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ON THE F-EXPANDING OF HOMOCLINIC CLASSES

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Abstract We establish a closing property for thin trapped (see Definition 1.2) homoclinic classes. Taking advantage of this property, we prove that if a homoclinic class $H(f,p)$ admits a dominated splitting $T_{H(f,p)}M = E \oplus_{<} F$, where the subbundle E is thin trapped with $\dim E = \mathrm{Ind}(p)$ and all periodic points homoclinically related to p are uniformly F-expanding at the period (see Definition 1.1), then the subbundle F is uniformly expanding.

Keywords Homoclinic classes, dominated splitting, thin trapped, periodic points, uniformly expanding.

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1. Introduction

In dynamical systems, a basic research method is to split the tangent bundle of some invariant sets into invariant subbundle, such as dominated splitting, hyperbolic splitting and Oseledec splitting (see [15, 17]). People often hope to prove uniform hyperbolicity of a subbundle under dominated splitting. In our paper, uniform hyperbolicity of a subbundle on a homoclinic class can be obtained under weak conditions. Let f be a diffeomorphism on a compact manifold M with metric d. A point is called *hyperbolic periodic point*, if there exists a hyperbolic splitting on its periodic orbit. Given a hyperbolic periodic point p, denoted by $W^s(\mathcal{O}rb(f,p))$, $W^u(\mathcal{O}rb(f,p))$ the stable and unstable manifolds of the orbit of p, respectively. A hyperbolic periodic point q is said to be homoclinically related to p, denoted by $p \sim q$, if

$$W^s(\mathcal{O}rb(f,p)) \pitchfork W^u(\mathcal{O}rb(f,q)) \neq \emptyset, \quad W^s(\mathcal{O}rb(f,q)) \pitchfork W^u(\mathcal{O}rb(f,p)) \neq \emptyset.$$

The homoclinic class of a hyperbolic periodic point p is defined as

$$H(f,p) \triangleq \overline{\{q : q \in P(f), q \sim p\}},$$

where $P(f)$ denotes the set of all hyperbolic periodic points of f.

Homoclinic classes were introduced by Newhouse in [18] as a generalization of the basic sets in Smale Decomposition Theorem (see [22, Theorem 6.2]). For *Axiom A* diffeomorphisms, homoclinic classes are exactly the hyperbolic basic sets in Smale Decomposition Theorem. For generic C^1 diffeomorphisms, Carballo, Morales and Pacifico [3, Theorem A] proved that homoclinic classes are maximal transitive

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sets and pairwise disjoint. In general case, Díaz and Santoro [8, Theorem A] gave a example that distinct homoclinic classes may intersect each other. Therefore, homoclinic classes may fail to cover the entire closure of the set of periodic points. In general, the hyperbolicity of periodic points contained in a compact invariant set is not enough to get that of the invariant set. For example, Kupka-Smale Theorem [19, pp.91] affirms that every periodic orbit of C^r-generic ($r \geq 1$) diffeomorphisms is hyperbolic and that the stable and unstable manifolds of those periodic orbits are pairwise transverse. It means that homoclinic class is usually not hyperbolic although it contains many hyperbolic invariant sets: $H(f,p)$ can be accumulated by uniformly hyperbolic horseshoes (finite union of periodic orbits homoclinically related to p).

Given a compact invariant set Λ, for a hyperbolic splitting $T_\Lambda M = E \oplus_< F$, the subbundle E is uniformly contracting on Λ, the subbundle F is uniformly expanding on Λ. For a hyperbolic periodic point p, the index $\mathrm{Ind}(p)$ is defined as the dimension of the stable manifolds $W^s(\mathcal{O}rb(f,p))$ of the orbit of p.

Definition 1.1. Let $H \triangleq \{q_n\}_{n\in\mathbb{N}}$ be a sequence of hyperbolic periodic points of a diffeomorphism f. For the dominated splitting $T_H M = E \oplus_< F$, f is uniformly F-expanding at the period on H if there are two constants $C > 0$, $\lambda \in (0,1)$ such that for any $q_n \in H$, one has that

$$\prod_{j=1}^{\pi(q_n)} \left\| Df^{-1}\mid_{F_{f^j(q_n)}} \right\| \leq C\lambda^{\pi(q_n)},$$

where $\pi(q_n)$ is the period of the periodic point q_n. The subbundle F is said to be uniformly λ-expanding at the period on H.

For a dominated splitting $T_\Lambda M = E \oplus_< F$, a plaque family tangent to the subbundle E is a family of continuous maps $\mathcal{W}$ from the linear subbundle E to M satisfying that:

(i) for each $x \in \Lambda$, the map $\mathcal{W}_x : E_x \to M$ is a C^1-embedding that satisfies $\mathcal{W}_x(0) = x$ and whose image is tangent to E_x at x;

(ii) $(\mathcal{W}_x)_{x\in\Lambda}$ is a continuous family of C^1-embeddings.

Let $\mathcal{W}(x)$ be the image of embedding $\mathcal{W}_x$. Fix $\varepsilon > 0$, denoted by $\mathcal{W}_\varepsilon(x)$ the image which is centered at x with size 2ε. A plaque family $\mathcal{W}$ is locally invariant, if there is $\delta > 0$ such that for every $x \in \Lambda$, one has that $f \circ \mathcal{W}_x(B(0,\delta)) \subseteq \mathcal{W}(fx)$, where $B(0,\delta) \subseteq E_x$ stands for the ball centered at 0 with radius δ. Plaque Family Theorem [12, Theorem 5.5] shows that there always exists a locally invariant plaque family tangent to E. A plaque family is called *trapped*, if for every $x \in \Lambda$, one has that

$$f(\overline{\mathcal{W}(x)}) \subseteq \mathcal{W}(fx).$$

The notion *thin trapped* was introduced by Crovisier in [4] and [5]. In the research about C^1 diffeomorphisms far away from tangencies and heterodimensional cycles, Crovisier and Pujals [6, Section 3] studied the properties of *thin trapped* subbundles (also see [23, Introduction]).

Definition 1.2. Assume that Λ is a compact invariant set which admits a dominated splitting $T_\Lambda M = E \oplus_< F$, the subbundle E is thin trapped if for any neighborhood U of the section 0 in E, there is

(i) a continuous family $\{\varphi_x\}_{x\in\Lambda}$ of C^1-diffeomorphisms of the spaces $\{E_x\}_{x\in\Lambda}$ supported in U;

(ii) a constant $\delta > 0$ such that $f(\overline{\mathcal{W}_x \circ \varphi_x(B(0,\delta))}) \subseteq \mathcal{W}_{fx} \circ \varphi_{fx}(B(0,\delta))$ for any $x \in \Lambda$.

Bonatti, Gan and Yang [2, Main Theorem] gave a sufficient criterion for the hyperbolicity of a homoclinic class. They proved that if a homoclinic class $H(f,p)$ admits a partially hyperbolic splitting $T_{H(f,p)}M = E \oplus_< F$, where the subbundle E is uniformly contracting with dim$E$ = Ind(p) and all periodic points homoclinically related to p are uniformly F-expanding at the period, then $H(f,p)$ is a hyperbolic set. They also raised a question: Can we obtain the hyperbolicity of an invariant compact set by using the hyperbolicity of those periodic orbits in the set? In our paper, we consider a weak topological condition that the subbundle E is *thin trapped*. Then, we get some "hyperbolicity"of $H(f,p)$. The obvious difference is that one can not get any differential nature in thin trapped of subbundles E. Now, we introduce our main result.

Main Theorem. Let f be a diffeomorphism on a compact Riemannian manifold M, p be a hyperbolic periodic point. Assume that $H(f,p)$ admits a dominated splitting $T_{H(f,p)}M = E \oplus_< F$ and the subbundle E is thin trapped with dim$E$ = Ind (p). If f is uniformly F-expanding at the period on all periodic points homoclinically related to p, then the subbundle F is uniformly expanding on $H(f,p)$.

The *weak periodic points* mean that they have a Lyapunov exponent arbitrarily close to zero. Crovisier, Sambarino and Yang [7, Theorem 1.1 and Corollary 1.4] proved that there exist weak periodic points in some homoclinic classes of generic diffeomorphisms far from homoclinic tangencies. This Main Theorem also gives a criterion for getting weak periodic points in some special homoclinic classes.

Theorem 1.1. *Let p be a hyperbolic periodic point of a diffeomorphism f on a compact Riemannian manifold M. If the homoclinic class $H(f,p)$ satisfies that:*

- *the homoclinic class $H(f,p)$ admits a partially hyperbolic splitting $T_{H(f,p)}M = E^s \oplus E^c \oplus E^u$ with that the subbundle E^s is thin trapped, the subbundle E^u is uniformly expanding and dim$E^c = 1$;*
- *dimE^s =Ind (p) and $H(f,p)$ is not hyperbolic.*

then for every $\varepsilon > 0$, one can find a periodic point q homoclinically related to p such that

$$\frac{1}{\pi(q)} \log\left(\left\|Df^{\pi(q)}|_{E^c_q}\right\|\right) \leq \varepsilon.$$

In Main Theorem, we do not perturb diffeomorphism f and assume any robust property. Therefore, Liao's selecting lemma [14] and Mañé's ergodic closing lemma [16] do not imply directly Main Theorem. Compare with the work of Bonatti, Gan and Yang [2], that the subbundle E is thin trapped is weaker than that subbundle E is uniformly contracting.

To prove Main Theorem, we should consider the question: how to establish the relations between non-periodic points in the compact set and periodic points? The Anosov Closing Lemma [13, pp.269, Theorem 6.4.15] implies that for any point in a hyperbolic set whose orbit nearly returns to itself, there is a periodic orbit closely shadowing this nearly-returning orbit. Gan [10, Theorem 1.1] showed that any quasi-hyperbolic pseudoorbit with recurrence can be shadowed by a periodic orbit. But in our assumptions, the homoclinic class is not a hyperbolic set and since subbundles E is thin trapped, we can not get the quasi-hyperbolic pseudoorbit with respect to the dominated splitting on the homoclinic class. Even though the recurrent orbits (non-periodic) can be shadowed by periodic orbits, we also need to consider that: how to extend the property of periodic orbits to other non-periodic orbits. Our paper is organized as follows. In Scetion 2, we introduce "hyperbolic time"and find infinite periodic points with large period. In Scetion 3, for a thin trapped homoclinic class, we find a dense subset which has long stable and unstable manifolds. We establish a closing property for a thin trapped homoclinic class in Scetion 4. The proof of Main Theorem and Theorem 1.1 is finished in Scetion 5.

2. Hyperbolic time

Let Λ be a compact invariant set with a dominated splitting $T_\Lambda M = E \oplus_< F$. By [1, pp.289, Appendix B], one can fix an admissible compact neighborhood U of Λ such that the dominated spliting $E \oplus_< F$ can be extended in a unique way to the maximal invariant set $M(f,U) \triangleq \bigcap_{i\in\mathbb{Z}} f^i U \subseteq U$. For every $x \in M$ and $n \in \mathbb{N}^+$, an orbit segment (x,n) is defined as:

$$(x,n) \triangleq \{x, f(x), \cdots, f^{n-1}(x)\}.$$

Definition 2.1. Given $\lambda \in (0,1)$ and $n \in \mathbb{N}^+$, for $x \in M(f,U)$, an orbit segment (x,n) is a uniform λ-string, if

$$\prod_{j=k+1}^{n} \left\| Df^{-1}|_{F_{f^j(x)}} \right\| \le \lambda^{n-k}, \quad \text{for } k = 0, 1, \cdots, n-1.$$

This n is called λ-hyperbolic time of x.

Denoted by $HT(x,\lambda)$ the set of all λ-hyperbolic times of x and the n-th λ-hyperbolic time of x is denoted by $\phi_n(x,\lambda)$. For a periodic point p, denoted by $\Gamma_1(p,\lambda)$ the largest λ-hyperbolic time which is less than period $\pi(p)$ and the smallest λ-hyperbolic time which is larger than the period $\pi(p)$ is denoted by $\Gamma_2(p,\lambda)$. Lemma 2.1, given by Pliss ([21, The preceding Lemma of Theorem 4.1]), gives us a tool to find many hyperbolic times.

Lemma 2.1 (Pliss Lemma [21])**.** *Given constants A, $C_2 < C_1 < 0$ with $A \ge |C_2|$, there is $\theta = \theta(C_1, C_2) \in (0,1)$ such that for any real numbers $\{a_j\}_{j=1}^N$ with that:*

(i) $|a_j| \le A$, *for* $j = 1, 2, \cdots, N$;

(ii) $\sum_{j=1}^{N} a_j \le NC_2$,

there is an integer $l \ge \theta N$ and a sequence of numbers $1 \le n_1 < n_2 < \cdots < n_l \le N$ such that

$$\sum_{j=n+1}^{n_i} a_j \le (n_i - n)C_1, \quad \textit{for any} \;\; 0 \le n < n_i, \;\; i = 1, 2, \cdots, l.$$

Lemma 2.2. *Fix two numbers $\mu < \lambda < 0$, for any sequence of numbers $\{a_i\}_{i=1}^{\infty}$ with $\sum_{i=1}^{m} a_i \leq m\mu$ and $a_{i+m} = a_i$, for every $i \in \mathbb{N}$ and some integer m, there are $\theta = \theta(\mu,\lambda) \in (0,1)$ and $N \in \mathbb{N}$ such that for any $k \geq N$, there exists an integer $l \geq k\theta$ and a sequence of numbers $1 \leq n_1 < n_2 < \cdots < n_l \leq k$ such that*

$$\sum_{i=n+1}^{n_j} a_i \leq (n_i - n)\lambda, \quad \textit{for any} \ \ 0 \leq n < n_j, \ j = 1, 2, \cdots, l.$$

Proof. Since $a_{i+m} = a_i$, for $i \in \mathbb{N}$ and some integer m, one has that $|a_i| \leq A$, for $i \in \mathbb{N}$, where $A \triangleq \max\{|a_1|, |a_2|, \cdots, |a_m|\}$. Therefore,

$$r \triangleq \max\left\{a_1, a_1 + a_2, \cdots, \sum_{i=1}^{m} a_i\right\} \leq mA < +\infty.$$

For any $k \in \mathbb{N}$, there are integers $n, \ r_0$ such that $k = nm + r_0$, where $0 \leq r_0 < m$. Hence,

$$\sum_{j=1}^{k} a_j \leq nm\mu + r.$$

Therefore,

$$\limsup_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} a_i \leq \limsup_{k\to\infty} \frac{nm\mu}{k} + \limsup_{k\to\infty} \frac{r}{k} = \mu.$$

Thus, for $\eta = \dfrac{\mu + \lambda}{2}$, there is $N \in \mathbb{N}$ such that $\sum\limits_{i=1}^{k} a_i \leq n\eta$, for any $k \geq N$. By Lemma 2.1, there is $1 \leq n_1 < n_2 < \cdots < n_l \leq k$ with $l \geq k\theta$, such that

$$\sum_{i=n+1}^{n_j} a_i \leq (n_j - n)\ \lambda, \quad \text{for any} \ \ 0 \leq n < n_j, \ j = 1, 2, \cdots, l.$$

$\square$

Definition 2.2. Given $\lambda \in (0,1]$ and $x \in M(f,U)$, an orbit segment (x,n) is called a λ-obstruction orbit segment, if

$$\prod_{j=1}^{k} \left\| Df^{-1}|_{F_{f^j(x)}} \right\| \geq \lambda^k, \ \text{for} \ k = 1, \cdots, n.$$

The point x is a λ-obstruction point, if (x,n) is a λ-obstruction orbit segment for any $n \in \mathbb{N}^{+}$.

For two consecutive hyperbolic times, we can not give the estimation as the obstruction orbit segment. The following content is a simple fact that can help us deal with the obstruction orbit segment.

Lemma 2.3. *For any $r \in (0,1)$ and $\varepsilon > 0$, there exists $N_0 = N_0(r,\varepsilon)$ such that for some $n \geq N_0$, if (x,n) is an r-obstruction orbit segment, then*

$$d(x, \Lambda(r)) < \varepsilon,$$

where $\Lambda(r)$ is the set of all r-obstruction point.

Proof. Let $\Lambda_N(r)$ be the set of points such that the orbit segment (x, N) is an r-obstruction orbit segment. Then $\Lambda(r) = \bigcap_{N>0} \Lambda_N(r)$. By Definition 2.2, one has that $\Lambda_N(r) \supset \Lambda_{N+1}(r)$. Therefore, this is a decreasing intersection of compact sets. For any $\varepsilon > 0$, taking $N_0 = N_0(r, \varepsilon)$ such that $\Lambda_{N_0}(r)$ is contained in the ε-neighborhood of $\Lambda(r)$. Then, $d(y, \Lambda(r)) < \varepsilon$, for any $y \in \Lambda_{N_0}(r)$. Given $n \geq N_0$, if (x, n) is an r-obstruction orbit segment, by Definition 2.2, then (x, N_0) is an r-obstruction orbit segment. Thus, $x \in \Lambda_{N_0}(r)$. Hence,

$$d(x, \Lambda(r)) < \varepsilon.$$

□

If there is an obstruction point in homoclinic class, then one can find a sequence of periodic points $\{q_n\}$ such that the first hyperbolic time of $\{q_n\}$ tend to infinity. The precise statament is as Lemma 2.4.

Lemma 2.4. *Let $H(f, p)$ be a homoclinic class which admits a dominated splitting $T_{H(f,p)}M = E \bigoplus_< F$, where the subbundle F is uniformly λ-expanding at the period on the set of all periodic points homoclinically related to p. For $r \in (\lambda, 1)$, if there is an r-obstruction point $b \in H(f, p)$, then there exists a sequence of periodic points $\{q_n : n \in \mathbb{N}^+\} \subset H(f, p)$ homoclinically related to p, such that for any $\mu \in (\lambda, r)$, one has that*

$$\lim_{n\to\infty} q_n = b, \quad \phi_1(q_n, \mu) \to \infty \text{ when } n \to \infty.$$

Moreover, $\Gamma_2(q_n, \mu) - \Gamma_1(q_n, \mu)$ tends to infinity as n tends to infinity.

Proof. Since $b \in H(f, p)$, by the definition of Homoclinic class, there is a sequence of periodic points $\{q_n : n \in \mathbb{N}^+\} \subset H(f, p)$ homoclinically related to p such that $\lim\limits_{n\to\infty} q_n = b$. Since the subbundle F is uniformly λ-expanding at the period on the set of all periodic points homoclinically related to p, by Definition 1.1, there is a constant $C > 0$, such that

$$\prod_{j=1}^{\pi(q_n)} \left\| Df^{-1} \mid_{F_{f^j(q_n)}} \right\| \leq C\lambda^{\pi(q_n)}, \text{ for every } n \in \mathbb{N}^+.$$

Let $a_i \triangleq \log\left(\left\| Df^{-1}|_{F_{f^i(q_n)}} \right\|\right)$, one has that

$$\limsup_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} a_i \leq \log \lambda.$$

For any $\mu \in (\lambda, r)$, by Lemma 2.2, one deduces that q_n has infinitely many μ-hyperbolic times.

Now, we prove that $\phi_1(q_n, \mu) \to \infty$ when $n \to \infty$ by contradiction. Otherwise, suppose that there exists a constant $C_1 > 0$ such that $\phi_1(q_n, \mu) \leq C_1$, for all n. Then there is a subsequence $\{q_{n_k}\}$ of $\{q_n\}$, such that $\phi_1(q_{n_k}, \mu)$ are constant for all k, denoted by L. Hence, one has that

$$\prod_{i=1}^{L} \left\| Df^{-1}|_{F_{f^i(q_{n_k})}} \right\| \leq \mu^L, \quad \text{for all } q_{n_k}.$$

By the continuous of Df^{-1}, one concludes that

$$\prod_{i=1}^{L}\left\|Df^{-1}|_{F_{f^i(b)}}\right\| = \lim_{k\to\infty}\prod_{i=1}^{L}\left\|Df^{-1}|_{F_{f^i(q_{n_k})}}\right\| \le \mu^L \le r^L.$$

This means that b is not an r-obstraction point which contradicts our assumption. Since

$$\Gamma_2(q_n,\mu)-\Gamma_1(q_n,\mu)\ge \Gamma_2(q_n,\mu)-\pi(q_n)\ge \phi_1(q_n,\mu),$$

one has that

$$\Gamma_2(q_n,\mu)-\Gamma_1(q_n,\mu)\to\infty,\ \text{when } n\to\infty.$$

□

3. Exponential properties in homoclinic class

In this section, our goals are to find many hyperbolic periodic points (homoclinically related to p) with long stable and unstable manifolds. At the begining, we introduce some properties of C^1 diffeomorphisms.

Lemma 3.1. *Let f be a C^1 diffeomorphism on a compact manifold M. For any $x\in M$, if there are $C=C(x)>0$ and $\mu_1,\ \lambda_1\in(0,1)$ with $\mu_1<\lambda_1$, such that*

$$C\mu_1^n \le \prod_{i=0}^{n-1}\left\|Df_{f^ix}\right\| \le C\lambda_1^n,\ \textit{for every } n\in\mathbb{N}^+,$$

then for any $\mu_2,\lambda_2\in(0,1)$ with $\mu_2<\mu_1<\lambda_1<\lambda_2$, there are $C_0=C_0(x)$, $r=r(\mu_1,\mu_2,\lambda_1,\lambda_2)$ such that

$$C_0\mu_2^n \le \prod_{i=0}^{n-1}\left\|Df_{f^iy}\right\| \le C_0\lambda_2^n,\quad \textit{for every } y\in B(x,r) \textit{ and any } n\in\mathbb{N}^+.$$

Proof. Since f is C^1 diffeomorphism, for $0<\mu_2<\mu_1<\lambda_1<\lambda_2<1$, there is $r_1>0$ such that

$$\frac{\mu_2}{\mu_1}\le\frac{\|Df_{\tilde y}\|}{\|Df_{\tilde x}\|}\le\frac{\lambda_2}{\lambda_1},\ \ \text{for any points } \tilde x,\ \tilde y \text{ with } d(\tilde x,\tilde y)\le r_1.$$

Given $x\in M$ which satisfies that

$$C\mu_1^n \le \prod_{i=0}^{n-1}\left\|Df_{f^ix}\right\| \le C\lambda_1^n,\ \text{for every } n\in\mathbb{N}^+.$$

For any $y\in B(x,r_1)$, by Mean Value Theorem, there exists $\xi\in B(x,r_1)$ such that

$$d(f(x),f(y))\le\|Df_\xi\|\cdot d(x,y).$$

Since $\xi\in B(x,r_1)$, one has that

$$d(f(x),f(y))\le\frac{\lambda_2}{\lambda_1}\|Df_x\|\cdot d(x,y)\le C\frac{\lambda_2}{\lambda_1}\lambda_1 r_1 = C\lambda_2 r_1.$$

Claim. Taking $r = \min\{r_1, r_1/C\}$, $C_0 = \max\{C, 1/C\}$, one has that

$$d(f^j(x), f^j(y)) \leq r_1, \text{ for every } y \in B(x,r) \text{ and every } j \in \mathbb{N}^+.$$

Proof. We prove the claim by induction. Assume that

$$d(f^j(x), f^j(y)) \leq r_1, \text{ for any } y \in B(x,r) \text{ and every } j = 1, 2, \cdots, m.$$

Then, for any $y \in B(x,r)$, by Mean Value Theorem, there is $\eta \in B(x,r)$ such that

$$\begin{aligned} d(f^{m+1}(x), f^{m+1}(y)) &\leq \|Df_\eta^{m+1}\| d(x,y) \leq \|Df_{f^m\eta}\| \|Df_{f^{m-1}\eta}\| \cdots \|Df_\eta\| d(x,y) \\ &\leq \left(\frac{\lambda_2}{\lambda_1}\|Df_{f^m x}\|\right) \cdot \left(\frac{\lambda_2}{\lambda_1}\|Df_{f^{m-1}x}\|\right) \cdots \left(\frac{\lambda_2}{\lambda_1}\|Df_x\|\right) \cdot d(x,y) \\ &\leq \frac{\lambda_2^{m+1}}{\lambda_1^{m+1}} \prod_{i=0}^{m} \left\|Df_{f^i x}\right\| r \leq C \frac{\lambda_2^{m+1}}{\lambda_1^{m+1}} \lambda_1^{m+1} r = C\lambda_2^{m+1} r < r_1. \end{aligned}$$

Therefore, $f^{m+1}(B(x,r)) \subset B(f^{m+1}(x), r_1)$. This proves our claim. □

Therefore, taking r as the claim, for $y \in B(x,r)$ and $n \in \mathbb{N}^+$, one has that

$$\frac{\mu_2^{n+1}}{\mu_1^{n+1}} \leq \frac{\prod_{i=0}^{n} \|Df_{f^i y}\|}{\prod_{i=0}^{n} \|Df_{f^i x}\|} \leq \frac{\lambda_2^{n+1}}{\lambda_1^{n+1}}.$$

Consequently, taking C_0 as the claim, one has that

$$C_0 \mu_2^n \leq \prod_{i=0}^{n-1} \| Df_{f^i y} \| \leq C_0 \lambda_2^n, \text{ for } y \in B(x,r) \text{ and } n \in \mathbb{N}^+.$$

□

Theorem 3.1. *Let $H(f,p)$ be a homoclinic class which admits a dominated splitting $T_{H(f,p)}M = E \oplus_< F$ with dimE =Ind (p). If f is uniformly F-expanding at the period on all periodic points homoclinically related to p, then there exists constant $N \in \mathbb{N}^+$ such that for any hyperbolic periodic points homoclinically related to p with period larger than N, any plaque family tangent to the subbundle F are the unstable manifolds.*

Proof. Since f is uniformly F-expanding, by Definition 1.1, there are two constants $C > 0$ and $\lambda \in (0,1)$ such that for any hyperbolic periodic point x homoclinically related to p, one has that

$$\prod_{j=1}^{\pi(x)} \left\| Df^{-1} \mid_{F_{f^j(x)}} \right\| \leq C\lambda^{\pi(x)}.$$

Let $N \triangleq \min\{n \in \mathbb{N}^+ : C\lambda^n < 1\}$, define the set $\mathcal{U}$ as

$$\mathcal{U} \triangleq \{x \in H(f,p) : x \sim p \text{ with } \pi(x) \geq N\}.$$

For the dominated splitting $T_{H(f,p)}M = E \oplus_< F$, by Plaque Family Theorem [12, Theorem 5.5], there always exists an invariant plaque family tangent to the subbundles E and F. For any $x \in \mathcal{U}$, denoted by $\mathcal{W}^F(x)$ the plaque family tangent to the subbundle F at point x. For the hyperbolic splitting $T_x M = E^s \bigoplus F^u$ at hyperbolic periodic point x, there are constants $C_1 > 0$, $\lambda_1 \in (0,1)$ such that for every $n \in \mathbb{N}^+$, one has that

$$\|Df^n |_{E^s}\| \leq C_1\lambda_1^n, \quad \|Df^{-n} |_{F^u}\| \leq C_1\lambda_1^n.$$

Since the plaque family tangent to the subbundle F^u are the unstable manifolds, it suffices to prove that the hyperbolic splitting $T_xM = E^s \bigoplus F^u$ and dominated splitting $T_xM = E \bigoplus_< F$ are coincide. Since $\dim E = \mathrm{Ind}\,(p)$, one has that $\dim F = \dim F^u$. It is sufficient to show that $F \subseteq F^u$. Conversely, suppose that $F \nsubseteq F^u$. Then there is a non-zero vector $v \in F$ such that $v \notin F^u$. Thus, one can obtain a decomposition $v = v^s \oplus v^u$, where $0 \neq v^s \in E^s$ and $v^u \in F^u$. Therefore, for any $m \in \mathbb{N}^+$, one has that

$$C_1^{-1}\lambda_1^{-m\pi(x)}\|v^s\| \leq \|Df^{-m\pi(x)}v\| \leq \left(\prod_{j=1}^{m\pi(x)} \|Df^{-1} |_{F_{f^j(x)}}\|\right) \|v\| \leq (C\lambda^{\pi(x)})^m\|v\|.$$

Taking m large enough such that

$$C_1^{-1}\lambda_1^{-m\pi(x)}\|v^s\| > 1, \qquad (C\lambda^{\pi(x)})^m\|v\| < 1.$$

Therefore,

$$1 < C_1^{-1}\lambda_1^{-m\pi(x)}\|v^s\| \leq \left(\prod_{j=1}^{m\pi(x)} \|Df^{-1} |_{F_{f^j(x)}}\|\right) \|v\| \leq \left(C\lambda^{\pi(x)}\right)^m \|v\| < 1.$$

This means that our assumption that $F \nsubseteq F^u$ is fault. Consequently, $F \subseteq F^u$. $\square$

We introduce that for some special homoclinic class $H(f,p)$, there is a dense set such that every point in this set has stable manifolds of uniformly size. Given $\varepsilon > 0$, a sequence of points $\{x_0, \cdots, x_m\}$ is called a *periodic ε-orbit or periodic pseudoorbit*, if $x_m = x_0$ and

$$d(f(x_i), x_{i+1}) < \varepsilon, \text{ for } i = 0, \cdots, m-1.$$

Definition 3.1. Fix $\delta > 0$ and $k \in \mathbb{N}^+$, a sequence of points $\{x_0, x_1, \cdots, x_k\}$ is called δ-shadowed by a periodic point x, if $k = \pi(x)$ and

$$d(f^n(x), x_n) < \delta, \text{ for every } 0 \leq n \leq k.$$

Theorem 3.2 ([13, Theorem 6.4.15], Anosov Closing Lemma)**.** *For a hyperbolic set Λ of diffeomorphism f, there is an open neighborhood U of Λ and two constants $C > 0$, $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, any periodic ε-orbit $\{x_0, \cdots, x_m\} \subset U$ can be $C\varepsilon$-shadowed by a periodic point $y \in U$.*

Crovisier and Pujals [6, Lemma 3.8 and Lemma 3.9] proved that a chain hyperbolic homoclinic class (see [6, Definition 2.10]) contains a dense set of hyperbolic periodic points with long stable and unstable manifolds. Here, we prove that if a homoclinic class $H(f,p)$ admits a dominated splitting $T_{H(f,p)}M = E \bigoplus_< F$, where the subbundle E is thin trapped and $\dim(E)$ =Ind(p), then the homoclinic class contains a dense set of hyperbolic periodic points with long stable manifolds.

Theorem 3.3. *Let $H(f,p)$ be a homoclinic class which admits a dominated splitting $T_{H(f,p)}M = E\bigoplus_{<} F$, where the subbundle E is thin trapped and dim(E) = Ind(p). For $\varepsilon > 0$ small enough, there is a ε-dense set $\mathcal{P} \subseteq H(f,p)$ of hyperbolic periodic points homoclinically related to p, such that every point $q \in \mathcal{P}$ has stable manifolds of uniformly size.*

Proof. Assume that p is a hyperbolic fixed point (otherwise, consider $g = f^{\pi(p)}$). Since $\dim(E)$ =Ind(p) and E is thin trapped, the dominated splitting $T_pM = E_p \bigoplus_{<} F_p$ is the hyperbolic splitting of p. By [11, Theorem 1], taking suitable Riemann norm, there exists $\lambda_1 \in (0,1)$ such that

$$\|Df|_{E_p}\| \le \lambda_1, \quad \|Df^{-1}|_{F_p}\| \le \lambda_1.$$

Hereafter, we fix the numbers $0 < \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < 1$. Since f is a C^1 diffeomorphism, for $\lambda_2 \in (\lambda_1, 1)$, there is $r > 0$ such that for any $x, y \in H(f,p)$ with $d(x,y) < r$, one has that

$$\frac{\|Df|_{E_x}\|}{\|Df|_{E_y}\|} \le \frac{\lambda_2}{\lambda_1}, \qquad \frac{\|Df^{-1}|_{F_x}\|}{\|Df^{-1}|_{F_y}\|} \le \frac{\lambda_2}{\lambda_1}.$$

An equivalence definition of the homoclinic class [1, pp.199] is

$$H(f,p) \triangleq \overline{W^s(\mathcal{O}rb(f,p)) \pitchfork W^u(\mathcal{O}rb(f,p))}.$$

By this characterization of $H(f,p)$, for any $\varepsilon < r$, one can take a $\varepsilon/2$-dense subset

$$B \triangleq \{x : x \in W^s(\mathcal{O}rb(f,p)) \pitchfork W^u(\mathcal{O}rb(f,p))\}$$

of $H(f,p)$.

Claim. For every $x \in B$, $\Lambda_x \triangleq \mathcal{O}rb(f,p) \bigcup \mathcal{O}rb(f,x)$ is a hyperbolic set.

Proof. Since $x \in W^s(\mathcal{O}rb(f,p)) \pitchfork W^u(\mathcal{O}rb(f,p))$, there exists $n_0 \in \mathbb{N}$ such that for any $n \ge n_0$, one has that

$$d(f^n(x), p) < r, \quad d(f^{-n}(x), p) < r.$$

Therefore,

$$\frac{\|Df|_{E_{f^n(x)}}\|}{\|Df|_{E_p}\|} \le \frac{\lambda_2}{\lambda_1}, \quad \frac{\|Df^{-1}|_{F_{f^{-n}(x)}}\|}{\|Df^{-1}|_{F_p}\|} \le \frac{\lambda_2}{\lambda_1}, \quad \text{for any } n \ge n_0.$$

Let $C_1 = \max\limits_{z \in H(f,p)} \left\{ \frac{\|Df_z\|}{\lambda_2}, \cdots, \frac{\|Df_z^{2n_0}\|}{\lambda_2^{2n_0}}, \frac{\|Df_z^{-1}\|}{\lambda_2}, \cdots, \frac{\|Df_z^{-2n_0}\|}{\lambda_2^{2n_0}} \right\}$, $C_2 = \max\{1, C_1\}$, for any $y \in \Lambda_x$ and any $n \in \mathbb{N}^+$, one has that

$$\|Df^n|_{E_y}\| \le C_2\lambda_2^n, \quad \|Df^{-n}|_{F_y}\| \le C_2\lambda_2^n.$$

Hence, $\Lambda_x = \mathcal{O}rb(f,p) \bigcup \mathcal{O}rb(f,x)$ is a hyperbolic set. □

Fix $x \in B$, by Theorem 3.2, for the hyperbolic set Λ_x, there exist an open neighborhood U of Λ_x and C_3, $\varepsilon_0 > 0$, such that for any $\varepsilon_1 \in (0, \varepsilon_0)$, any periodic ε_1-orbit $\{x_0, \cdots, x_m\} \subset U$ can be $C_3\varepsilon$-shadowed by a periodic point $y \in U$. It means that

$$d(f^i(y), x_i) < C_3\varepsilon_1, \text{ for } i = 0, 1, \cdots, m.$$

Taking $\varepsilon_2 = \min\{\varepsilon/2C_3,\ \varepsilon/2,\ r\}$, since $x \in W^s(\mathcal{O}rb(f,p)) \pitchfork W^u(\mathcal{O}rb(f,p))$, there exists $m_0 \in \mathbb{N}$ such that for any $m \geq m_0$, one has that

$$d(f^m(x), p) < \varepsilon_2/2, \quad d(f^{-m}(x), p) < \varepsilon_2/2.$$

Let $K = \sup_{z\in H(f,p)} \{\|Df_z\|,\ 2,\ \|Df_z^{-1}\|\}$, for $\lambda_2 < \lambda_3 < 1$, taking an integer $m_1 \geq \dfrac{C}{2\log(\lambda_3/\lambda_2)}$, where $C = (2m_0 - 1)\log K - (2m_0 + 1)\log \lambda_2$, one can obtain a periodic ε_2-orbit:

$$\{f^{m_1}(x), f^{-m_1+1}(x), \cdots, f^{-m_0}(x), f^{-m_0+1}(x), \cdots, f^{-1}(x), x, f(x), \cdots, \\ \cdots, f^{m_0}(x), \cdots, f^{m_1}(x)\}.$$

By Theorem 3.2, this periodic ε_2-orbit is $\varepsilon/2$-shadowed by a periodic point q_x with $\pi(q_x) = 2m_1$. Therefore,

$$\prod_{i=0}^{\pi(q_x)-1} \left\| Df|_{E_{f^i(q_x)}} \right\| \leq K^{2m_0-1} \cdot \lambda_2^{2m_1-2m_0+1} \leq \lambda_3^{\pi(q_x)}.$$

Hence, for any $n \in \mathbb{N}$, one has that

$$\prod_{i=0}^{n\pi(q_x)-1} \left\| Df|_{E_{f^i(q_x)}} \right\| \leq \lambda_3^{n\pi(q_x)}.$$

One can deduce the similar estimation on the subbundle F. Next, we define the set $\mathcal{P} \triangleq \{q_x : x \in B\}$, then $\mathcal{P} \subseteq H(f,p)$ is a set of hyperbolic periodic points homoclinically related to p. For $\lambda_3 < \lambda_4$, any $q \in \mathcal{P}$ and any $n \in \mathbb{N}$, by Lemma 2.1, there are $\theta = \theta(\lambda_3, \lambda_4) \in (0,1)$ and positive integers $n_1 < n_2 < \cdots < n_l \leq n$ with $l \geq \theta n\pi(q)$, such that

$$\prod_{i=k}^{n_j-1} \left\| Df|_{E_{f^i(q)}} \right\| \leq \lambda_4^{n_j-k}, \text{ for any } k = 0, 1, \cdots, n_j - 1,\ j = 1, 2, \cdots, l.$$

Since q is a periodic point, if $n \to \infty$, then there is a point q' which is a iteration of q, such that

$$\prod_{i=0}^{m-1} \left\| Df|_{E_{f^i(q')}} \right\| \leq \lambda_4^m, \text{ for any } m \in \mathbb{N}.$$

Therefore, the point q' has stable manifolds of δ-size, where δ is only related to λ_4. For this δ, since the subbundle E is thin trapped, for every $y \in \{q', \cdots, f^{\pi(q')-1}(q')\}$, one has that

$$f^i(\mathcal{W}^s_\delta(y)) \subset \mathcal{W}^s_\delta(f^i y) = \mathcal{W}^s_\delta(q'), \quad \text{for some } i \in \{0, 1, \cdots, \pi(q') - 1\}.$$

Therefore, every point $q \in \mathcal{P}$ has stable manifolds of uniformly size.

For any $b \in H(f,p)$, by the choice of B, there is $x \in B$ such that $d(b,x) < \frac{\varepsilon}{2}$. For this point x, there is a $q_x \in \mathcal{P}$ such that $d(x, q_x) < \frac{\varepsilon}{2}$. Then, $d(b, q_x) < \varepsilon$. Therefore, $\mathcal{P}$ is a ε-dense subset of $H(f,p)$. $\square$

4. The closing property in thin trapped homoclinic classes

It is well-known that pseudoorbits near a hyperbolic set can be shadowed by a real orbit. This is called Pseudo-Orbit Tracing Property. This property plays an important role in the study of stability of dynamical systems (see [24], [9], [25] and [20]). Gan [10, Theorem 1.1] showed that quasi-hyperbolic pseudoorbits can be shadowed by a real orbit. In this section, we introduce the Pseudo-Orbit Tracing Property in thin trapped $H(f,p)$. Before heading to the main block, we clarify some notations and identify some constants.

Sun and Yang [23, Lemma 2.2] affirmed that chain hyperbolic homoclinic classes have local product structures. Hereafter, we assume that $H(f,p)$ admits a dominated splitting $T_{H(f,p)}M = E \bigoplus_{<} F$ with $\dim(E) = \mathrm{Ind}(p)$, the subbundle E is thin trapped and f is uniformly F-expanding at the period on all periodic points homoclinically related to p. Then the homoclinic class has local product structures. What is important is that one can also choose dense hyperbolic periodic points, which have long stable and unstable manifolds, from thin trapped homoclinic class.

Proposition 4.1. *For $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in H(f,p)$ with $d(x,y) < \delta$, $\mathcal{W}^{cs}_\varepsilon(x)$ and $\mathcal{W}^{cu}_\varepsilon(y)$ transversally intersect at a single point belonging to $H(f,p)$, where $\mathcal{W}^{*}_\varepsilon(x) \subset \mathcal{W}^{*}(x)$ is centered at x with length 2ε, $* = cs$ or cu.*

Proof. From Sun and Yang [23, Lemma 2.2], the proof is completed by showing that the homoclinic class $H(f,p)$ is a chain hyperbolic homoclinic class. Under our assumption, what is left is to show that the subbundle F is trapped for f^{-1}. By Theorem 3.1, for the hyperbolic periodic points with large enough period, any plaque family tangent to F are unstable manifolds. Thus, the subbundle F at those points is trapped for f^{-1}. Since $T_{H(f,p)}M = E \bigoplus_{<} F$ is a dominated splitting, by the uniqueness and continuity of dominated splitting, any plaque family tangent to F are unstable manifolds. Thus, the subbundle F is trapped for f^{-1}. □

For a chain hyperbolic homoclinic class, Crovisier and Pujals [6, Lemma 3.9] found that $H(f,p)$ contains a dense set of "well"periodic points, which is defined as Lemma 4.1.

Lemma 4.1 ([6, Lemma 3.9]). *For any small enough $\delta > 0$, there is a dense set $\mathcal{P}_0 \subset H(f,p)$ of periodic points homoclinically related to p with the properties:*

(i) The modulus of the Lyapunov exponents of any point $q \in \mathcal{P}_0$ are larger than δ;

(ii) The plaques $\mathcal{W}^{cs}_q$ and $\mathcal{W}^{cu}_q$ for any point $q \in \mathcal{P}_0$ contained in the stable and in the unstable manifolds of q respectively.

Furthermore, the ε-dense subset $\mathcal{P}$ in Theorem 3.3 can be the subset $\mathcal{P}_0$ when choose a suitable δ. Hereafter, we always consider $\mathcal{P}_0$ when no confusion can arise.

Lemma 4.2. *There exists a number α such that for any $x \in \mathcal{P}_0$, the plaque family $\mathcal{W}^{cs}_\alpha(x)$ and $\mathcal{W}^{cu}_\alpha(x)$ are the stable and unstable manifolds of x, respectively.*

Proof. By Theorem 3.3, there is a number α_1 such that for any $x \in \mathcal{P}_0$, the plaque family $\mathcal{W}^{cs}_{\alpha_1}(x)$ are the stable manifolds of x. From Lemma 4.1, since the modulus of the Lyapunov exponents of any point $x \in \mathcal{P}_0$ are larger than δ, there is constant

C such that for any point $x \in \mathcal{P}_0$ and every $n \in \mathbb{N}^+$, one has that

$$\|Df^n|_{F_x}\| \geq Ce^{n\delta}.$$

Thus, there is a number α_2 such that for any $x \in \mathcal{P}_0$, the plaque family $\mathcal{W}^{cu}_{\alpha_2}(x)$ are the unstable manifolds of x. Taking $\alpha = \min\{\alpha_1, \alpha_2\}$, the lemma follows. □

Definition 4.1. Let $T_\Lambda M = E \bigoplus_< F$ be a dominated splitting on the compact invariant set Λ. For $x \in \Lambda$, $n \in \mathbb{N}^+$ and $\lambda \in (0,1)$, an orbit segment (x, n) is called λ-thin trapped, if subbundle E is thin trapped and subbundle F satisfies that

$$\prod_{j=1}^{k} \left\|Df^{-1}|_{F_{f^{n-j}(x)}}\right\| \leq \lambda^k, \text{ for } k = 1, \cdots, n-1.$$

Definition 4.2. Given $\lambda \in (0,1)$ and $\gamma > 0$, a finite number of λ-thin trapped orbit segment $\{(x_i, n_i)\}_{i=0}^m$ is called a (λ,γ)-thin trapped closed pseudoorbit, if the orbit segment (x_i, n_i) is λ-thin trapped and

$$d(f^{n_i-1}(x_i), x_{i+1}) \leq \gamma, \quad \text{for } i = 0, 1, \cdots, m-1 \text{ and } d(f^{n_m-1}x_m, x_0) < \gamma.$$

Definition 4.3. Given $n_0 \in \mathbb{N}$, $\lambda \in (0,1)$ and $\gamma > 0$, a (λ,γ)-thin trapped closed pseudoorbit $\{(x_i, n_i)\}_{i=0}^m$ is called an (n_0, λ, γ)-closed pseudoorbit, if the integers $n_i \geq n_0$, for $i = 0, 1, \cdots, m$.

Sun and Yang [23, Theorem 1.7] gave a closing property for some special chain hyperbolic homoclinic class. Here, we establish a closing property for the dense subset $\mathcal{P}_0$ in thin trapped homoclinic class.

Theorem 4.1. *For any $\eta \in (0, \alpha)$, where α is given by Lemma 4.2, there are $\gamma_0 = \gamma_0(\eta)$ and $n_0 = n_0(\eta) \in \mathbb{N}$, such that for any $\gamma \in (0, \gamma_0)$, if a finite number of orbit segment $\{(x_i, n_i + 1)\}_{i=0}^m \subset \mathcal{P}_0$ is a (n_0, λ, γ)-closed pseudoorbit, then the (n_0, λ, γ)-closed pseudoorbit can be η-shadowed by a periodic point.*

Proof. For any $\eta \in (0, \alpha)$, by Proposition 4.1, there exists $\beta \in (0, \eta)$ such that for any $x, y \in H(f, p)$ with $d(x, y) < \beta$, one has that

$$\mathcal{W}^{cs}_\eta(x) \pitchfork \mathcal{W}^{cu}_\eta(y) \neq \emptyset, \quad \mathcal{W}^{cu}_\eta(x) \pitchfork \mathcal{W}^{cs}_\eta(y) \neq \emptyset.$$

Due to Lemma 4.2, for any $x \in \mathcal{P}_0$, the plaque $\mathcal{W}^{cs}_\eta(x)$ are the stable manifolds of x, denoted by $\mathcal{W}^{s}_\eta(x)$ and the plaque family $\mathcal{W}^{cu}_\eta(x)$ are the unstable manifolds of x, denoted by $\mathcal{W}^{u}_\eta(x)$.

According to Lemma 4.1, by the choice of $\mathcal{P}_0$, there is $C > 0$ such that for any $x \in \mathcal{P}_0$ and any $n \in \mathbb{N}^+$, one has that

$$\| Df^n|_{E_x} \| \leq Ce^{-n\delta}.$$

Let $n_0 \triangleq \min\{n \in \mathbb{N} : Ce^{-n\delta}\eta \leq \beta/2\}$ and $\gamma_0 \triangleq \beta/4$, then for any $\gamma \in (0, \gamma_0)$, if a finite number of orbit segment $\{(x_i, n_i + 1)\}_{i=0}^m \subset \mathcal{P}_0$ is a (n_0, λ, γ)-closed pseudoorbit, then we can construct a sequence of points which are the intersection points of some stable manifolds and $\mathcal{W}^{cu}$ plaques.

Since $\{(x_i, n_i + 1)\}_{i=0}^m \subset \mathcal{P}_0$ is a (n_0, λ, γ)-closed pseudoorbit, by Proposition 4.1, there exists

$$z_1 \in \mathcal{W}^{cu}_\eta(f^{n_1}x_1) \pitchfork \mathcal{W}^{s}_\eta(x_2) \neq \emptyset.$$

Therefore,

$$d(f^{n_2}(x_2), f^{n_2}(z_1)) \le Ce^{-n_2\delta} d(x_2, z_1) \le Ce^{-n_0\delta} d(x_2, z_1) \le Ce^{-n_0\delta}\eta \le \beta/2.$$

Consequently,

$$d(f^{n_2}(z_1), x_3) \le d(f^{n_2}(x_2), f^{n_2}(z_1)) + d(f^{n_2}(x_2), x_3) \le \beta.$$

Applying to Proposition 4.1 again, there exists

$$z_2 \in \mathcal{W}^{cu}_\eta(f^{n_2} z_1) \pitchfork \mathcal{W}^s_\eta(x_3) \neq \emptyset.$$

Similarly, one obtains that

$$z_i \in \mathcal{W}^{cu}_\eta(f^{n_i} z_{i-1}) \pitchfork \mathcal{W}^s_\eta(x_{i+1}) \neq \emptyset, \text{ for } i = 3, 4, \cdots, m-1 \ ,$$
$$z_m \in \mathcal{W}^{cu}_\eta(f^{n_m} z_{m-1}) \pitchfork \mathcal{W}^s_\eta(x_1) \neq \emptyset.$$

By the compactness of $H(f,p)$, there is a point $z \in H(f,p)$ such that

$$z = \lim_{k\to+\infty} f^{k\left(\sum\limits_{i=1}^m n_i\right)}(z_m).$$

Due to

$$f^{\sum\limits_{i=1}^m n_i}(z) = f^{\sum\limits_{i=1}^m n_i}\left(\lim_{k\to+\infty} f^{k\left(\sum\limits_{i=1}^m n_i\right)}(z_m)\right) = \lim_{k\to+\infty} f^{(k+1)\left(\sum\limits_{i=1}^m n_i\right)}(z_m) = z,$$

the point z is a periodic point. From our construction, the sequence of points are the intersection points of some stable manifolds and $\mathcal{W}^{cu}$ plaques. By the properties of the (n_0, λ, γ)-closed pseudoorbit, the (n_0, λ, γ)-closed pseudoorbit can be η-shadowed by the periodic point z. □

5. Proof of Main Theorem and Theorem 1.1

We prove Main Theorem by contradiction under the assumptions that

- The point p is a hyperbolic periodic point;
- The homoclinic class $H(f,p)$ admits a dominated splitting $T_{H(f,p)}M = E \bigoplus_< F$ with $\dim(E) = \mathrm{Ind}(p)$;
- The subbundle E is thin trapped and f is uniformly F-expanding at the period on the set of all periodic points homoclinically related to p;
- The subbundle F is not uniformly expanding on $H(f,p)$.

Building closed pseudoorbit. Bonatti, Gan and Yang [2, Lemma 3.1] proved the existence of obstruction point. We give a similar conclusion for thin trapped homoclinic class.

Lemma 5.1. *There exists a point b in $H(f,p)$, such that*

$$\prod_{j=1}^{n} \left\| Df^{-1}|_{f^j(b)} \right\| \ge 1, \quad \textit{for any } n \in \mathbb{N}^+.$$

Therefore, b is a 1-obstruction point.

Proof. Suppose that the lemma were false. Then for any $x \in H(f,p)$, there exists $n = n(x) > 0$ such that

$$\prod_{i=1}^{n} \left\| Df^{-1}|_{F(f^i(x))} \right\| < 1.$$

Taking $r(x) \in (0,1)$ and the neighborhood $U(x)$ of x such that

$$\prod_{i=1}^{n(x)} \left\| Df^{-1}|_{F(f^i(y))} \right\| < r(x)^{n(x)}, \text{ for any } y \in U(x) \cap H(f,p).$$

By the compactness of $H(f,p)$, there exist finite points $\{x_j\}_{j=1}^{m}$ such that $H(f,p) \subset \bigcup_{i=1}^{m} U(x_i)$. Let $r \triangleq \max_{1 \le i \le m}\{r(x_i)\}$, $N \triangleq \max_{1 \le i \le m}\{n(x_i)\}$ and

$$C \triangleq \max_{1 \le i \le m} \left\{ \max \left\{ \frac{\|Df^{-1}|_{F(f(x_i))}\|}{r(x_i)}, \cdots, \frac{\prod_{j=1}^{N} \|Df^{-1}|_{F(f^j(x_i))}\|}{r(x_i)^N} \right\} \right\},$$

for any $x \in H(f,p)$ and $n \in \mathbb{N}^+$, by splitting every orbit segment $(x, f^n x)$ in segments of the form $(f^i x, f^{n(f^i x)}(f^i x))$, one has that

$$\prod_{i=1}^{n} \left\| Df^{-1}|_{F(f^i(x))} \right\| \le C r^n.$$

Hence, the subbundle F is uniformly expanding. This contradicts our assumption that the subbundle F is not uniformly expanding on $H(f,p)$. □

Now we construct a closed pseudoorbit P as follows. Hereafter, we fix a sequence of numbers $0 < \lambda < r_4 < r_3 < r_2 < r_1 \le 1$, $\varepsilon > 0$ and dense subset $\mathcal{P}_0$ given by Lemma 4.1. By Lemma 4.2, every point belonging to $\mathcal{P}_0$ has stable and unstable manifolds of uniformly size. According to Lemma 5.1, there is an r_1-obstruction point $b_1 \in H(f,p)$. By Lemma 2.4, there is a sequence of periodic points homoclinically related to p such that the periods of these periodic points tend to infinity and the first hyperbolic time of these also tend to infinity. Thus, without loss of generality, we assume that the period of every periodic point in the dense subset $\mathcal{P}_0 \subseteq H(f,p)$ is large enough.

Step 1. One can find a sequence of points $\{q_n : q_n \sim p\}$ such that $\lim\limits_{n\to\infty} q_n = b_1$. For $\varepsilon/2$, $r_2 < \lambda_1 < r_1 \le 1$, by Lemma 2.3, there exists $N_1 = N_1(\varepsilon, \lambda_1)$ such that if $(x, f^{N_1}(x))$ is a λ_1-obstruction segment, then $d(x, \Lambda(\lambda_1)) < \varepsilon/2$. Taking $x_1 \in \mathcal{P}_0$ with $\Gamma_2(x_1,\lambda_1) - \Gamma_1(x_1,\lambda_1) - 1 \ge N_1$, we obtain uniform λ_1-strings $(x_1, f^{\phi_1(x_1,\lambda_1)}(x_1))$, $(f^{\phi_1(x_1,\lambda_1)}(x_1), \ f^{\Gamma_1(x_1,\lambda_1)}(x_1))$ and λ_1-obstruction segment $(f^{\Gamma_1(x_1,\lambda_1)}(x_1), f^{\Gamma_2(x_1,\lambda_1)-1}(x_1))$. Then, there exists a λ_1-obstruction point $b_2 \in H(f,p)$ such that $d(b_2, f^{\Gamma_1(x_1,\lambda_1)}(x_1)) < \varepsilon/2$.

Step 2. For the λ_1-obstruction point $b_2 \in H(f,p)$, one can find a sequence of points $\{q_n' : q_n' \sim p\}$ such that $\lim\limits_{n\to\infty} q_n' = b_2$. For $\varepsilon/2$, $r_2 < \lambda_2 < \lambda_1 < r_1$, by Lemma 2.3, there exists $N_2 = N_2(\lambda_2, \varepsilon)$ such that if $(x, f^{N_2}(x))$ is a λ_2-obstruction segment, then $d(x, \Lambda(\lambda_2)) < \varepsilon/2$. By Lemma 2.4, $\phi_1(q_n', \lambda_2) \to \infty$ as $n \to \infty$. Let B be the subset of $\mathcal{P}_0$ such that for every $q_n' \in B$, one has that

$$r_3^{\phi_1(q_n',\lambda_2)-1} \cdot \zeta \cdot \zeta^{\Gamma_1(x_1,\lambda_1)-\phi_1(x_1,\lambda_1)} \ge r_4^{\phi_1(q_n',\lambda_2)+\Gamma_1(x_1,\lambda_1)-\phi_1(x_1,\lambda_1)},$$

where $\zeta \triangleq \inf_{z\in H(f,p)} \|Df^{-1}|_{F(z)}\|$. Take $x_2 \in B$ with $\Gamma_2(x_2,\lambda_2)-\Gamma_1(x_2,\lambda_2)-1 \geq N_2$, we obtain uniform λ_2-strings $(x_2, f^{\phi_1(x_2,\lambda_2)}(x_2))$, $(f^{\phi_1(x_2,\lambda_2)}(x_2),\ f^{\Gamma_1(x_2,\lambda_2)}(x_2))$ and λ_2-obstruction segment $(f^{\Gamma_1(x_2,\lambda_2)}(x_2), f^{\Gamma_2(x_2,\lambda_2)-1}(x_2))$. Therefore, there exists a λ_2-obstruction point $b_3 \in H(f,p)$ such that $d(b_3, f^{\Gamma_1(x_2,\lambda_2)}(x_2)) < \varepsilon/2$.

Step 3. For $\varepsilon > 0$, take λ_j with $r_4 < r_3 < r_2 < \cdots < \lambda_j < \lambda_{j-1} < \cdots < \lambda_2 < \lambda_1 < r_1$, where $j = 1, 2, \cdots$, by repeating Step 2, we obtain a sequence of points $\{x_n\}$ with properties:

- $(x_j, f^{\phi_1(x_j,\lambda_j)}(x_j))$, $(f^{\phi_1(x_j,\lambda_j)}(x_j), f^{\Gamma_1(x_j,\lambda_j)}(x_j))$ are uniform λ_j-strings;
- $(f^{\Gamma_1(x_j,\lambda_2)}(x_j), f^{\Gamma_2(x_j,\lambda_j)-1}(x_j))$ are λ_j-obstruction segment;
- $d(x_j, f^{\Gamma_1(x_{j-1},\lambda_{j-1})}(x_{j-1})) \leq \varepsilon$;
- $r_3^{\phi_1(x_j,\lambda_j)-1}\cdot\zeta\cdot\zeta^{\Gamma_1(x_{j-1},\lambda_{j-1})-\phi_1(x_{j-1},\lambda_{j-1})} \geq r_4^{\phi_1(x_j,\lambda_j)+\Gamma_1(x_{j-1},\lambda_{j-1})-\phi_1(x_{j-1},\lambda_{j-1})}$.

From the above step, we construct a pseudoorbit which is not closed. Since $H(f,p)$ is a compact set, there exist two positive integers m_0 and k, such that

$$d(f^{\phi_1(x_{m_0},\lambda_{m_0})}(x_{m_0}), f^{\phi_1(x_{m_0+k},\lambda_{m_0+k})}(x_{m_0+k})) < \varepsilon.$$

Let $K_j \triangleq \phi_1(x_{m_0+j}, \lambda_{m_0+j}),\ j = 1, 2, \cdots, k$ and $y_{m_0+j} \triangleq f^{\phi_1(x_{m_0+j},\lambda_{m_0+j})}(x_{m_0+j})$, $L_j \triangleq \Gamma_1(x_{m_0+j}, \lambda_{m_0+j}) - \phi_1(x_{m_0+j}, \lambda_{m_0+j}),\ j = 0, 1, \cdots, k-1$. Therefore, we have a closed pseudoorbit P which is the union of uniform λ_1-strings as

$$(y_{m_0}, f^{L_0}(y_{m_0})), (x_{m_0+1}, f^{K_1}(x_{m_0+1})), (y_{m_0+1}, f^{L_1}(y_{m_0+1})), (x_{m_0+2}, f^{K_2}(x_{m_0+2})),$$

$$\cdots, (y_{m_0+k-1}, f^{L_{k-1}}(y_{m_0+k-1})), (x_{m_0+k}, f^{K_k}(x_{m_0+k})),$$

where $y_{m_0+j} = f^{K_j}(x_{m_0+j})$ and $d(f^{L_j}(y_{m_0+j}), x_{m_0+j+1}) < \varepsilon,\ j = 0, 1, \cdots, k-1$.

Estimation about periodic orbits. From the construction of closed pseudoorbit, by Theorem 4.1, for $\eta \in (0, \alpha)$, there exist $\gamma_0 = \gamma_0(\eta) > 0$, such that for any $\gamma \in (0, \gamma_0]$, there exists a periodic point η-shadows (n_0, λ, γ)-thin trapped closed pseudoorbit.

Lemma 5.2. *For the fixed $r_4 < r_3 < r_1$, there is a constant $\delta_0 > 0$ such that all those periodic points which δ_0-shadows (n_0, λ, γ)-thin trapped closed pseudoorbit have stable and unstable manifolds of uniformly size.*

Proof. For the dominated splitting $T_{H(f,p)}M = E \bigoplus_< F$, by [11, Theorem 1], one can take an suitable norm such that

$$\left\|Df|_{E(x)}\right\| \cdot \left\|Df^{-1}|_{F(f(x))}\right\| \leq \lambda_0,$$

where $0 < \lambda_0 < \lambda < 1$. For the fixed $r_4 < r_3 < r_1$, by Lemma 3.1, there is a constant δ_1 such that for any $x \in B(q_n, \delta_1)$, one has that

$$\prod_{i=1}^{\Gamma_1(q_n,\lambda_n)} \left\|Df^{-1}|_{F(f^i x)}\right\| \geq r_4^{\Gamma_1(q_n,\lambda_n)}, \qquad \prod_{i=1}^{\Gamma_1(q_n,\lambda_n)} \left\|Df^{-1}|_{F(f^i x)}\right\| \leq r_1^{\Gamma_1(q_n,\lambda_n)}.$$

By Theorem 4.1, let $\delta_0 = \min\{\varepsilon, \delta_1, \alpha\}$, then there exist $\gamma_0 = \gamma_0(\delta_0) > 0$, such that for any $\gamma \in (0, \gamma_0]$, the (n_0, λ, γ)-thin trapped closed pseudoorbit can be δ_0-shadowed by a periodic point y. This means that $y \in B(q_n, \delta_0)$ for any q_n in (n_0, λ, γ)-thin trapped closed pseudoorbit. Therefore,

$$\prod_{i=1}^{\pi(y)} \left\|Df^{-1}|_{F(f^i(y))}\right\| \geq r_4^{\pi(y)}, \quad \prod_{i=1}^{\pi(y)} \left\|Df^{-1}|_{F(f^i(y))}\right\| \leq r_1^{\pi(y)}.$$

Hence,

$$\prod_{i=0}^{\pi(y)-1} \left\|Df|_{E(f^i(y))}\right\| \leq \frac{\lambda_0^{\pi(y)}}{\prod\limits_{i=1}^{\pi(y)} \left\|Df^{-1}|_{F(f^i(y))}\right\|} \leq \left(\frac{\lambda_0}{\lambda}\right)^{\pi(y)}.$$

Taking $\widetilde{\lambda} = \max\{\lambda_0/\lambda,\ r_1\}$, one has that

$$\prod_{i=0}^{\pi(y)-1} \left\|Df|_{E(f^i(y))}\right\| \leq \widetilde{\lambda}^{\pi(y)}, \quad \prod_{i=1}^{\pi(y)} \left\|Df^{-1}|_{F(f^i(y))}\right\| \leq \widetilde{\lambda}^{\pi(y)}.$$

Since the subbundle E is thin trapped with $\dim E$ = Ind (p) and f is uniformly F-expanding at the period on all periodic points homoclinically related to p, from Theorem 3.1 and Theorem 3.3, those periodic points have stable and unstable manifolds of uniformly size. □

According to Theorem 3.3, by the choice of $\mathcal{P}_0$, hyperbolic periodic points belonging to $\mathcal{P}_0$ have stable and unstable manifolds of uniformly size. By Lemma 5.2, those periodic points which δ_0-shadows (n_0, λ, γ)-thin trapped closed pseudoorbit have stable and unstable manifolds of uniformly size. Therefore, there is a $\delta' > 0$ such that if periodic point δ'-shadows (n_0, λ, γ)-thin trapped closed pseudoorbit, then periodic point is homoclinically related to the hyperbolic periodic points that given in the construction of the closed pseudoorbit. Then, for $\eta = \min\{\delta_0, \delta', \varepsilon, \alpha\}$, where δ_0 is given by Lemma 5.2, by Theorem 4.1, the closed pseudoorbit P can be η-shadowed by a periodic point $\widetilde{p}$ which satisfies that

$$W_\eta^s(\mathcal{O}rb(f, \widetilde{p})) \pitchfork W_\eta^u(\mathcal{O}rb(f, p)) \neq \emptyset, \quad W_\eta^s(\mathcal{O}rb(f, p)) \pitchfork W_\eta^u(\mathcal{O}rb(f, \widetilde{p})) \neq \emptyset.$$

This means that $\widetilde{p}$ is homoclinically related to p. By Lemma 5.2, one deduces that

$$\prod_{i=1}^{\pi(\widetilde{p})} \left\|Df^{-1}|_{F(f^i(\widetilde{p}))}\right\| \geq r_4^{\pi(\widetilde{p})}.$$

This contradicts that f is uniformly F-expanding at the period on all periodic points homoclinically related to p. Thus, the assumption that F is not uniformly expanding on $H(f, p)$, is invalid. Therefore, F is uniformly expanding on $H(f, p)$. Here, we finish the proof of Main Theorem.

Now, we give the proof of Theorem 1.1 under Main Theorem.

Proof. In the assumption of Theorem 1.1, for the dominated splitting $T_{H(f,p)} = E \oplus_< F$, we may assume that the splitting F splits in $F = E^c \oplus E^u$. Therefore, Main Theorem shows that f is not uniformly F-expanding at the period on all periodic points homoclinically related to p.

For every $\varepsilon > 0$, taking $r < 1$ such that $\log(r^{-1}) < \varepsilon$. Since f is not uniformly F-expanding at the period on all periodic points homoclinically related to p, there is a periodic point q homoclinically related to p such that

$$\prod_{i=1}^{\pi(q)} \left\|Df^{-1}|_{F(f^i(q))}\right\| \geq r^{\pi(q)}.$$

Since $\dim E^c = 1$, one has that

$$\left\|Df^{\pi(q)}|_{E^c(q)}\right\|^{-1} = \prod_{i=1}^{\pi(q)} \left\|Df^{-1}|_{E^c(f^i(q))}\right\| \geq \prod_{i=1}^{\pi(q)} \left\|Df^{-1}|_{F(f^i(q))}\right\| \geq r^{\pi(q)}.$$

Therefore,

$$\frac{1}{\pi(q)} \log\left(\left\|Df^{\pi(q)}|_{E^c(q)}\right\|\right) \leq \varepsilon.$$

□

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