$$\begin{aligned}\Psi(t) &= \int_{-\infty}^{t} T(t-s)F_1(s)dW(s) + \int_{-\infty}^{t} T(t-s)F_2(s)dW(s)\\ &=: \Psi_1(t) + \Psi_2(t).\end{aligned}$$

Next we only need to verify $\Psi_1 \in AP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$ and $\Psi_2 \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$. Thus, the following verification procedure is divided into three steps.

Step 1. $\Psi_1 \in UPC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$.

Let $t', t'' \in (t_i, t_{i+1})$, $i \in \mathbb{Z}$, $t'' < t'$. By $\{T(t)\}_{t\geq 0}$ is an exponentially stable analytic semigroup, for any $\varepsilon > 0$, there exists $0 < \xi < (1 - \frac{2p\alpha}{p-2})^{\frac{p-2}{p}}(\frac{\varepsilon}{2\tilde{F}_1})^{\frac{p}{p-2-p\alpha)}}$ such that $0 < t' - t'' < \xi$, we have for $p > 2$,

$$\parallel T(t'-t'') - I \parallel^p \leq \frac{\tilde{\delta}_2\varepsilon}{2\tilde{F}_1},$$

where $\tilde{F}_1 = 2^{p-1}M_\alpha^p C_p \parallel F_1 \parallel_{S^p}^p$, $\tilde{\delta}_2 = [(\Gamma(1 - \frac{2p\alpha}{p-2})(2\delta)^{\frac{p\alpha}{p-2}-1})^{\frac{p-2}{p}} \frac{e^{2\delta}}{e^{2\delta}-1}]^{-1}$. Using

Hölder's inequality and the Itô integral [10], we have

$$
\begin{aligned}
& E \parallel \Psi_1(t') - \Psi_1(t'') \parallel_\alpha^p \\
\leq\ & 2^{p-1} E \left\| \int_{-\infty}^{t''} T(t''-s)[T(t'-t'')-I]F_1(s)dW(s) \right\|_\alpha^p \\
& + 2^{p-1} E \left\| \int_{t''}^{t'} T(t'-s)F_1(s)dW(s) \right\|_\alpha^p \\
& + 2^{p-1} M_\alpha^p C_p E \Bigg[\int_{-\infty}^{t''} (t''-s)^{-2\alpha} e^{-2\delta(t''-s)} \\
& \times \parallel T(t'-t'') - I \parallel^2 \parallel F_1(s) \parallel_{L_2^0}^2 \, ds \Bigg]^{p/2} \\
& + 2^{p-1} M_\alpha^p C_p \left(\int_{t''}^{t'} (t'-s)^{-\frac{2p}{p-2}\alpha} e^{-2\delta(t'-s)} \right)^{\frac{p-2}{p}} \\
& \times \left(\int_{t''}^{t'} e^{-2\delta(t'-s)} E \parallel F_1(s) \parallel_{L_2^0}^p \, ds \right) \\
\leq\ & 2^{p-1} M_\alpha^p C_p \parallel T(t'-t'') - I \parallel^p \left(\int_{-\infty}^{t''} (t''-s)^{-\frac{2p\alpha}{p-2}} e^{-2\delta(t''-s)} ds \right)^{\frac{p-2}{p}} \\
& \times \left(\sum_{n=1}^{\infty} \int_{t''-n}^{t''-n+1} e^{-2\delta(t''-s)} E \parallel F_1(s) \parallel_{L_2^0}^p \, ds \right) \\
& + 2^{p-1} M_\alpha^p C_p \left(\int_{t''}^{t'} (t'-s)^{-\frac{2p\alpha}{p-2}} e^{-2\delta(t'-s)} \right)^{\frac{p-2}{p}} \\
& \times \left(\int_{t''}^{t'} e^{-2\delta(t'-s)} E \parallel F_1(s) \parallel_{L_2^0}^p \, ds \right) \\
\leq\ & 2^{p-1} M_\alpha^p C_p \parallel T(t'-t'') - I \parallel^p \left(\Gamma(1 - \frac{2p\alpha}{p-2})(2\delta)^{\frac{2p\alpha}{p-2}-1} \right)^{\frac{p-2}{p}} \\
& \times \left[\sum_{n=1}^{\infty} e^{-2\delta(n-1)} \int_{n-1}^{n} E \parallel F_1(t''-s) \parallel_{L_2^0}^p \, ds \right] \\
& + 2^{p-1} M_\alpha^p C_p \left(\int_{t''}^{t'} (t'-s)^{-\frac{2p\alpha}{p-2}} \right)^{\frac{p-2}{p}} \left(\int_{t''}^{t'} E \parallel F_1(s) \parallel_{L_2^0}^p \, ds \right) \\
\leq\ & 2^{p-1} M_\alpha^p C_p \parallel T(t'-t'') - I \parallel^p \\
& \times \left(\Gamma(1 - \frac{2p\alpha}{p-2})(2\delta)^{\frac{2p\alpha}{p-2}-1} \right)^{\frac{p-2}{p}} \frac{e^{2\delta}}{e^{2\delta}-1} \parallel F_1 \parallel_{S^p}^p \\
& + 2^{p-1} M_\alpha^p C_p \left(1 - \frac{2p\alpha}{p-2} \right)^{\frac{2-p}{p}} \parallel F_1 \parallel_{S^p}^p (t'-t'')^{\frac{p-2-p\alpha}{p}} \\
<\ & 2^{p-1} M_\alpha^p C_p \frac{\tilde{\delta}_2 \varepsilon}{2\tilde{F}_1} \left(\Gamma(1 - \frac{2p\alpha}{p-2})(2\delta)^{\frac{p\alpha}{p-2}-1} \right)^{\frac{p-2}{p}} \frac{e^{2\delta}}{e^{2\delta}-1} \parallel F_1 \parallel_{S^p}^p \\
& + 2^{p-1} M_\alpha^p C_p \left(1 - \frac{2p\alpha}{p-2} \right)^{\frac{2-p}{p}} \parallel F_1 \parallel_{S^p}^p \left[\left(\frac{\varepsilon}{2\tilde{F}_1} \right)^{\frac{p}{p-2-p\alpha}} \right]^{\frac{p-2-p\alpha}{p}} \\
=\ & \varepsilon.
\end{aligned}
$$

For $p=2$. Let $\varepsilon>0$, there exists $0<\xi<(\frac{\varepsilon}{2\tilde{F}_1})^{-\frac{1}{2\alpha}}$ such that $0<t'-t''<\xi$, we have

$$\| T(t'-t'')-I \|^p \leq \frac{\tilde{\delta}_2\varepsilon}{2\tilde{F}_1},$$

where $\tilde{F}_1=2M_\alpha^p \| F_1 \|_{S^p}^2$, $\tilde{\delta}_2=[\Gamma(1-2\alpha)(2\delta)^{2\alpha-1}]^{-1}$. Similar to the above discussion, one has

$$\begin{aligned}
& E \| \Psi_1(t')-\Psi_1(t'') \|_\alpha^2 \\
& \leq 2M^2 \| T(t'-t'')-I \|^2 \sum_{n=1}^{\infty}(n-1)^{-2\alpha}e^{-2\delta(n-1)} \\
& \times \int_{n-1}^{n} E \| F_1(t''-s) \|_{L_2^0}^2 \, ds + 2M^2(t'-t'')^{-2\alpha}\left(\int_{t''}^{t'} E \| F_1(s) \|_{L_2^0}^2 \, ds\right) \\
& < 2M_\alpha^p \| F_1 \|_{S^p}^p \frac{\tilde{\delta}_2\varepsilon}{5\tilde{F}_1}\Gamma(1-2\alpha)(2\delta)^{2\alpha-1} + 2M_\alpha^p \| F_1 \|_{S^p}^2 \left[\left(\frac{\varepsilon}{2\tilde{F}_1}\right)^{-2\alpha}\right]^{-\frac{1}{2\alpha}} \\
& = \varepsilon.
\end{aligned}$$

Consequently, $\Psi_1 \in UPC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$.

Step 2. $\Psi_1 \in AP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$.

Consider for each $n=1,2,\ldots$, the integrals

$$\Psi_1^{(n)}(t)=\int_{t-n}^{t-n+1} T(t-s)F_1(s)dW(s)$$

for each $t \in (t_i, t_{i+1})$, $i \in \mathbb{N}$. Set

$$\Psi_1^{(n)}(t)=\int_{t-n}^{t-n+1} T(t-s)F_1(s)dWs)=\int_{n-1}^{n} T(s)F_1(t-s)dW(s)$$

for each $t \in (t_i, t_{i+1})$, $i \in \mathbb{N}$.

From (H4), Hölder's inequality and the Itô integral, it follows that

$$\begin{aligned}
E \| \Psi_1^{(n)}(t) \|_\alpha^p & \leq M_\alpha^p C_p E\left[\int_{n-1}^{n} s^{-2\alpha}e^{-2\delta s} \| F_1(t-s) \|_{L_2^0}^2 \, ds\right]^{p/2} \\
& \leq M_\alpha^p C_p \left(\int_{n-1}^{n} s^{-\frac{2p\alpha}{p-2}}e^{-2\delta s}ds\right)^{\frac{p-2}{p}} \\
& \quad \times \left(\int_{n-1}^{n} e^{-2\delta s}E \| F_1(t-s) \|_{L_2^0}^p \, ds\right) \\
& \leq M_\alpha^p C_p (n-1)^{-2p\alpha}\left(\int_{n-1}^{n} e^{-2\delta s}ds\right)^{p-1} e^{-2\delta(n-1)} \\
& \quad \times \left(\int_{n-1}^{n} E \| F_1(t-s) \|_{L_2^0}^p \, ds\right) \\
& \leq M_\alpha^p C_p (n-1)^{-2p\alpha}e^{-p\delta(n-1)} \| F_1 \|_{S^p}^p .
\end{aligned}$$

Since the series

$$\sum_{n=1}^{\infty}\int_{n-1}^{n}(n-1)^{-2p\alpha}e^{-2p\delta(n-1)}<\infty,$$

we deduce from the well-known Weirstrass test that the series $\sum_{n=1}^{\infty}\Psi_1^{(n)}(t)$ is convergent in the sense of the norm $\|\cdot\|_{S^p}$ uniformly on $\mathbb{R}$. Now let $\Psi_1(t)=\sum_{n=1}^{\infty}\Psi_1^{(n)}(t),\quad t\in\mathbb{R}$. Observe that

$$\Psi_1(t)=\int_{-\infty}^{t}T(t-s)F_1(s)dW(s).$$

and hence $\Psi_1(t)\in PC(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$. Moreover, for any $t\in\mathbb{R}$, we have

$$E\parallel \Psi_1(t)\parallel_\alpha^p\leq\sum_{n=1}^{\infty}E\parallel\Psi_1^{(n)}(t)\parallel_\alpha^p\leq C_2(M_\alpha,p,\alpha,\delta)\parallel F_1\parallel_{S^p}^p,$$

where $C_2(M_\alpha,p,\alpha,\delta)$ depends only on the fixed constants M and δ.

Now we show that each $\Psi_1^{(n)}\in AP_T(\mathbb{R},L^p(\mathbb{P},\mathbb{H}_\alpha))$. Indeed, by $F_1^b\in AP_T(\mathbb{R},L^p(0,1),L^p(\mathbb{P},\mathbb{H})))$, given $\varepsilon>0$, one can find $l(\varepsilon)>0$ such that any interval of length $l(\varepsilon)$ contains at least s' with the property that

$$\int_t^{t+1}E\parallel F_1(s+s')-F_1(s)\parallel_{L_2^0}^p dW(s)<\varepsilon \tag{3.5}$$

for all $t\in(t_i,t_{i+1})$, $i\in\mathbb{N}$. On the other hand, using the inequality (3.5), exponential stable of $T(t)_{t\geq0}$ and Höder's inequality, we obtain that

$$\begin{aligned}
&E\parallel\Psi_1^{(n)}(t+s')-\Psi_1^{(n)}(t)\parallel_\alpha^p\\
&\leq M_\alpha^pC_pE\left[\int_{n-1}^{n}s^{-2\alpha}e^{-2\delta s}\parallel F_1(t+s'-s)-F_1(t-s)\parallel_{L_2^0}ds\right]^{p/2}\\
&\leq M_\alpha^pC_p\left(\int_{n-1}^{n}s^{-\frac{2p\alpha}{p-2}}e^{-2\delta s}ds\right)^{\frac{p-2}{p}}\\
&\quad\times\left(\int_{n-1}^{n}e^{-2\delta s}E\parallel F_1(t+s'-s)-F_1(t-s)\parallel^2ds\right)\\
&\leq M_\alpha^pC_p(n-1)^{-2p\alpha}e^{-2p\delta(n-1)}\left(\int_{t-n}^{t-n+1}E\parallel F_1(s+s')-F_1(s)\parallel^p ds\right)\\
&< M_\alpha^pC_p(n-1)^{-2p\alpha}e^{-2p\delta(n-1)}\varepsilon.
\end{aligned}$$

Therefore, we deduce that $\Psi_1^{(n)}\in AP_T(\mathbb{R},L^p(\mathbb{P},\mathbb{H}_\alpha))$.

Step 3. $\Psi_2\in PAP_T^0(\mathbb{R},L^p(\mathbb{P},\mathbb{H}_\alpha))$.

We will prove that $\Psi_2^{(n)}\in PAP_T^0(\mathbb{R},L^p(\mathbb{P},\mathbb{H}_\alpha))$. It is obvious that $\Psi_2^{(n)}\in BC(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$, the left task is to show that

$$\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}E\parallel\Psi_2^{(n)}(t)\parallel^p dt=0$$

for each $t\in(t_i,t_{i+1})$, $i\in\mathbb{N}$, and $n=1,2,3,\ldots$. Then, by using the exponential stable of $T(t)_{t\geq0}$ and Höder's inequality, it follows that

$$\begin{aligned}
E\parallel\Psi_2^{(n)}(t)\parallel_\alpha^p&\leq E\left\|\int_{n-1}^{n}T(s)F_2(t-s)ds\right\|_\alpha^p\\
&\leq M_\alpha^pC_pE\left[\int_{n-1}^{n}s^{-2\alpha}e^{-2\delta s}\parallel F_2(t-s)\parallel_{L_2^0}ds\right]^{p/2}
\end{aligned}$$

$$\leq M_\alpha^p C_p \left(\int_{n-1}^{n} s^{-\frac{p\alpha}{p-2}} e^{-2\delta s} ds \right)^{\frac{p-2}{p}}$$
$$\times \left(\int_{n-1}^{n} e^{-2\delta s} E \parallel F_2(t-s) \parallel_{L_2^0}^p ds \right)$$
$$\leq M_\alpha^p C_p (n-1)^{-2p\alpha} \left(\int_{n-1}^{n} e^{-2\delta s} ds \right)^{\frac{p-2}{p}}$$
$$\times e^{-2\delta(n-1)} \left(\int_{n-1}^{n} E \parallel F_2(t-s) \parallel_{L_2^0}^p ds \right)$$
$$\leq M_\alpha^p C_p (n-1)^{-2p\alpha} e^{-2p\delta(n-1)} \left(\int_{n-1}^{n} E \parallel F_2(t-s) \parallel_{L_2^0}^p ds \right).$$

Then, for $r > 0$, we see that

$$\frac{1}{2r} \int_{-r}^{r} E \parallel \Psi_2^{(n)}(t) \parallel_\alpha^p dt$$
$$\leq M_\alpha^p C_p (n-1)^{-2p\alpha} e^{-2p\delta(n-1)} \frac{1}{2r} \int_{-r}^{r} \int_{n-1}^{n} E \parallel F_2(t-s) \parallel^p dW(s).$$

Since $F_2^b \in PAP_T^0(\mathbb{R}, L^p((0,1), L^p(\mathbb{P}, \mathbb{H}_\alpha)))$, the above inequality leads to $\Psi_2^{(n)} \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$ for each $n = 1, 2, \ldots$. The above inequality leads also to

$$E \parallel \Psi_2^{(n)}(t) \parallel_\alpha^p \leq M_\alpha^p C_p (n-1)^{-2p\alpha} e^{-2p\delta(n-1)} \parallel F_2 \parallel_{S^p}.$$

Since the series

$$\sum_{n=1}^{\infty} \int_{n-1}^{n} (n-1)^{-2p\alpha} e^{-2p\delta(n-1)} < \infty,$$

we deduce from the well-known Weirstrass test that the series $\sum_{n=1}^{\infty} \Psi_2^{(n)}(t)$ is convergent in the sense of the norm $\| \cdot \|_{S^p}$ uniformly on $\mathbb{R}$. Furthermore,

$$\Psi_2(t) = \sum_{n=1}^{\infty} \Psi_2^{(n)}(t)$$

for $(t_i, t_{i+1}), i \in \mathbb{N}$. Clearly, $\Psi_2(t) \in PC_T(R, L^p(\mathbb{P}, \mathbb{H}_\alpha))$. Applying $\Psi_2^{(n)} \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$ and the inequality

$$\frac{1}{2r} \int_{-r}^{r} E \parallel \Psi_2(t) \parallel_\alpha^p dt \leq \frac{1}{2r} \int_{-r}^{r} E \left\| \Psi_2(t) - \sum_{n=1}^{m} \Psi_2^{(n)}(t) \right\|_\alpha^p dt$$
$$+ \sum_{n=1}^{m} \frac{1}{2r} \int_{-r}^{r} E \parallel \Psi_2^{(n)}(t) \parallel_\alpha^p.$$

We deduce that the uniformly limit $\Psi_2(t) = \sum_{n=1}^{\infty} \Phi_2^{(n)}(t) \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$. □

Lemma 3.3. *If the assumptions (H1) and (H6) hold and if* Π *is the function, for* $x \in PAP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$, *defined by*

$$\Pi(t) := \sum_{\sigma < t_i < t} T(t - t_i) I_i(x_{t_i}) \tag{3.6}$$

for each $t \in \mathbb{R}$, *then* $\Pi \in PAP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$.

Proof. Let $x \in PAP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$. Setting $\gamma_i = I_i(x_{t_i})$ and using (H6) and Lemma 2.3, it follows that $\gamma_i \in PAPS_T^p(\mathbb{R},\ L^p(\mathbb{P}, \mathbb{H}))$. Let $\gamma_i = \gamma_{1,i} + \gamma_{2,i}$, where $\gamma_{1,i} \in AP(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$ and $\gamma_{2,i} \in PAP^0(\mathbb{Z}, L^p(\mathbb{P},\ \mathbb{H}_\alpha))$. Hence,

$$\Pi(t) = \sum_{t_i<t} T(t-t_i)\gamma_{1,i} + \sum_{t_i<t} T(t-t_i)\gamma_{2,i} =: \Pi_1(t) + \Pi_2(t).$$

Next we only need to verify $\Pi_1(t) \in AP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$ and $\Pi_2(t) \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$. Thus, the following verification procedure is divided into three steps.

Step 1. $\Pi_1 \in UPC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$.

Let $t', t'' \in (t_i, t_{i+1})$, $i \in \mathbb{Z}$, $t'' < t'$. By $\{T(t)\}_{t\geq 0}$ is an exponentially stable analytic semigroup, for any $\varepsilon > 0$, there exists $\xi > 0$ such that $0 < t' - t'' < \xi$, we have

$$\| T(t'-t'') - I \|^p \leq \frac{\tilde{\delta}_3 \varepsilon}{\tilde{\gamma}_1},$$

where $\tilde{\gamma}_1 = M_\alpha^p \| \gamma_{1,i} \|_\infty^p \gamma^{-p\alpha}$ and $\tilde{\delta}_3 = (1 - e^{-\delta\gamma})^p$. Using Hölder's inequality, we have

$$\begin{aligned}
&E \| \Pi_1(t') - \Pi_1(t'') \|_\alpha^p \\
&\leq M_\alpha^p \| T(t'-t'') - I \|^p \left(\sum_{t_i<t''} (t''-t_i)^{-\frac{p}{p-1}\alpha} e^{-\delta(t''-t_i)} \right)^{p-1} \\
&\quad \times \left(\sum_{t_i<t''} e^{-\delta(t''-t_i)} E \| \gamma_{1,i} \|^p \right) \\
&\leq M_\alpha^p \| T(t'-t'') - I \|^p \gamma^{-p\alpha} \left(\sum_{t_i<t''} e^{-\delta(t''-t_i)} \right)^p \sup_{i\in\mathbb{Z}} E \| \gamma_{1,i} \|^p \\
&< M_\alpha^p \| \gamma_{1,i} \|_\infty^p \gamma^{-p\alpha} \frac{\tilde{\delta}_3 \varepsilon}{5\tilde{\gamma}_1} \frac{1}{(1-e^{-\delta\gamma})^p} = \varepsilon.
\end{aligned}$$

Step 2. $\Pi_1 \in AP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$.

For any $\varepsilon > 0$, by Lemma 2.7, there exists relative dense sets of real numbers Ω_ε and integers Q_ε, for every $\tau \in \Omega_\varepsilon$, there exists at least one number $\tilde{q} \in Q_\varepsilon$ such that $|t^{\tilde{q}} - \tau| < \varepsilon, i \in \mathbb{Z}$ and $E \| \gamma_{1,i+q} - \gamma_{1,i} \|^p < \varepsilon, \tilde{q} \in Q_\varepsilon, i \in \mathbb{Z}$. Then,

$$\begin{aligned}
&E \| \Pi_1(t+\tau) - \Pi_1(t) \|_\alpha \\
&\leq M_\alpha^p E \Bigg[\left(\sum_{t_i<t} (t-t_i)^{-\frac{p}{p-1}\alpha} e^{-\delta(t-t_i)} \right)^{p-1} \\
&\quad \times \left(\sum_{t_i<t} e^{-\delta(t-t_i)} \| \gamma_{1,i+q} - \gamma_{1,i} \|^p \right) \Bigg] \\
&\leq M_\alpha^p \gamma^{-p\alpha} \left(\sum_{t_i<t} e^{-\delta(t-t_i)} \right)^p E \| \gamma_{1,i+q} - \gamma_{1,i} \|^p \\
&\leq \frac{M_\alpha^p \gamma^{-p\alpha} \varepsilon}{(1-e^{-\delta\gamma})^p}.
\end{aligned}$$

Hence, $\Pi_1 \in AP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$.

Step 3. $\Pi_2 \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$.

In fact, for $r > 0$, one has

$$\frac{1}{2r}\int_{-r}^{r} E \parallel \Pi_2(t) \parallel_\alpha^p dt \leq \frac{1}{2r}\int_{-r}^{r} E\left\| \sum_{t_i<t} T(t-t_i)\gamma_{2,i}\right\|_\alpha^p dt.$$

For a given $i \in \mathbb{Z}$, define the function $v(t)$ by $v(t) = T(t-t_i)\gamma_{2,i}, t_i < t \leq t_{i+1}$, then

$$\begin{aligned}\lim_{t\to\infty} E \parallel v(t) \parallel_\alpha^p &= \lim_{t\to\infty} E \parallel T(t-t_i)\gamma_{2,i} \parallel_\alpha^p \\ &\leq \lim_{t\to\infty} M_\alpha^p (t-t_i)^{-p\alpha} e^{-p\delta(t-t_i)} \sup_{i\in\mathbb{Z}} E \parallel \gamma_{2,i} \parallel^p \\ &\leq \lim_{t\to\infty} M_\alpha^p \gamma^{-p\alpha} e^{-p\delta(t-t_i)} \sup_{i\in\mathbb{Z}} E \parallel \gamma_{2,i} \parallel^p = 0.\end{aligned}$$

Thus $v \in PC_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha)) \subset PAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$. Define $v_j : \mathbb{R} \to L^p(\mathbb{P}, \mathbb{H}_\alpha)$ by

$$v_j(t) = T(t-t_{i-j})\gamma_{2,i-j},\ t_i < t \leq t_{i+1},\ j \in \mathbb{N}.$$

So $v_j \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$. Moreover,

$$\begin{aligned}E \parallel v_j(t) \parallel_\alpha^p &= E \parallel T(t-t_{i-j})\gamma_{2,i-j} \parallel_\alpha^p \\ &\leq M_\alpha^p (t-t_{i-j})^{-p\alpha} e^{-p\delta(t-t_{i-j})} \sup_{i\in\mathbb{Z}} E \parallel \gamma_{2,i} \parallel^p \\ &\leq M_\alpha^p \gamma^{-p\alpha} e^{-p\delta(t-t_i)} e^{-p\delta\gamma j} \sup_{i\in\mathbb{Z}} E \parallel \gamma_{2,i} \parallel^p .\end{aligned}$$

Therefore, the series $\sum_{j=0}^{\infty} v_j$ is uniformly convergent on $\mathbb{R}$. By Lemma 2.1, one has

$$\sum_{t_i<t} T(t-t_i)\gamma_{2,i} = \sum_{j=0}^{\infty} v_j(t) \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha)),$$

that is

$$\frac{1}{2r}\int_{-r}^{r} E\left\| \sum_{t_i<t} T(t-t_i)\gamma_{2,i}\right\|_\alpha^p dt \to 0 \quad \text{as } r \to \infty.$$

Using the Lebesgue's dominated convergence theorem, we have $\Pi_2 \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$. □

Theorem 3.1. *Assume that (H1)–(H6) are satisfied. Then system* (1.1)–(1.2) *has at least one p-mean piecewise pseudo almost periodic mild solution on* $\mathbb{R}$, *provided that*

$$2^{p-1}[\parallel A^{\alpha-\beta} \parallel^p L_g \varpi_0^p + M_\alpha^p \gamma^{-p\alpha} \frac{1}{(1-e^{-\delta\gamma})^p} \sup_{i\in\mathbb{Z}} c_i] < 1. \tag{3.7}$$

Proof. Let $\mathbb{Y} = PAP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha)) \cap UPC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$. Consider the operator $\Phi : \mathbb{Y} \to PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$ defined by

$$(\Phi x)(t) = (\Phi_1 x)(t) + (\Phi_2 x)(t),$$

where

$$(\Phi_1 x)(t) = -g(t, B_1 x(t)) + \sum_{t_i<t} T(t-t_i) I_i(x(t_i)), \quad t \in \mathbb{R}.$$

$$\begin{aligned}(\Phi_2 x)(t) &= \int_{-\infty}^{t} T(t-s) f(s, B_2 x(s)) ds \\ &\quad + \int_{-\infty}^{t} T(t-s) F(s, B_3 x(s)) dW(s), \quad t \in \mathbb{R}.\end{aligned}$$

Let $\rho_1 > 0$ be fixed. By (H4) it follows that there exist a positive constant ϱ such that, for all $t \in \mathbb{R}$ and $\psi \in L^p(\mathbb{P}, \mathbb{H})$ with $E \parallel \psi \parallel^p > \varrho$,

$$E \parallel f(t,\psi) \parallel_{S^p}^p + E \parallel F(t,\psi) \parallel_{S^p}^p \leq \rho_1 \Theta(\parallel \psi \parallel^p).$$

Let

$$\nu_0 = \sup_{t\in\mathbb{R}} \{E \parallel f(t,\psi) \parallel_{S^p}^p,\ E \parallel F(t,\psi) \parallel_{S^p}^p : \parallel \psi \parallel^p \leq \varrho\}.$$

Thus, we have for all $t \in \mathbb{R}$,

$$E \parallel f(t,\psi) \parallel_{S^p}^p + E \parallel F(t,\psi) \parallel_{S^p}^p \leq \rho_1 \Theta(E \parallel \psi \parallel^p) + \nu_0, \quad \psi \in L^p(\mathbb{P}, \mathbb{H}). \tag{3.8}$$

For L_g, ρ_1 sufficiently small, we can choose $r^* > 0$ such that

$$\frac{(1-L_0)r^*}{(\tilde{d}_1 + \tilde{d}_2)[\rho_1 \Theta(\varpi_0^p r^*) + \nu_0]} > 1 \tag{3.9}$$

for $p > 2$, and for $p = 2$,

$$\frac{(1-L_0)r^*}{(\tilde{a}_1 + \tilde{a}_2)[\rho_1 \Theta(\varpi_0^p r^*) + \nu_0]} > 1, \tag{3.10}$$

where $L_0 = 2^{p-1}[\parallel A^{\alpha-\beta} \parallel^p L_g \varpi_0^p + M_\alpha^p \gamma^{-p\alpha} \frac{1}{(1-e^{-\delta\gamma})^p} \sup_{i\in\mathbb{Z}} c_i]$, $\tilde{d}_1 = 4^{p-1} M_\alpha^p (\Gamma(1-\frac{p\alpha}{p-1}))^{p-1}\ \delta^{p(1-\alpha)}$, $\tilde{d}_2 = 4^{p-1} C_p M_\alpha^p (\Gamma(1-\frac{2p\alpha}{p-2}))^{\frac{p-2}{p}} (2\delta)^{\frac{2(p\alpha-p+1)}{p}}$ for $p > 2$, and $\tilde{a}_1 = 4M_\alpha^2 (\Gamma(1-2\alpha)\delta^{2(1-\alpha)}$, $\tilde{a}_2 = 4M_\alpha^2 \Gamma(1-2\alpha)(2\delta)^{2\alpha-1}$ for $p = 2$. In order to use Lemma 2.8, we will verify that Φ_1 is a contraction while Φ_2 is a completely continuous operator. For better readability, we break the proof into a sequence of steps.

Step 1. For every $x \in \mathbb{Y}$, $\Phi x \in \mathbb{Y}$.

Let $x(\cdot) \in \mathbb{Y}$, by (H2), (H4) and Lemmas 2.2, 3.1-3.3, we deduce that $g(\cdot, x(\cdot))$, $f(\cdot, x(\cdot)) \in PAPS_T^p(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$ and $I_i(x(t_i)) \in PAP(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$. Similarly as the proof of Theorem 3.1, one has $\Phi x \in \mathbb{Y}$.

Step 2. Φ_1 is a contraction on $\mathbb{Y}$.

For $t \in \mathbb{R}$, and $x^*, x^{**} \in \mathbb{Y}$. From (H3) and (H6), we have

$$\begin{aligned}
&E \parallel (\Phi_1 x^*)(t) - (\Phi_1 x^{**})(t) \parallel_\alpha^p \\
\leq\ & 2^{p-1} \parallel A^{\alpha-\beta} \parallel^p E \parallel A^\beta g(t, B_1 x^*(t)) - A^\beta g(t, B_1 x^{**}(t)) \parallel^p \\
&+ 2^{p-1} E \left\| \sum_{t_i<t} T(t-t_i)[I_i(x^*(t)) - I_i(x^{**}(t))] \right\|_\alpha^p \\
\leq\ & 2^{p-1} \parallel A^{\alpha-\beta} \parallel^p L_g \varpi_0^p \parallel x^*(t) - x^{**}(t) \parallel_\alpha^p \\
&+ 2^{p-1} M_\alpha^p \left(\sum_{t_i<t} (t-t_i)^{-\frac{p}{p-1}\alpha} e^{-\delta(t-t_i)} \right)^{p-1} \\
&\times \left(\sum_{t_i<t} e^{-\delta(t-t_i)} c_i \sup_{t\in\mathbb{R}} E \parallel x^*(t) - x^{**}(t) \parallel_\alpha^p \right) \\
\leq\ & 2^{p-1} \left[\parallel A^{\alpha-\beta} \parallel^p L_g \varpi_0^p + M_\alpha^p \gamma^{-p\alpha} \frac{1}{(1-e^{-\delta\gamma})^p} \sup_{i\in\mathbb{Z}} c_i \right] \\
&\times \parallel x^* - x^{**} \parallel_{\alpha,\infty}^p .
\end{aligned}$$

Taking supremum over t,

$$\| \Phi_1 x^* - \Phi_1 x^{**} \|_{\alpha,\infty}^p \leq L_0 \| x^* - x^{**} \|_{\alpha,\infty}^p,$$

where $L_0 < 1$. Hence, Φ_1 is a contraction on $\mathbb{Y}$.

Step 3. Φ_2 maps bounded sets into bounded sets in $\mathbb{Y}$.

Indeed, it is enough to show that there exists a positive constant $\mathcal{L}$ such that for each $x \in B_{r^*} = \{x \in \mathbb{Y} : \| x \|_{\alpha,\infty}^p < r^*\}$, $r^* > 0$, one has $\| \Phi_2 x \|_{\alpha,\infty}^p \leq \mathcal{L}$. Then, by (3.8), Hölder's inequality and the Itôintegral, we have for $x \in B_{r^*}$ and $p > 2$,

$$\begin{aligned}
&E \| (\Phi_2 x)(t) \|_\alpha^p \\
&\leq 2^{p-1} E \left\| \int_{-\infty}^t T(t-s) f(s, B_2 x(s)) ds \right\|_\alpha^p \\
&\quad + 2^{p-1} E \left\| \int_{-\infty}^t T(t-s) F(s, B_3 x(s)) dW(s) \right\|_\alpha^p \\
&\leq 2^{p-1} M_\alpha^p \left(\int_{-\infty}^t (t-s)^{-\frac{p}{p-1}\alpha} e^{-\delta(t-s)} ds \right)^{p-1} \\
&\quad \times \left(\sum_{n=1}^\infty \int_{t-n}^{t-n+1} e^{-\delta(t-s)} E \| f(s, B_2 x(s)) \|^p \, ds \right) \\
&\quad + 2^{p-1} C_p M_\alpha^p E \left(\int_{-\infty}^t (t-s)^{-2\alpha} e^{-2\delta(t-s)} \| F(s, B_3 x(s)) \|_{L_2^0}^2 \, ds \right)^{p/2} \\
&\leq 2^{p-1} M_\alpha^p \left(\Gamma(1 - \frac{p\alpha}{p-1}) \delta^{\frac{p\alpha}{p-1} - 1} \right)^{p-1} \\
&\quad \times \left(\sum_{n=1}^\infty e^{-\delta(n-1)} \int_{n-1}^n E \| f(t-s, B_2 x(t-s)) \|^p \, ds \right) \\
&\quad + 2^{p-1} M_\alpha^p \left(\int_{-\infty}^t (t-s)^{-\frac{2p}{p-2}\alpha} e^{-2\delta(t-s)} ds \right)^{\frac{p-2}{p}} \\
&\quad \times \left(\sum_{n=1}^\infty e^{-2\delta(n-1)} \int_{n-1}^n E \| F(t-s, B_3 x(t-s)) \|^p \, ds \right) \\
&\leq 2^{p-1} M_\alpha^p \left(\Gamma(1 - \frac{p\alpha}{p-1}) \delta^{\frac{p\alpha}{p-1} - 1} \right)^{p-1} \frac{e^\delta}{e^\delta - 1} [\rho_1 \Theta(\| B_2 x(s) \|^p) + \nu_0] \\
&\quad + 2^{p-1} M_\alpha^p \left(\int_{-\infty}^t (t-s)^{-\frac{2p}{p-2}\alpha} e^{-2\delta(t-s)} ds \right)^{\frac{p-2}{p}} \\
&\quad \times \frac{e^{2\delta}}{e^{2\delta} - 1} [\rho_1 \Theta(\| B_3 x(s) \|^p) + \nu_0] \\
&\leq 2^{p-1} M_\alpha^p \left(\Gamma(1 - \frac{p\alpha}{p-1}) \delta^{\frac{p\alpha}{p-1} - 1} \right)^{p-1} \frac{e^\delta}{e^\delta - 1} [\rho_1 \Theta(\varpi_0^p r^*) + \nu_0] \\
&\quad + 2^{p-1} C_p M_\alpha^p \left(\Gamma(1 - \frac{2p\alpha}{p-2}) (2\delta)^{\frac{p\alpha}{p-2} - 1} \right)^{\frac{p-2}{p}} \frac{e^{2\delta}}{e^{2\delta} - 1} [\rho_1 \Theta(\varpi_0^p r^*) + \nu_0] \\
&:= \mathcal{L}.
\end{aligned}$$

For $p=2$, we have

$$E \parallel (\Phi_2 x)(t) \parallel_\alpha^2 \leq 2M_\alpha^2(\Gamma(1-2\alpha)\delta^{2(1-\alpha)}\frac{e^\delta}{e^\delta-1}[\rho_1\Theta(\varpi_0^p r^*)+\nu_0] \\ + 2M_\alpha^2\Gamma(1-2\alpha)(2\delta)^{2\alpha-1}[\rho_1\Theta(\varpi_0^p r^*)+\nu_0] := \mathcal{L}.$$

Then for each $x \in B_{r^*}$, we have $\parallel \Phi_2 x \parallel_{\alpha,\infty}^p \leq \mathcal{L}$.

Step 4. Φ_2 is a compact operator.

(1) Φ_2 maps bounded sets into equicontinuous sets of $\mathbb{Y}$.

Since $T(\cdot)$ is analytic, the function $t \to A^\alpha T(t)$ is continuous in the uniform operator topology in $(0,\infty)$. Let $t_i < \varepsilon < t \leq t_{i+1}, i \in \mathbb{Z}$, and $\tilde{\delta} > 0$ such that $\parallel A^\alpha[T(\tilde{h}) - I \parallel^p < \varepsilon$ for every with $|\tilde{h}| < \tilde{\delta}$. For each $x \in B_{r^*}$, $0 < |\eta| < \tilde{\delta}, t+\eta \in (t_i, t_{i+1}], i \in \mathbb{Z}$, we have for $p > 2$,

$$\begin{aligned}
& E \parallel (\Phi_2 x)(t+\eta) - (\Phi_2 x)(t) \parallel_\alpha^p \\
& \leq 4^{p-1}E\left\| \int_{-\infty}^t T(t-s)[T(\eta)-I]f(s,B_2x(s))ds \right\|_\alpha^p \\
& \quad + 4^{p-1}E\left\| \int_t^{t+\eta} T(t+\eta-s)f(s,B_2x(s))ds \right\|_\alpha^p \\
& \quad + 4^{p-1}E\left\| \int_{-\infty}^t T(t-s)[T(\eta)-I]F(s,B_3x(s))dW(s) \right\|_\alpha^p \\
& \quad + 4^{p-1}E\left\| \int_t^{t+\eta} T(t+\eta-s)F(s,B_3x(s))dW(s) \right\|_\alpha^p \\
& \leq 4^{p-1}M^p \parallel A^\alpha[T(\eta)-I] \parallel^p \left(\int_{-\infty}^t (t-s)^{-\alpha}e^{-\delta(t-s)}ds \right)^{p-1} \\
& \quad \times \left(\sum_{n=1}^\infty e^{-\delta(n-1)} \int_{n-1}^n E \parallel f(t-s, B_2x(t-s)) \parallel^p ds \right) \\
& \quad + 4^{p-1}M_\alpha^p \left(\int_t^{t+\eta} (t+\eta-s)^{-\frac{p}{p-1}\alpha} e^{-\delta(t+\eta-s)}ds \right)^{p-1} \\
& \quad \times \left(\int_t^{t+\eta} E \parallel f(s,B_2x(s)) \parallel^p ds \right) \\
& \quad + 4^{p-1}M^pC_p \parallel A^\alpha[T(\eta)-I] \parallel^p \\
& \quad \times E\left[\int_{-\infty}^t (t-s)^{-2\alpha}e^{-2\delta(t-s)} \parallel F(s,B_3x(s)) \parallel_{L_2^0}^2 ds \right]^{p/2} \\
& \quad + 4^{p-1}M_\alpha^pC_pE\left[\int_t^{t+\eta} (t+\eta-s)^{-2\alpha}e^{-2\delta(t+\eta-s)} \right. \\
& \quad \left. \times \parallel F(s,B_3x(s)) \parallel_{L_2^0}^2 ds \right]^{p/2} \\
& \leq 4^{p-1}M^p\varepsilon\left(\Gamma(1-\frac{p\alpha}{p-1})\delta^{\frac{p\alpha}{p-1}-1} \right)^{p-1} \frac{e^\delta}{e^\delta-1}[\rho_1\Theta(\parallel B_1x(s) \parallel^p)+\nu_0] \\
& \quad + 4^{p-1}M_\alpha^p \left(\int_t^{t+\eta} (t+\eta-s)^{-\frac{p}{p-1}\alpha} e^{-\delta(t+\eta-s)}ds \right)^{p-1} \\
& \quad \times [\rho_1\Theta(\parallel B_2x(s) \parallel^p)+\nu_0]
\end{aligned}$$

$$
\begin{aligned}
&+ 4^{p-1}M^pC_p\varepsilon\bigg(\Gamma(1-\frac{2p\alpha}{p-2})(2\delta)^{\frac{p\alpha}{p-2}-1}\bigg)^{\frac{p-2}{p}}\frac{e^{2\delta}}{e^{2\delta}-1}[\rho_1\Theta(\| B_3x(s)\|^p)+\nu_0]\\
&+ 4^{p-1}M_\alpha^pC_p\bigg(\int_t^{t+\eta}(\tau_2-s)^{-\frac{2p}{p-2}\alpha}e^{-2\delta(t+\eta-s)}ds\bigg)^{\frac{p-2}{p}}[\rho_1\Theta(\| B_2x(s)\|^p)+\nu_0]\\
\le\;& 4^{p-1}M^p\varepsilon\bigg(\Gamma(1-\frac{p\alpha}{p-1})\delta^{\frac{p\alpha}{p-1}-1}\bigg)^{p-1}\frac{e^{\delta}}{e^{\delta}-1}[\rho_1\Theta(\varpi_0^pr^*)+\nu_0]\\
&+ 4^{p-1}M_\alpha^p\bigg(\int_t^{t+\eta}(t+\eta-s)^{-\frac{p}{p-1}\alpha}e^{-\delta(t+\eta-s)}ds\bigg)^{p-1}[\rho_1\Theta(\varpi_0^pr^*)+\nu_0]\\
&+ 4^{p-1}M^pC_p\varepsilon\bigg(\Gamma(1-\frac{2p\alpha}{p-2})(2\delta)^{\frac{p\alpha}{p-2}-1}\bigg)^{\frac{p-2}{p}}\frac{e^{2\delta}}{e^{2\delta}-1}[\rho_1\Theta(\varpi_0^pr^*)+\nu_0]\\
&+ 4^{p-1}M_\alpha^pC_p\bigg(\int_t^{t+\eta}(t+\eta-s)^{-\frac{2p}{p-2}\alpha}e^{-2\delta(t+\eta-s)}ds\bigg)^{\frac{p-2}{p}}[\rho_1\Theta(\varpi_0r^*)+\nu_0].
\end{aligned}
$$

For $p=2$, we have

$$
\begin{aligned}
&E\,\|\,(\Phi_2x)(t+\eta)-(\Phi_2x)(t))\,\|_\alpha^2\\
\le\;& 4M^p\varepsilon\Gamma(1-2\alpha)\delta^{2\alpha-1}\frac{e^{\delta}}{e^{\delta}-1}[\rho_1\Theta(\varpi_0^pr^*)+\nu_0]\\
&+ 4M_\alpha^2\bigg(\int_t^{t+\eta}(t+\eta-s)^{-2\alpha}e^{-\delta(t+\eta-s)}ds\bigg)[\rho_1\Theta(\varpi_0^pr^*)+\nu_0]\\
&+ 4^{p-1}M^p\varepsilon\Gamma(1-2\alpha)(2\delta)^{2\alpha-1}[\rho_1\Theta(\varpi_0^pr^*)+\nu_0]\\
&+ 4^{p-1}M_\alpha^2\gamma^{-2\alpha}[\rho_1\Theta(\varpi_0^pr^*)+\nu_0].
\end{aligned}
$$

The right-hand side of the above inequality is independent of $x\in B_{r^*}$ and tends to zero as $\eta\to 0$, and sufficiently small positive number ε. Thus, Φ_2 maps B_{r^*} into an equicontinuous family of functions.

(2) Φ_2 maps B_{r^*} into a relatively set in $L^p(\mathbb{P},\mathbb{H}_\alpha)$.

For each $x\in B_{r^*}$, there exists $0\le\alpha<q<1$, we have for $p>2$,

$$
\begin{aligned}
&E\,\|\,A^q(\Phi_2x)(t)\,\|^p\\
\le\;& 2^{p-1}E\bigg\|\int_{-\infty}^tA^qT(t-s)f(s,B_2x(s))ds\bigg\|^p\\
&+ 2^{p-1}E\bigg\|\int_{-\infty}^tA^qT(t-s)F(s,B_3x(s))dW(s)\bigg\|^p\\
\le\;& 2^{p-1}M_q^p\bigg(\int_{-\infty}^t(t-s)^{-\frac{p}{p-1}q}e^{-\delta(t-s)}ds\bigg)^{p-1}\\
&\times\bigg(\sum_{n=1}^\infty e^{-\delta(n-1)}\int_{n-1}^nE\,\|\,f(t-s,B_2x(t-s))\,\|^p\,ds\bigg)\\
&+ 2^{p-1}C_pM_q^pE\bigg(\int_{-\infty}^t(t-s)^{-2q}e^{-2\delta(t-s)}\,\|\,F(s,B_3x(s))\,\|_{L_2^0}^2\,ds\bigg)^{p/2}\\
\le\;& 2^{p-1}M_q^p\bigg(\Gamma(1-\frac{pq}{p-1})\delta^{\frac{pq}{p-1}-1}\bigg)^{p-1}\frac{e^{\delta}}{e^{\delta}-1}[\rho_1\Theta(\| B_2x(s)\|^p)+\nu_0]ds\bigg)
\end{aligned}
$$

$$+ 2^{p-1}C_pM_q^p\bigg(\int_{-\infty}^{t}(t-s)^{-\frac{2p}{p-2}q}e^{-2\delta(t-s)}ds\bigg)^{\frac{p-2}{p}}$$
$$\times\bigg(\sum_{n=1}^{\infty}e^{-\delta(n-1)}\int_{n-1}^{n}E\parallel F(t-s,B_3x(t-s))\parallel^p ds\bigg)$$
$$\leq 2^{p-1}M_q^p\bigg(\Gamma(1-\frac{pq}{p-1})\delta^{\frac{pq}{1-p}-1}\bigg)^{p-1}[\rho_1\Theta(\varpi_0^p r^*)+\nu_0]$$
$$+ 3^{p-1}C_pM_q^p\bigg(\Gamma(1-\frac{2pq}{p-2})(2\delta)^{\frac{pq}{p-2}-1}\bigg)^{\frac{p-2}{p}}[\rho_1\Theta(\varpi_0^p r^*)+\nu_0].$$

For $p=2$, we have

$$E\parallel(\Phi_2x)(t)\parallel_q^2\leq 2M_q^2(\Gamma(1-2q)\delta^{2q-1}\frac{e^\delta}{e^\delta-1}[\rho_1\Theta(\varpi_0^p r^*)+\nu_0]$$
$$+ 2M_q^2\Gamma(1-2q)(2\delta)^{2q-1}[\rho_1\Theta(\varpi_0^p r^*)+\nu_0],$$

which implies $A^q(\Phi_2x)(t)$ is bounded in $\mathbb{H}$. It is known that $A^{-q}:\mathbb{H}\to\mathbb{H}_\alpha$ is compact for $0\leq\alpha<q<1$. Then $(\Phi_2x)(t)$ is precompact in $\mathbb{H}_\alpha$ for each $t\in\mathbb{R}$. Since $\{\Phi_2x:x\in B_{r^*}\}\subset PC_h^0(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$, by Lemma 2.5 it suffices to show that $\{\Phi_2x:x\in B_{r^*}\}$ is a relatively compact set. Therefore, we conclude that operator Φ_2 is also a compact map.

Step 5. $\Phi_2:\mathbb{Y}\to\mathbb{Y}$ is continuous.

Let $\{x^{(n)}\}\subseteq B_{r^*}$ with $x^{(n)}\to x(n\to\infty)$ in $\tilde{\mathbb{Y}}$, then there exists a bounded subset $K\subseteq L^p(\mathbb{P},\mathbb{K})$ such that $\mathbb{R}(x)\subseteq K,\mathbb{R}(x^n)\subseteq K,n\in\mathbb{N}$. By the assumption (H4), for any $\varepsilon>0$, there exists $\xi>0$ such that $x,y\in K$ and $E\parallel x-y\parallel^p<\xi$ implies that

$$E\parallel f(s,B_2x(s))-f(s,B_2y(s))\parallel_{S^p}^p<\varepsilon\quad\text{for all}\quad t\in\mathbb{R},$$
$$E\parallel F(s,B_3x(s))-F(s,B_3y(s))\parallel_{S^p}^p<\varepsilon\quad\text{for all}\quad t\in\mathbb{R}.$$

For the above ξ there exists n_0 such that $E\parallel x^{(n)}(t)-x(t)\parallel^p<\varepsilon$ for $n>n_0$ and $t\in\mathbb{R}$, then for$n>n_0$, we have

$$E\parallel f(s,B_2x^{(n)}(s))-f(s,B_2x(s))\parallel_{S^p}^p<\varepsilon\quad\text{for all}\quad t\in\mathbb{R},$$
$$E\parallel F(s,B_3x^{(n)}(s))-F(s,B_3x(s))\parallel_{S^p}^p<\varepsilon\quad\text{for all}\quad t\in\mathbb{R}.$$

Then, by Hölder's inequality, we have that for $p>2$,

$$E\parallel(\Phi_2x^{(n)})(t)-(\Phi_2x)(t)\parallel_\alpha^p$$
$$\leq 2^{p-1}E\bigg\|\int_{-\infty}^{t}T(t-s)[f(s,B_2x^{(n)}(s))-f(s,B_2x(s))]ds\bigg\|_\alpha^p$$
$$+ 2^{p-1}E\bigg\|\int_{-\infty}^{t}T(t-s)[F(s,B_3x^{(n)}(s))-F(s,B_3x(s))]dW(s)\bigg\|_\alpha^p$$
$$\leq 2^{p-1}M_\alpha^p\bigg(\int_{-\infty}^{t}(t-s)^{-\frac{p}{p-1}\alpha}e^{-\delta(t-s)}ds\bigg)^{p-1}$$
$$\times\bigg(\sum_{n=1}^{\infty}e^{-\delta(n-1)}\int_{n-1}^{n}E\parallel f(t-s,B_2x^{(n)}(t-s))$$

$$\begin{aligned}
&- f(t-s, B_2x(t-s)) \parallel^p ds\Big) \\
&+ 2^{p-1}C_pM^p\Big(\int_{-\infty}^t (t-s)^{-2\alpha}e^{-2\delta(t-s)}E \parallel F(s, B_3x^{(n)}(s)) \\
&- F(s, B_3x(s)) \parallel_{L_2^0}^2 ds\Big)^{p/2} \\
\leq\, & 2^{p-1}M^p\Big(\Gamma(1-\frac{p\alpha}{p-1})\delta^{\frac{p\alpha}{p-1}-1}\Big)^{p-1}\frac{e^\delta}{e^\delta-1}\varepsilon \\
&+ 2^{p-1}C_pM_\alpha^p\Big(\int_{-\infty}^t (t-s)^{\frac{2p}{p-2}\delta}e^{-2\delta(t-s)}ds\Big)^{\frac{p-2}{p}}\Big(\sum_{n=1}^{\infty}e^{-2\delta(n-1)} \\
&\times \int_{n-1}^n E \parallel F(t-s, B_3x^{(n)}(t-s)) - F(t-s, B_3x(t-s)) \parallel_{L_2^0}^p ds\Big) \\
\leq\, & 2^{p-1}\Big[M_\alpha^p\Big(\Gamma(1-\frac{p\alpha}{p-1}\delta^{\frac{p\alpha}{p-1}-1})\Big)^{p-1}\frac{e^\delta}{e^\delta-1} \\
&+ C_pM_\alpha^p\Big(\Gamma(1-\frac{2p\alpha}{p-2})(2\delta)^{\frac{2p\alpha}{p-2}-1}\Big)^{\frac{p-2}{p}}\frac{e^{2\delta}}{e^{2\delta}-1}\Big]\varepsilon.
\end{aligned}$$

For $p = 2$, we have

$$\begin{aligned}
&E \parallel (\Phi_2x^{(n)})(t) - (\Phi_2x)(t) \parallel_\alpha^2 \\
&\leq 2\Big[M_\alpha^2\Gamma(1-2\alpha)\delta^{2(1-\alpha)}\frac{e^\delta}{e^\delta-1} + M_\alpha^2\Gamma(1-2\alpha)(2\delta)^{2\alpha-1}\Big]\varepsilon.
\end{aligned}$$

Thus Φ_2 is continuous.

Step 6. We shall show the set $G = \{x \in \mathbb{Y} : \lambda\Phi_1(\frac{x}{\lambda}) + \lambda\Phi_2(x) = x$ for some $\lambda \in (0,1)\}$ is bounded on $\mathbb{R}$.

To do this, we consider the following nonlinear operator equation

$$x(t) = \lambda\Phi x(t), \quad 0 < \lambda < 1, \tag{3.11}$$

where Φ is already defined. Next we gives a priori estimate for the solution of the above equation. Indeed, let $x \in \mathbb{Y}$ be a possible solution of $x = \lambda\Phi(x)$ for some $0 < \lambda < 1$. This implies by (3.11) that for each $t \in \mathbb{R}$ we have

$$\begin{aligned}
x(t) = \lambda(\Phi x)(t) = -\lambda g(t, B_1x(t)) + \lambda\int_{-\infty}^t T(t-s)f(s, B_2x(s))ds \\
+ \lambda\int_{-\infty}^t T(t-s)F(s, B_3x(s))dW(s) + \lambda\sum_{t_i<t}T(t-t_i)I_i(x(t_i)).
\end{aligned}$$

Then, by (H1)-(H6), Hölder's inequality and the Itô integral, we have for $p > 2$,

$$\begin{aligned}
E \parallel x(t) \parallel_\alpha^p \leq 4^{p-1}E \parallel g(t, B_1x(t)) \parallel_\alpha^p + 4^{p-1}E\Big\| \int_{-\infty}^t T(t-s)f(s, B_2x(s))ds\Big\|_\alpha^p \\
+ 4^{p-1}E\Big\| \int_{-\infty}^t T(t-s)F(s, B_3x(s))dW(s)\Big\|_\alpha^p
\end{aligned}$$

$$
\begin{aligned}
&+ 4^{p-1}E\Big\| \sum_{t_i<t} T(t-t_i)I_i(x(t_i))\Big\|_\alpha^p \\
\leq& 4^{p-1} \parallel A^{\alpha-\beta} \parallel^p E \parallel A^\beta g(t, B_1 x(t)) \parallel^p \\
&+ 4^{p-1}M_\alpha^p \Big(\int_{-\infty}^t (t-s)^{-\frac{p}{p-1}\alpha} e^{-\delta(t-s)} ds\Big)^{p-1} \\
&\times \Big(\sum_{n=1}^\infty e^{-\delta(n-1)} \int_{n-1}^n E \parallel f(t-s, B_2 x(t-s)) \parallel^p ds\Big) \\
&+ 4^{p-1}C_p M_\alpha^p E\Big(\int_{-\infty}^t (t-s)^{-2\alpha} e^{-2\delta(t-s)} \parallel F(s, B_3 x(s)) \parallel_{L_2^0}^2 ds\Big)^{p/2} \\
&+ 4^{p-1}M_\alpha^p E\Big[\Big(\sum_{t_i<t} (t-t_i)^{-\frac{p}{p-1}\alpha} e^{-\delta(t-t_i)}\Big)^{p-1} \\
&\times \Big(\sum_{t_i<t} e^{-\delta(t-t_i)} \parallel I_i(x(t_i)) \parallel^p\Big)\Big] \\
\leq& 4^{p-1} \parallel A^{\alpha-\beta} \parallel^p L_g(E \parallel B_1 x(t) \parallel^p + 1) \\
&+ 4^{p-1}M_\alpha^p \Big(\Gamma(1-\frac{p\alpha}{p-1})\delta^{\frac{p\alpha}{p-1}-1}\Big)^{p-1} \frac{e^\delta}{e^\delta - 1}[\rho_1 \Theta(\parallel B_2 x(t) \parallel^p) + \nu_0] \\
&+ 4^{p-1}C_p M_\alpha^p \Big(\int_{-\infty}^t (t-s)^{-\frac{2p}{p-2}\alpha} e^{-2\delta(t-s)} ds\Big)^{\frac{p-2}{p}} \\
&\times \Big(\sum_{n=1}^\infty e^{-2\delta(n-1)} \int_{n-1}^n E \parallel F(t-s, B_3 x(t-s)) \parallel_{L_2^0}^p ds\Big) \\
&+ 4^{p-1}M_\alpha^p \gamma^{-p\alpha} \frac{1}{(1-e^{-\delta\alpha})^{p-1}} \Big(\sum_{t_i<t} e^{-\delta(t-t_i)} c_i (E \parallel x(t_i) \parallel^p + 1)\Big) \\
\leq& 4^{p-1} \parallel A^{\alpha-\beta} \parallel^p L_g(\varpi_0^p \sup_{t\in\mathbb{R}} E \parallel x(t) \parallel_\alpha^p + 1) \\
&+ 4^{p-1}M_\alpha^p \Big(\Gamma(1-\frac{p\alpha}{p-1})\delta^{\frac{p\alpha}{p-1}-1}\Big)^{p-1} \frac{e^\delta}{e^\delta - 1} \\
&\times [\rho_1 \Theta(\varpi_0^p \sup_{t\in\mathbb{R}} \parallel x(t) \parallel_\alpha^p) + \nu_0] \\
&+ 4^{p-1}C_p M_\alpha^p \Big(\Gamma(1-\frac{2p\alpha}{p-2})(2\delta)^{\frac{2p\alpha}{p-2}-1}\Big)^{\frac{p-2}{p}} \frac{e^{2\delta}}{e^{2\delta}-1} \\
&\times [\rho_1 \Theta(\varpi_0^p \sup_{t\in\mathbb{R}} \parallel x(t) \parallel_\alpha^p) + \nu_0] \\
&+ 4^{p-1}M_\alpha^p \gamma^{-p\alpha} \frac{1}{(1-e^{-\delta\gamma})^p}[\sup_{i\in\mathbb{Z}} c_i \sup_{t\in\mathbb{R}} E \parallel x(t) \parallel_\alpha^p + 1].
\end{aligned}
$$

For $p = 2$, we have

$$
\begin{aligned}
E \parallel x(t) \parallel_\alpha^2 \leq& 4 \parallel A^{\alpha-\beta} \parallel^2 L_g \varpi_0^p (\sup_{t\in\mathbb{R}} E \parallel x(t) \parallel_\alpha^2 + 1) \\
&+ 4M_\alpha^2 (\Gamma(1-2\alpha)\delta^{2\alpha-1} \frac{e^\delta}{e^\delta - 1}[\rho_1 \Theta(\varpi_0^2 \sup_{t\in\mathbb{R}} \parallel x(s) \parallel_\alpha^2) + \nu_0]
\end{aligned}
$$

$$+ 4M_\alpha^2\Gamma(1-2\alpha)(2\delta)^{2\alpha-1}[\rho_1\Theta(\varpi_0^2 \sup_{t\in\mathbb{R}} \| x(s) \|_\alpha^2) + \nu_0]$$
$$+ 4M_\alpha^2\gamma^{-2\alpha}\frac{1}{(1-e^{-\delta\gamma})^2}[\sup_{i\in\mathbb{Z}} c_i \sup_{t\in\mathbb{R}} E \| x(t) \|_\alpha^2 + 1].$$

Consequently, we have for $p > 2$,

$$\frac{(1-L_0) \| x \|_{\alpha,\infty}^p}{(\tilde{d}_1+\tilde{d}_2)[\rho_1\Theta(\varpi_0^p \| x \|_{\alpha,\infty}^p) + \nu_0]} \le 1, \tag{3.12}$$

and for $p = 2$,

$$\frac{(1-L_0) \| x \|_{\alpha,\infty}^p}{(\tilde{a}_1+\tilde{a}_2)[\rho_1\Theta(\varpi_0^p \| x \|_{\alpha,\infty}^p) + \nu_0]} \le 1. \tag{3.13}$$

Then by (3.9) and (3.10), there exists r^* such that $\| x \|_{\alpha,\infty}^p \neq r^*$. This indicates that G is bounded on $\mathbb{R}$. As a consequence of Lemma 2.8, we deduce that $\Phi_1 + \Phi_2$ has a fixed point $x(\cdot) \in \mathbb{Y}$, which is a mild solution of the system (1.1)–(1.2). □

4. Existence of stochastic optimal controls

In this section we consider a control problem and present a result on the existence of stochastic optimal controls. let Y is a separable reflexive Hilbert space from which the controls u take the values. Operator $B \in L^\infty(\mathbb{R}, L(Y,\mathbb{H}))$, $\| B \|_{L^\infty}$ stands for the norm of operator B on Banach space $L^\infty(\mathbb{R}, L(Y,\mathbb{H}))$, where $L^\infty(\mathbb{R}, L(Y,H))$ denote the space of operator valued functions which are measurable in the strong operator topology. Let $L_{\mathcal{F}}^p(\mathbb{R}, Y)$ is the closed subspace of $L_{\mathcal{F}}^p(\mathbb{R}\times\Omega, Y)$, consisting of all $\mathcal{F}_t$-progressively measurable, Y-valued stochastic processes satisfying the condition $E\int_{-\infty}^t \| u(s) \|_Y^p\, ds < \infty$, and endowed with the norm $\| u \|_{L_{\mathcal{F}}^p(\mathbb{R},Y)} = (\sup_{t\in\mathbb{R}} E\int_{-\infty}^t \| u(s) \|_Y^p\, ds)^{1/p}$.

Let $\tilde{U}$ be a nonempty closed bounded convex subset of Y. We define the admissible control set

$$U_{ad} = \{\varpi(\cdot) \in L_{\mathcal{F}}^p(\mathbb{R}, Y); \varpi(t) \in \tilde{U} \text{ a.e.} \quad t \in \mathbb{R}\}.$$

Consider the following controlled stochastic partial differential equations of the form

$$\begin{aligned} d[x(t) + g(t, B_1x(t))] =& A[x(t) + g(t, B_1x(t))]dt + [f(t, B_2x(t)) + B(t)u(t)]dt \\ &+ F(t, B_3x(t))dW(t), \quad t \in \mathbb{R}, u \in U_{ad}, t \neq t_i, i \in \mathbb{Z}, \end{aligned} \tag{4.1}$$

$$\Delta x(t_i) = x(t_i^+) - x(t_i^-) = I_i(x(t_i)), \quad i \in \mathbb{Z}. \tag{4.2}$$

We will assume that

(S) $Bu \in PAPS_T^p(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$ for $u \in U_{ab}$.

By Theorem 3.1, we have the following result.

Theorem 4.1. *Assume that assumptions of Theorem 3.1 hold and, in addition, the assumption (S) is satisfied. For every $u \in U_{ad}$, the system* (4.1)–(4.2) *has a* *pseudo almost periodic in distribution mild solution corresponding to u given by the solution of the following integral equation*

$$\begin{aligned}x^u(t) = -g(t, B_1x(t)) &+ \int_{-\infty}^{t} T(t-s)f(s, B_2x(s))ds \\ &+ \int_{-\infty}^{t} T(t-s)B(s)u(s)ds + \int_{-\infty}^{t} T(t-s)F(s, B_3x(s))dW(s) \\ &+ \sum_{t_i<t} T(t-t_i)I_i(x(t_i)), \quad t \in \mathbb{R}.\end{aligned}$$

Proof. Consider the space $\mathbb{Y}$ endowed with the uniform convergence topology and define the operator $\tilde{\Phi} : \mathbb{Y} \to \mathbb{Y}$ by

$$\begin{aligned}(\tilde{\Phi}x)(t) = -g(t, B_1x(t)) &+ \int_{-\infty}^{t} T(t-s)f(s, B_2x(s))ds \\ &+ \int_{-\infty}^{t} T(t-s)B(s)u(s)ds + \int_{-\infty}^{t} T(t-s)F(s, B_3x(s))dW(s) \\ &+ \sum_{t_i<t} T(t-t_i)I_i(x(t_i)), \quad t \in \mathbb{R}.\end{aligned}$$

Using Hölder's inequality, we have

$$\begin{aligned}&E\Big\| \int_{-\infty}^{t} T(t-s)B(s)u(s)ds \Big\|_{\alpha}^{p} \\ &\le M_{\alpha}^{p} E\Big[\int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\delta(t-s)} \parallel B(s)u(s) \parallel ds \Big]^{p} \\ &\le M_{\alpha}^{p} \parallel B \parallel_{\infty}^{p} \left(\int_{-\infty}^{t} (t-s)^{-\frac{p}{p-1}\alpha} e^{-\frac{p}{p-1}\delta(t-s)} ds \right)^{p-1} E\int_{-\infty}^{t} \parallel u(s) \parallel_{Y}^{p} ds \\ &\le M_{\alpha}^{p} \parallel B \parallel_{\infty}^{p} \left(\Gamma(1 - \frac{p\alpha}{p-1})(\frac{p}{p-1}\delta)^{\frac{p\alpha}{p-1}-1} \right)^{p-1} \parallel u \parallel_{L_{\mathcal{F}}^{p}(\mathbb{R},Y)}^{p} .\end{aligned}$$

Then from Bochner's Theorem, it follows that $T(t-s)B(s)u(s)$ are integrable on $(-\infty, t)$, where $\parallel B \parallel_{\infty}$ is the norm of operator B in Banach space $L_{\infty}(\mathbb{R}, L(Y, \mathbb{H}))$. Hence we conclude that $\tilde{\Phi}$ is a well-defined operator from $\mathbb{Y}$ into $\mathbb{Y}$. The proofs of the other steps are similar to those in Theorem 3.1. Therefore, we omit the details. □

Let x^u denote the mild solution of system (4.1)–(4.2) corresponding to the control $u \in U_{ad}$. We consider the Lagrange problem (P): find an optimal pair $(x^0, u^0) \in \mathbb{Y} \times U_{ad}$ such that

$$\mathcal{J}(x^0, u^0) \le \mathcal{J}(x^u, u) \text{ for all } u \in U_{ad},$$

where the cost function

$$\mathcal{J}(x^u, u) = \sup_{t\in\mathbb{R}} E \int_{-\infty}^{t} \vartheta(s, x^u(s), u(s))ds.$$

We introduce the following assumptions on ϑ.

(D1) The functional $\vartheta : \mathbb{R} \times L^p(\mathbb{P}, \mathbb{H}) \times Y \to \mathbb{R} \cup \{\infty\}$ is Borel measurable.

(D2) $\vartheta(t, \cdot, \cdot)$ is sequentially lower semicontinuous on $L^p(\mathbb{P}, \mathbb{H}) \times Y$ for almost all $t \in \mathbb{R}$.

(D3) $\vartheta(t, x, \cdot)$ is convex on Y for each $x \in L^p(\mathbb{P}, \mathbb{H})$ and almost all $t \in \mathbb{R}$.

(D4) There exist constants $d_1 \geq 0, d_2 > 0$, $\tilde{\varsigma}_0$ is nonnegative and $\tilde{\varsigma}_0 \in L^1(\mathbb{R}, \mathbb{R})$ such that $\vartheta(t, x, u) \geq \tilde{\varsigma}_0(t) + d_1 \parallel y \parallel + d_2 \parallel u \parallel_Y^p$.

To prove the existence of solution for problem (P), we need the following important lemma.

Lemma 4.1. *Operator $\tilde{\Phi} : L^p(\mathbb{R}, Y) \to \mathbb{Y}$ given by*

$$(\tilde{\Psi}u)(\cdot) = \int_{-\infty}^{t} T(t-s)B(s)u(s)ds$$

is completely continuous.

Proof. Suppose that $u^n \subseteq L^p_{\mathcal{F}}(\mathbb{R}, Y)$ is bounded, we define $\Theta_n(t) = (\tilde{\Psi}u^n)(t), t \in \mathbb{R}$. Similar to the proof of Theorem 3.1, one can know that for any fixed $t \in \mathbb{R}$ and, $E \parallel \Theta_n(t) \parallel_\alpha^p$ is bounded. By using (S), it is ease to verify that $\Theta_n(t)$ is relatively compact in $L^p(\mathbb{P}, \mathbb{H}_\alpha)$ and is also equicontinuous. Due to Lemma 2.5 again, $\{\Theta_n(t)\}$ is compact in $L^p(\mathbb{P}, \mathbb{H}_\alpha)$. Obviously, $\tilde{\Psi}$ is linear and continuous. Hence, $\tilde{\Psi}$ is a completely continuous operator. □

Next we can give the following result on existence of optimal controls for problem (P).

Theorem 4.2. *If the assumptions (S), (D1)-(D4) and the assumptions of Theorem 3.1 hold. Then the Lagrange problem (P) admits at least one optimal pair on $\mathbb{Y} \times U_{ad}$.*

Proof. Without loss of generality, we assume that $\inf\{\mathcal{J}(x^u, u) | u \in U_{ad}\} = \varepsilon < +\infty$. Otherwise, there is nothing to prove. Using assumptions (D1)-(D4), we have

$$\begin{aligned}\mathcal{J}(x^u, u) &\geq \int_{-\infty}^{t} \tilde{\varsigma}_0(s)ds + d_1 \int_{-\infty}^{t} \parallel x^u(s) \parallel ds + d_2 \int_{-\infty}^{t} \parallel u(s) \parallel_Y^p ds \\ &\geq -\tilde{\eta} > -\infty,\end{aligned}$$

where $\tilde{\eta} > 0$ is a constant. Hence, $\varepsilon \geq -\tilde{\eta} > -\infty$. On the other hand, by using definition of infimum there exists a minimizing sequence of feasible pair $\{(x^m, u^m)\} \subset A_{ad}$, where $\mathcal{A}_{ad} = \{(x, u) | x$ is an Stepanov-like pseudo almost periodic mild solution of system (4.1)–(4.2) corresponding to $u \in U_{ad}\}$, such that $\mathcal{J}(x^m, u^m) \to \varepsilon$ as $m \to +\infty$. For $\{u^m\} \subseteq U_{ad}$, $\{u^m\}$ is bounded in $L^p_{\mathcal{F}}(\mathbb{R}, Y)$, so there exists a subsequence, relabeled as $\{u^m\}$, and $u^0 \in L^p_{\mathcal{F}}(\mathbb{R}, Y)$ such that

$$u^m \xrightarrow{w} u^0 \ \text{ in } \ L^p_{\mathcal{F}}(\mathbb{R}, Y) \quad \text{as } m \to \infty.$$

Since U_{ad} is closed and convex, by Mazur's lemma, we conclude that $u^0 \in U_{ad}$.

Now we suppose that x^m are the mild solutions of system (4.1)–(4.2) corresponding to $u^m (m = 0, 1, 2, \ldots)$, and x^m satisfied the following integral equation

$$\begin{aligned} x^m(t) = & -g(t, B_1 x^m(t)) + \int_{-\infty}^{t} T(t-s) f(s, B_2 x^m(s)) ds \\ & + \int_{-\infty}^{t} T(t-s) B(s) u^m(s) ds + \int_{-\infty}^{t} T(t-s) F(s, B_3 x^m(s)) dW(s) \\ & + \sum_{t_i < t} T(t-t_i) I_i(x^m(t_i)), \quad t \in \mathbb{R}. \end{aligned}$$

Let $f^m(s) \equiv f(s, B_2 x^m(s))$, $F^m(s) \equiv F(s, B_3 x^m(s))$. Then for each $x^m \in B_{r^*} \subset \mathbb{Y}$, by (3.8), we obtain that $f^m : \mathbb{R} \to L^p(\mathbb{P}, \mathbb{H})$, $F^m : \mathbb{R} \to L^p(\mathbb{P}, L_2^0)$ are bounded continuous operators. Hence, $f^m(\cdot) \in PAPS_T^p(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$, $F^m(\cdot) \in PAPS_T^p(\mathbb{R}, L^p(\mathbb{P}, L_2^0))$. Furthermore, $\{f^m(\cdot)\}, \{F^m(\cdot)\}$ are bounded in $L^p(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$, in $L^p(\mathbb{R}, L^p(\mathbb{P}, L_2^0))$ and there are subsequences, relabeled as $\{f^m(\cdot)\}, \{F^m(\cdot)\}$, and $\widehat{f}(\cdot) \in PAPS_T^p(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H})), \widehat{F}(\cdot) \in PAPS_T^p(\mathbb{R}, L^p(\mathbb{P}, L_2^0))$ such that

$$f^m(\cdot) \xrightarrow{w} \widehat{f}(\cdot) \text{ in } PAPS_T^p(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H})) \quad \text{as } m \to \infty,$$

$$F^m(\cdot) \xrightarrow{w} \widehat{F}(\cdot) \text{ in } PAPS_T^p(\mathbb{R}, L^p(\mathbb{P}, L_2^0)) \quad \text{as } m \to \infty.$$

By Lemma 4.1, we have

$$\tilde{\Phi} f^m \to \tilde{\Phi} \widehat{f}, \quad \tilde{\Phi} F^m \to \tilde{\Phi} \widehat{F}.$$

Next we turn to consider the following controlled system

$$\begin{aligned} d[x(t) + g(t, B_1 x(t))] = & A[x(t) + g(t, B_1 x(t))]dt \\ & + [\widehat{f}(t) + B(t) u^0(t)]dt + \widehat{F}(t) dW(t), \\ & t \in \mathbb{R},\ u \in U_{ad},\ t \neq t_i,\ i \in \mathbb{Z}, \end{aligned} \tag{4.3}$$

$$\Delta x(t_i) = x(t_i^+) - x(t_i^-) = I_i(x(t_i)), \quad i \in \mathbb{Z}. \tag{4.4}$$

By Theorem 4.1, it is easy to see that system (4.3)–(4.4) has a mild solution

$$\begin{aligned} \widehat{x}(t) = & -g(t, B_1 \widehat{x}(t)) + \int_{-\infty}^{t} T(t-s) \widehat{f}(s) ds + \int_{-\infty}^{t} T(t-s) B(s) u^0(s) ds \\ & + \int_{-\infty}^{t} T(t-s) \widehat{F}(s) dW(s) + \sum_{t_i < t} T(t-t_i) I_i(\widehat{x}(t_i)), \quad t \in \mathbb{R}. \end{aligned}$$

Then we have for all $t \in \mathbb{R}$,

$$E \parallel x^m(t) - \widehat{x}(t) \parallel_\alpha^p \leq \sum_{j=1}^{4} \nu_m^{(j)}(t),$$

where

$$\nu_m^{(1)}(t) = 5^{p-1}E \parallel g(t, B_1 x^m(t))) - g(t, B_1 \widehat{x}(t))) \parallel_\alpha^p,$$
$$\nu_m^{(2)}(t) = 5^{p-1}E \left\| \sum_{t_i<t} T(t-t_i)[I_i(x^m(t_i)) - I_i(\widehat{x}(t_i))] \right\|_\alpha^p,$$
$$\nu_m^{(3)}(t) = 5^{p-1}E \left\| \int_{-\infty}^t T(t-s)B(s)[u^m(s) - u^0(s)]ds \right\|_\alpha^p,$$
$$\nu_m^{(4)}(t) = 5^{p-1}E \left\| \int_{-\infty}^t T(t-s)[f^m(s) - \widehat{f}(s)]ds \right\|_\alpha^p,$$
$$\nu_m^{(5)}(t) = 5^{p-1}E \left\| \int_{-\infty}^t T(t-s)[F^m(s) - \widehat{F}(s)]dw(s) \right\|_\alpha^p.$$

By (H3) and (H6), we can obtain

$$\nu_m^{(1)}(t) + \nu_m^{(2)}(t) \le L_0 \parallel x^m - \widehat{x} \parallel_{\alpha,\infty}^p,$$

where L_0 as in Theorem 3.1. Let $\varepsilon > 0$, $t \in (t_i, t_{i+1})$, $i \in \mathbb{Z}$ and $0 < q < 1$ such that $0 < \alpha + q < 1$, By the compact operator property, there exists $\tilde{N}_0 > 0$ such that

$$E \parallel A^{-q}B(s)[u^m(s) - u^0(s)] \parallel_{S^p}^p < \varepsilon,$$
$$E \parallel A^{-q}[f^m(s) - \widehat{f}(s)] \parallel_{S^p}^p < \varepsilon,$$
$$E \parallel A^{-q}[F^m(s) - \widehat{F}(s)] \parallel_{S^p}^p < \varepsilon$$

for $m > \tilde{N}_0$. Using Hölder's inequality, we have

$$\begin{aligned}
\nu_m^{(3)}(t) &\le 5^{p-1}E\Big[\int_{-\infty}^t \parallel A^{\alpha+q}T(t-s)A^{-q}B(t-s) \\
&\quad \times [u^m(t-s) - u^0(t-s)] \parallel ds\Big]^p \\
&\le 5^{p-1}M_{\alpha+q}^p\Big(\int_{-\infty}^t (t-s)^{-\frac{p}{p-1}(\alpha+q)}e^{-\delta(t-s)}ds\Big)^{p-1} \\
&\quad \times \Big(\sum_{n=1}^\infty e^{-\delta(n-1)}\int_{n-1}^n E \parallel A^{-q}B(s)[u^m(s) - u^0(s)] \parallel^p ds\Big) \\
&\le 5^{p-1}M_{\alpha+q}^p\Big(\Gamma(1 - \frac{p(\alpha+q)}{p-1})\delta^{\frac{p(\alpha+q)}{p-1}-1}\Big)^{p-1}\frac{e^\delta}{e^\delta - 1}\varepsilon,
\end{aligned}$$

and

$$\begin{aligned}
\nu_m^{(4)}(t) &\le 5^{p-1}E\Big[\int_{-\infty}^t \parallel A^{\alpha+q}T(t-s)A^{-q}[f^m(s) - \widehat{f}(s)] \parallel ds\Big]^p \\
&\le 5^{p-1}M_{\alpha+q}^p\Big(\int_{-\infty}^t (t-s)^{-\frac{p}{p-1}(\alpha+q)}e^{-\delta(t-s)}ds\Big)^{p-1} \\
&\quad \times \Big(\sum_{n=1}^\infty e^{-\delta(n-1)}\int_{n-1}^n E \parallel A^{-q}[f^m(t-s) - \widehat{f}(t-s)] \parallel^p ds\Big) \\
&\le 5^{p-1}M_{\alpha+q}^p\Big(\Gamma(1 - \frac{p(\alpha+q)}{p-1})\delta^{\frac{p(\alpha+q)}{p-1}-1}\Big)^{p-1}\frac{e^\delta}{e^\delta - 1}\varepsilon.
\end{aligned}$$

Using the Itô integral, we have for $p > 2$,

$$\begin{aligned}
\nu_m^{(5)}(t) &\le 5^{p-1} C_p \left[\int_{-\infty}^{t} [A^{\alpha+q} T(t-s) A^{-q} \parallel [F^m(s) - \widehat{F}(s)] \parallel_{L_2^0}^p]^{2/p} ds \right]^{p/2} \\
&\le 5^{p-1} C_p M_{\alpha+q}^p \left(\int_{-\infty}^{t} (t-s)^{-\frac{2p}{p-2}(\alpha+q)} e^{-2\delta(t-s)} ds \right)^{\frac{p-2}{p}} \\
&\quad \times \left(\sum_{n=1}^{\infty} e^{-2\delta(n-1)} \int_{n-1}^{n} E \parallel A^{-q} [F^m(s) - \widehat{F}(s)] \parallel_{L_2^0}^p ds \right) \\
&\le 5^{p-1} C_p M_{\alpha+q}^p \left(\Gamma(1 - \frac{2p\alpha}{p-2}) (2\delta)^{\frac{2p\alpha}{p-2} - 1} \right)^{\frac{p-2}{p}} \frac{e^{2\delta}}{e^{2\delta} - 1} \varepsilon.
\end{aligned}$$

For $p = 2$, we have

$$\begin{aligned}
\nu_m^{(5)}(t) &\le 5 \left[\int_{-\infty}^{t} E \parallel A^{\alpha+q} T(t-s) A^{-q} [[F^m(s) - \widehat{F}(s)] \parallel_{L_2^0}^2] ds \right] \\
&\le 5 M_\alpha^2 \Gamma(1 - 2\alpha)(2\delta)^{2\alpha-1} \varepsilon.
\end{aligned}$$

This implies that

$$\nu_m^{(j)}(t) \to 0 \quad \text{as } m \to \infty, \ j = 3, 4, 5.$$

Thus we can infer that

$$x^m \to \widehat{x} \text{ in } \mathbb{Y} \quad \text{as } m \to \infty.$$

Further, by (H4) and (H5), we can obtain

$$f^m(\cdot) \to f(\cdot, B_2 x(\cdot)), \quad F^m(\cdot) \to F(\cdot, B_3 x(\cdot)) \text{ in } \mathbb{Y} \quad \text{as } m \to \infty.$$

Using the uniqueness of limit, we have

$$\widehat{f}(s) = f(s, B_2 \widehat{x}(s)), \quad \widehat{F}(s) = F(s, B_3 \widehat{x}(s)).$$

Therefore, $\widehat{x}$ can be given by

$$\begin{aligned}
\widehat{x}(t) &= -g(t, B_1 \widehat{x}(t)) + \int_{-\infty}^{t} T(t-s) f(s, B_2 \widehat{x}(s)) ds \\
&\quad + \int_{-\infty}^{t} T(t-s) B(s) u^0(s) ds + \int_{-\infty}^{t} T(t-s) F(s, B_3 \widehat{x}(s)) dW(s) \\
&\quad + \sum_{t_i < t} T(t - t_i) I_i(\widehat{x}(t_i)), \quad t \in \mathbb{R},
\end{aligned}$$

which is just a mild solution of system (4.1)–(4.2) corresponding to u^0. Since $\mathbb{Y} \hookrightarrow L^1(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$, using (D1)–(D4) and Balder's theorem (see [2]), we can obtain

$$\begin{aligned}
\varepsilon &= \lim_{m \to \infty} \sup_{t \in \mathbb{R}} E \int_{-\infty}^{t} \vartheta(s, x^m(s), u^m(s)) ds \\
&\ge \sup_{t \in \mathbb{R}} E \int_{-\infty}^{t} \vartheta(s, \widehat{x}(s), u^0(s)) ds = \mathcal{J}(\widehat{x}, u^0) \ge \varepsilon.
\end{aligned}$$

This shows that $\mathcal{J}$ attains its minimum at $(\widehat{x}, u^0) \in \mathbb{Y} \times U_{ad}$. □

5. Applications

Consider following partial stochastic differential equations of the form

$$
\begin{aligned}
& d\Big[z(t,x)+\iota_1(t)\int_0^\pi b_1(x,y)\frac{\partial^2}{\partial y^2}z(t,y)dy\Big] \\
&= \frac{\partial^2}{\partial x^2}\Big[z(t,x)+\iota_1(t)\int_0^\pi b_1(x,y)\frac{\partial^2}{\partial y^2}z(t,y)dy\Big]dt \\
& +\iota_2(t)\Big[\int_0^\pi b_2(x,y)\frac{\partial^2}{\partial y^2}z(t,y)dy+\frac{1}{2}u(t,x) \\
& +\int_0^\pi b_3(x,y)\frac{\partial^2}{\partial y^2}z(t,y)dydW(t)\Big], \\
& \quad t\in\mathbb{R}, t\neq t_i, i\in\mathbb{Z}, u\in\tilde{U}_{ab}, x\in[0,\pi],
\end{aligned} \tag{5.1}
$$

$$\Delta z(t_i,x)=\iota_1(i)z(t_i,x),\quad i\in\mathbb{Z}, x\in[0,\pi], \tag{5.2}$$

$$z(t,0)=z(t,\pi)=0,\quad t\in\mathbb{R}, \tag{5.3}$$

where $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega,\mathcal{F},\mathbb{P},\mathcal{F}_t)$. In this system, $t_i=i+\frac{1}{4}|\sin i+\sin\sqrt{2}i|$, $\{t_i^j\}, i\in\mathbb{Z}, j\in\mathbb{Z}$ are equipotentially almost periodic and $\gamma=\inf_{i\in\mathbb{Z}}(t_{i+1}-t_i)>0$, one can see [21] for more details.

Let $\mathbb{H}=Y=L^2([0,\pi])$ with the norm $\|\cdot\|$ and define the operators $A:A(D)\subset\mathbb{H}\to\mathbb{H}$ by $A\omega=\omega''$ with the domain $D(A):=\{\omega\in\mathbb{H}:\omega''\in\mathbb{H},\omega(0)=\omega(\pi)=0\}$. Then A is the infinitesimal generator of an analytic semigroup $T(t)$ on $\mathbb{H}$ and $\| T(t)\|\le e^{-t}$ for $t\ge0$. Furthermore, A has a discrete spectrum with eigenvalues of the form $-n^2, n\in\mathbb{N}$ and normalized eigenfunctions given by $e_n(\xi):=(\frac{2}{\pi})^{\frac{1}{2}}\sin(n\xi)$. The following properties hold:

(a) If $\omega\in D(A)$, then $A\omega=-\sum_{n=1}^\infty n^2\langle\omega,e_n\rangle e_n$.

(b) For $\omega\in\mathbb{H}$ $T(t)\omega=\sum_{n=1}^\infty e^{-n^2t}\langle\omega,e_n\rangle e_n$, $(-A)^{-\frac{1}{2}}\omega=\sum_{n=1}^\infty\frac{1}{n}\langle\omega,e_n\rangle e_n$.

(c) The operator $A^{\frac{1}{2}}:D(A^{\frac{1}{2}})\subseteq\mathbb{H}\to\mathbb{H}$ given by $A^{\frac{1}{2}}\omega=\sum_{n=1}^\infty n\langle\omega,e_n\rangle e_n$ on the space $D(A^{\frac{1}{2}})=\{\omega(\cdot)\in\mathbb{H}:\sum_{n=1}^\infty n\langle\omega,e_n\rangle e_n\in\mathbb{H}\}$.

Moreover, $\| A^{-\frac{1}{2}}\|=1$, and

$$\| A^{\frac{1}{2}}T(t)\omega\|^2=\frac{1}{t}\sum_{n=1}^\infty n^2te^{-2n^2t}|\langle\omega,e_n\rangle e_n|^2\le\frac{1}{2et}\|\omega\|^2,\ \omega\in\mathbb{H},$$

where we have used the property that the function $\tilde{\mu}(x)=xe^{-x}$ takes its maximum e^{-1} at $x=1$. Therefore, we can choose $M_{\frac{1}{2}}=\frac{1}{\sqrt{2e}}$. Let $H_{\frac{1}{2}}:=(D(A^{\frac{1}{2}}),\|\cdot\|_{\frac{1}{2}})$, where $\|\cdot\|_{\frac{1}{2}}:=\| A^{\frac{1}{2}}x\|$ for each $x\in D(A^{\frac{1}{2}})$. For $\tilde{d}>0$, we define the admissible control set $\tilde{U}_{ad}=\{u(\cdot,y)|\mathbb{R}\to Y$ measurable, $\mathcal{F}_t$-adapted stochastic processes, and $\| u\|_{L^p_{\mathcal{F}}(\mathbb{R},Y)}\le\tilde{d}\}$. Find the control $u(t,x)$ that minimises the performance index

$$\mathcal{J}(z,u)(x)=\sup_{t\in\mathbb{R}}E\int_{-\infty}^t\Big[\int_{[0,\pi]}|z^u(\tau,x)|^2dx+\int_{[0,\pi]}|u(\tau,x)|^2dx\Big]d\tau.$$

Define the operators B_j by $B_j = A$ with $D(B_j) = D(A)$, then $B_j : L^p(\mathbb{P}, \mathbb{H}) \to L^p(\mathbb{P}, \mathbb{H})$ are bounded and $\| B_j \|_{L(L^p(\mathbb{P},\mathbb{H}))} = 1, j = 1, 2, 3$.

We assume that the following conditions hold.

(i) The functions $b_1(\cdot)$, $\frac{\partial^2}{\partial x^2} b_1(x, y)$, are (Lebesgue) measurable, $b_1(0, y) = b_1(\pi, y) = 0$ for every $y \in [0, \pi]$, and

$$\tilde{L}_g = \max\left\{\left(\int_0^\pi \int_0^\pi \left(\frac{\partial^j}{\partial x^j} b_1(x, y)\right)^2 dxdy\right)^{1/2} : j = 0, 1\right\} < \infty.$$

(ii) The functions $b_j : [0, \pi] \times [0, \pi] \to \mathbb{R}, j = 2, 3$, are continuous functions, and

$$\left(\int_0^\pi \int_0^\pi b_j^2(x, y) dydx\right)^{1/2} < \infty.$$

(iii) The functions $\iota_1 : \mathbb{R} \to \mathbb{R}$ is piecewise pseudo almost periodic functions and $\iota_2 : \mathbb{R} \to \mathbb{R}$ is piecewise Stepanov-like pseudo almost periodic functions, and there exist $\tilde{l}_0, \tilde{l}_1 > 0$ such that $|\iota_1(t) - \iota_1(s)|^p \le \tilde{l}_0 |t - s|$ and $|\iota_1(t)|^p \le \tilde{l}_0$, $|\iota_2(t)|^p \le \tilde{l}_1$ for all $t, s \in \mathbb{R}$.

For $t \in \mathbb{R}, x \in [0, \pi]$. Defining the maps $g, f, F : \mathbb{R} \times L^p(\mathbb{P}, \mathbb{H}) \to L^p(\mathbb{P}, \mathbb{H})$, $I_i : L^p(\mathbb{P}, \mathbb{H}) \to L^p(\mathbb{P}, \mathbb{H})$ by

$$g(t, \psi)(x) = \iota_1(t) \int_0^\pi b_1(x, y) \frac{\partial^2}{\partial y^2} \psi(y) dy,$$

$$f(t, \psi)(x) = \iota_2(t) \int_0^\pi b_2(x, y) \frac{\partial^2}{\partial y^2} \psi(y) dy,$$

$$F(t, \psi)(x) = \iota_2(t) \int_0^\pi b_3(x, y) \frac{\partial^2}{\partial y^2} \psi(y) dy,$$

and

$$I_i(\psi)(\cdot) = \beta_i \psi(t_i, \cdot), \quad i \in \mathbb{Z}.$$

For all $u \in L^2(\mathbb{R} \times [0, \pi])$, we define an operator B as follows:

$$(Bu)(t)(x) = \iota_2(t)(\frac{1}{2} u(t, x)).$$

Then, the above equation (5.1)–(5.3) can be written in the abstract form as the system (1.1)–(1.2). It is then easy see that g, I_i are piecewise pseudo almost periodic functions and f, F piecewise Stepanov-like pseudo almost periodic functions. Moreover, assumption (i) implies that g is $D(A^{\frac{1}{2}})$-valued. In fact, for any $\psi \in L^p(\mathbb{P}, \mathbb{H})$, we have by assumption (i)

$$\begin{aligned}\langle g(t, \psi), \omega_n \rangle &= \int_0^\pi \omega_n(x) \left(\iota_1(t) \int_0^\pi b_1(x, y) \frac{\partial^2}{\partial y^2} \psi(y) dy\right) dx \\ &= \frac{\iota_1(t)}{n} \left\langle \int_0^\pi \frac{\partial}{\partial x} \left(\int_0^\pi b_1(x, y) \frac{\partial^2}{\partial y^2} \psi(y) dy\right) dx, \sqrt{\frac{2}{\pi}} \cos(nx) \right\rangle,\end{aligned}$$

and

$$\begin{aligned}
& E \parallel g(t,\psi) - g(t_1,\psi_1) \parallel_{\frac{1}{2}}^p \\
= & E\bigg[\bigg(\iota_1(t)\int_0^\pi\bigg(\int_0^\pi b_1(x,y)\frac{\partial^2}{\partial y^2}\psi(y)dy \\
& - \bigg(\iota_1(t_1)\int_0^\pi\bigg(\int_0^\pi b_1(x,y)\frac{\partial^2}{\partial y^2}\psi_1(y)dy\bigg)^2 dx\bigg)^{1/2}\bigg]^p \\
\le & E\bigg[\bigg(\int_0^\pi\int_0^\pi\bigg(\frac{\partial b_1(x,y)}{\partial x}\bigg)^2 dydx\bigg)^{1/2} \\
& \times\bigg(|\iota_1(t)-\iota_1(t_1)| + |\iota_1(t_1)| \parallel \psi-\psi_1 \parallel\bigg)\bigg]^p \\
\le & 2^{p-1}\tilde{L}_g^p[|\iota_1(t)-\iota_1(t_1)|^p + |\iota_1(t_1)|^p \parallel \psi-\psi_1 \parallel^p] \\
\le & L_g[|t-t_1| + \parallel \psi-\psi_1 \parallel^p]
\end{aligned}$$

and $E \parallel g(t,\psi) \parallel_{\frac{1}{2}}^p \le L_g \parallel \psi \parallel^p$ for all $t, t_1 \in \mathbb{R}, \psi, \psi_1 \in L^p(\mathbb{P},\mathbb{H})$, where $L_h = 2^{p-1}\tilde{L}_h^p\tilde{l}_0$. From assumption (ii), we have

$$\begin{aligned}
& E \parallel f(t,\psi) - f(t_1,\psi_1) \parallel^p + E \parallel F(t,\psi) - F(t_1,\psi_1) \parallel^p \\
\le & \tilde{l}_1 E\bigg[\bigg(\int_0^\pi\bigg(\int_0^\pi b_2(x,y)\frac{\partial^2}{\partial y^2}[\psi(y)-\psi_1(y)]dy\bigg)^2 dx\bigg)^{1/2}\bigg]^p \\
& + \tilde{l}_1 E\bigg[\bigg(\int_0^\pi\bigg(\int_0^\pi b_3(x,y)\frac{\partial^2}{\partial y^2}[\psi(y)-\psi_1(y)]dy\bigg)^2 dx\bigg)^{1/2}\bigg]^p \\
\le & \tilde{l}_1\bigg[\bigg(\int_0^\pi\int_0^\pi b_1^2(x,y)dydx\bigg)^{p/2} + \bigg(\int_0^\pi\int_0^\pi b_3^2(x,y)dydx\bigg)^{p/2}\bigg] \\
& \times \parallel \psi-\psi_1 \parallel^p \\
= & L_{f,F} \parallel \psi-\psi_1 \parallel^p
\end{aligned}$$

and $E \parallel f(t,\psi) \parallel_{S^p}^p + E \parallel F(t,\psi) \parallel_{S^p}^p \le L_{f,F} \parallel \psi \parallel^p$ for all $t, t_1 \in \mathbb{R}, \psi, \psi_1 \in L^p(\mathbb{P},\mathbb{H})$, where $L_{f,F} = \tilde{l}_1[\int_0^\pi\int_0^\pi b_2^2(x,y)dydx)^{p/2} + \int_0^\pi\int_0^\pi b_3^2(x,y)dydx)^{p/2}]$. Further, $\iota_1(i) \in PAP(\mathbb{Z},\mathbb{R})$ implies that $I_i \in PAP(\mathbb{Z}, L^p(\mathbb{P},\mathbb{H}))$, $i \in \mathbb{Z}$, and $E \parallel I_i(x_1) - I_i(x_2) \parallel^p \le c_i \parallel x_1 - x_2 \parallel^p \le c_i \parallel x_1 - x_2 \parallel_{\frac{1}{2}}^p$ and $E \parallel I_i(x_1) \parallel^p \le c_i \parallel x_1 \parallel_{\frac{1}{2}}^p$ for all $x_1, x_2 \in L^p(\mathbb{P},\mathbb{H}_{\frac{1}{2}})$, where $c_i = |\iota_1(i)|^p$, $i \in \mathbb{Z}$. Therefore, the assumptions (H2)-(H6) all hold in Section 3. It is obvious that $(Bu)(t)(x)$ is measurable in $t \in \mathbb{R}, x \in [0,\pi]$. For $u \in L^2(\mathbb{R}\times[0,\pi])$, we have $\parallel Bu \parallel_{L^2(\mathbb{R}\times[0,\pi])}^p \le L_u \parallel u \parallel_{L^2(\mathbb{R}\times[0,\pi])}^p$, where $L_u = \frac{\tilde{l}_1}{2^p}$. Then, we can conclude that $B \in L^\infty(\mathbb{R}, L(\mathbb{H}))$. Further, all the conditions stated in Theorem 4.2 satisfied. Hence by Theorems 4.2, the system (5.1)–(5.3) has at least one optimal pair.

6. Conclusion

In this paper, we studied the Stepanov-like pseudo almost periodic periodicity for a class of impulsive partial stochastic differential equations in Hilbert spaces. More precisely, by using stochastic analysis, analytic semigroup, fractional powers of closed operators and the Krasnoselskii-Schaefer type fixed point theorem along with a new composition theorem, we discussed the existence of p-mean piecewise Stepanov-like pseudo almost periodic mild solutions for these equations under non-Lipschitz conditions. Then, we investigated the existence of optimal pairs of the impulsive stochastic control system. Finally, an application is provided to illustrate the applicability of the new results.

There are two direct issues which require further study. First, we will investigate the p-mean piecewise Stepanov-like weighted pseudo almost periodicity in distribution and optimal control for impulsive partial stochastic differential equations with infinite delay in Hilbert spaces. Second, we will devote our efforts to the study of the time optimal control of impulsive partial stochastic differential equations and inclusions.

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References

[1] P. Balasubramaniam and P. Tamilalagan, *The solvability and optimal controls for impulsive fractional stochastic integro-differential equations via resolvent operators*, J. Optim. Theory Appl., 2017, 174(1), 139–155.

[2] E. Balder, *Necessary and sufficient conditions for L_1-strong-weak lower semi-continuity of integral functional*, Nonlinear Anal. Real World Appl., 1987, 11, 1399–1404.

[3] M. Baroun, K. Ezzinbi, K. Khalil and L. Maniar, *Pseudo almost periodic solutions for some parabolic evolution equations with Stepanov-like pseudo almost periodic forcing terms*, J. Math. Anal. Appl., 2018, 462(1), 233–262.

[4] P.H. Bezandry and T. Diagana, *Existence of S^2-almost periodic solutions to a class of nonautonomous stochastic evolution equations*, Electron. J. Qual. Theory Differ. Equ., 2008, 35, 1–19.

[5] P.H. Bezandry and T. Diagana, *Almost Periodic Stochastic Processes*, Springer-Verlag, New York Inc., 2011.

[6] C. Burgos, J.-C. Cortés, A. Debbouche, L. Villafuerte and R.-J. Villanueva, *Random fractional generalized Airy differential equations: A probabilistic analysis using mean square calculus*, Appl. Math. Comput., 2019, 352, 15–29.

[7] T.A. Burton and C. Kirk, *A fixed point theorem of Krasnoselski-Schaefer type*, Math. Nachr., 1998, 189, 23–31.

[8] J. Cao, A. Debbouche and Y. Zhou, *Asymptotically almost periodicity for a class of Weyl-Liouville fractional evolution equations*, Mediterr. J. Math., 2018, 15(4), 1–22.

[9] J. Cao, A. Debbouche and Y. Zhou, *Asymptotic almost-periodicity for a class of Weyl-Like fractional difference equations*, Mathematics, 2019, 7(7), 592–616.

[10] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions, Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, Cambridge, 1992.

[11] T. Diagana, *Stepanov-like pseudo-almost periodicity and its applications to some nonautonomous differential equations*, Nonlinear Anal., 2008, 69(12), 4277–4285.

[12] M.A. Diop, K. Ezzinbi and M.M. Mbaye, *Measure theory and S^2-pseudo almost periodic and automorphic process: application to stochastic evolution equations*, Afrika Mat., 2015, 26(5–6), 779–812.

[13] A. Harrat, J.J. Nieto and A. Debbouche, *Solvability and optimal controls of impulsive Hilfer fractional delay evolution inclusions with Clarke subdifferential*, J. Comput. Appl. Math., 2018, 344, 725–737.

[14] Z. Hu and Z. Jin, *Stepanov-like pseudo almost periodic mild solutions to nonautonomous neutral partial evolution equations*, Nonlinear Anal., 2012, 75(1), 244–252.

[15] L. Hu and Y. Ren, *Existence results for impulsive neutral stochastic functional integro-differential equations with infinite delays*, Acta Appl. Math., 2010, 111, 303–317.

[16] J. Liu and C. Zhang, *Composition of piecewise pseudo almost periodic functions and applications to abstract impulsive differential equations*, Adv. Differ. Equ., 2013, 2013(11), 1–21.

[17] E. B. Davies, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Bulletin of the London Mathematical Society, 1984, 16(6), 650-651.

[18] L. Peng, Y. Zhou and A. Debbouche, *Approximation techniques of optimal control problems for fractional dynamic systems in separable Hilbert spaces*, Chaos Solitons Fractals, 2019, 118, 234–241.

[19] Y. Ren, X. Jia and R. Sakthivel, *The p-th moment stability of solutions to impulsive stochastic differential equations driven by G-Brownian motion*, Appl. Anal., 2017, 96(6), 988–1003.

[20] R. Sakthivel, P. Revathi and Y. Ren, *Existence of solutions for nonlinear fractional stochastic differential equations*, Nonlinear Anal., 2013, 81, 70–86.

[21] A.M. Samoilenko and N.A. Perestyuk, *Impulsive Differential Equations*, World Scientific, Singapore, 1995.

[22] N. Song, H. Li and C. Chen, *Piecewise weighted pseudo almost periodic functions and applications to impulsive differential equations*, Math. Slovaca, 2017, 66(5), 1139–1156.

[23] G.T. Stamov and I.M. Stamova, *Almost periodic solutions for impulsive fractional differential equations*, Dyn. Syst., 2014, 29(1), 119–132.

[24] J. Wang, M. Fečkan and A. Debbouche, *Time optimal control of a system governed by non-instantaneous impulsive differential equations*, J. Optim. Theory Appl., 2019, 182(2), 573–587.

[25] Z. Xia, *Pseudo almost periodic mild solution of nonautonomous impulsive integro-differential equations*, Mediterr. J. Math., 2016, 13(3), 1065–1086.

[26] Z. Xia and D. Wang, *Piecewise weighted pseudo almost periodic solutions of impulsive integro-differential equations via fractional operators*, Electron. J. Differential Equations, 2015, 2015, 1–18.

[27] Z. Yan and F. Lu, *Existence results for a new class of fractional impulsive partial neutral stochastic integro-differential equations with infinite delay*, J. Appl. Anal. Comput., 2015, 5(3), 329–346.

[28] Z. Yan and F. Lu, *Existence and exponential stability of pseudo almost periodic solutions for impulsive nonautonomous partial stochastic evolution equations*, Adv. Differ. Equ., 2016, 2016(294), 1–37.

[29] Z. Yan and F. Lu, *The optimal control of a new class of impulsive stochastic neutral evolution integro-differential equations with infinite delay*, Int. J. Control, 2016, 89(8), 1592–1612.

[30] Z. Yan, and F. Lu, *Solvability and optimal controls of a fractional impulsive stochastic partial integro-differential equation with state-dependent delay*, Acta Appl. Math., 2018, 155(1), 57–84.

[31] Z. Yan and H. Zhang, *Existence of Stepanov-like square-mean pseudo almost periodic solutions to partial stochastic neutral differential equations*, Ann. Funct. Anal., 2015, 6(1), 116–138.