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OUTPUT CONTROLLABILITY AND OPTIMAL OUTPUT CONTROL OF POSITIVE FRACTIONAL ORDER LINEAR DISCRETE SYSTEM WITH MULTIPLE DELAYS IN STATE, INPUT AND OUTPUT

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Abstract The article concerns output controllability and optimal output control of positive fractional order discrete linear systems with multiple delays in state, input and output. Necessary and sufficient conditions for output reachability (output controllability from zero initial conditions) and null output controllability (output controllability to zero final output) are given and proven. We also prove that the positive system is output controllable if it is output reachable and null output controllable with the output reachability index is equal or less than the null output controllability index. Sufficient conditions for the solvability of the optimal output control problem are given. Numerical examples are presented to illustrate the theoretical results.

Keywords Fractional order, positive linear systems, output reachability, null output controllability, output controllability, optimal output control.

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1. Introduction

In positive systems state variables and outputs are constrained to be positive, or at least nonnegative for all time whenever the initial conditions and inputs are nonnegative [13]. Since the state variables and outputs of many real-world processes represent quantities that may not have meaning unless they are nonnegative because they measure concentrations of substances, population levels, and so on, positive systems arise frequently in chemistry, biology, ecology, pharmacology, medicine, management sciences, economics, social sciences, etc. An excellent survey of positive systems with an emphasis on their applications in the areas of management and social sciences is given by Luenberger in [28]. The more recent monographs by Farina and Rinaldi in [9] and Kaczorek in [13] are devoted entirely to positive linear systems and some of their applications. Since positive systems are confined within

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a cone located in the positive orthant rather than in the whole space [2, 27, 35], their analysis and synthesis are more complicated and more challenging.

The notion of controllability due to Kalman [19] is important in the mathematical control theory [5, 20, 22]. Controllability continually appears as a sufficient condition for the existence of solutions to many control problems, for example, stabilization of unstable linear system by feedback. Basically a system is controllable if its state can be driven from any initial state to any final state using only certain admissible controls.

The systems described by fractional order differential or difference equations have been investigated in many areas in science and engineering [7, 18, 29–31, 33, 34]. The reachability and controllability of fractional discrete linear systems have been considered in [12, 21]. The reachability and controllability of fractional discrete linear systems with delayed state were analyzed in [4]. The minimum energy control problem for fractional discrete linear systems with and without delays has been formulated and solved in [4, 21, 23]. Works on reachability, controllability to zero and minimum energy control of positive fractional order discrete linear systems has been investigated in [16, 17, 36].

It is worthwhile to note that controllability is defined for states instead of outputs. In most engineering applications, it is needed to direct the output toward some desired value. In fact, having control over the output of the system has a significant importance if not more than the states. For example, the control of a multilink cable-driven manipulator, where the task is typically defined in terms of end effector pose, rather than the joint positions and velocities which can define the system's state [25]. Under such a situation, it is natural to consider output controllability [1, 6, 8, 10, 11, 24, 26].

Output controllability is a property of the impulse response matrix of a linear invariant-time system which reflects the dominant ability of an external input to move the output from any initial condition to any final condition in a finite time. The necessary and sufficient criterion for output controllability of linear time-invariant systems is addressed in, for example, [10]. The output reachability of positive linear discrete systems is discussed in [14]. The problem of output reachability of positive discrete linear systems with state delay has been studied in [15]. The output controllability of positive linear discrete systems with delays in state, input and output was considered in [32].

In this paper the output reachability, null output controllability, output controllability and optimal output control problems for the positive fractional order discrete linear systems with multiple delays in state, input and output will be formulated and solved.

The remainder of this paper is organized as follows. In the next section some mathematical preliminaries of fractional order positive linear discrete systems with delays in state, input and output are presented. We investigate the output controllability in Section 2. Section 3 gives the formulation and solution to the optimal output control problem.

Notations. $\mathbb{N}$ the set of nonnegative integers, $\mathbb{N}_+$ the set of positive integers, $\sigma_s^k = \{s, s+1, \dots, k\}$ the finite subset of $\mathbb{N}$ with $s \leq k$, $\mathbb{R}^n$ the set of real vectors with n components, $\mathbb{R}^n_+$ the set of vectors in $\mathbb{R}^n$ with nonnegative components, i.e.,

$$\mathbb{R}^n_+ = \left\{ x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n : \ x_i \geq 0, \quad i \in \sigma_1^n \right\},$$

where T denotes the transpose, $\mathbb{R}^{n\times m}$ the set of real constant matrices of dimension $n\times m$ $\left(\mathbb{R}^n=\mathbb{R}^{n\times 1}\right)$, I_n the identity matrix in $\mathbb{R}^{n\times n}$, A^{-1} the inverse of $A\in\mathbb{R}^{n\times n}$, diag $\left[a_1\cdots a_n\right]$ the matrix formed with $(a_i)_{i\in\sigma_1^n}$ in the diagonal and zero else.

2. Positive fractional delay systems

Using Grünwald-Letnikov approach, a definition of fractional discrete approximation of the derivative is given as follows.

Definition 2.1 ([30,33]). The discrete fractional difference is defined by

$$\Delta^\alpha x_i=\frac{1}{h^\alpha}\sum_{j=0}^{i}(-1)^j\begin{pmatrix}\alpha\\ j\end{pmatrix}x_{i-j}, \tag{2.1}$$

where $0<\alpha<1$ is the fractional order, h is the sampling time (taken equal to 1 in all that follows) and $i\in\mathbb{N}$ is the number of sample for which the approximation of the derivative is calculated.

The binomial coefficients $\begin{pmatrix}\alpha\\ j\end{pmatrix}$ can be obtained from the following relation

$$\begin{pmatrix}\alpha\\ j\end{pmatrix}=\begin{cases}1 & \text{for } j=0\\ \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!} & \text{for } j\in\mathbb{N}_+.\end{cases}$$

In this work we shall consider the fractional discrete linear delay system described by

$$\begin{cases}\Delta^\Upsilon x_{i+1}=\sum\limits_{j=0}^{p}A_jx_{i-j}+\sum\limits_{j=0}^{q}B_ju_{i-j},\\ \qquad y_i=\sum\limits_{j=0}^{l}C_jx_{i-j}+\sum\limits_{j=0}^{v}D_ju_{i-j}, \quad i\in\mathbb{N},\end{cases} \tag{2.2}$$

the initial conditions for (2.2) are given by

$$u_{-j}\in\mathbb{R}^m \text{ for } j\in\sigma_1^{\max\{q,v\}} \text{ and } x_{-j}\in\mathbb{R}^n \text{ for } j\in\sigma_0^{\max\{p,l\}}, \tag{2.3}$$

where

$$\Delta^\Upsilon x_{i+1}=\begin{bmatrix}\Delta^{\alpha_1}x_{1,i+1}\\ \vdots\\ \Delta^{\alpha_n}x_{n,i+1}\end{bmatrix}, \tag{2.4}$$

with $0<\alpha_j<1$ for $j\in\sigma_1^n$, $x_i=\begin{bmatrix}x_{1,i}\\ \vdots\\ x_{n,i}\end{bmatrix}\in\mathbb{R}^n$ is the system state, $u_i\in\mathbb{R}^m$ the control, $y_i\in\mathbb{R}^r$ the output, A_j, B_j, C_j and D_j real constant matrices with appropriate dimensions and p, q and v, and l the nonnegative integer maximal values of delays on state, input and output, respectively.

Using Definition 2.1 we may write the equation (2.4) in the equivalent form

$$\Delta^{\Upsilon} x_{i+1} = \begin{bmatrix} \Delta^{\alpha_1} x_{1,i+1} \\ \vdots \\ \Delta^{\alpha_n} x_{n,i+1} \end{bmatrix} = \begin{bmatrix} \sum\limits_{j=0}^{i+1} (-1)^j \begin{pmatrix} \alpha_1 \\ j \end{pmatrix} x_{1,i+1-j} \\ \vdots \\ \sum\limits_{j=0}^{i+1} (-1)^j \begin{pmatrix} \alpha_n \\ j \end{pmatrix} x_{n,i+1-j} \end{bmatrix} = \sum_{j=0}^{i+1} (-1)^j \Upsilon_j x_{i+1-j}, \tag{2.5}$$

with

$$\Upsilon_j = \operatorname{diag} \left[\begin{pmatrix} \alpha_1 \\ j \end{pmatrix} \cdots \begin{pmatrix} \alpha_n \\ j \end{pmatrix} \right].$$

By equations (2.5) and (2.2) we deduce that

$$x_{i+1} - \sum_{j=1}^{i+1} (-1)^{1+j} \Upsilon_j x_{i+1-j} = \sum_{j=0}^{p} A_j x_{i-j} + \sum_{j=0}^{q} B_j u_{i-j}. \tag{2.6}$$

Let $\bar{A}_j = (-1)^{1+j} \Upsilon_j$. Equation (2.6) can be rewritten as

$$x_{i+1} = (A_0 + \Upsilon_1) x_i + \sum_{j=1}^{p} A_j x_{i-j} + \sum_{j=2}^{i+1} \bar{A}_j x_{i+1-j} + \sum_{j=0}^{q} B_j u_{i-j},$$

then

$$x_{i+1} = \mathcal{A}_0 x_i + \sum_{j=1}^{p} A_j x_{i-j} + \sum_{j=1}^{i} \bar{A}_{1+j} x_{i-j} + \sum_{j=0}^{q} B_j u_{i-j}, \tag{2.7}$$

where $\mathcal{A}_0 = A_0 + \Upsilon_1$.

Remark 2.1. The matrices $\bar{A}_j$ ($j \in \mathbb{N}$) satisfy the relation

$$\bar{A}_{1+j} = \frac{1}{1+j} (j I_n - \bar{A}_1) \bar{A}_j.$$

Now, we define the positivity of system (2.2) using the following

Definition 2.2. System (2.2) is said to be positive if $x_i \in \mathbb{R}_+^n$ and $y_i \in \mathbb{R}_+^r$, $i \in \mathbb{N}$, for any initial states $x_{-j} \in \mathbb{R}_+^n$ ($j \in \sigma_0^{\max\{p,l\}}$), for any initial inputs $u_{-j} \in \mathbb{R}_+^m$ ($j \in \sigma_1^{\max\{q,v\}}$) and all inputs $u_i \in \mathbb{R}_+^m$, $i \in \mathbb{N}$.

Definition 2.3. A matrix $A = (a_{ij})$ in $\mathbb{R}^{n \times m}$ is said to be nonnegative, and denoted by $A \in \mathbb{R}_+^{n \times m}$, if all of its elements are nonnegative, i.e., $a_{ij} \geq 0$ for all $i \in \sigma_1^n$, $j \in \sigma_1^m$.

Since $0 < \alpha_j < 1$ $(j \in \sigma_1^n)$, then $(-1)^{1+i}\begin{pmatrix}\alpha_j\\ i\end{pmatrix} > 0$ for all $i \in \mathbb{N}_+$ (see [18]), and consequently the diagonal matrix $\bar{A}_i$ $(i \in \mathbb{N}_+)$ is composed of positive diagonal elements. Hence, necessary and sufficient conditions for positivity of the retarded system (2.2) are given by the following theorem.

Theorem 2.1. *System (2.2) is positive if and only if*

$$\mathcal{A}_0 \in \mathbb{R}_+^{n\times n},\ A_j \in \mathbb{R}_+^{n\times n}\ \ (j \in \sigma_1^p),\ B_j \in \mathbb{R}_+^{n\times m}\ \ (j \in \sigma_0^q), \tag{2.8}$$

$$C_j \in \mathbb{R}_+^{r\times n}\ \ \left(j \in \sigma_0^l\right),\ D_j \in \mathbb{R}_+^{r\times m}\ \ (j \in \sigma_0^v). \tag{2.9}$$

Proof. (Sufficiency) If the condition (2.8) is satisfied, then from (2.7), for $i = 0$, we have

$$x_1 = \mathcal{A}_0 x_0 + \sum_{j=1}^{p} A_j x_{-j} + \sum_{j=0}^{q} B_j u_{-j} \in \mathbb{R}_+^n,$$

since $x_{-j} \in \mathbb{R}_+^n$ $(j \in \sigma_0^p)$ and $u_{-j} \in \mathbb{R}_+^m$ $(j \in \sigma_0^q)$. Assume that $x_k \in \mathbb{R}_+^n$ for $k \in \sigma_1^i$. From (2.7) we have

$$x_{i+1} = \mathcal{A}_0 x_i + \sum_{j=1}^{p} A_j x_{i-j} + \sum_{j=1}^{i} \bar{A}_{1+j} x_{i-j} + \sum_{j=0}^{q} B_j u_{i-j} \in \mathbb{R}_+^n,$$

since (2.8) holds, $x_{i-j} \in \mathbb{R}_+^n$ $(j \in \sigma_0^p)$, $u_{-j} \in \mathbb{R}_+^m$ $(j \in \sigma_1^q)$ and $u_i \in \mathbb{R}_+^m$, $i \in \mathbb{N}$. Hence $x_i \in \mathbb{R}_+^n$ for any $i \in \mathbb{N}$. Consequently, if the condition (2.9) is satisfied, we get that $y_i \in \mathbb{R}_+^r$ for every $i \in \mathbb{N}$ since $x_{-j} \in \mathbb{R}_+^n$ $\left(j \in \sigma_0^l\right)$, $u_{-j} \in \mathbb{R}_+^m$ $(j \in \sigma_1^v)$ and $u_i \in \mathbb{R}_+^m$, $i \in \mathbb{N}$.

(Necessity) Assuming that the system (2.2) is positive, let $u_{-j} = 0$ for $j \in \sigma_0^{\max\{q,v\}}$. Then for $i = 0$, we have

$$x_1 = \mathcal{A}_0 x_0 + \sum_{j=1}^{p} A_j x_{-j} = A\bar{x}_0 \in \mathbb{R}_+^n \ \text{ and } y_0 = \sum_{j=0}^{l} C_j x_{-j} = C\bar{x}_1 \in \mathbb{R}_+^r,$$

with

$$A = \begin{bmatrix}\mathcal{A}_0 \ A_1 \ \cdots \ A_p\end{bmatrix} \in \mathbb{R}^{n\times n(p+1)},\ C = \begin{bmatrix}C_0 \ C_1 \ \cdots \ C_l\end{bmatrix} \in \mathbb{R}^{r\times n(l+1)},$$

and

$$\bar{x}_0 = \begin{bmatrix}x_0 \ x_{-1} \ \cdots \ x_{-p}\end{bmatrix}^T \in \mathbb{R}^{n(p+1)},\ \bar{x}_1 = \begin{bmatrix}x_0 \ x_{-1} \ \cdots \ x_{-l}\end{bmatrix}^T \in \mathbb{R}^{n(l+1)}.$$

Hence $A \in \mathbb{R}_+^{n\times n(p+1)}$, i.e., $\mathcal{A}_0 \in \mathbb{R}_+^{n\times n}$, $A_j \in \mathbb{R}_+^{n\times n}$ $(j \in \sigma_1^p)$ and $C \in \mathbb{R}_+^{r\times n(l+1)}$, i.e., $C_j \in \mathbb{R}_+^{r\times n}$ $\left(j \in \sigma_0^l\right)$ since $\bar{x}_0 \in \mathbb{R}_+^{n(p+1)}$ and $\bar{x}_1 \in \mathbb{R}_+^{n(l+1)}$ are arbitrary.

Now, assume that $x_{-j} = 0$ for $j \in \sigma_0^{\max\{p,l\}}$, and for $i = 0$, we obtain

$$x_1 = \sum_{j=0}^{q} B_j u_{-j} = B\bar{u}_0 \in \mathbb{R}_+^n \ \text{ and } y_0 = \sum_{j=0}^{v} D_j u_{-j} = D\bar{u}_1 \in \mathbb{R}_+^r,$$

with

$$B = \begin{bmatrix} B_0 \; B_1 \; \cdots \; B_q \end{bmatrix} \in \mathbb{R}^{n\times m(q+1)},\; D = \begin{bmatrix} D_0 \; D_1 \; \cdots \; D_v \end{bmatrix} \in \mathbb{R}^{r\times m(v+1)},$$

and

$$\bar{u}_0 = \begin{bmatrix} u_0 \; u_{-1} \; \cdots \; u_{-q} \end{bmatrix}^T \in \mathbb{R}^{m(q+1)},\; \bar{u}_1 = \begin{bmatrix} u_0 \; u_{-1} \; \cdots \; u_{-v} \end{bmatrix}^T \in \mathbb{R}^{m(v+1)}.$$

Which implies that $B \in \mathbb{R}_+^{n\times m(q+1)}$, i.e., $B_j \in \mathbb{R}_+^{n\times m}$ $(j \in \sigma_0^q)$ and $D \in \mathbb{R}_+^{r\times m(v+1)}$, i.e., $D_j \in \mathbb{R}_+^{r\times m}$ $(j \in \sigma_0^v)$ since $\bar{u}_0 \in \mathbb{R}_+^{m(q+1)}$ and $\bar{u}_1 \in \mathbb{R}_+^{m(v+1)}$ are arbitrary. This completes the proof. $\square$

In all the sequel, we assume that the system (2.2) is positive.

3. Output Controllability

Definition 3.1. System (2.2) is said to be output controllable in N steps if, for any desired final output $y_d \in \mathbb{R}_+^r$, any initial state sequence $x_{-j} \in \mathbb{R}_+^n$ $(j \in \sigma_0^{\max\{p,l\}})$ and any initial input sequence $u_{-j} \in \mathbb{R}_+^m$ $(j \in \sigma_1^{\max\{q,v\}})$, there exist an input sequence $u_i \in \mathbb{R}_+^m$, $i \in \sigma_0^{N-1}$, which steers the output of the system from x_{-j} to y_d, i.e., $y_{N-1} = y_d$. We say that system (2.2) is output reachable in N steps (null output controllable in N steps) if it is output controllable in N steps from zero initial conditions (if it is output controllable in N steps to zero final output).

Definition 3.2. System (2.2) is said to be output controllable if, for any desired final output $y_d \in \mathbb{R}_+^r$, any initial state sequence $x_{-j} \in \mathbb{R}_+^n$ $(j \in \sigma_0^{\max\{p,l\}})$ and any initial input sequence $u_{-j} \in \mathbb{R}_+^m$ $(j \in \sigma_1^{\max\{q,v\}})$, there exist a positive integer N and an input sequence $u_i \in \mathbb{R}_+^m$, $i \in \sigma_0^{N-1}$ such that the output of the system is driven from x_{-j} to y_d, i.e., $y_{N-1} = y_d$. We say that system (2.2) is output reachable (null output controllable) if it is output controllable from zero initial conditions (if it is output controllable to zero final output).

Clearly, if a system is output controllable then it is output reachable and null output controllable. The aim of this section is to establish a sufficient condition for the output controllability of system (2.2).

Similarly to the case of classical discrete system with delays (see [3]), we use the $\mathcal{Z}$-transform method to show that the general formula of the state of system (2.2) has the form

$$\begin{aligned} x_i = G_i x_0 &+ \sum_{j=1}^{p}\sum_{k=1}^{p-j+1} G_{i-k}A_{k-1+j}x_{-j} + \sum_{j=1}^{q}\sum_{k=1}^{q-j+1} G_{i-k}B_{k-1+j}u_{-j} \\ &+ \sum_{j=0}^{i-1}\sum_{k=0}^{q} G_{i-1-j-k}B_k u_j, \quad i \in \mathbb{N}, \end{aligned} \tag{3.1}$$

where the transition matrix $G_i \in \mathbb{R}^{n\times n}$ $(i \in \mathbb{N})$ is determined by the recurrence relation

$$G_i = \begin{cases} I_n & \text{for } \; i = 0, \\ \mathcal{A}_0 G_{i-1} + \sum_{k=1}^{p} A_k G_{i-1-k} + \sum_{k=1}^{i-1} \bar{A}_{k+1} G_{i-1-k} & \text{for } \; i \in \mathbb{N}_+, \end{cases}$$

with the assumption

$$G_i = 0 \quad \text{for} \quad i < 0.$$

For any integer i, we put $H_i^0 = G_i$ and, for all $i \in \mathbb{N}_+$, we pose

$$\begin{cases} H_i^j = \sum\limits_{k=1}^{p-j+1} H_{i-k}^0 A_{k-1+j}, & j \in \sigma_1^p, \\ L_i^j = \sum\limits_{k=1}^{q-j+1} H_{i-k}^0 B_{k-1+j}, & j \in \sigma_1^q, \end{cases} \tag{3.2}$$

with $H_i^j = 0$ $(j \in \sigma_1^p)$ and $L_i^j = 0$ $(j \in \sigma_1^q)$ for $i \leq 0$.

Remark 3.1. For all $i \in \mathbb{N}$, we have

$$\begin{cases} H_{i+1}^j = H_i^{j+1} + H_i^0 A_j, & j \in \sigma_1^{p-1}, \\ H_{i+1}^p = H_i^0 A_p, \end{cases}$$

and

$$\begin{cases} L_{i+1}^j = L_i^{j+1} + H_i^0 B_j, & j \in \sigma_1^{q-1}, \\ L_{i+1}^q = H_i^0 B_q. \end{cases}$$

Moreover, for $i \in \mathbb{N}$, we put

$$K_i = \sum_{k=0}^{q} H_{i-k}^0 B_k, \tag{3.3}$$

with $K_i = 0$ for $i < 0$.

Clearly by (3.2) and (3.2), the solution (3.1) is given by the following new formula

$$x_i = H_i^0 x_0 + \sum_{j=1}^{p} H_i^j x_{-j} + \sum_{j=1}^{q} L_i^j u_{-j} + \sum_{j=0}^{i-1} K_{i-1-j} u_j, \quad i \in \mathbb{N}.$$

In the remainder of this section, and without loss of generality, we assume that $p \geq l$, $q \geq v$ and $l = v$. Indeed, for example, if $l > p$ we can set $A_j = 0$ for $j \in \sigma_{p+1}^l$.

Now, we introduce a matrices sequence as follows

$$\begin{aligned} \mathcal{H}_i^j &= \sum_{k=0}^{l} C_k H_{i-k}^j, \quad j \in \sigma_0^p, \quad i \in \mathbb{N}, \\ \tilde{\mathcal{H}}_i^j &= \mathcal{H}_i^j + C_{i+j}, \quad j \in \sigma_1^{l-i}, \quad i \in \sigma_0^{l-1}, \\ \mathcal{L}_i^j &= \sum_{k=0}^{l} C_k L_{i-k}^j, \quad j \in \sigma_1^q, \quad i \in \mathbb{N}, \\ \tilde{\mathcal{L}}_i^j &= \mathcal{L}_i^j + D_{i+j}, \quad j \in \sigma_1^{l-i}, \quad i \in \sigma_0^{l-1}, \\ \mathcal{K}_i &= \sum_{k=0}^{l} C_k K_{i-k}, \quad i \in \mathbb{N}, \\ \tilde{\mathcal{K}}_i &= \mathcal{K}_i + D_{i+1}, \quad i \in \sigma_0^{l-1}. \end{aligned}$$

For $0 \leq i < l$, we have

$$y_i = \sum_{k=0}^{i} C_k x_{i-k} + \sum_{k=i+1}^{l} C_k x_{i-k} + \sum_{k=0}^{i} D_k u_{i-k} + \sum_{k=i+1}^{l} D_k u_{i-k}$$

$$
\begin{aligned}
&= \sum_{k=0}^{i} C_k H_{i-k}^0 x_0 + \sum_{k=0}^{i} C_k \sum_{j=1}^{p} H_{i-k}^j x_{-j} + \sum_{k=0}^{i} C_k \sum_{j=1}^{q} L_{i-k}^j u_{-j} + \sum_{k=i+1}^{l} C_k x_{i-k} \\
&\quad + \sum_{k=i+1}^{l} D_k u_{i-k} + \sum_{k=0}^{i} C_k \sum_{j=0}^{i-k-1} K_{i-k-1-j} u_j + \sum_{k=0}^{i} D_k u_{i-k} \\
&= \sum_{k=0}^{l} C_k H_{i-k}^0 x_0 + \sum_{k=0}^{l} C_k \sum_{j=1}^{p} H_{i-k}^j x_{-j} + \sum_{k=0}^{l} C_k \sum_{j=1}^{q} L_{i-k}^j u_{-j} + \sum_{k=i+1}^{l} C_k x_{i-k} \\
&\quad + \sum_{k=i+1}^{l} D_k u_{i-k} + \sum_{k=0}^{i-1} C_k \sum_{j=0}^{i-k-1} K_{i-k-1-j} u_j + \sum_{k=0}^{i} D_k u_{i-k} \\
&= \left(\sum_{k=0}^{l} C_k H_{i-k}^0 \right) x_0 + \sum_{j=1}^{p} \left(\sum_{k=0}^{l} C_k H_{i-k}^j \right) x_{-j} + \sum_{j=1}^{q} \left(\sum_{k=0}^{l} C_k L_{i-k}^j \right) u_{-j} \\
&\quad + \sum_{j=i+1}^{l} C_j x_{i-j} + \sum_{j=i+1}^{l} D_j u_{i-j} + \sum_{j=0}^{i-1} \left(\sum_{k=0}^{i-j-1} C_k K_{i-j-1-k} \right) u_j + \sum_{j=0}^{i-1} D_{i-j} u_j + D_0 u_i \\
&= \mathcal{H}_i^0 x_0 + \sum_{j=1}^{p} \mathcal{H}_i^j x_{-j} + \sum_{j=1}^{q} \mathcal{L}_i^j u_{-j} + \sum_{j=1}^{l-i} C_{i+j} x_{-j} + \sum_{j=1}^{l-i} D_{i+j} u_{-j} \\
&\quad + \sum_{j=0}^{i-1} \left(\mathcal{K}_{i-j-1} + D_{i-j} \right) u_j + D_0 u_i \\
&= \mathcal{H}_i^0 x_0 + \sum_{j=1}^{l-i} (\mathcal{H}_i^j + C_{i+j}) x_{-j} + \sum_{j=l-i+1}^{p} \mathcal{H}_i^j x_{-j} + \sum_{j=1}^{l-i} (\mathcal{L}_i^j + D_{i+j}) u_{-j} \\
&\quad + \sum_{j=l-i+1}^{q} \mathcal{L}_i^j u_{-j} + \sum_{j=0}^{i-1} \tilde{\mathcal{K}}_{i-j-1} u_j + D_0 u_i \\
&= \mathcal{H}_i^0 x_0 + \sum_{j=1}^{l-i} \tilde{\mathcal{H}}_i^j x_{-j} + \sum_{j=l-i+1}^{p} \mathcal{H}_i^j x_{-j} + \sum_{j=1}^{l-i} \tilde{\mathcal{L}}_i^j u_{-j} + \sum_{j=l-i+1}^{q} \mathcal{L}_i^j u_{-j} \\
&\quad + \sum_{j=0}^{i-1} \tilde{\mathcal{K}}_{i-j-1} u_j + D_0 u_i.
\end{aligned}
$$

Hence

$$
y_i = \mathcal{Q}_{i+1} \tilde{x}_0 + \mathcal{R}_{i+1} u_0^{i+1}, \tag{3.4}
$$

with

$$
\mathcal{Q}_{i+1} = \left[\mathcal{M}_{i+1} \; \mathcal{O}_{i+1} \right] \in \mathbb{R}_+^{r \times (n(p+1)+mq)},
$$

where

$$
\mathcal{M}_{i+1} = \left[\mathcal{H}_i^0 \; \tilde{\mathcal{H}}_i^1 \; \cdots \; \tilde{\mathcal{H}}_i^{l-i} \; \mathcal{H}_i^{l-i+1} \; \cdots \; \mathcal{H}_i^p \right] \in \mathbb{R}_+^{r \times n(p+1)},
$$

and

$$
\mathcal{O}_{i+1} = \left[\tilde{\mathcal{L}}_i^1 \; \cdots \; \tilde{\mathcal{L}}_i^{l-i} \; \mathcal{L}_i^{l-i+1} \; \cdots \; \mathcal{L}_i^q \right] \in \mathbb{R}_+^{r \times mq},
$$

$$\tilde{x}_0 = \begin{bmatrix} x_0 \\ x_{-1} \\ \vdots \\ x_{-p} \\ u_{-1} \\ \vdots \\ u_{-q} \end{bmatrix} \in \mathbb{R}_+^{n(p+1)+mq},$$

$$\mathcal{R}_{i+1} = \left[D_0 \ \tilde{\mathcal{K}}_0 \ \tilde{\mathcal{K}}_1 \ \cdots \ \tilde{\mathcal{K}}_{i-2} \ \tilde{\mathcal{K}}_{i-1} \right] \in \mathbb{R}_+^{r\times(i+1)m},$$

and

$$u_0^{i+1} = \begin{bmatrix} u_i \\ u_{i-1} \\ \vdots \\ u_0 \end{bmatrix} \in \mathbb{R}_+^{(i+1)m}.$$

For $i \geq l$, we have

$$\begin{aligned}
y_i &= \sum_{k=0}^{l} C_k H_{i-k}^0 x_0 + \sum_{k=0}^{l} C_k \sum_{j=1}^{p} H_{i-k}^j x_{-j} + \sum_{k=0}^{l} C_k \sum_{j=1}^{q} L_{i-k}^j u_{-j} \\
&\quad + \sum_{k=0}^{l} C_k \sum_{j=0}^{i-k-1} K_{i-k-1-j} u_j + \sum_{k=0}^{l} D_k u_{i-k} \\
&= \left(\sum_{k=0}^{l} C_k H_{i-k}^0 \right) x_0 + \sum_{j=1}^{p} \left(\sum_{k=0}^{l} C_k H_{i-k}^j \right) x_{-j} + \sum_{j=1}^{q} \left(\sum_{k=0}^{l} C_k L_{i-k}^j \right) u_{-j} \\
&\quad + \sum_{j=0}^{i-l-1} \left(\sum_{k=0}^{l} C_k K_{i-j-1-k} \right) u_j + \sum_{j=i-l}^{i-1} \left(\sum_{k=0}^{i-j-1} C_k K_{i-j-1-k} \right) u_j + \sum_{j=0}^{l} D_j u_{i-j} \\
&= \left(\sum_{k=0}^{l} C_k H_{i-k}^0 \right) x_0 + \sum_{j=1}^{p} \left(\sum_{k=0}^{l} C_k H_{i-k}^j \right) x_{-j} + \sum_{j=1}^{q} \left(\sum_{k=0}^{l} C_k L_{i-k}^j \right) u_{-j} \\
&\quad + \sum_{j=0}^{i-1} \left(\sum_{k=0}^{l} C_k K_{i-j-1-k} \right) u_j + \sum_{j=0}^{l} D_j u_{i-j} \\
&= \mathcal{H}_i^0 x_0 + \sum_{j=1}^{p} \mathcal{H}_i^j x_{-j} + \sum_{j=1}^{q} \mathcal{L}_i^j u_{-j} + \sum_{j=0}^{i-1} \mathcal{K}_{i-j-1} u_j + \sum_{j=0}^{l} D_j u_{i-j} \\
&= \mathcal{H}_i^0 x_0 + \sum_{j=1}^{p} \mathcal{H}_i^j x_{-j} + \sum_{j=1}^{q} \mathcal{L}_i^j u_{-j} + \sum_{j=0}^{i-l-1} \mathcal{K}_{i-j-1} u_j + \sum_{j=i-l}^{i-1} \left(\mathcal{K}_{i-j-1} + D_{i-j} \right) u_j + D_0 u_i \\
&= \mathcal{H}_i^0 x_0 + \sum_{j=1}^{p} \mathcal{H}_i^j x_{-j} + \sum_{j=1}^{p} \mathcal{L}_i^j u_{-j} + \sum_{j=0}^{i-l-1} \mathcal{K}_{i-j-1} u_j + \sum_{j=i-l}^{i-1} \tilde{\mathcal{K}}_{i-j-1} u_j + D_0 u_i.
\end{aligned}$$

Then, we get

$$y_i = \mathcal{Q}_{i+1}\tilde{x}_0 + \mathcal{R}_{i+1}u_0^{i+1}, \tag{3.5}$$

with

$$\mathcal{Q}_{i+1} = \begin{bmatrix} \mathcal{H}_i^0 \ \mathcal{H}_i^1 \ \cdots \ \mathcal{H}_i^p \ \mathcal{L}_i^1 \ \cdots \ \mathcal{L}_i^q \end{bmatrix} \in \mathbb{R}_+^{r\times(n(p+1)+mq)},$$

and

$$\mathcal{R}_{i+1} = \begin{bmatrix} D_0 \ \tilde{\mathcal{K}}_0 \cdots \tilde{\mathcal{K}}_{l-1} \ \mathcal{K}_l \ \cdots \ \mathcal{K}_{i-2} \ \mathcal{K}_{i-1} \end{bmatrix} \in \mathbb{R}_+^{r\times(i+1)m}.$$

A column with exactly one of its components is positive and all the others are zero is called monomial or i-monomial if the positive component is in the ith position. A monomial matrix consists of linearly independent monomial columns.

Lemma 3.1 ([13]). *Let $A \in \mathbb{R}_+^{n\times n}$. Then A^{-1} exists and is nonnegative if and only if A is a monomial matrix. Furthermore, A^{-1} is equal to the transpose matrix A^T in which every nonzero element is replaced by its inverse.*

Then, a necessary and sufficient condition for the output reachability of system (2.2) is given by

Theorem 3.1. *The system (2.2) is output reachable if and only if for some $N \in \mathbb{N}_+$, the output reachability matrix $\mathcal{R}_N$ contains a $r \times r$ monomial submatrix (so $r \leq Nm$).*

Proof. (Sufficiency) Let $y_d \in \mathbb{R}_+^r$ be the desired output to be reached. From (3.4) or (3.5), we have

$$y_{N-1} = \mathcal{Q}_N\tilde{x}_0 + \mathcal{R}_N u_0^N.$$

With $\tilde{x}_0 = 0$, this gives

$$y_{N-1} = \mathcal{R}_N u_0^N.$$

The matrix $\mathcal{R}_N$ includes a $r \times r$ monomial submatrix, and without loss of generality, we can assume that

$$\mathcal{R}_N = \begin{bmatrix} R_1 \ R_2 \end{bmatrix}$$

such that $R_1 \in \mathbb{R}_+^{r\times r}$ is a monomial matrix and $R_2 \in \mathbb{R}_+^{r\times(Nm-r)}$. Hence by Lemma 3.1, we have $R_1^{-1} \in \mathbb{R}_+^{r\times r}$. Thus for

$$u_0^N = \begin{bmatrix} R_1^{-1}y_d \\ 0 \end{bmatrix} \in \mathbb{R}_+^{Nm},$$

we get

$$y_{N-1} = \begin{bmatrix} R_1 \ R_2 \end{bmatrix} \begin{bmatrix} R_1^{-1}y_d \\ 0 \end{bmatrix} = y_d,$$

i.e., the system (2.2) is output reachable.

(Necessity) If the system (2.2) is output reachable, then, in particular for $y_d = e_k$ with e_k being the kth column of I_r, there exists $N_k \in \mathbb{N}_+$ and an input $u_0^{N_k} \in \mathbb{R}_+^{N_k m}$ such that

$$e_k = \mathcal{R}_{N_k} u_0^{N_k},$$

with $\mathcal{R}_{N_k} = (r_{ij})_{i\in\sigma_1^r, j\in\sigma_1^{N_k m}}$ and $u_0^{N_k} = (\mu_j)_{j\in\sigma_1^{N_k m}}$. Hence

$$\sum_{j=1}^{N_k m} r_{kj}\mu_j = 1, \tag{3.6}$$

and for $i \in \sigma_1^r$ with $i \neq k$, we have

$$\sum_{j=1}^{N_k m} r_{ij}\mu_j = 0. \tag{3.7}$$

So by (3.6), there exists $s \in \sigma_1^{N_k m}$ such that $\mu_s \neq 0$, and consequently by the equation (3.7) we have $r_{is} = 0$ for all $i \in \sigma_1^r$ with $i \neq k$. Hence, if $r_{ks} \neq 0$, then the sth column of $\mathcal{R}_{N_k}$ is monomial. If $r_{ks} = 0$, then the sth column of $\mathcal{R}_{N_k}$ is null, which implies that

$$\begin{cases} \sum\limits_{j=1}^{N_k m} r_{kj}\mu_j = 1, \quad j \neq s, \\ \sum\limits_{j=1}^{N_k m} r_{ij}\mu_j = 0, \quad i \in \sigma_1^r \text{ with } i \neq k. \end{cases}$$

The same reasoning gives the existence of a k-monomial column or another null column of $\mathcal{R}_{N_k}$. Since the columns of $\mathcal{R}_{N_k}$ are not all null, then $\mathcal{R}_{N_k}$ has at least one k-monomial column. Let $N = \max\limits_{k\in\sigma_1^r} N_k$. Since every column vector of $\mathcal{R}_{N_k}$ is also a column vector of $\mathcal{R}_N$, then $\mathcal{R}_N$ contains a $r \times r$ monomial submatrix. The theorem is proved. □

Corollary 3.1. *System (2.2) is output reachable in N steps if and only if the output reachability matrix $\mathcal{R}_N$ includes a monomial submatrix of order $r \times r$ $(r \leq Nm)$.*

Proof. It follows directly from the proof of Theorem 3.1. □

Definition 3.3. If the system (2.2) is output reachable, then the minimum positive integer N such that $\mathcal{R}_N$ contains a $r \times r$ monomial submatrix is said the output reachability index.

Example 3.1. Suppose that we are given the fractional system

$$\begin{cases} \Delta^{\Upsilon} x_{i+1} = \sum\limits_{j=0}^{p} A_j x_{i-j} + \sum\limits_{j=0}^{q} B_j u_{i-j}, \\ \qquad y_i = \sum\limits_{j=0}^{l} C_j x_{i-j} + \sum\limits_{j=0}^{v} D_j u_{i-j}, \quad i \in \mathbb{N}, \end{cases}$$

with $p = q = l = v = 2$ and the matrices

$$A_0 = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & -0.4 & 0 \\ 1 & 0 & 0.3 \end{bmatrix}, \ A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ A_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0.8 & 0 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$C_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}, \ C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.7 & 0.2 \end{bmatrix}, \ C_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0.5 & 0 & 0 \end{bmatrix},$$

$$D_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ D_1 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \ D_2 = \begin{bmatrix} 1 \\ 0.3 \end{bmatrix}.$$

If $\alpha_1 = 0.5$, $\alpha_2 = 0.4$, $\alpha_3 = 0.7$, then for $N = 5$ we obtain the output reachability matrix

$$\begin{aligned} \mathcal{R}_5 &= \begin{bmatrix} D_0 \ \tilde{\mathcal{K}}_0 \ \tilde{\mathcal{K}}_1 \ \mathcal{K}_2 \ \mathcal{K}_3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0.5 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0.9 & 1.14 \end{bmatrix}. \end{aligned}$$

The matrix $\mathcal{R}_5$ includes a monomial submatrix. Therefore, by Theorem 3.1, the system is output reachable, with 5 is the output reachability index.

Now, if we choose $\alpha_1 = 0.8$, $\alpha_2 = 0.7$, $\alpha_3 = 0.7$, then we obtain the following output reachability matrix

$$\mathcal{R}_5 = \begin{bmatrix} 1 & 0.5 & 1 & 1 & 0.3 \\ 0 & 1 & 1 & 1.18 & 3.339 \end{bmatrix}.$$

Hence the system is not output reachable in 5 steps.

Remark 3.2. System (2.2) is output reachable if for some $N \in \mathbb{N}_+$, the matrix $\mathcal{R}_N$ has full row rank, i.e., $\operatorname{rank} \mathcal{R}_N = r$ and

$$\mathcal{R}_N^T(\mathcal{R}_N\mathcal{R}_N^T)^{-1} \in \mathbb{R}_+^{Nm\times r}. \tag{3.8}$$

The nonnegative input sequence $u_i \in \mathbb{R}_+^m$, $i \in \sigma_0^{N-1}$ which steers the output of the system from $x_{-j} = 0$, $j \in \sigma_0^p$, to any desired output $y_d \in \mathbb{R}_+^r$, with $u_{-j} = 0$ for $j \in \sigma_1^q$, can be computed by the formula

$$u_0^N = \mathcal{R}_N^T(\mathcal{R}_N\mathcal{R}_N^T)^{-1}y_d. \tag{3.9}$$

Indeed, if $\operatorname{rank} \mathcal{R}_N = r$ then the matrix $\mathcal{R}_N\mathcal{R}_N^T$ is invertible and, if (3.8) holds and $y_d \in \mathbb{R}_+^r$, then $u_0^N \in \mathbb{R}_+^{Nm}$ with

$$y_{N-1} = \mathcal{R}_N u_0^N = \mathcal{R}_N\mathcal{R}_N^T(\mathcal{R}_N\mathcal{R}_N^T)^{-1}y_d = y_d.$$

Now, a characterization of the null output controllability of system (2.2) is given by the following theorem.

Theorem 3.2. *The system (2.2) is null output controllable if and only if for some $N \in \mathbb{N}_+$, the null output controllability matrix $\mathcal{Q}_N$ is null.*

Proof. (Sufficiency) From (3.4) or (3.5), at the step $i = N-1$, we have

$$y_{N-1} = \mathcal{Q}_N \tilde{x}_0 + \mathcal{R}_N u_0^N,$$

since $\mathcal{Q}_N = 0$, then for $u_0^N = 0$, we have $y_{N-1} = 0$, i.e., the system (2.2) is null output controllable.

(Necessity) If the system (2.2) is null output controllable, then, in particular for $\tilde{x}_0 = (1, 1, \ldots, 1)^T \in \mathbb{R}_+^{n(p+1)\ +mq}$, there exists $N \in \mathbb{N}_+$ and an input $u_0^N \in \mathbb{R}_+^{Nm}$ such that

$$\mathcal{Q}_N \tilde{x}_0 + \mathcal{R}_N u_0^N = 0,$$

with $\mathcal{Q}_N = (q_{ij})_{i \in \sigma_1^r, j \in \sigma_1^{n(p+1)+mq}}$, $\mathcal{R}_N = (r_{ij})_{i \in \sigma_1^r, j \in \sigma_1^{Nm}}$ and $u_0^N = (\mu_j)_{j \in \sigma_1^{Nm}}$. Thus $\sum\limits_{j=1}^{n(p+1)\ +mq} q_{ij} = -\sum\limits_{j=1}^{Nm} r_{ij}\mu_j$. But $r_{ij}\mu_j \geq 0$, implies that $\sum\limits_{j=1}^{n(p+1)\ +mq} q_{ij} \leq 0$. Hence $q_{ij} = 0$ for $i \in \sigma_1^r$, $j \in \sigma_1^{n(p+1)+mq}$, which finishes the proof. □

Corollary 3.2. *System (2.2) is null output controllable in N steps if and only if the null output controllability matrix $\mathcal{Q}_N$ is null.*

Proof. It follows directly from the proof of Theorem 3.2. □

Lemma 3.2. *For all $i \geq 2$, the diagonal elements of H_i^0 are nonzero.*

Proof. For all $i \geq 2$, we have

$$H_i^0 = \mathcal{A}_0 H_{i-1}^0 + \sum_{k=1}^{p} A_k H_{i-1-k}^0 + \sum_{k=1}^{i-2} \bar{A}_{k+1} H_{i-1-k}^0 + \bar{A}_i,$$

with $\bar{A}_i = (-1)^{1+i}\, \Upsilon_i$ whose diagonal elements are nonzero. □

Example 3.2. Consider the fractional system

$$\begin{cases} \Delta^{\Upsilon} x_{i+1} = \sum\limits_{j=0}^{p} A_j x_{i-j} + \sum\limits_{j=0}^{q} B_j u_{i-j}, \\ \qquad\quad y_i = \sum\limits_{j=0}^{l} C_j x_{i-j} + \sum\limits_{j=0}^{v} D_j u_{i-j}, \quad i \in \mathbb{N}, \end{cases}$$

with $p = l = v = 1$, $q = 2$ and the matrices

$$A_0 = \begin{bmatrix} -0.3 & 0 & 0 \\ 0 & -0.3 & 0 \\ 1 & 0.5 & -0.6 \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0.4 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

$$C_0 = 0,\ C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 0 & 0 \end{bmatrix},\ D_0 = \begin{bmatrix} 0 \\ 0.7 \end{bmatrix},\ D_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We first choose $\alpha_1 = 0.3$, $\alpha_2 = 0.3$, $\alpha_3 = 0.6$, then the system is null output controllable because the null output controllability matrix in tree steps

$$\mathcal{Q}_3 = \begin{bmatrix} \mathcal{H}_2^0 \ \mathcal{H}_2^1 \ \mathcal{L}_2^1 \ \mathcal{L}_2^2 \end{bmatrix}$$

is null.

Now, we choose $\alpha_1 = 0.6$, $\alpha_2 = 0.5$, $\alpha_3 = 0.7$, then $\mathcal{H}_2^0 \neq 0$, and hence $\mathcal{Q}_3 \neq 0$. On the other hand, according to Lemma 3.2, the diagonal elements of H_{N-2}^0 are nonzero for all $N \geq 4$, and since $C_1 \neq 0$, then $\mathcal{H}_{N-1}^0 = C_1 H_{N-2}^0 \neq 0$, thus $\mathcal{Q}_N \neq 0$, which implies that system is not null output controllable.

Remark 3.3. The system considered in the above example is null output controllable in tree steps for $\alpha_1 = \alpha_2 = 0.3$, $\alpha_3 = 0.6$ but it is not in $N \geq 4$ steps because $\mathcal{Q}_N \neq 0$. On the contrary, in the case of traditional discrete positive linear systems, if the system is null output controllable in N steps then it is null output controllable in every step $\tilde{N} \geq N$ [32].

In the rest of this section, we assume that the matrices C_j $(j \in \sigma_0^l)$ are not all null.

Lemma 3.3. *If system (2.2) is null output controllable in N steps, then $\mathcal{Q}_K \neq 0$ for all $K > N$.*

Proof. The system (2.2) is null output controllable in N steps, then $\mathcal{Q}_N = 0$. If $N \leq l$, we have

$$\tilde{\mathcal{H}}_{N-1}^j = \mathcal{H}_{N-1}^j + C_{N-1+j} = 0 \ \text{ for } \ j \in \sigma_1^{l-N+1},$$

which implies that $C_j = 0$ for $j \in \sigma_N^l$. On the other hand, we have

$$\mathcal{H}_{N-1}^0 = \sum_{j=0}^{N-1} C_j H_{N-1-j}^0 = C_{N-1} + C_{N-2} H_1^0 + C_{N-3} H_2^0 + \cdots + C_0 H_{N-1}^0 = 0,$$

since the diagonal elements of H_i^0 are nonzero for all $i \geq 2$, then $C_j = 0$ for $j \in \sigma_0^{N-1}$ with $j \neq N-2$ which ensures that $C_{N-2} \neq 0$. Thus $\mathcal{H}_{N-1+k}^0 = C_{N-2} H_{k+1}^0 \neq 0$ for all $k \geq 1$, then $\mathcal{Q}_{N+k} \neq 0$ for all $k \geq 1$. Similarly, we prove that $\mathcal{Q}_K \neq 0$ for all $K > N$ if $N \geq 1 + l$. □

Definition 3.4. If system (2.2) is null output controllable, then the positive integer N such that $\mathcal{Q}_N = 0$ is said the null output controllability index.

Theorem 3.3. *The system (2.2) is output controllable if it is output reachable and null output controllable with the null output controllability index is equal or greater than the output reachability index.*

Proof. Since the system (2.2) is output reachable, then according to Theorem 3.1, $\mathcal{R}_{N_1}$ includes a monomial submatrix of order $r \times r$, with N_1 is the output reachability index. On the other hand, the system (2.2) is null output controllable, hence according to Theorem 3.2, $\mathcal{Q}_{N_2} = 0$, with N_2 is the null output controllability index. Then the matrix

$$\mathcal{R}_{N_2} = \left[\mathcal{R}_{N_1} \ \tilde{\mathcal{R}} \right],$$

contains a monomial submatrix of order $r \times r$, with $\tilde{\mathcal{R}} \in \mathbb{R}_+^{r\times(N_2-N_1)m}$. Hence, by proof of Theorem 3.1, for any $y_d \in \mathbb{R}_+^r$, there exists a nonnegative input $u_0^{N_2} \in \mathbb{R}_+^{N_2 m}$ such that

$$y_d = \mathcal{R}_{N_2} u_0^{N_2}.$$

And since $\mathcal{Q}_{N_2} = 0$, then for any $\tilde{x}_0 \in \mathbb{R}_+^{n(p+1)+mq}$, we get that

$$y_{N_2-1} = \mathcal{Q}_{N_2}\tilde{x}_0 + \mathcal{R}_{N_2} u_0^{N_2} = y_d,$$

i.e., the system (2.2) is output controllable. □

Example 3.3. The system in Example 3.2 is output reachable for $\alpha_1 = \alpha_2 = 0.3$, $\alpha_3 = 0.6$, with 2 is the output reachability index since

$$\mathcal{R}_2 = \left[D_0 \ \tilde{\mathcal{K}}_0 \right] = \begin{bmatrix} 0 & 1 \\ 0.7 & 0 \end{bmatrix}$$

is monomial and null output controllable with 3 is the null output controllability index, so by Theorem 3.3, the system is output controllable.

4. Optimal output control

The optimal output control problem, considered in this section, consists of clarifying the conditions for the existence of an optimal control $u^* \in \mathbb{R}_+^{Nm}$ which will solve the problem

$$\mathcal{P} : \min_{u\in\mathcal{U}_+} J(u)$$

where $\mathcal{U}_+$ is the set of nonnegative controls which steer the output of system (2.2) from zero initial conditions (2.3) to a desired final output $y_d \in \mathbb{R}_+^r$ in N steps and the objective functional J is defined by

$$\begin{aligned} J: \quad & \mathcal{U}_+ \qquad \to \mathbb{R}_+ \\ & u = (u_i)_{i\in\sigma_0^{N-1}} \mapsto \sum_{i=0}^{N-1} u_i^T P u_i \end{aligned} \tag{4.1}$$

$P \in \mathbb{R}^{m\times m}$ is a symmetric positive definite matrix.

To solve the problem we define the matrix

$$\mathcal{M} = \mathcal{R}_N P_N^{-1} \mathcal{R}_N^T \in \mathbb{R}^{r\times r},$$

with

$$P_N = diag \left[\underbrace{P \ P \cdots \ P}_{N\text{-times}} \right] \in \mathbb{R}^{Nm\times Nm}.$$

Remark 4.1. If $\operatorname{rank} \mathcal{R}_N = r$, i.e., a necessary condition of output reachability in N steps of system (2.2), then the matrix $\mathcal{M}$ is invertible.

Now, we prove the following result.

Theorem 4.1. *If system (2.2) is output reachable in N steps and*

$$u^* := P_N^{-1}\mathcal{R}_N^T\mathcal{M}^{-1}y_d \in \mathbb{R}_+^{Nm}, \tag{4.2}$$

then u^* is the solution of the problem $\mathcal{P}$, where $y_d \in \mathbb{R}_+^r$ is the desired output.

Proof. We have

$$y_{N-1}(u^*) = \mathcal{R}_N u^* = \mathcal{R}_N P_N^{-1}\mathcal{R}_N^T\mathcal{M}^{-1}y_d = \mathcal{M}\mathcal{M}^{-1}y_d = y_d,$$

where $y_{N-1}(u^*)$ is the output of system (2.2) corresponding to the control u^* with the initial conditions (2.3) equal to zero.

If $u \in \mathcal{U}_+$, then we have

$$y_{N-1}(u) = y_{N-1}(u^*),$$

thus

$$\mathcal{R}_N(u - u^*) = 0,$$

which implies

$$(u - u^*)^T\mathcal{R}_N^T = 0.$$

Hence, by (4.2), we have

$$(u - u^*)^T P_N u^* = (u - u^*)^T\mathcal{R}_N^T\mathcal{M}^{-1}y_d = 0. \tag{4.3}$$

On the other hand, we have

$$\begin{aligned}(u - u^*)^T P_N(u - u^*) &= (u - u^*)^T P_N u - (u - u^*)^T P_N u^* \\ &= (u - u^*)^T P_N u = u^T P_N u - u^{*T} P_N u.\end{aligned}$$

According to (4.3), we obtain

$$u^{*T} P_N u^* = u^T P_N u^* = u^{*T} P_N u,$$

then

$$(u - u^*)^T P_N(u - u^*) = J(u) - J(u^*) \geq 0,$$

which ends the proof. □

Remark 4.2. Using (4.2) we establish that

$$\begin{aligned}J(u^*) &= u^{*T} P_N u^* \\ &= y_d^T(\mathcal{M}^{-1})^T\mathcal{R}_N(P_N^{-1})^T P_N P_N^{-1}\mathcal{R}_N^T\mathcal{M}^{-1}y_d \\ &= y_d^T(\mathcal{M}^{-1})^T\mathcal{R}_N(P_N^{-1})^T\mathcal{R}_N^T\mathcal{M}^{-1}y_d \\ &= y_d^T\mathcal{M}^{-1}\mathcal{R}_N P_N^{-1}\mathcal{R}_N^T\mathcal{M}^{-1}y_d \\ &= y_d^T\mathcal{M}^{-1}\mathcal{M}\mathcal{M}^{-1}y_d \\ &= y_d^T\mathcal{M}^{-1}y_d.\end{aligned}$$

Remark 4.3. If system (2.2) is output reachable in N steps and

$$P_N^{-1}\mathcal{R}_N^T\mathcal{M}^{-1} \in \mathbb{R}_+^{Nm\times r},$$

then the optimal output control u^* which steers the output of the system from zero initial conditions (2.3) to any desired output $y_d \in \mathbb{R}_+^r$ is given by

$$u^* = P_N^{-1}\mathcal{R}_N^T\mathcal{M}^{-1}y_d.$$

Remark 4.4. If $P = I_m$ and (3.8) holds, then the nonnegative input u_0^N computed from (3.9) is an optimal output control with

$$J\left(u_0^N\right) = \sum_{i=0}^{N-1} u_i^T u_i = y_d^T(\mathcal{R}_N\mathcal{R}_N^T)^{-1}y_d.$$

Example 4.1. Suppose that we are given the fractional system

$$\begin{cases} \Delta^\Upsilon x_{i+1} = \sum\limits_{j=0}^{p} A_j x_{i-j} + \sum\limits_{j=0}^{q} B_j u_{i-j}, \\ \qquad y_i = \sum\limits_{j=0}^{l} C_j x_{i-j} + \sum\limits_{j=0}^{v} D_j u_{i-j}, \quad i\in\mathbb{N}, \end{cases}$$

with $p = q = l = v = 2$ and the matrices

$$A_0 = \begin{bmatrix} -0.2 & 0 & 0 \\ 0 & -0.5 & 0 \\ 0 & 0 & 0.4 \end{bmatrix},\ A_1 = \begin{bmatrix} 0.5 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0.5 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix},$$

$$C_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ C_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ C_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

$$D_0 = 0,\ D_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \end{bmatrix},\ D_2 = \begin{bmatrix} 0 & 0 \\ 0.5 & 1 \end{bmatrix}.$$

In this example, for every $0<\alpha_i<1$, $i\in\sigma_1^3$, such that $\mathcal{A}_0 = A_0 + \operatorname{diag}\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3\end{bmatrix} \in \mathbb{R}_+^{3\times 3}$, we obtain the output reachability matrix in four steps

$$\begin{aligned} \mathcal{R}_4 &= \begin{bmatrix} D_0 & \tilde{\mathcal{K}}_0 & \tilde{\mathcal{K}}_1 & \mathcal{K}_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0.5 & 0.5 & 1 & 0 & 0 \end{bmatrix}. \end{aligned}$$

The matrix $\mathcal{R}_4$ includes a monomial submatrix. Then the system is output reachable in four steps.

Now, we shall find the optimal control u^* that transfers the output of the system from zero initial conditions to the final desired output $y_d = \begin{bmatrix} 1 \; 1 \end{bmatrix}^T$ in four steps and minimizes the functional (4.1) with

$$P = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

We have $P^{-1} = \begin{bmatrix} 2 \; 1 \\ 1 \; 1 \end{bmatrix}$ and $\mathcal{M} = \mathcal{R}_4 P_4^{-1} \mathcal{R}_4^T = \begin{bmatrix} 5 & 0 \\ 0 & 2.75 \end{bmatrix}$. Consequently

$$P_4^{-1} \mathcal{R}_4^T \mathcal{M}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0.6 & 0.4 \\ 0 & 0 & \frac{2}{11} & \frac{2}{11} & \frac{8}{11} & \frac{6}{11} & 0 & 0 \end{bmatrix}^T \in \mathbb{R}_+^{8\times 2}.$$

Hence the optimal control sequence has the form

$$u^* = \begin{bmatrix} u_0^* \\ u_1^* \\ u_2^* \\ u_3^* \end{bmatrix} = P_4^{-1} \mathcal{R}_4^T \mathcal{M}^{-1} y_d,$$

with

$$u_0^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; u_1^* = \begin{bmatrix} \frac{2}{11} \\ \frac{2}{11} \end{bmatrix}, \; u_2^* = \begin{bmatrix} \frac{8}{11} \\ \frac{6}{11} \end{bmatrix}, \; u_3^* = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix},$$

and

$$J(u^*) = 0.53.$$

5. Conclusion

The output controllability and optimal output control for fractional order positive discrete linear systems with delays in state, input and output has been formulated and solved. Necessary and sufficient conditions for the positivity have been established (Theorem 2.1). Criteria for the output reachability (Theorem 3.1) and null output controllability (Theorem 3.2) have been proved. Sufficient conditions for the output controllability have been established and proved (Theorem 3.3). Solution to the optimal output control problem has been given (Theorem 4.1). We verified the theoretical results stated in this paper with numerical examples.

We think that the techniques used in this paper can be useful to investigate the output controllability and optimal output control problems for different positive dynamical systems such as switched systems, fractional switched systems, stochastic systems, etc.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References

[1] A. Babiarz, A. Czornik and M. Niezabitowski, *Output controllability of discrete-time linear switched systems*, Nonlinear Analysis: Hybrid Systems, 2016, 21, 1–10.

[2] L. Benvenuti and L. Farina, *Positive and compartmental systems*, IEEE Transactions on Automatic Control, 2002, 47(2), 370–373.

[3] M. Busłowicz, *On some properties of the solution of state equation of discrete-time systems with delays*, Zesz. Nauk. Polit. Bial., Elektrotechnika, 1983, 1, 17–29 (in Polish).

[4] M. Busłowicz, *Controllability, reachability and minimum energy control of fractional discrete-time linear systems with multiple delays in state*, Bulletin of the Polish Academy of Sciences: Technical Sciences, 2014, 62(2), 233–239.

[5] C. T. Chen, *Introduction to Linear System Theory*, Holt, Rinehart and Winston Inc., New York, 1970.

[6] L. Deng, S. Fu, Y. Li and P. Zhu, *Robust output controllability analysis and control design for incomplete boolean networks with disturbance inputs*, Mathematical Problems in Engineering, 2018. DOI: 10.1155/2018/2940105.

[7] A. Dzieliński, D. Sierociuk and G. Sarwas, *Some applications of fractional order calculus*, Bulletin of the Polish Academy of Sciences: Technical Sciences, 2010, 58(4), 583–592.

[8] J. Eden, Y. Tan, D. Lau and D. Oetomo, *On the positive output controllability of linear time invariant systems*, Automatica, 2016, 71, 202–209.

[9] L. Farina and S. Rinaldi, *Positive Linear Systems: Theory and Applications*, J. Wiley & Sons, New York, 2000.

[10] M. I. García-Planas and J.L. Domínguez-García, *Alternative tests for functional and pointwise output-controllability of linear time-invariant systems*, Systems & Control Letters, 2013, 62(5), 382–387.

[11] V. Glizer, *Euclidean space output controllability of singularly perturbed systems with small state delays*, Journal of Applied Mathematics and Computing, 2018, 57(1-2), 1–38.

[12] S. Guermach, S. Djennoune and M. Bettayeb, *Controllability and observability of linear discrete-time fractional-order systems*, International Journal of Applied Mathematics and Computer Science, 2008, 18(2), 213–222.

[13] T. Kaczorek, *Positive 1D and 2D Systems*, Springer-Verlag, London, 2002.

[14] T. Kaczorek, *Output-reachability of positive linear discrete-time systems*, Proc. of 7th Int. Workshop "Computational Problems of Electrical Engineering" CPEE'06, Odessa, Ukraine, 2006, 64–68.

[15] T. Kaczorek, *Output-reachability of positive linear discrete-time systems with delays*, Archives of Control Sciences, 2006, 16(3), 247–255.

[16] T. Kaczorek, *Reachability and controllability to zero of positive fractional discrete-time systems*, Machine Intelligence and Robotic Control, 2007, 6(4), 139–143.

[17] T. Kaczorek, *Minimum energy control problem of positive fractional discrete-time systems*, In : Proc. 22ndEur. Conf. on Modeling and Simulation 1, 2008.

[18] T. Kaczorek, *Selected Problems of Fractional Systems Theory*, Springer-Verlag, Berlin, 2011.

[19] R. E. Kalman, *On the general theory of control systems*, Proc. of the First Intern, Congress on Automatic Control, Butterworth, London, 1960, 1, 481–493.

[20] J. Klamka, *Controllability of Dynamical Systems*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1991.

[21] J. Klamka, *Controllability and minimum energy control problem of fractional discrete-time systems*, In D. Baleanu, Z. B. Guvenc and J. A.T. Machado (Eds.) New Trends in Nanotechnology and Fractional Calculus Applications, Springer-Verlag, New York, 2010, 503–509.

[22] J. Klamka, *Controllability of dynamical systems. A survey*, Bulletin of the Polish Academy of Sciences: Technical Sciences, 2013, 61(2), 335–342.

[23] R. Kociszewski, *Minimum energy control of fractional discrete-time linear systems with delays in state and control*, In: R. Szewczyk, C. Zieliński, M. Kaliczyńska (Eds.) Recent Advances in Automation, Robotics and Measuring Techniques. Advances in Intelligent Systems and Computing, Springer, cham, 2014, 267, 127–136.

[24] R. Kociszewski, *Output controllability problem of fractional discrete-time systems with multiple delays in state and control vector*, In: R. Szewczyk, C. Zieliński, M. Kaliczyńska (Eds) Automation 2018. Advances in Intelligent Systems and Computing, Springer, Cham, 2018, 743, 203–212.

[25] D. Lau, D. Oetomo and S. K. Halgamuge, *Generalized modeling of multilink cable-driven manipulators with arbitrary routing using the cable-routing matrix*, IEEE Transactions on Robotics, 2013, 29(5), 1102–1113.

[26] M. Lhous, M. Rachik, J. Bouyaghroumni and A. Tridane, *On the output controllability of a class of discrete nonlinear distributed systems: a fixed point theorem approach*, International Journal of Dynamics and Control, 2018, 6(2), 768–777.

[27] X. Liu, L. Wang, W. Yu and S. Zhong, *Constrained control of positive discrete-time systems with delays*, IEEE Transactions on Circuits and System II, 2008, 55(2), 193–197.

[28] D. G. Luenberger, *Introduction to Dynamic Systems: Theory, Models and Applications*, John Wiley & Sons, New York, 1979.

[29] Y. Luo and Y.Q. Chen, *Fractional Order Motion Controls*, John Wiley & Sons Ltd, Chichester, 2013.

[30] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differenctial Equations*, John Wiley & Sons, New York, 1993.

[31] C. A. Monje, Y. Q. Chen, B. M. Vinagre, D. Xue and V. Feliu-Batlle, *Fractional-Order Systems and Controls Fundamentals and Applications*, Springer, London, 2010.

[32] M. Naim, F. Lahmidi, A. Namir and M. Rachik, *On the output controllability of positive discrete linear delay systems*, Abstract and Applied Analysis, 2017. DOI: 10.1155/2017/3651271.

[33] K. B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, New York, 1974.

[34] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[35] J. Shen and J. Lam, *On l_∞ and L_∞ gains for positive systems with bounded time-varying delays*, International Journal of Systems Science, 2015, 46(11), 1953–1960.

[36] W. Trzasko, *Reachability and controllability of positive fractional-order discrete-time systems*, Pomiary Automatyka Robotyka, 2013, 17(2), 365–370.