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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A FRACTIONAL DIFFERENTIAL EQUATION WITH MULTI-POINT BOUNDARY VALUE PROBLEMS*

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Abstract In this paper, we study the existence and uniqueness solutions of a fractional differential equation with multi-point boundary value problems. By using the fixed point theorems, some new results are established and two examples are given to demonstrate the application of main results.

Keywords Fractional differential equation, existence, uniqueness, multi-point, fixed point theorem.

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1. Introduction

This paper is concerned with the existence and uniqueness of solutions to the following boundary value problem (BVP) for fractional differential equation

$$\begin{cases} D_{0+}^{\alpha}u(t)+f(t,u(t),I_{0+}^{\beta}u(t))=0, \quad 0\le t\le 1, \\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, \quad D_{0+}^{p}u(1)=\displaystyle\sum_{i=1}^{m}a_iD_{0+}^{q}u(t)|_{t=\xi_i}, \end{cases} \tag{1.1}$$

where D_{0+}^{α} is the Riemann-Liouville fractional derivative of order $n-1<\alpha\le n$, $n\ge 2$, $0<\beta<1$, $p\in[1,n-2], q\in[0,p]$, $0<\xi_1<\xi_2<\cdots<\xi_m<1$, $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ is a continuous function and $a_i>0(i=1,2,\cdots,m)$.

Fractional differential equations describe many phenomena in various fields of engineering and scientific disciplines such as physics, biophysics, chemistry, biology, economics, control theory, signal and image processing, aerodynamics, viscoelasticity, electromagnetics and other fields(see [26, 28]). In the last decade, a variety of results concerning the existence of solutions of fractional BVPs has been developed, based on various analytic techniques, such as fixed point theorems [1–3, 5, 6, 10–13, 18–25, 27, 31, 32, 42–44], topological degree method [4, 33, 34, 41], iterative techniques [9, 16, 17, 37], upper and lower solution method [8, 35, 36] and

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variational methods [15, 38–40]. In [29], Salem investigated the nonlinear m-point BVP of fractional type

$$\begin{cases} D^{\alpha}_{0+}u(t) + q(t)f(t,u(t)) = 0,\ 0<t<1, n-1<\alpha\le n, n\ge 2, \\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, \quad u(1)=\sum_{i=1}^{m-2}\zeta_i x(\eta_i), \end{cases} \tag{1.2}$$

where $0<\eta_1<\eta_2<\cdots<\eta_{m-2}<1$, $\zeta_i>0$ with $\sum_{i=1}^{m-2}\zeta_i\eta_i^{\alpha-1}<1$. It is assumed that q is a real-valued continuous function and f is a nonlinear Pettis integrable function. By means of the fixed point theorem attributed to D. O'Regan, a criterion was established for the existence of at least one Pseudo solution for the problem (1.2).

El-Shahed and Nieto [11] studied the nonlinear m-point BVP of fractional type:

$$\begin{cases} {}_RD^{\alpha}_{0+}u(t) + f(t,u(t)) = 0, \quad t\in[0,1], \alpha\in(n-1,n], n\in\mathbb{N}, \\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, \quad u(1)=\sum_{i=1}^{m-2}a_i u(\eta_i), \end{cases}$$

where $n\ge 2$, $a_i>0(i=1,2,\cdots,m-2)$, $0<\eta_1<\eta_2<\cdots<\eta_{m-2}<1$, $f\in C([0,1]\times\mathbb{R},\mathbb{R})$. The authors using the Caputo fractional derivative also considered the analogous problem:

$$\begin{cases} {}_CD^{\alpha}_{0+}u(t) + f(t,u(t)) = 0, \quad t\in[0,1], \alpha\in(n-1,n], n\in\mathbb{N}, \\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, \quad u(1)=\sum_{i=1}^{m-2}a_i u(\eta_i). \end{cases}$$

Several sufficient conditions for the existence of nontrivial solution are obtained by using the Leray-Schauder nonlinear alternative under certain growth conditions on the nonlinearity.

Goodrich [12] considered the following nonlinear fractional BVP

$$\begin{cases} D^{\nu}_{0+}u(t) + f(t,u(t)) = 0,\ 0<t<1, n-1<\nu\le n, n\ge 3, \\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, \quad [D^{\alpha}_{0+}u(t)]_{t=1}=0, 1<\alpha\le n-2, \end{cases} \tag{1.3}$$

where where D^{ν}_{0+} is the Riemann-Liouville fractional derivative of order $n-1<\nu\le n$, $n\ge 3$. By means of Krasnoselskii's fixed point theorem on cone expansion and compression to show the existence of positive solutions for BVP (1.3). The higher order fractional BVP (1.3) are also studied in [13], the existence of positive solutions of the problem are established. In [34], Xu et al. also investigated problems (1.3) with $h(t)f(t,u(t))$ instead of $f(t,u(t))$, the existence and uniqueness of positive solutions are obtained by means of the fixed point index theory in cones.

Inspired by the work of the above papers, the aim of this paper is to establish the existence and uniqueness of the solutions for the fractional differential equation multi-point BVP (1.1). The multi-point boundary value condition and the method makes our results are new and meaningful. By using the Banach's fixed point theorem, the Krasnosel'skii fixed-point theorem, the Nonlinear alternative for single valued maps and Boyd and Wong fixed point theorem, some existence and unique results of solutions are obtained.

The rest of this paper is organized as follows. In Sect. 2, we present some basic concepts of fractional calculus and transform a given problem into an equivalent integral equation problem. In Sect. 3, the existence and uniqueness results are established based on the fixed point theorems. Two illustrative examples are presented to support our main results in Sect. 4 and concluded in Sect. 5.

2. Preliminaries

For reader's convenience, we will present some preliminaries and lemmas of fractional calculus theory, which can be founded in [26, 28].

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

where $n-1 < \alpha < n$, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined as

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds,$$

where $n = [\alpha] + 1$, $[\alpha]$denotes the integer part of the number α, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2.3. *A* mapping F acting in a Banach space U is said to be a nonlinear contraction, if there exists a continuous nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(0) = 0$, $\phi(\zeta) < \zeta$ for all $\zeta > 0$ and that $\|Fu - Fv\| \leq \phi(\|u - v\|)$, $\forall\, u, v \in U$.

Lemma 2.1 ([19]). *For $h \in C(0,1) \cap L^1(0,1)$, the solution of the linear fractional differential equation*

$$D_{0+}^{\alpha} u(t) + h(t) = 0,$$

supplemented with the boundary conditions

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad D_{0+}^{p} u(1) = \sum_{i=1}^{m} a_i D_{0+}^{q} u(t)|_{t=\xi_i}$$

is equivalent to the integral equation

$$\begin{aligned} u(t) = &-\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds + \frac{t^{\alpha-1}}{\Delta \Gamma(\alpha-p)} \int_0^1 (1-s)^{\alpha-p-1} h(s) ds \\ &- \frac{t^{\alpha-1}}{\Delta \Gamma(\alpha-q)} \sum_{i=1}^{m} a_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-q-1} h(s) ds, \quad t \in [0,1], \end{aligned}$$

where

$$\Delta = \frac{\Gamma(\alpha)}{\Gamma(\alpha-p)} - \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} \sum_{i=1}^{m} a_i \xi_i^{\alpha-q-1} \neq 0. \tag{2.1}$$

Let $U = C[0,1]$, then U is a Banach space with the norm $\|u\| = \sup_{0\le t\le 1}|u(t)|$. We define an operator $A : U \to U$ as follows:

$$\begin{aligned}(Au)(t) = &-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u(s),I_{0^+}^{\beta}u(s))ds \\ &+\frac{t^{\alpha-1}}{\Delta\Gamma(\alpha-p)}\int_0^1 (1-s)^{\alpha-p-1} f(s,u(s),I_{0^+}^{\beta}u(s))ds \\ &-\frac{t^{\alpha-1}}{\Delta\Gamma(\alpha-q)}\sum_{i=1}^m a_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-q-1} f(s,u(s),I_{0^+}^{\beta}u(s))ds, \quad t\in[0,1].\end{aligned} \tag{2.2}$$

Observe that problem (1.1) has solutions if the operator A has fixed points.

For the sake of computational convenience, we set

$$\Lambda = \frac{1}{\Gamma(\alpha+1)} + \frac{1}{|\Delta|}\left(\frac{1}{\Gamma(\alpha-p+1)} + \sum_{i=1}^m \frac{a_i\xi_i^{\alpha-q}}{\Gamma(\alpha-q+1)}\right), \tag{2.3}$$

$$L_1 = 1 + \frac{1}{\Gamma(\beta+1)}. \tag{2.4}$$

3. Main results

Theorem 3.1. *Let $f : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ be a continuous function satisfying the condition:*
(H_1) There exists a positive constant $L > 0$ such that,

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \le L(|x_1-x_2| + |y_1-y_2|),$$

for any $t\in[0,1]$, $x_i, y_i \in \mathbb{R}(i=1,2)$.
Then BVP (1.1) has a unique solution if $LL_1\Lambda < 1$, where Λ and L_1 are given by (2.3) and (2.4), respectively.

Proof. Take $M = \sup_{t\in[0,1]}|f(t,0,0)|$ and $r > \frac{M\Lambda}{1-LL_1\Lambda}$. Then, $A(B_r)\subset B_r$, where $B_r = \{u\in U : \|u\|\le r\}$. In fact, for $u\in B_r$, by (H_1), we have

$$\begin{aligned}|f(t,u(t),I_{0^+}^{\beta}u(t))| &= |f(t,u(t),I_{0^+}^{\beta}u(t)) - f(t,0,0)| + |f(t,0,0)| \\ &\le L[|u(t)| + |I_{0^+}^{\beta}u(t)|] + M \\ &\le L\left[\|u\| + \frac{1}{\Gamma(\beta+1)}\|u\|\right] + M \\ &= LL_1\|u\| + M \le LL_1 r + M, \quad t\in[0,1],\end{aligned}$$

therefore, we have

$$\begin{aligned}\|Au\| \le \sup_{t\in[0,1]}\Bigg\{&\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s),I_{0^+}^{\beta}u(s))|ds \\ &+\frac{t^{\alpha-1}}{|\Delta|}\Bigg[\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|f(s,u(s),I_{0^+}^{\beta}u(s))|ds \\ &+\sum_{i=1}^m a_i\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-q-1}}{\Gamma(\alpha-q)}|f(s,u(s),I_{0^+}^{\beta}u(s))|ds\Bigg]\Bigg\}\end{aligned}$$

$$
\begin{aligned}
&\le (LL_1 r+M)\sup_{t\in[0,1]}\left\{\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds+\frac{t^{\alpha-1}}{|\Delta|}\left[\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}ds\right.\right.\\
&\quad\left.\left.+\sum_{i=1}^m a_i\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-q-1}}{\Gamma(\alpha-q)}ds\right]\right\}\\
&\le (LL_1 r+M)\sup_{t\in[0,1]}\left\{\frac{t^\alpha}{\Gamma(\alpha+1)}\right.\\
&\quad\left.+\frac{t^{\alpha-1}}{|\Delta|}\left[\frac{1}{\Gamma(\alpha-p+1)}+\sum_{i=1}^m a_i\frac{\xi_i^{\alpha-q}}{\Gamma(\alpha-q+1)}\right]\right\}\\
&\le (LL_1 r+M)\Lambda\le r.
\end{aligned}
$$

This shows that A maps B_r into itself.

Now, for $u,v\in B_r$, we obtain

$$
\begin{aligned}
\|Au-Av\|\le&\sup_{t\in[0,1]}\left\{\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s),I_{0+}^\beta u(s))-f(s,v(s),I_{0+}^\beta v(s))|ds\right.\\
&+\frac{t^{\alpha-1}}{|\Delta|}\left[\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|f(s,u(s),I_{0+}^\beta u(s))-f(s,v(s),I_{0+}^\beta v(s))|ds\right.\\
&\left.\left.+\sum_{i=1}^m a_i\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-q-1}}{\Gamma(\alpha-q)}|f(s,u(s),I_{0+}^\beta u(s))-f(s,v(s),I_{0+}^\beta v(s))|ds\right]\right\}\\
\le& L\left(\|u-v\|+\frac{\|u-v\|}{\Gamma(\beta+1)}\right)\sup_{t\in[0,1]}\left\{\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right.\\
&\left.+\frac{t^{\alpha-1}}{|\Delta|}\left[\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}ds+\sum_{i=1}^m a_i\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-q-1}}{\Gamma(\alpha-q)}ds\right]\right\}\\
\le& LL_1\Lambda\|u-v\|.
\end{aligned}
$$

Since $LL_1\Lambda<1$, the operator A is a contraction. By Banach's fixed point theorem, BVP (1.1) has a unique solution in $C[0,1]$. □

Lemma 3.1 ([7]). *(Boyd and Wong) Let U be a Banach Space, and let $F:U\to U$ be a nonlinear contraction. Then F has a unique fixed point in U.*

Theorem 3.2. *Let $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ be a continuous function satisfying the condition:*
$|f(t,x_1,y_1)-f(t,x_2,y_2)|\le g_1(t)\frac{|x_1-x_2|}{H^*+1}+g_2(t)\frac{|y_1-y_2|}{H^*+1}$, $t\in(0,1)$, $x_i,y_i\in\mathbb{R}, i=1,2,$ *where $g_1,g_2:(0,1)\to\mathbb{R}^+$ are continuous function with $H^*=\left(\|g_1\|+\frac{\|g_2\|}{\Gamma(\gamma+1)}\right)\Lambda$, where Λ is given by (2.3). Then BVP (1.1) has a unique solution.*

Proof. Consider the operator $A:U\to U$. Let the continuous nondecreasing function $\phi:\mathbb{R}^+\to\mathbb{R}^+$ be defined by

$$\phi(\zeta)=\frac{H^*\zeta}{H^*+1},\quad \forall\zeta\ge 0.$$

Observe that $\phi(0) = 0$ and $\phi(\zeta) < \zeta$ for all $\zeta > 0$. For any $u, v \in U$, we can get

$$\begin{aligned}|f(s,u(s),I_{0+}^{\beta}u(s)) - f(s,v(s),I_{0+}^{\beta}v(s))| \\ \leq & g_1(s)\frac{|u(s)-v(s)|}{H^*+1} + g_2(s)\frac{|I_{0+}^{\beta}u(s) - I_{0+}^{\beta}v(s)|}{H^*+1} \\ \leq & g_1(s)\frac{\|u-v\|}{H^*+1} + g_2(s)\frac{\|u-v\|}{(H^*+1)\Gamma(\beta+1)} \\ \leq & \left(\frac{\|g_1\|}{H^*} + \frac{\|g_2\|}{H^*\Gamma(\beta+1)}\right)\phi(\|u-v\|).\end{aligned}$$

Then for any $u, v \in U$, we have

$$\begin{aligned}\|Au - Av\| \leq & \sup_{t\in[0,1]}\left\{\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s),I_{0+}^{\beta}u(s)) - f(s,v(s),I_{0+}^{\beta}v(s))|ds\right. \\ & + \frac{t^{\alpha-1}}{|\Delta|}\left[\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|f(s,u(s),I_{0+}^{\beta}u(s)) - f(s,v(s),I_{0+}^{\beta}v(s))|ds\right. \\ & \left.\left.+ \sum_{i=1}^{m} a_i\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-q-1}}{\Gamma(\alpha-q)}|f(s,u(s),I_{0+}^{\beta}u(s)) - f(s,v(s),I_{0+}^{\beta}v(s))|ds\right]\right\} \\ \leq & \left(\frac{\|g_1\|}{H^*} + \frac{\|g_2\|}{H^*\Gamma(\beta+1)}\right)\phi(\|u-v\|)\Lambda.\end{aligned}$$

So we get $\|Au - Av\| \leq \phi(\|u-v\|)$ and A is a nonlinear contraction, by Lemma 3.1, A has a unique fixed point in U and BVP (1.1) has a unique solution. □

Lemma 3.2 ([30]). *(Krasnoselskii) Let Q be a closed, convex, bounded and nonempty subset of a Banach space E. Let G_1, G_2 be operators such that (i) $G_1u_1 + G_2u_2 \in Q$ whenever $u_1, u_2 \in Q$; (ii) G_1 is compact and continuous; (iii) G_2 is a contraction mapping. Then there exists $v \in Q$ such that $v = G_1v + G_2v$.*

Theorem 3.3. *Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function satisfying (H_1). In addition, the following assumption holds:*
(H_2) $|f(t,x,y)| \leq \omega(t)$, $\forall (t,x,y) \in [0,1] \times \mathbb{R}^2$, and $\omega \in C([0,1], \mathbb{R}^+)$.
Then BVP (1.1) has at least one solution in $C[0,1]$ provided

$$\frac{LL_1}{|\Delta|}\left(\frac{1}{\Gamma(\alpha-p+1)} + \sum_{i=1}^{m}\frac{a_i\xi_i^{\alpha-q}}{\Gamma(\alpha-q+1)}\right) < 1. \tag{3.1}$$

Proof. Let $B_r = \{u \in U : \|u\| \leq r\}$, where $r > \|\omega\|\Lambda$, ($\|\omega\| = \sup_{t\in[0,1]}|\omega(t)|$). Define the operators A_1 and A_2 on B_r as:

$$\begin{aligned}(A_1u)(t) = & -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,u(s),I_{0+}^{\beta}u(s))ds, \\ (A_2u)(t) = & \frac{t^{\alpha-1}}{\Delta\Gamma(\alpha-p)}\int_0^1 (1-s)^{\alpha-p-1}f(s,u(s),I_{0+}^{\beta}u(s))ds \\ & - \frac{t^{\alpha-1}}{\Delta\Gamma(\alpha-q)}\sum_{i=1}^{m} a_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-q-1}f(s,u(s),I_{0+}^{\beta}u(s))ds.\end{aligned}$$

For any $x, y \in B_r$, easily we can prove that $\|A_1x + A_2y\| \le \|\omega\|\Lambda \le r$, where Λ is given by (2.3). So, $A_1x + A_2y \in B_r$.

Now, we claim that the operator A_2 is a contraction. In fact, for any $u, v \in B_r$, we have

$$\begin{aligned}
&\|A_2u - A_2v\| \\
&\le \sup_{t\in[0,1]} \left\{ \frac{t^{\alpha-1}}{|\Delta|} \left[\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} |f(s,u(s),I_{0+}^{\beta}u(s)) - f(s,v(s),I_{0+}^{\beta}v(s))|ds \right.\right. \\
&\quad \left.\left. + \sum_{i=1}^{m} a_i \int_0^{\xi_i} \frac{(\xi_i-s)^{\alpha-q-1}}{\Gamma(\alpha-q)} |f(s,u(s),I_{0+}^{\beta}u(s)) - f(s,v(s),I_{0+}^{\beta}v(s))|ds \right] \right\} \\
&\le LL_1\|u-v\| \sup_{t\in[0,1]} \left\{ \frac{t^{\alpha-1}}{|\Delta|} \left[\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} ds + \sum_{i=1}^{m} a_i \int_0^{\xi_i} \frac{(\xi_i-s)^{\alpha-q-1}}{\Gamma(\alpha-q)} ds \right] \right\} \\
&\le \frac{LL_1}{|\Delta|} \left(\frac{1}{\Gamma(\alpha-p+1)} + \sum_{i=1}^{m} \frac{a_i\xi_i^{\alpha-q}}{\Gamma(\alpha-q+1)} \right) \|u-v\|.
\end{aligned}$$

This together with (3.1) show that A_2 is a contraction operator.

Next, we shall show that A_1 is continuous and compact. It follows f is continuous that the operator A_1 is continuous. And, since $\|A_1x\| \le \frac{\|\omega\|}{\Gamma(\alpha+1)}$, so A_1 is uniformly bounded on B_r. Moreover, for any $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, we have

$$\begin{aligned}
|(A_1u)(t_2) - (A_1u)(t_1)| \le& \left| \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),I_{0+}^{\beta}u(s))ds \right. \\
&\left. - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),I_{0+}^{\beta}u(s))ds \right| \\
\le& \frac{\|\omega\|}{\Gamma(\alpha+1)} \left(|t_2^{\alpha} - t_1^{\alpha}| + |2(t_2-t_1)^{\alpha}| \right),
\end{aligned}$$

which tends to zero independent of u as $t_2 \to t_1$. Hence, A_1 is relatively compact on B_r. By the Arzela-Ascoli theorem, the operator A_1 is compact on B_r. Krasnoselskii's fixed point theorem 3.2 implies that there exists a solution for BVP (1.1) in $C[0,1]$. □

Lemma 3.3 ([30]). *Let E be a Banach space. Assume that $A : E \to E$ is a completely continuous operator and the set $B = \{x \in E : x = \xi Tx, 0 < \xi < 1\}$ is bounded. Then A has a fixed point in E.*

Theorem 3.4. *Assume that there exists a positive constant L_2 such that $|f(t,x,y)| \le L_2$ for all $t \in [0,1], x, y \in \mathbb{R}$. Then there exists at least one solution for BVP (1.1) in $C[0,1]$.*

Proof. Firstly, we will show that the operator A is completely continuous. Obviously, continuity of f implies the continuity of A. Suppose $B \subset U$ is bounded, then, $\forall\ u \in B$, one can easily obtain $|(Au)(t)| \le L_2\Lambda = L_3$, where Λ is given by

(2.3). Furthermore, for any $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, we have

$$\begin{aligned}|(Au)(t_2) - (Au)(t_1)| \leq & L_2 \Bigg[\frac{|t_2^\alpha - t_1^\alpha| + 2(t_2 - t_1)^\alpha}{\Gamma(\alpha+1)} \\ & + \frac{|t_2^{\alpha-1} - t_1^{\alpha-1}|}{|\Delta|} \left(\frac{1}{\Gamma(\alpha-p+1)} + \sum_{i=1}^m \frac{a_i \xi_i^{\alpha-q}}{\Gamma(\alpha-q+1)} \right) \Bigg],\end{aligned}$$

which tends to zero independent of u as $(t_2 - t_1) \to 0$. Therefore, A is equicontinuous on $[0,1]$. According to Arzela-Ascoli theorem, we can get that A is completely continuous.

Next, define a set $N = \{u \in U : u = \rho Au, 0 < \rho < 1\}$, and we show that N is bounded. For any $u \in U$, we have

$$\begin{aligned}u(t) = & -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s), I_{0^+}^\beta u(s)) ds \\ & + \frac{t^{\alpha-1}}{\Delta\Gamma(\alpha-p)} \int_0^1 (1-s)^{\alpha-p-1} f(s, u(s), I_{0^+}^\beta u(s)) ds \\ & - \frac{t^{\alpha-1}}{\Delta\Gamma(\alpha-q)} \sum_{i=1}^m a_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-q-1} f(s, u(s), I_{0^+}^\beta u(s)) ds, \quad t \in [0,1].\end{aligned}$$

Similarly, we can obtain $|u(t)| = \rho|(Au)(t)| \leq L_2 \Lambda = L_3$, which implies that $\|u\| \leq L_3$, for any $u \in N, t \in [0,1]$. Hence, N is bounded. Consequently, by Lemma 3.3, BVP (1.1) has at least one solution in $C[0,1]$. $\square$

Lemma 3.4 ([14]). *(Nonlinear alternative for single valued maps) Let X be a Banach space, X_1 a closed, convex subset of X, H an open subset of X_1 and $0 \in H$. Suppose that $A : \overline{H} \to X_1$ is a continuous, compact (that is, $A(\overline{H})$ is a relatively compact subset of X_1) map. Then either A has a fixed point in $\overline{H}$ or there is an $x \in \partial H$ (the boundary of H in X_1) and $\vartheta \in (0,1)$ with $x = \vartheta A(x)$.*

Theorem 3.5. *Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function and satisfying the following assumption:*
(H_3) There exists a function $\psi \in C([0,1], \mathbb{R}^+)$ and a nondecreasing, subhomogeneous (that is, $\varphi(mu) \leq m\varphi(u)$, for all $m \geq 1$ and $u \in \mathbb{R}^+$) function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|f(t,u,v)| \leq \psi(t)\varphi(\|u\| + \|v\|), \quad for\ all\ (t,u,v) \in [0,1] \times \mathbb{R}^2.$$

(H_4) There exists a constant $C > 0$ such that

$$C \left[\|\psi\| L_1 \varphi(C) \left(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{|\Delta|} \left(\frac{1}{\Gamma(\alpha-p+1)} + \sum_{i=1}^m \frac{a_i \xi_i^{\alpha-q}}{\Gamma(\alpha-q+1)} \right) \right) \right]^{-1} > 1,$$

where where Δ and L_1 are given by (2.1) and (2.4), respectively.
Then BVP (1.1) has at least one solution in $C[0,1]$.

Proof. Consider the operator $A : U \to U$ by defined by (2.2). Firstly, we will show that A maps bounded sets into bound sets in U. For $\eta > 0$, let $B_\eta = \{u \in$

$U : \|u\| \leq \eta\}$ be a bounded set in U. Then, for $u \in B_\eta$, in view of (H_3), we get

$$\begin{aligned}|(Au)(t)| \leq &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(\psi(s)\varphi(L_1\|u\|))ds \\ &+ \frac{t^{\alpha-1}}{|\Delta|}\left[\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}(\psi(s)\varphi(L_1\|u\|))ds \right.\\ &\left.+ \sum_{i=1}^m a_i \int_0^{\xi_i} \frac{(\xi_i-s)^{\alpha-q-1}}{\Gamma(\alpha-q)}(\psi(s)\varphi(L_1\|u\|))ds\right], \quad t \in [0,1],\end{aligned}$$

it follows that

$$\|Au\| \leq \|\psi\|\varphi(L_1\eta)\left[\frac{1}{\Gamma(\alpha+1)} + \frac{1}{|\Delta|}\left(\frac{1}{\Gamma(\alpha-p+1)} + \sum_{i=1}^m a_i \frac{\xi_i^{\alpha-q}}{\Gamma(\alpha-q+1)}\right)\right].$$

Next, we show that A maps bounded sets into equicontinuous sets of U. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, and $u \in B_\eta$, then we obtain

$$\begin{aligned}|(Au)(t_2) - (Au)(t_1)| \leq L_1\|\psi\|\varphi(\eta)&\left[\frac{|t_2^\alpha - t_1^\alpha| + 2(t_2-t_1)^\alpha}{\Gamma(\alpha+1)}\right.\\ &\left.+ \frac{|t_2^{\alpha-1} - t_1^{\alpha-1}|}{|\Delta|}\left(\frac{1}{\Gamma(\alpha-p+1)} + \sum_{i=1}^m \frac{a_i\xi_i^{\alpha-q}}{\Gamma(\alpha-q+1)}\right)\right].\end{aligned}$$

Obviously the right-hand side of the above inequalities tends to zero independently of $u \in B_\eta$ as $(t_2 - t_1) \to 0$. Thus, by the Arzela-Ascoli theorem, the operator $A : U \to U$ is completely continuous.

Let u be a solution. Then, for $\eta_1 \in (0,1)$, together with that A is bounded, we obtain

$$\begin{aligned}|u(t)| =&|\eta_1(Au)(t)| \\ \leq&\|\psi\|\varphi(\|u\| + \frac{1}{\Gamma(\beta+1)}\|u\|) \\ &\times\left[\frac{1}{\Gamma(\alpha+1)} + \frac{1}{|\Delta|}\left(\frac{1}{\Gamma(\alpha-p+1)} + \sum_{i=1}^m a_i\frac{\xi_i^{\alpha-q}}{\Gamma(\alpha-q+1)}\right)\right] \\ \leq&L_1\|\psi\|\varphi(\|u\|)\left[\frac{1}{\Gamma(\alpha+1)} + \frac{1}{|\Delta|}\left(\frac{1}{\Gamma(\alpha-p+1)} + \sum_{i=1}^m a_i\frac{\xi_i^{\alpha-q}}{\Gamma(\alpha-q+1)}\right)\right],\end{aligned}$$

which yields

$$\|u\|\left[\|\psi\|L_1\varphi(\|u\|)\left(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{|\Delta|}\left(\frac{1}{\Gamma(\alpha-p+1)} + \sum_{i=1}^m \frac{a_i\xi_i^{\alpha-q}}{\Gamma(\alpha-q+1)}\right)\right)\right]^{-1} \leq 1.$$

In view of (H_4), there exists $C > 0$, such that $\|u\| \neq C$. Choose $D = \{u \in U : \|u\| \leq C+1\}$. Note that the operator $A : \overline{D} \to U$ is continuous and completely continuous. From the choice of D, there is no $u \in \partial D$ such that $u = \eta_1 A(u)$ for some $\eta_1 \in (0,1)$. Consequently, by Lemma 3.4 we deduce that A has a fixed point $u \in \overline{D}$ which is a solution of BVP (1.1). □

4. Examples

Example 4.1. Consider the following problem

$$\begin{cases} D_{0+}^{\frac{7}{2}}u(t)+f(t,u(t),I_{0+}^{\frac{1}{2}}u(t))=0, \quad 0\le t\le 1,\\ u(0)=u'(0)=u''(0)=0,\\ D_{0+}^{2}u(1)=\sum_{i=1}^{3}a_iD_{0+}^{1}u(t)|_{t=\xi_i}, \end{cases} \tag{4.1}$$

where

$$\alpha=\frac{7}{2},\quad \beta=\frac{1}{2},\quad p=2,\quad q=1,\quad a_1=\frac{1}{2},$$
$$a_2=\frac{1}{4},\quad a_3=\frac{1}{3},\quad \xi_1=\frac{1}{3},\quad \xi_2=\frac{1}{2},\quad \xi_3=\frac{1}{5}.$$

By a simple computation, we have

$$\Delta=\frac{\Gamma(\alpha)}{\Gamma(\alpha-p)}-\left(\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}\right)\sum_{i=1}^{m}a_i\xi_i^{\alpha-q-1}=3.214,$$
$$\Lambda=\frac{1}{\Gamma(\alpha+1)}+\frac{1}{|\Delta|}\left(\frac{1}{\Gamma(\alpha-p+1)}+\sum_{i=1}^{m}\frac{a_i\xi_i^{\alpha-q}}{\Gamma(\alpha-q+1)}\right)=0.3277,$$
$$L_1=1+\frac{1}{\Gamma(\beta+1)}=2.1283.$$

We consider

$$\begin{aligned} f(t,u(t),I_{0+}^{\frac{1}{2}}u(t))=&\frac{1}{\sqrt{t+169}}\left(\frac{1}{\sqrt{t+4}}u(t)+\frac{1}{\sqrt{t^2+9}}\tan^{-1}(u(t))\right)\\ &+\frac{5}{78}I_{0+}^{\frac{1}{2}}u(t)+\sin(\frac{\pi t}{2}). \end{aligned}$$

Obviously,

$$|f(t,u(t),I_{0+}^{\frac{1}{2}}u(t))-f(t,v(t),I_{0+}^{\frac{1}{2}}v(t))|\le\frac{5}{78}(\|u-v\|+\|I_{0+}^{\frac{1}{2}}u-I_{0+}^{\frac{1}{2}}v\|),$$

and $L=\frac{5}{78}$. Further, $LL_1\Lambda\approx 0.0447<1$.

Therefore, all conditions of Theorem 3.1 are satisfied. Thus, by Theorem 3.1, we conclude that problem (4.1) has a unique solution.

Example 4.2. Consider the problem (4.1) with

$$f(t,u(t),I_{0+}^{\frac{1}{2}}u(t))=\frac{1}{30+t}\left(2u(t)\sin(u(t))+\sqrt{\pi}I_{0+}^{\frac{1}{2}}u(t)+4\right).$$

Obviously, $|f(t,u(t),I_{0+}^{\frac{1}{2}}u(t))|\le\frac{4}{30+t}(\|u\|+1)$ with $\psi(t)=\frac{4}{30+t}$, $\|\psi\|=\frac{2}{15}$, $\varphi(\|u\|)=1+\|u\|$, we find that $C>0.1025$.

Therefore, all conditions of Theorem 3.5 are satisfied. Thus, by Theorem 3.5, we conclude that problem (4.1) exists at least one solution.

5. Conclusion

In this paper, we obtained several sufficient conditions for the existence and unique of solutions for a class of fractional-order multi-point boundary value problem. Our results will be a useful contribution to the existing literature on the topic of fractional-order nonlocal differential equations. The results of the existence and uniqueness are demonstrated on two relevant examples.

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