

NONUNIFORM DICHOTOMY SPECTRUM INTERVALS: THEOREM AND COMPUTATION*

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Abstract In this paper, on the condition of nonuniformly bounded growth, we give the relationship between the nonuniform exponential dichotomy spectrum and the other two classical spectrums (the Lyapunov spectrum and Sacker-Sell spectrum), present their stability under small linear perturbations. The main goal of this paper is to discuss the theory for the computation of these spectrums on the condition of nonuniformly bounded growth, and this extends the work of Dieci and Vleck [22], which computes the Lyapunov spectrum and Sacker-Sell spectrum under the condition of bounded. Finally, the numerical examples are given to illustrate and verify the theoretical results.

Keywords Lyapunov exponents, nonuniform exponential dichotomy spectrum, spectrum Intervals, weak integral separation.

MSC(2010) 34D08, 34D09.

1. Introduction

Lyapunov exponent was introduced by Lyapunov himself, reprinted in [28]. In this paper, Lyapunov exponent was generalized for illustrating the characterization of exponential growth rates of time varying matrix functions. For an n -dimensional (time varying) differential equations, there are n Lyapunov exponents, Lyapunov exponents are natural to be of our interest for the reason that Lyapunov spectrum is discussed in this paper, which characterizes stability properties of the solution trajectory of nonautonomous systems. Since then, different characterizations of spectrums for linear nonautonomous differential equation have been proposed. Among them, one of the most famous spectrums is dichotomy spectrum (or called Sacker-Sell spectrum, dynamical spectrum), which was introduced by Sacker and Sell in [34, 35] defined by exponential dichotomies to study the linear skew product flows. Afterwards, a lot of researches have been done to understand and extend this fruitful concept in various ways. For example, a spectral theory about linear difference equations has been studied in [2–5]. Reducibility and normal forms for

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*The first author was supported by the NSF of China (No. 11671118), China Postdoctoral Science Foundation funded project (No. 2016M591697), NSF of Anhui Province of China(No. gxfxZD2016090 and No. KJ2018A0437). The second author was supported by the NSF of China (No. 11871043, No. 11771298 and No. 11671251) and China Postdoctoral Science Foundation(No.2015M571538).

nonautonomous differential equations by using dichotomy spectrum has been given in [37, 38]. For more results about dichotomy spectrum, see [24, 25, 32, 33, 36] and the references therein.

In the computation of spectral intervals, both for dichotomy spectrum $\Sigma_{ED}(A)$ and Lyapunov spectrum $\Sigma_L(A)$, SVD and QR methods have been proposed by Dieci and Vleck [21, 22] to study the computation methods for these spectrums. After that, further research on this topic has been proposed by Dieci and his collaborators (see [18–20, 23] for details). For more information about the theoretical and numerical analysis of dichotomy spectrum, one can refer to [26, 27] and the references cited therein.

Meanwhile, as Barreira and Valls mentioned in [10], the classical notion of exponential dichotomy substantially restricts some dynamics, from the point of view of ergodic theory, almost all linear variational equations have a nonuniform exponential behavior. During the last several years, a more generalized concept: nonuniform exponential dichotomy has been introduced and investigated by Barreira and Valls (see e.g., [7–9, 42]). Based on the study of exponential dichotomy, the nonuniform dichotomy spectral theory was introduced in [16, 40] for linear nonautonomous system with the coefficients being nonuniformly bounded growth (see Definition 4.1 below).

Here we mention that the numerical methods proposed by Dieci and Vleck in [21, 22] demand the coefficients of the linear systems to be bounded. Otherwise, the numerical technique for computing Sacker-Sell spectrum, which is based on the condition of integral separateness, is not quite right. For example, we consider the following two dimensional diagonal system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 t \sin t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{1.1}$$

where, $\omega_1 > \omega_2 > 0$ are real paraments. One can see that the coefficients of (1.1) are not bounded, and (1.1) is not integrally separated (see [41]). Moreover, one can prove that the dichotomy spectrum Σ_{ED} of (1.1) is trivial, i.e., $\Sigma_{ED} = \mathbb{R}$, and the nonuniformly dichotomy spectrum is $\Sigma_{NED} = \{\omega_1\} \cup [-\omega_2, \omega_2]$ (see Example 2.1 in [16] for details. Remark 4.1 below also presents an explanation from the point of view of numerical analysis).

This work, inspired by both the classical notion of dichotomy spectrum [34, 35] and the notion of nonuniform dichotomy spectrum introduced by [16, 40], is an attempt to discuss the relationship of three different spectrums: $\Sigma_L(A)$, $\Sigma_{ED}(A)$ and $\Sigma_{NED}(A)$, and extend the numerical technique developed by Dieci and Vleck [21, 22] for studying linear nonautonomous system with the coefficients being nonuniformly bounded growth.

An outline of the paper is as follows. In Section 2, the basic definitions and properties of $\Sigma_L(A)$, $\Sigma_{ED}(A)$ and $\Sigma_{NED}(A)$ will be presented. Section 3 discusses the relationship of $\Sigma_L(A)$, $\Sigma_{ED}(A)$ and $\Sigma_{NED}(A)$, and summarizes the stability of these spectrums under small linear perturbations.

In Section 4, we first establish necessary and sufficient condition of Steklov function and weak integral separateness under the condition of nonuniformly bounded growth, and then we can use this relationship to show the numerical methods for $\Sigma_{ED}(A)$ and $\Sigma_{NED}(A)$. An example will be given in Section 5 to illustrate and verify the theoretical results.

2. Lyapunov, exponential dichotomy, and nonuniform exponential dichotomy spectrum

Given an n -dimensional linear system

$$\dot{x} = A(t)x, \quad (2.1)$$

where $x(t) \in \mathbb{R}^n$ and $A(t)$: is an $n \times n$ matrix with real entries depending continuously on $t \geq 0$. Considering the trivial solution of (2.1), it is well known that if the matrix function $A(t)$ is constant, i.e., $A(t) = A$ for all $t \geq 0$, then the zero solution of (2.1) is asymptotically (and indeed, exponentially) stable if and only if the real part of every eigenvalue of the matrix A is negative. A similar result holds in the case when the matrix function $A(t)$ is periodic by using the Floquet theory. For the general (nonautonomous) case, we need to consider the spectral intervals instead of eigenvalues, so in this section we first recall the definitions of the next two classical concepts of spectrum: Lyapunov spectrum $\Sigma_L(A)$, and exponential dichotomy spectrum $\Sigma_{ED}(A)$, then we introduce a third related one, the nonuniform exponential dichotomy spectrum, $\Sigma_{NED}(A)$.

2.1. Lyapunov spectrum.

Given a fundamental matrix solution $\Phi(t)$ of (2.1), define $\lambda_j, j = 1, \dots, n$, as

$$\lambda_j(\Phi(t)) := \limsup_{t \rightarrow +\infty} \frac{1}{t} \ln \|\Phi(t)e_j\|,$$

where the vector norm is the 2-norm (invariant under orthogonal transformations), and the e_j is the unit column-vector in the x_j direction, i.e.,

$$e_j = (\underbrace{0, \dots, 0}_j, 1, 0, \dots, 0)^T.$$

When the sum of the numbers λ_j is minimized as we vary over all possible fundamental matrix solutions of the system, i.e.,

$$\sum_{j=1}^n \lambda_j^s := \inf \left\{ \sum_{j=1}^n \lambda_j(\Phi(t)) : \Phi(t) \text{ is a fundamental matrix of (2.1)} \right\}$$

then the numbers $\lambda_j^s, j = 1, \dots, n$ are called (*upper*) *Lyapunov exponents*, and the corresponding basis is called *normal*.

Now we consider the linear differential equation which is dual to (2.1)

$$\dot{y} = -A^T(t)y. \quad (2.2)$$

Similarly, one can have the upper Lyapunov exponents $\lambda_j^i, j = 1, \dots, n$ of (2.2), which are the lower Lyapunov exponents of (2.1) (see e.g., [22] for details). Let λ_j^s, λ_j^i be ordered: $\lambda_1^s \geq \lambda_2^s \geq \dots \geq \lambda_n^s$ and $\lambda_1^i \geq \lambda_2^i \geq \dots \geq \lambda_n^i$. Considering such a fact that $\lambda_j^i \leq \lambda_j^s$, then the Lyapunov spectrum can be defined as

$$\Sigma_L := \bigcup_{j=1}^n [\lambda_j^i, \lambda_j^s].$$

Especially, the system is called *regular* while $\lambda_j^i = \lambda_j^s$ for all $j = 1, \dots, n$.

2.2. Exponential dichotomy spectrum.

Recall that (2.1) admits an *exponential dichotomy* if there exists an invariant projection P , and constants $\alpha > 0$, $M > 0$ such that

$$\|\Phi(t)P\Phi^{-1}(s)\| \leq Me^{-\alpha(t-s)}, \quad \text{for } 0 \leq s \leq t,$$

and

$$\|\Phi(t)Q\Phi^{-1}(s)\| \leq Me^{\alpha(t-s)}, \quad \text{for } 0 \leq t \leq s,$$

where $Q = I_n - P$ is the complementary projection. Furthermore, for any fixed $\gamma \in \mathbb{R}$, write a shifted system

$$\dot{x} = [A(t)x - \gamma I_n]x. \tag{2.3}$$

Then the *exponential dichotomy spectrum* of (2.1) is given by the set

$$\Sigma_{NED}(A) = \{\gamma \in \mathbb{R} : (2.3) \text{ admits no exponential dichotomy}\},$$

and the resolvent set $\rho_{ED}(A) = \mathbb{R} \setminus \Sigma_{ED}(A)$ is its complements.

In [35,36], it has been shown that $\Sigma_{ED}(A)$ is at most a disjoint union of n closed intervals. This means that $\Sigma_{ED}(A) = \emptyset$ or $\Sigma_{ED}(A) = \mathbb{R}$ or $\Sigma_{ED}(A)$ is in one of the four cases

$$\Sigma_{ED}(A) = \left\{ \begin{array}{c} [a_1, b_1] \\ \text{or} \\ (-\infty, b_1] \end{array} \right\} \cup [a_2, b_2] \cup \dots \cup [a_{k-1}, b_{k-1}] \cup \left\{ \begin{array}{c} [a_k, b_k] \\ \text{or} \\ [a_k, \infty) \end{array} \right\}$$

for some $k : 1 \leq k \leq n$.

2.3. Nonuniform exponential dichotomy spectrum.

In [7,8], Barreira and Valls propose a new notion called nonuniform, which extends the notion of dichotomy of uniform. Later, [16] presents a new spectrum for (2.1) based upon the nonuniform exponential dichotomy.

Recall that (2.1) admits a *nonuniform exponential dichotomy* if there exists an invariant projection P , constants $\alpha > 0$, $M > 0$, and $\varepsilon \in [0, \alpha)$ such that

$$\|\Phi(t)P\Phi^{-1}(s)\| \leq Me^{-\alpha(t-s)}e^{\varepsilon s}, \quad \text{for } 0 \leq s \leq t, \tag{2.4}$$

and

$$\|\Phi(t)Q\Phi^{-1}(s)\| \leq Me^{\alpha(t-s)}e^{\varepsilon s}, \quad \text{for } 0 \leq t \leq s, \tag{2.5}$$

where $Q = I_n - P$ is the complementary projection. Then the *nonuniform exponential dichotomy spectrum* of (2.1) is given by the set

$$\Sigma_{NED}(A) = \{\gamma \in \mathbb{R} : (2.3) \text{ admits no nonuniform exponential dichotomy}\},$$

and the resolvent set $\rho_{NED}(A) = \mathbb{R} \setminus \Sigma_{NED}(A)$ is its complements.

Similarly, it has been shown in [16] that $\Sigma_{NED}(A)$ is at most a disjoint union of n closed intervals. This means that $\Sigma_{NED}(A) = \emptyset$ or $\Sigma_{NED}(A) = \mathbb{R}$ or $\Sigma_{NED}(A)$ is in one of the four cases

$$\Sigma_{NED}(A) = \left\{ \begin{array}{c} [a_1, b_1] \\ or \\ (-\infty, b_1] \end{array} \right\} \cup [a_2, b_2] \cup \dots \cup [a_{k-1}, b_{k-1}] \cup \left\{ \begin{array}{c} [a_k, b_k] \\ or \\ [a_k, \infty) \end{array} \right\}$$

for some $k : 1 \leq k \leq n$.

3. Relationship of spectrums $\Sigma_L(A)$, $\Sigma_{ED}(A)$ and $\Sigma_{NED}(A)$.

It is well known that the notion of Lyapunov exponents, exponential dichotomy together with some of their variants, extensions, and modifications, plays a central role in the study of general theory of dynamical systems. To gain insight into the behavior of the dynamical approaches of (2.1), several aspects are discussed in this section to illustrate the relationship of spectrums $\Sigma_L(A)$, $\Sigma_{ED}(A)$ and $\Sigma_{NED}(A)$.

We first present the relation of the inclusion of these three spectrums.

Proposition 3.1. *For an n -dimensional linear system (2.1), we have the following chain of implications*

$$\Sigma_L(A) \subset \Sigma_{NED}(A) \subset \Sigma_{ED}(A).$$

Proof. Clearly $\Sigma_{NED}(A) \subset \Sigma_{ED}(A)$ due to the fact $\varepsilon \geq 0$ in (2.4)-(2.5) (see [16] for details). Now we prove that $\Sigma_L(A) \subset \Sigma_{NED}(A)$. Obviously, $\Sigma_L(A) \subset \Sigma_{NED}(A)$ if $\Sigma_{NED}(A) = \mathbb{R}$. Conversely, if $\Sigma_{NED}(A) = \emptyset$, then $\rho_{NED}(A) = \mathbb{R}$. This means that for any $\lambda \in \mathbb{R}$, there exists an invariant projection P , constants $\alpha > 0$, $M > 0$, and $\varepsilon \in [0, \alpha)$ such that

$$\|\Phi_\lambda(t)P\Phi_\lambda^{-1}(s)\| \leq Me^{-\alpha(t-s)}e^{\varepsilon s}, \quad \text{for } 0 \leq s \leq t,$$

or equivalently,

$$\|\Phi(t)P\Phi^{-1}(s)\| \leq Me^{(\lambda-\alpha)(t-s)}e^{\varepsilon s}, \quad \text{for } 0 \leq s \leq t.$$

Set $s = 0$, the inequality above implies that

$$\|\Phi(t)P\Phi^{-1}(0)x_0\| \leq M\|x_0\|e^{(\lambda-\alpha)t} \quad \text{for } 0 \leq t.$$

with any initial point $(t, x(t))|_{t=0} = (0, x_0) \in \mathbb{R} \times \mathbb{R}^n$. It is easy to see that $\lambda_1^s \rightarrow -\infty$ since λ is arbitrary one in \mathbb{R} , then we have $\emptyset = \Sigma_L(A) = \Sigma_{NED}(A)$.

Now, we prove the theorem for the nontrivial case ($\Sigma_{NED}(A) \neq \emptyset$ and $\Sigma_{NED}(A) \neq \mathbb{R}$). Choosing $\gamma \in \rho_{NED}(A)$, define

$$S_\gamma := \left\{ (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n : \sup_{t \geq \tau} \{\|\Phi(t, \tau)\xi\|e^{-\gamma t}\}e^{-\varepsilon \tau} < \infty \right\},$$

and

$$U_\gamma := \left\{ (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n : \sup_{t \leq \tau} \{\|\Phi(t, \tau)\xi\|e^{-\gamma t}\}e^{-\varepsilon \tau} < \infty \right\}.$$

Then for any $\gamma_j \in \rho_{NED}(A)$, i.e.,

$$b_j < \gamma_j < a_{j+1}, \quad \text{for } j = 1, \dots, n-1,$$

the intersection

$$W_j = U_{\gamma_{j-1}} \cap S_{\gamma_j}, \quad \text{for } j = 1, \dots, n-1$$

forms a linear integral manifold of (2.1) with $\dim W_j \geq 1$ (see [16] for details). Let λ be an arbitrary point in (b_j, a_{j+1}) , thus $\lambda \in \rho_{NED}(A)$, and there exist an invariant projection P , constants $\alpha > 0$, $M > 0$, and $\varepsilon \in [0, \alpha)$ such that

$$\|\Phi_\lambda(t)P\Phi_\lambda^{-1}(s)\| \leq Me^{-\alpha(t-s)}e^{\varepsilon s}, \quad \text{for } 0 \leq s \leq t,$$

or equivalently,

$$\|\Phi(t)P\Phi^{-1}(s)\| \leq Me^{(\lambda-\alpha)(t-s)}e^{\varepsilon s}, \quad \text{for } 0 \leq s \leq t.$$

Set $s = 0$, the inequality above implies that

$$\|\Phi(t)P\Phi^{-1}(0)x_0\| \leq M\|x_0\|e^{(\lambda-\alpha)t} \quad \text{for } t \geq 0$$

with any initial point $(t, x(t))|_{t=0} = (0, x_0) \in W_j$. Now we can prove that $\lambda_j^s \leq b_j$ since λ is arbitrary one in (b_j, a_{j+1}) .

Similarly, choosing $\lambda \in (b_{j-1}, a_j)$, thus $\lambda \in \rho_{NED}(A)$, and there exists an invariant projection $Q = I_n - P$, constants $\alpha > 0$, $M > 0$, and $\varepsilon \in [0, \alpha)$ such that

$$\|\Phi_\lambda(t)Q\Phi_\lambda^{-1}(s)\| \leq Me^{\alpha(t-s)}e^{\varepsilon s}, \quad \text{for } 0 \leq t \leq s,$$

or equivalently,

$$\|\Phi(t)Q\Phi^{-1}(s)\| \leq Me^{(\lambda+\alpha)(t-s)}e^{\varepsilon s}, \quad \text{for } 0 \leq t \leq s.$$

Set $t = 0$, the inequality above implies that

$$\|x_0\| = \|\Phi(0)Q\Phi^{-1}(s)x(s)\| \leq M\|x(s)\|e^{-(\lambda+\alpha-\varepsilon)s}, \quad \text{for } s \geq 0$$

with any initial point $(t, x(t))|_{t=0} = (0, x_0) \in W_j$. Then we have

$$\|x(s)\| \geq e^{(\lambda+\alpha-\varepsilon)s}\|x_0\|, \quad \text{for } s \geq 0,$$

which means that $\lambda_j^i \geq \lambda + \alpha - \varepsilon \geq \lambda$ due to the fact $\varepsilon \in [0, \alpha)$. Now we can prove that $\lambda_j^i \geq a_j$ since λ is arbitrary one in (b_{j-1}, a_j) . \square

The next connection concentrates on the perturbation results of spectrums $\Sigma_L(A)$, $\Sigma_{ED}(A)$ and $\Sigma_{NED}(A)$. It is well known that exponential dichotomy of (2.1) remains unchanged with a small perturbation, which is called *roughness* (see e.g., [17, pp. 34] for details), i.e., for a perturbed system

$$\dot{x} = (A(t) + B(t))x \tag{3.1}$$

with $\|B(t)\| \leq \delta$ for some sufficiently small $\delta > 0$, the perturbed equation (3.1) has also an exponential dichotomy. Thus, $\Sigma_{ED}(A)$ is stable under small perturbation, since the shifted system does not change the stability of exponential dichotomy. In [9], Barreira and Valls show that the perturbation with the coefficient matrix

being exponentially decaying, i.e., the linear perturbed system (3.1) has also a nonuniform exponential dichotomy, while $\|B(t)\| \leq \delta e^{-\varepsilon t}$ for some sufficiently small $\delta > 0$, $\varepsilon \in [0, \alpha)$. Thus, $\Sigma_{NED}(A)$ is stable with the perturbation of the coefficient matrix being exponentially decaying.

The stability theory of Lyapunov spectrum $\Sigma_L(A)$ is more complicated than $\Sigma_{ED}(A)$ and $\Sigma_{NED}(A)$, and we first mention that it is not enough to ensure the stability of Lyapunov exponents for a general system even if for a regular system with different Lyapunov exponents. Example from [1, p. 171] shows that a two dimension system

$$\dot{x}_1 = \left(1 + \frac{\pi}{2} \sin(\pi\sqrt{t})\right)x_1, \quad \dot{x}_2 = 0$$

has distinct Lyapunov exponents $\lambda_1 = 1$ and $\lambda_2 = 0$. However, the Lyapunov exponents of this system are not stable.

A general condition called integral separateness (see, e.g., [1]), which is introduced and improved by Bylov, Vinograd, Izobov, Grobman, Millionščikov and several others [6, 12–15, 29, 30, 39], is generally used to guarantee the stability of Lyapunov exponents. Now we introduce the definitions of weak integral separateness, which extend the concept of integral ones.

Definition 3.1 (see [41, Def. 2.2]). The continuous functions $g_i, i = 1, \dots, n$, are said to be weakly integrally separated if for $i = 1, \dots, n-1$, there exist some constants $a, b \geq 0$ and $d \in \mathbb{R}$ such that

$$\int_s^t (g_{i+1}(\tau) - g_i(\tau))d\tau \geq a(t-s) - bs + d, \quad t \geq s \geq 0.$$

Definition 3.2 (see [41, Def. 2.3]). Let $\Phi(t) = (\Phi_1(t), \dots, \Phi_n(t))$ be a fundamental matrix solution of (2.1). Then, system (2.1) is said to be weakly integrally separated if for $i = 1, \dots, n-1$, there exist some constants $a, b \geq 0$ and $D > 0$ such that

$$\frac{\|\Phi_{i+1}(t)\|}{\|\Phi_{i+1}(s)\|} \cdot \frac{\|\Phi_i(s)\|}{\|\Phi_i(t)\|} \geq D e^{a(t-s)-bs}, \quad t \geq s \geq 0. \quad (3.2)$$

(2.1) is called integrally separated if $a > 0$ and $b = 0$ in (3.2) (see e.g. [1, Definition 5.3.2] and [22]). Obviously, integral separateness implies weak integral separateness due to the fact $b \geq 0$, but not vice versa. Indeed, (1.1) is weakly integrally separated but not integrally separated.

The following two theorems present the necessary and sufficient conditions of the stability of Lyapunov exponents, and therefore the corresponding stability of $\Sigma_L(A)$.

Theorem 3.1 (see [1, Thm. 5.4.7] and [14]). *Assuming that the system (2.1) has distinct Lyapunov exponents $\lambda_1 > \dots > \lambda_n$, they are stable if and only if there exists a fundamental matrix solution with integrally separated columns.*

Theorem 3.2 (see [41]). *Assuming that the system (2.1) with nonuniformly bounded growth has distinct Lyapunov exponents $\lambda_1 > \dots > \lambda_n$, they are stable with the perturbations of the coefficient matrix being exponentially decaying i.e., for a perturbed system (3.1) with $\|B(t)\| \leq \delta e^{-\varepsilon t}$ for some $\delta > 0$, $\varepsilon \in [0, \alpha)$, the Lyapunov exponents of system (2.1) are stable if and only if there exists a fundamental matrix solution with weakly integrally separated columns.*

From the analysis above, we have the following perturbation results about $\Sigma_L(A)$, $\Sigma_{ED}(A)$ and $\Sigma_{NED}(A)$.

Proposition 3.2. *For an n -dimensional linear system (2.1).*

1. *Given a sufficiently small parameter $\delta > 0$, such that $\|B(t)\| \leq \delta$, then the perturbed system (3.1) having the following properties:*
 - (a) $\Sigma_{ED}(A)$ is stable under the perturbation $\|B(t)\| \leq \delta$;
 - (b) (2.1) is integrable separated $\Leftrightarrow \Sigma_L(A)$ is stable under the perturbation $\|B(t)\| \leq \delta$;
2. *Given a sufficiently small parameter $\delta > 0$, and $\varepsilon \in [0, \alpha)$, such that $\|B(t)\| \leq \delta e^{-\varepsilon t}$, then the perturbed system (3.1) having the following properties:*
 - (a) $\Sigma_{NED}(A)$ is stable under the perturbation $\|B(t)\| \leq \delta e^{-\varepsilon t}$;
 - (b) (2.1) is weakly integrable separated $\Leftrightarrow \Sigma_L(A)$ is stable under the perturbation $\|B(t)\| \leq \delta e^{-\varepsilon t}$;

From the Proposition 3.2, one can find perturbation results of $\Sigma_L(A)$, $\Sigma_{ED}(A)$ and $\Sigma_{NED}(A)$ and some connection among these three spectrums, which can be used to assure the correctness of the numerical works in the next section. To further explore the relationship of these three spectrums, we focus on a special case: full spectrum. In [11], Bodine and Sacker presented that the system (2.1) with full exponential dichotomy spectrum is integrally separated. The converse does not hold (see [31, pp. 193] for details). Thus the following relationship holds:

$$\Sigma_{ED}(A) \text{ is full} \implies \text{Integral Separation}$$

More recently, we have proved in [41] that the system (2.1) with full nonuniform exponential dichotomy spectrum is weakly integrally separated, and the converse can hold true if we consider some additional condition (see [41, Theorem 1.2] for details), which can also be used to prove the converse part from integrally separated to full exponential dichotomy spectrum. This means that

$$\Sigma_{NED}(A) \text{ is full} \implies \text{Weakly integral Separation}$$

Note that $\Sigma_{NED}(A) \subset \Sigma_{ED}(A)$, and integral separateness always implies weak integral separateness. Combine these relationship with the approaches above, we will have the following chain of implications

$$\begin{array}{ccc} \Sigma_{ED}(A) \text{ is full} & \Rightarrow & \text{Integral Separation} \\ \downarrow & & \downarrow \\ \Sigma_{NED}(A) \text{ is full} & \Rightarrow & \text{Weakly integral Separation} \end{array}$$

Remark 3.1. If $\Sigma_{ED}(A)$ is full, the perturbation results of $\Sigma_L(A)$ and $\Sigma_{ED}(A)$ are the same. Similarly, if $\Sigma_{NED}(A)$ is full, the perturbation results of $\Sigma_L(A)$ and $\Sigma_{NED}(A)$ are the same.

4. Numerical computation of $\Sigma_{ED}(A)$ and $\Sigma_{NED}(A)$ without bounded condition.

Recall that (2.1) has *bounded growth* (see [36] and [17, pp. 8]) if and only if there exist constants $K > 0, \tilde{a} > 0$ such that

$$\|\Phi(t)\Phi^{-1}(s)\| \leq Ke^{\tilde{a}|t-s|}, \quad \text{for } t, s \geq 0. \tag{4.1}$$

However, the notion of bounded growth demands considerably from the dynamics and it is of considerable interest to look for more general types of hyperbolic behavior. We now present an example without uniform bounded growth.

Example 4.1. The scalar equation

$$\dot{x} = t(\sin t + 1)x \tag{4.2}$$

has no uniform bounded growth.

Proof. It is easy to verify that

$$\begin{aligned} \exp\left(\int_s^t \tau(\sin \tau + 1)d\tau\right) &= \exp(-t \cos t + s \cos s + \sin t - \sin s + t - s) \\ &= \exp(2(t - s) - t(\cos t + 1) + s(\cos s + 1) + (\sin t - \sin s)). \\ &\leq \exp(2(t - s) + 2s + 2). \end{aligned}$$

Furthermore, if $t = 2k\pi + \pi$ and $s = 2k\pi$ with $k \in \mathbb{N}$, then

$$\exp\left(\int_s^t \tau(\sin \tau + 1)d\tau\right) = \exp(2(t - s) + 2s). \tag{4.3}$$

Thus (4.2) has no uniform bounded growth due to the fact that the perturbation $2s$ in (4.3) can not be eliminated. This means that bounded growth (4.1) is not satisfied. \square

Remark 4.1. The numerical method proposed in [21,22] is not quite right without bounded condition. In fact, the computational procedure to approximate dichotomy spectrum is based on the separateness of Steklov function, which is equivalent to integral separateness under the condition of bounded (See Lemma 5.4.1 in [1]).

For example, considering the following scalar equation

$$\dot{x} = a(t)x \tag{4.4}$$

with $a(t)$ is continuous, the computational procedure to approximate dichotomy spectrum of (4.4) is as follows. Given $H > 0$, and $T > t_0 > 0$. Let $a_H = \frac{1}{H} \int_t^{t+H} a(\tau)d\tau$ for $t \in [t_0, T]$. One can compute

$$\bar{a} = \sup_{t_0 \leq t \leq T-H} a_H \quad \text{and} \quad \underline{a} = \inf_{t_0 \leq t \leq T-H} a_H$$

and use $[\underline{a}, \bar{a}]$ as an approximation to dichotomy spectrum of (4.4).

Let $a(t) = t(\sin t + 1)$ as in (4.2), $t_1 = 2k_1\pi$, and $H_1 = 2k_2\pi + \pi$ with $k_1, k_2 \in \mathbb{N}$, it follows easily from (4.3) that

$$\frac{1}{H_1} \int_{t_1}^{t_1+H_1} \tau(\sin \tau + 1)d\tau = 2 + \frac{t_1}{H_1}.$$

Similarly, let $t_1 = 2k_1\pi + \pi$, and $H_1 = 2k_2\pi$ with $k_1, k_2 \in \mathbb{N}$, we have

$$\frac{1}{H_2} \int_{t_2}^{t_2+H_2} \tau(\sin \tau + 1)d\tau = -\frac{t_2}{H_2}.$$

Hence, the dichotomy spectrum of (4.2) is \mathbb{R} , due to the fact that $a(t) = t(\sin t + 1)$ is not bounded.

In order to present the numerical computation of spectral intervals $\Sigma_{ED}(A)$ and $\Sigma_{NED}(A)$, we need to introduce the following definition to extend the known results of bounded growth to nonuniformly bounded growth.

Definition 4.1 (see [16, Def. 2.9]). We say that (2.1) has a nonuniformly bounded growth if there exist constants $K > 0$, $\tilde{a} > 0$ and $\tilde{b} \geq 0$ such that

$$\|\Phi(t)\Phi^{-1}(s)\| \leq Ke^{\tilde{a}|t-s|}e^{\tilde{b}s}, \quad \text{for } t, s \geq 0,$$

where $\Phi(t)$ is a fundamental matrix of (2.1) .

Recall that the function

$$f^H(t) = \frac{1}{H} \int_t^{t+H} f(\tau)d\tau \tag{4.5}$$

is defined as *Steklov function* or *Steklov average* (see [1, Def. 5.4.1]) with step $H > 0$. Inspired by the result

$$\int_s^t \tau(\sin \tau + 1)d\tau \leq 2(t - s) + 2s + 2$$

in Example 4.1, the following lemma, unlike the work in [1, Lemma 5.4.1], is to investigate the necessary and sufficient condition of weak integral separateness under the condition of nonuniform bounded growth, i.e.,

$$\int_s^t |f(\tau)|d\tau \leq \tilde{a}|t - s| + \tilde{b}s + \tilde{d} \quad t, s \geq 0, \tag{4.6}$$

with $\tilde{a}, \tilde{b} > 0$ and $\tilde{d} \in \mathbb{R}$, since the fundamental matrix solution of $\dot{x} = f(t)x$ satisfies

$$|\Phi(t)\Phi^{-1}(s)| \leq e^{\int_s^t |f(\tau)|d\tau} \leq e^{\tilde{d}}e^{\tilde{a}|t-s|+\tilde{b}s}.$$

Lemma 4.1. *Assuming that $f_1(t)$ and $f_2(t)$ are nonuniformly bounded growth functions, i.e., (4.6) holds true, then $f_1(t)$ and $f_2(t)$ are weakly integrally separated with $a, b > 0$ if and only if for sufficiently large $H \gg t$, the Steklov functions are separated in the standard sence:*

$$f_2^H(t) - f_1^H(t) \geq M \tag{4.7}$$

for some constant $M > 2\tilde{b} > 0$;

Proof. From the proof of Lemma 5.4.1 in [1], we know that the equality

$$\int_s^t f^H(\tau)d\tau = \int_s^t f(\tau)d\tau + I(t) - I(s) \tag{4.8}$$

holds with

$$I(t) = \frac{1}{H} \int_t^{t+H} f(y) dy \int_{y-H}^t dx.$$

Using (4.6), we have

$$|I(t)| \leq \frac{1}{H} \int_t^{t+H} (t - y + H) |f(y)| dy \leq \tilde{a}H + \tilde{b}t + \tilde{d}.$$

Thus it follows from (4.7) and (4.8) that

$$\begin{aligned} \int_s^t (f_2(\tau) - f_1(\tau)) d\tau &= \int_s^t (f_2^H(\tau) - f_1^H(\tau)) d\tau - I_2(t) + I_2(s) + I_1(t) - I_1(s) \\ &\geq (M - 2\tilde{b})(t - s) - 4\tilde{b}s - 4(\tilde{a}H + \tilde{d}), \end{aligned}$$

this implies that the functions $f_1(t)$ and $f_2(t)$ are weakly integrally separated.

Conversely, assuming that the functions $f_1(t)$ and $f_2(t)$ are weakly integrally separated, then we have

$$\int_s^t (f_2(\tau) - f_1(\tau)) d\tau \geq a(t - s) - bs + D, \quad t \geq s \geq 0$$

for $a, b > 0$ and $D \in \mathbb{R}$. Thus the difference of Steklov functions is

$$f_2^H(t) - f_1^H(t) = \frac{1}{H} \int_t^{t+H} (f_2(\tau) - f_1(\tau)) d\tau \geq a - b\frac{t}{H} + \frac{D}{H}. \quad (4.9)$$

Hence (4.7) holds with $H \gg t$. \square

Remark 4.2. From the proof of Lemma 4.1, H in Steklov function (4.5) must be chosen such that $H \gg t$, or else, $\frac{t}{H}$ in (4.9) can not be ignored, which is completely different from those in [1, Lemma 5.4.1].

Moreover, in the actual calculation process, we need the condition $t \gg H$ to find the effect of the nonuniform item. In fact, it follows from Definition 3.1 that

$$\frac{H}{t} |f_2^H(t) - f_1^H(t)| = \frac{1}{t} \left| \int_t^{t+H} (f_2(\tau) - f_1(\tau)) d\tau \right| \geq b - a\frac{H}{t} - \frac{|D|}{t} \geq 0$$

with $t \gg H$. This means that for sufficiently large $t \gg H$, the Steklov functions satisfy the inequality

$$\frac{H}{t} |f_2^H(t) - f_1^H(t)| \geq N \quad (4.10)$$

for some constant $N > 0$ if the nonuniform term does exist, and this effect does not appear in [1, Lemma 5.4.1] with $f_1(t)$ and $f_2(t)$ are bounded, or even if $f_1(t)$ and $f_2(t)$ are uniformly bounded growth functions.

In this paper, we always assume that (2.1) is weakly integrally separated. Note that a weakly integrally separated system is invariant under Lyapunov transformation, and a weakly integrally separated system is kinematically similar to a diagonal one by using the Lyapunov transformation (see [41]). So for a diagonal system, or for any system which can be reduced to a diagonal system through a Lyapunov transformation, our approach for approximating $\Sigma_L(A)$ of (2.1) under the condition of

nonuniform bounded growth is the same as A is bounded in [22]. Hence, on a finite time interval, our computational procedure for $\Sigma_L(A)$ is as follows. Considering a diagonal system

$$\dot{y} = \text{diag}[a_1(t), \dots, a_n(t)]y, \tag{4.11}$$

where $T_1, T_2 > 0$ are given constants, such that $t \in [T_1, T_2]$ with $T_2 \gg T_1 > 0$. Let $\lambda_j(t) = \frac{1}{t} \int_0^t a_j(\tau) d\tau$, and compute

$$\bar{\lambda}_j = \sup_{T_1 \leq t \leq T_2} \lambda_j^t \quad \text{and} \quad \underline{\lambda}_j = \inf_{T_1 \leq t \leq T_2} \lambda_j^t$$

and use $[\underline{\lambda}_j, \bar{\lambda}_j]$ as an approximation to $[\lambda_j^t, \lambda_j^s]$.

However, under the condition of nonuniform bounded growth, the procedure for approximation of $\Sigma_{NED}(A)$ is essentially different from the approximation of $\Sigma_{ED}(A)$ in [22]. Indeed, the nonuniform item can cause catastrophic failure in the computation when the approximation scheme in [22] is applied here since the nonuniform item can not be eliminated (see Remark 4.1 for details).

Definition 4.2. the weak integral separation spectrum is

$$\Sigma_{WIS} = \bigcup_{j=1}^n \Lambda_j,$$

where $\Lambda_j = \Lambda_j^+ \cap \Lambda_j^-$ is a closed interval, with Λ_j^+ corresponding to the j th diagonal planar systems

$$\dot{y}_j = \begin{pmatrix} \lambda & 0 \\ 0 & a_j(t) \end{pmatrix} y_j, \tag{4.12}$$

and Λ_j^- corresponding to the j th diagonal planar systems

$$\dot{y}_j = \begin{pmatrix} a_j(t) & 0 \\ 0 & \lambda \end{pmatrix} y_j \tag{4.13}$$

for each $j = 1, \dots, n$, which are given by

$$\Lambda_j^+ = \{ \lambda \in \mathbb{R} : (4.12) \text{ is not weakly integrally separated} \}$$

and

$$\Lambda_j^- = \{ \lambda \in \mathbb{R} : (4.13) \text{ is not weakly integrally separated} \}.$$

The following theorem mimics the classical one about integral separation spectrum [21, Theorem 2.29], but for the variation of spectrum has been extended from uniform item to the nonuniform ones.

Theorem 4.1. For (4.11), $\Sigma_{WIS} = \Sigma_{NED}$.

Proof. Given $\lambda \in \mathbb{R}$, if $\lambda \notin \Sigma_{NED}$. It follows from (2.4)-(2.5) that there exist constants $\alpha > 0$, $M > 0$, and $\varepsilon \in [0, \alpha)$ such that either

$$e^{\int_s^t a_j(\tau) d\tau} e^{-\lambda(t-s)} \leq M e^{-\alpha(t-s)} e^{\varepsilon s}, \quad \text{for } 0 \leq s \leq t, \tag{4.14}$$

or

$$e^{\int_s^t a_j(\tau) d\tau} e^{-\lambda(t-s)} \leq M e^{\alpha(t-s)} e^{\varepsilon s}, \quad \text{for } 0 \leq t \leq s. \tag{4.15}$$

If (4.14) holds then (4.13) is weakly integrally separated. If (4.15) holds then (4.12) is weakly integrally separated. This means that $\lambda \notin \Sigma_{NED} \Rightarrow \lambda \notin \Sigma_{WIS}$. Conversely, if $\lambda \notin \Sigma_{WIS}$, then for all $j = 1, \dots, n$, either (4.12) or (4.13) is weakly integrally separated and hence either (4.14) or (4.15) holds. \square

Theorem 4.2. *Considering the diagonal system (4.11) with nonuniformly bounded growth, i.e., for $j = 1, \dots, n$,*

$$\int_s^t |a_j(\tau)|d\tau \leq \tilde{a}(t - s) + \tilde{b}s + \tilde{d} \quad t \geq s \geq 0, \tag{4.16}$$

with $\tilde{a}, \tilde{b} > 0$ and $\tilde{d} \in \mathbb{R}$, let

$$\alpha_j^H = \inf_t \frac{1}{H} \int_t^{t+H} a_j(\tau)d\tau \quad \text{and} \quad \beta_j^H = \sup_t \frac{1}{H} \int_t^{t+H} a_j(\tau)d\tau$$

with any given $H > 0$, then, for each $j = 1, \dots, n$, $\Lambda_j \subseteq [\alpha_j^H, \beta_j^H]$. Moreover, assuming that (4.12) and (4.13) are weakly integrally separated respectively with $a, b > 0$, then for $H \gg t$ sufficiently large, $[\alpha_j^H, \beta_j^H] \subseteq \Lambda_j$ for $j = 1, \dots, n$.

Proof. First, let $\lambda > \beta_j^H$. Thus, there exists $M_j > 0$ such that

$$\int_t^{t+H} (\lambda - a_j(\tau))d\tau \geq M_j H, \quad \forall t. \tag{4.17}$$

In order to prove that λ and a_j are weakly integrally separated, it suffices to present that there exist some constants $a, b > 0$ and $D \in \mathbb{R}$ such that

$$\int_s^t (\lambda - a_j(\tau))d\tau \geq a(t - s) - bs - D, \quad t \geq s \geq 0. \tag{4.18}$$

We will verify (4.18) with $a = M_j, b = \tilde{b}, D = H(\lambda + \tilde{a}) + \tilde{b}H + \tilde{d}$. In fact, because of the inequality (4.17), it's easy to see that (4.18) holds with $a = M_j$ for all t and s with $t = s + H$. Now considering the case with $t < s + H$, we can rewrite the left hand side of (4.18) as

$$\int_s^t (\lambda - a_j(\tau))d\tau = \int_s^{s+H} (\lambda - a_j(\tau))d\tau - \int_t^{s+H} (\lambda - a_j(\tau))d\tau,$$

and thus it follows from $s \leq t < s + H$ and (4.16) that

$$\begin{aligned} \int_t^{s+H} (\lambda - a_j(\tau))d\tau &\leq (\lambda + \tilde{a})(s + H - t) + \tilde{b}(s + H) + \tilde{d} \\ &\leq bs + D, \end{aligned}$$

and hence,

$$\int_s^t (\lambda - a_j(\tau))d\tau \geq aH - bs - D \geq a(t - s) - bs - D.$$

Next, let $t > s + H$. Then, for some integer $k > 1, t = s + kH + \rho$ with $\rho \in [0, H)$. Then we have

$$\int_s^t (\lambda - a_j(\tau))d\tau = \sum_{j=0}^k \int_{t-(j+1)H}^{t-jH} (\lambda - a_j(\tau))d\tau - \int_{t-(k+1)H}^s (\lambda - a_j(\tau))d\tau$$

$$\begin{aligned} &\geq M_j(k+1)H - ((\lambda + \tilde{a})(H - \rho) + \tilde{d}) - \tilde{b}s \\ &\geq a(t - s) - bs - D. \end{aligned}$$

Therefore, λ and a_j are weakly integrally separated, and this means that $\lambda \notin \Lambda_j$. Similarly, we can prove that $\lambda \notin \Lambda_j$ for any $\lambda < \alpha_j^H$, and so $\Lambda_j \subseteq [\alpha_j^H, \beta_j^H]$ for any given $H > 0$.

Conversely, assuming that $\lambda \notin \Lambda_j$, then λ and a_j or a_j and λ are weakly integrally separated. Supposing that λ and a_j are weakly integrally separated with constants $a, b > 0$ and $D \in \mathbb{R}$, thus for any given $t \in \mathbb{R}$, choose $H \gg t$ such that

$$\frac{1}{H} \int_t^{t+H} (\lambda - a_j(\tau))d\tau \geq a - b\frac{t}{H} - \frac{D}{H} > \frac{a}{2} > 0, \tag{4.19}$$

and so $\lambda > \beta_j^H$. Similarly, we can prove that $\lambda < \alpha_j^H$. Therefore, for $H \gg t$ sufficiently large, $[\alpha_j^t, \beta_j^t] \subseteq \Lambda_j$ for $j = 1, \dots, n$. \square

Unlike the result of [22], $[\alpha_j^H, \beta_j^H] \subseteq \Lambda_j$ does not hold in general for the nonuniform bounded case. In fact, it follows from (4.19) that α_j^H and/or β_j^H can be unbounded if $t \gg H$ (see Remark 4.1 for details).

Now, to obtain a computational procedure on a finite time interval for Σ_{NED} out of Theorem 4.2, we need to verify whether the condition (4.10) holds or not. If (4.10) holds with $N > 0$, then the nonuniform term in (4.11) does exist, and this means that $\Sigma_{ED} = \mathbb{R}$ since the nonuniform term can not be eliminated (see Example 2.1 in [16] for details). Otherwise, $N = 0$ in (4.10) means that there is no nonuniform term in (4.11), that is, $\Sigma_{NED} = \Sigma_{ED}$.

Hence, on a finite time interval, our computational procedure is as follows. First, following the ideas in Remark 4.2, we compute the size of bias of nonuniform item b in (4.18). Given any $H > 0$, there exist constants $T_1, T_2 > 0$, such that $t \in [T_1, T_2]$ and $T_1 \gg H$. Let $b_j^t = \frac{1}{t} \int_t^{t+H} a_j(\tau)d\tau$ for $t \in [T_1, T_2]$. Then we can compute

$$\bar{b}_j = \sup_{T_1 \leq t \leq T_2} |b_j^t|, \tag{4.20}$$

and use \bar{b}_j to represent the bias of the nonuniform item b in (4.18). if $0 < \bar{b}_j < \epsilon \ll 1$ for some $\epsilon > 0$, there is no nonuniform term in (4.11), or else, the nonuniform term in (4.11) does exist.

Now we compute the spectrum Σ_{NED} . If the nonuniform item b in (4.18) is far away from zero, otherwise, we can follow the idea in [22] to compute the spectrum Σ_{ED} since there is no nonuniform item b in (4.18). Thus, it follows from Theorem 4.2 that a computational procedure for Σ_{NED} on a finite time interval is as follows. Given constants $T_1, T_2 > 0$, and $H > 0$, such that $t \in [T_1, T_2]$ and $H \gg T_2$, let $a_j^t = \frac{1}{H} \int_t^{t+H} a_j(\tau)d\tau$ for $t \in [T_1, T_2]$, then we compute

$$\bar{a}_j = \sup_{T_1 \leq t \leq T_2} a_j^t \quad \text{and} \quad \underline{a}_j = \inf_{T_1 \leq t \leq T_2} a_j^t \tag{4.21}$$

and use $[\underline{a}_j, \bar{a}_j]$ as an approximation to $[\alpha_j^H, \beta_j^H]$.

5. Numerical Simulation

In this Section, we consider a planar problem, which satisfies the condition of nonuniform bounded growth. In this case, we approximate the spectral intervals of $\Sigma_L(A)$, $\Sigma_{ED}(A)$ and $\Sigma_{NED}(A)$ and compute the bias of nonuniform item.

Example 5.1. Consider a planar system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \sin(\ln(t)) + \cos(\ln(t)) & 0 \\ 0 & \omega_1 - \omega_2 t \sin t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{5.1}$$

with $\omega_1 > \omega_2 > 0$. The problem is designed so that the coefficient matrix of (5.1) is nonuniform bounded growth. Note that the solution of (5.1) is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \exp(t \sin(\ln(t)) - t_0 \sin(\ln(t_0))) \cdot x_1(t_0) \\ \exp(\omega_1(t - t_0) + \omega_2 t \cos t - \omega_2 t_0 \cos t_0 - \omega_2 \sin t + \omega_2 \sin t_0) \cdot x_2(t_0) \end{pmatrix}.$$

Hence, $\Sigma_L = [-1, 1] \cup [\omega_1 - \omega_2, \omega_1 + \omega_2]$. Moreover, it follows from Example 6.2 of [22] and Example 2.1 of [16] that $\Sigma_{ED} = [-\sqrt{2}, \sqrt{2}] \cup \mathbb{R}$ and $\Sigma_{NED} = [-\sqrt{2}, \sqrt{2}] \cup [\omega_1 - \omega_2, \omega_1 + \omega_2]$. In the actual computation, here we choose $\omega_1 = 4$, and $\omega_2 = 2$ for (5.1).

Table 1. Approximation of Σ_L

| T_1 | T_2 | $[\underline{\lambda}_1, \bar{\lambda}_1]$ | $[\underline{\lambda}_2, \bar{\lambda}_2]$ |
|-------|-------|--|--|
| 1.E2 | 1.E4 | $[-1.0098, 1.0004]$ | $[2.0019, 6.0000]$ |
| 1.E2 | 1.E6 | $[-1.0060, 1.0004]$ | $[2.0000, 6.0000]$ |
| 1.E4 | 1.E6 | $[-1.0000, 0.9487]$ | $[2.0000, 6.0000]$ |

Table 2. Bias of Nonuniform item

| H | T_1 | T_2 | \bar{b}_1 | \bar{b}_2 |
|------|-------|-------|-------------------------|-------------|
| 1.E2 | 1.E4 | 1.E5 | 0.0013 | 1.0949 |
| 1.E3 | 1.E6 | 1.E7 | 1.2284×10^{-4} | 1.8760 |
| 1.E4 | 1.E6 | 1.E7 | 1.2882×10^{-5} | 1.0500 |

Table 3. Approximation of Σ_{ED}

| H | T_1 | T_2 | $[\underline{a}_1, \bar{a}_1]$ | $[\underline{a}_2, \bar{a}_2]$ |
|------|-------|-------|--------------------------------|---|
| 1.E3 | 1.E6 | 1.E8 | $[-1.4142, 1.2645]$ | $[-1.8707 \times 10^5, 1.8707 \times 10^5]$ |
| 1.E5 | 1.E6 | 1.E8 | $[-1.4142, 1.2323]$ | $[-3.9964 \times 10^3, 4.0044 \times 10^3]$ |
| 1.E4 | 1.E5 | 1.E8 | $[-1.4142, 1.4142]$ | $[-3.9510 \times 10^4, 3.9511 \times 10^4]$ |

Our numerical results of Σ_L are listed in Table 1. In this table we specify the values of T_1 and T_2 , and calculate the approximations of the two spectral intervals of Σ_L . The results in Table 1 show that the approximations are quite accurate for these three time intervals $[T_1, T_2]$.

In Table 2 we use the computational procedure outlined in Section 4, and report on numerical results which calculate the bias of nonuniform item based on (4.20).

Table 4. Approximation of Σ_{NED}

| H | T_1 | T_2 | $[a_2, \bar{a}_2]$ |
|------|-------|-------|--------------------|
| 1.E4 | 1.E2 | 1.E3 | [1.6649, 6.3694] |
| 1.E6 | 1.E2 | 1.E3 | [1.9999, 5.9985] |
| 1.E8 | 1.E3 | 1.E4 | [1.9998, 5.9986] |

It can be seen that the nonuniform items are sufficiently small for the first equation of (5.1) and far away from zero in the second equation of (5.1), which means that the first equation of (5.1) admits a uniform exponential dichotomy, while the second equation of (5.1) admits a nonuniform ones. Then, by calculating the the approximations of the two spectral intervals of Σ_{ED} in Table 3, we can find that the second spectral interval of Σ_{ED} is large enough while time intervals $[T_1, T_2]$ tend to infinity, which agrees with the theoretical result. At last, we just present the second spectral interval of Σ_{NED} based upon (4.21) in Table 4, since the first one does not have nonuniform one. The results in this table shows that spectral interval Σ_{NED} can be approximated accurately by letting $H > 0$ large enough such that $H \gg T_2$.

Acknowledgements

The authors thank two anonymous referees for their helpful comments.

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