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GLOBAL ANALYSIS OF AN AGE-STRUCTURED SEIR MODEL WITH IMMIGRATION OF POPULATION AND NONLINEAR INCIDENCE RATE*

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Abstract Epidemic models with infection age of infectious individuals have been extensively studied, however, most of the existing works ignore the combined effects of immigration and nonlinear incidence. In this paper, we incorporate both the effects of immigration and nonlinear incidence, based on which we formulate an SEIR epidemic model. We give a rigorous mathematical analysis on some necessary technical materials. Then, by constructing a Lyapunov functional, we show that the endemic equilibrium is globally asymptotically stable. Numerical simulations of an application are given to support our theoretical results.

Keywords Lyapunov functional, global stability, infection age, immigration, nonlinear incidence.

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1. Introduction

Mathematical modelling of natural phenomena in epidemiology has been widely used in the last ten decades. In 1911, with the milestone foundations on the approach to epidemiology based on compartmental models, Ross proposed and studied a malaria model [32]. In 1927, Kermack and McKendrick established the remarkable epidemic model which is known as susceptible (S)-infectious (I)-recovered (R) model [18]. After this research, a very large number of models have been studied, which include SIS models, SEIR models with or without delays (see, for example, [20, 24, 35]).

The incidence rate in epidemic models plays an important role in the disease dynamics [2]. Traditionally, the incidence rate of an infectious disease in most of the literature is assumed to be of mass action form $\beta S(t)I(t)$ [2]. But since the disease transmission process is generally unknown [21], some nonlinear incidence rates have been introduced and studied (to name a few, [9, 15, 27, 39]). For more general cases, Capasso et al. [5] and Li et.al [23] considered an incidence rate with

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the form $f(I)S$; Korobeinikov and Maini studied models with incidence rate of the form $g(I)h(S)$ [21], where the global asymptotic stability of an SIR and an SEIR model were shown by constructing suitable Lyapunov functions.

Infection age is also an important factor in epidemiology. Recently, Rost and Wu [33] formulated an SEIR model with infection-age structure. They rewrote the model as a delay differential equation and investigated the local stability of both disease-free and endemic equilibria. In 2010, Magal et al. [28] prosed the following SIR model with infection age:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{dt}} = \Lambda_s - \mu_s S(t) - \displaystyle\int_0^\infty \beta(a)S(t)i(t,a)\mathrm{d}a, \\ \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a}\right) i(t,a) = -(\mu_i(a) + \delta(a))i(t,a), \\ i(t,0) = \displaystyle\int_0^\infty \beta(a)S(t)i(t,a)\mathrm{d}a, \\ S(0) = S_0 > 0, \quad i(0,\cdot) = i_0 \in L_1^+(0,+\infty), \end{cases}$$

where $S(t)$ is the numbers of susceptible populations at time t, while $i(t,a)$ denote the densities at time t of infectious individuals who have been infectious for duration a. The biological interpretation of all coefficients are shown in Table 1. The authors show that the unique endemic equilibrium of system (1) is globally stable amongst solutions for which disease transmission occurs by a suitable Lyapunov functional. For some recent works on models with infection age, we refer readers to the papers [1, 7, 10–12, 17, 19, 25, 26, 37, 40, 45] and the monographs [16, 44].

Moreover, because of the rapid globalization during the last decades, movements among different regions or countries have become more and more frequently. Indeed, people can arrive at any place on this planet within days, which brings new challenge of controlling the global spread of infectious diseases. For example, international traveling significantly accelerated the transmission of the 2003 SARS pandemic [31] and the outbreak of avian-origin influenza A(H7N9) [13]. Due to these facts, some researchers introduced immigration into infectious disease models [4, 41]. In these works, immigration of population was always supposed to be of constant rates. However, in the real world, age-dependent immigration rate seems more realistic [30, 43], for example, the children immigration rates of different ages could not be constant. Thus it is meaningful for us to investigate the models that consider the effects of immigration of infectious individuals.

Based on the above motivations, in this paper, we extend the model in [30] by considering general nonlinear incidence rate. Mathematically, both age-dependent immigration rate and general nonlinear incidence bring nontrivial challenges in analysis, especially in the well-posedness problem and in the construction of suitable Lyapunov functionals. So it is worthwhile for us to study the properties of this kind of models.

The rest of this paper is organized as follows. In Section 2, we formulate our model and give the assumptions. In Section 3, we identify the dissipativeness and positivity of the model. In Section 4, the asymptotic smoothness is established. Section 5 is devoted to the existence and local stability of the equilibrium while its global stability is established in Section 6 by employing the approach of Lyapunov functionals. In Section 7, we introduce an age-structured SEIR model with saturation incidence rate and immigration, which can be regarded as special cases of the general model studied in Section 2, and we perform numerical simulation to verify the validity of our main theoretical results. The paper ends with a brief discussion.

2. The model and preliminaries

By considering nonlinear incidence rate, we study the following model (since we have assumed that the recovered populations have gained permanent immunity),

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda_s - \mu_s S(t) - \displaystyle\int_0^\infty \beta(a) f(S(t)) h(i(t,a)) \mathrm{d}a, \\ \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a}\right) e(t,a) = \Lambda_e(a) - (\mu_e(a) + \gamma(a)) e(t,a), \\ \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a}\right) i(t,a) = \Lambda_i(a) - (\mu_i(a) + \delta(a)) i(t,a), \end{cases} \tag{2.1}$$

with the boundary conditions

$$\begin{cases} e(t,0) = \displaystyle\int_0^\infty \beta(a) f(S(t)) h(i(t,a)) \mathrm{d}a, \\ i(t,0) = \displaystyle\int_0^\infty \gamma(a) e(t,a) \mathrm{d}a \end{cases} \tag{2.2}$$

and the initial condition

$$x_0 = (S(0), e(0,\cdot), i(0,\cdot)) = (S_0, e_0, i_0) \in \mathcal{X}_+, \tag{2.3}$$

where $S(t)$ denotes the number of susceptible populations at time t, while $e(t,a)$ and $i(t,a)$ denote the densities at time t of exposed and infectious individuals who have been exposed and infectious for duration a, respectively. All coefficients are assumed to be positive and the biological interpretation of coefficients is listed in Table 1.

Table 1. The biological interpretation of coefficients of model (2.1).

Coefficient	Interpretation
Λ_s	Recruitment through birth and immigration for the susceptible
$\Lambda_e(a)$	Recruitment into the exposed class at age a
$\Lambda_i(a)$	Recruitment into the infectious class at age a
μ_s	The per capita death rate of the susceptible
$\mu_e(a)$	The per capita death rate of the exposed at age a
$\mu_i(a)$	The per capita death rate of the infectious at age a
$\beta(a)$	Disease transmission rate between the susceptible and infectious at age a
$\gamma(a)$	The rate of progression from the exposed to the infectious occurring at age a
$\delta(a)$	The recovery rate of the infectious at age a

Let $\mathcal{X}_+ = \mathbb{R}_+ \times L^1_+(0,\infty) \times L^1_+(0,\infty)$, which is the nonnegative cone of the banach space $\mathcal{X} = \mathbb{R} \times L^1(0,\infty) \times L^1(0,\infty)$ equipped with the norm

$$\|(x,\varphi,\phi)\|_{\mathcal{X}} = |x| + \int_0^\infty |\varphi(a)|\mathrm{d}a + \int_0^\infty |\phi(a)|\mathrm{d}a, \qquad (x,\varphi,\phi) \in \mathcal{X}.$$

In the sequel, we always assume the initial condition (S_0, e_0, i_0) satisfies

$$e_0(0) = \int_0^\infty \beta(a) f(S_0) h(i_0(a))\mathrm{d}a \quad \text{and} \quad i_0(0) = \int_0^\infty \gamma(a) e_0(a)\mathrm{d}a.$$

Following the standard theory in [44], (2.1) has a unique nonnegative solution on $\mathbb{R}_+$. Then we get a continuous semiflow associated with (2.1), that is, $\Phi : \mathbb{R}_+ \times \mathcal{X}_+ \to \mathcal{X}_+$ defined by

$$\Phi(t, x_0) = (S(t), e(t,\cdot), i(t,\cdot)), \qquad t \in \mathbb{R}_+,\ x_0 \in \mathcal{X}_+.$$

To further the study, we make the following assumptions on the parameters and the incidence rate.

Assumption 2.1. Assume that

(i) Λ_s, $\mu_s > 0$; Λ_e, $\Lambda_i \in L^1_+(0,\infty)$; μ_e, μ_i, β, γ, $\delta \in L^\infty_+(0,\infty)$.

(ii) β and γ are Lipschitz continuous with Lipschitz constants M_β and M_γ, respectively.

(iii) The support of each β, γ, and $\Lambda_e + \Lambda_i$ has a positive measure.

Assumption 2.2. For $x \in \mathbb{R}$

(i) $f(x) \geq 0$ and $h(x) \geq 0$ with $f(x) = 0$ or $h(x) = 0$ if and only if $x = 0$.

(ii) $f'(x) > 0$, $h'(x) > 0$ and $f''(x) < 0$, $h''(x) < 0$.

Functions f and h satisfying Assumption 2.2 are quite general. Chen et al. [8] gave a summary of such functions, which include the bilinear incidence rate with $f(S) = S$ and $h(I) = I$, the saturated incidence rate with $h(I) = \frac{I}{1+\alpha I}$ by [5], the saturated nonlinear incidence rate with $h(I) = \frac{I}{1+\alpha I^p}(0 < p < 1)$ by [27], and so on. Assumption 2.2 also implies that $f(x)$ and $h(x)$ are Lipschitz continuous on $\mathbb{R}_+$. Denote their corresponding Lipschitz constants by M_f and M_h, respectively.

We denote $\bar{\mu}_e$, $\bar{\mu}_i$, $\bar{\beta}$, $\bar{\gamma}$, and $\bar{\delta}$ to be the essential infimums of μ_e, μ_i, β, γ, and δ, respectively, while $\hat{\mu}_e$, $\hat{\mu}_i$, $\hat{\beta}$, $\hat{\gamma}$, and $\hat{\delta}$ to be the corresponding essential supremums. We also denote $\tilde{\Lambda}_e = \int_0^\infty \Lambda_e(a)\mathrm{d}a$ and $\tilde{\Lambda}_i = \int_0^\infty \Lambda_i(a)\mathrm{d}a$.

3. Dissipativeness

For convenience, we define two notations

$$\Omega(a) = e^{-\int_0^a (\mu_e(\theta)+\gamma(\theta))\mathrm{d}\theta}, \tag{3.1}$$

$$\Gamma(a) = e^{-\int_0^a (\mu_i(\theta)+\delta(\theta))\mathrm{d}\theta}. \tag{3.2}$$

From equations 2.2 and (2.3), using the method in [22] to integrate the second and the third equations in (2.1) along the characteristic lines $t - a = const.$, we have

$$e(t,a) = \begin{cases} e(t-a,0)\Omega(a) + \displaystyle\int_0^a \Lambda_e(\epsilon)\frac{\Omega(a)}{\Omega(\epsilon)}\mathrm{d}\epsilon, & 0 \leq a \leq t, \\ e(0,a-t)\dfrac{\Omega(a)}{\Omega(a-t)} + \displaystyle\int_{a-t}^a \Lambda_e(\epsilon)\frac{\Omega(a)}{\Omega(\epsilon)}\mathrm{d}\epsilon, & 0 \leq t \leq a, \end{cases} \tag{3.3}$$

$$i(t,a)=\begin{cases} i(t-a,0)\Gamma(a)+\displaystyle\int_0^a \Lambda_i(\epsilon)\frac{\Gamma(a)}{\Gamma(\epsilon)}\mathrm{d}\epsilon, & 0\le a\le t,\\ i(0,a-t)\dfrac{\Gamma(a)}{\Gamma(a-t)}+\displaystyle\int_{a-t}^a \Lambda_i(\epsilon)\frac{\Gamma(a)}{\Gamma(\epsilon)}\mathrm{d}\epsilon, & 0\le t\le a.\end{cases} \tag{3.4}$$

Now, we concern with the boundedness of solutions to (2.1).

Proposition 3.1. *Denote $\Lambda=\Lambda_s+\tilde{\Lambda}_e+\tilde{\Lambda}_i$ and $\mu=\min\{\mu_s,\bar{\mu}_e,\bar{\mu}_i\}$. For (2.1), we have the following statements.*

(i) $\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi(t,x_0)\|_{\mathcal{X}}\le\Lambda-\mu\|\Phi(t,x_0)\|_{\mathcal{X}}$ *for all* $t\in\mathbb{R}_+$.

(ii) $\|\Phi(t,x_0)\|_{\mathcal{X}}\le\max\{\frac{\Lambda}{\mu},\frac{\Lambda}{\mu}+e^{-\mu t}(\|x_0\|_{\mathcal{X}}-\frac{\Lambda}{\mu})\}\le\max\{\frac{\Lambda}{\mu},\|x_0\|_{\mathcal{X}}\}$ *for all* $t\in\mathbb{R}_+$.

(iii) $\limsup_{t\to\infty}\|\Phi_t(x_0)\|_{\mathcal{X}}\le\frac{\Lambda}{\mu}$.

(iv) Φ *is point dissipative, that is, there is a bounded set that attracts all points in* $\mathcal{X}_+$.

Proof. For any function $g(\tau,a)$, the following identity plays an important role in the discussion,

$$\int_0^t\int_0^a g(\tau,a)\mathrm{d}\tau\mathrm{d}a+\int_t^\infty\int_{a-t}^a g(\tau,a)\mathrm{d}\tau\mathrm{d}a=\int_0^\infty\int_\tau^{\tau+t} g(\tau,a)\mathrm{d}a\mathrm{d}\tau, \tag{3.5}$$

which is obtained by interchanging the order of integration.

Note that

$$\begin{aligned}\int_0^\infty e(t,a)\mathrm{d}a&=\int_0^t e(t,a)\mathrm{d}a+\int_t^\infty e(t,a)\mathrm{d}a\\ &=\int_0^t e(t-a,0)\Omega(a)\mathrm{d}a+\int_t^\infty e(0,a-t)\frac{\Omega(a)}{\Omega(a-t)}\mathrm{d}a\\ &\quad+\int_0^t\int_0^a\Lambda_e(\epsilon)\frac{\Omega(a)}{\Omega(\epsilon)}\mathrm{d}\epsilon\mathrm{d}a+\int_t^\infty\int_{a-t}^a\Lambda_e(\epsilon)\frac{\Omega(a)}{\Omega(\epsilon)}\mathrm{d}\epsilon\mathrm{d}a.\end{aligned}$$

Using (3.5) for the double integrals and making change of integration variable for the two single integrals give

$$\begin{aligned}\int_0^\infty e(t,a)\mathrm{d}a&=\int_0^t e(\tau,0)\Omega(t-\tau)\mathrm{d}\tau+\int_0^\infty e(0,\tau)\frac{\Omega(t+\tau)}{\Omega(\tau)}\mathrm{d}\tau\\ &\quad+\int_0^\infty\int_\epsilon^{\epsilon+t}\Lambda_e(\epsilon)\frac{\Omega(a)}{\Omega(\epsilon)}\mathrm{d}a\mathrm{d}\epsilon.\end{aligned}$$

Then

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty e(t,a)\mathrm{d}a\\ &=e(t,0)+\int_0^t e(\tau,0)\frac{\mathrm{d}}{\mathrm{d}t}\Omega(t-\tau)\mathrm{d}\tau\\ &\quad+\int_0^\infty e(0,\tau)\frac{\frac{\mathrm{d}}{\mathrm{d}t}\Omega(t+\tau)}{\Omega(\tau)}\mathrm{d}\tau+\int_0^\infty\Lambda_e(\epsilon)\frac{\Omega(\epsilon+t)}{\Omega(\epsilon)}\mathrm{d}\epsilon.\end{aligned}$$

With the help of [30, Equation 12], we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty e(t,a)\mathrm{d}a = e(t,0) - \int_0^\infty (\mu_e(a)+\gamma(a))e(t,a)\mathrm{d}a + \tilde{\Lambda}_e.$$

Similarly,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty i(t,a)\mathrm{d}a = i(t,0) - \int_0^\infty (\mu_i(a)+\delta(a))i(t,a)\mathrm{d}a + \tilde{\Lambda}_i.$$

Therefore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi(t,X_0)\|_{\mathcal{X}} &= \Lambda_s - \mu_s S(t) - \int_0^\infty \beta(a) f(S(t))h(i(t,a))\mathrm{d}a \\ &\quad + e(t,0) - \int_0^\infty (\mu_e(a)+\gamma(a))e(t,a)\mathrm{d}a + \tilde{\Lambda}_e \\ &\quad + i(t,0) - \int_0^\infty (\mu_i(a)+\delta(a))i(t,a)\mathrm{d}a + \tilde{\Lambda}_i \\ &\le \Lambda - \mu\|\Phi(t,X_0)\|_{\mathcal{X}}.\end{aligned}$$

This shows statement (i). Using the variation of constants formula yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi(t,X_0)\|_{\mathcal{X}} \le \frac{\Lambda}{\mu} - e^{-\mu t}\left(\frac{\Lambda}{\mu} - \|X_0\|_{\mathcal{X}}\right).$$

Then the remaining three statements follow immediately and hence the proof is complete. □

From Proposition 3.1, we can easily get the following result.

Proposition 3.2. *If $x_0 \in \mathcal{X}_+$ and $\|x_0\|_{\mathcal{X}} \le M$ with some constant $M \ge \frac{\Lambda}{\mu}$, then the following statements hold for $t \in \mathbb{R}_+$.*

(i) $0 \le S(t)$, $\int_0^\infty e(t,a)\mathrm{d}a$, $\int_0^\infty i(t,a)\mathrm{d}a \le M$.

(ii) $e(t,0) \le \hat{\beta} f'(0)h'(0)M^2$, $i(t,0) \le \hat{\gamma}\hat{\beta} f'(0)h'(0)M^2$.

The following proposition provides a positive asymptotic lower bound for $S(t)$.

Proposition 3.3. *If $x_0 \in \mathcal{X}_+$ then*

$$\liminf_{t\to\infty} S(t) \ge \frac{\Lambda_s}{\mu_s + \hat{\beta} f'(0)h'(0)\frac{\Lambda}{\mu}}.$$

Proof. For any $\epsilon > 0$, it follows from Proposition 3.1 that there exists a $t_0 \in \mathbb{R}_+$ such that $\int_0^\infty i(t,a)\mathrm{d}a \le \frac{\Lambda}{\mu} + \epsilon$ for $t \ge t_0$. Then, for $t \ge t_0$,

$$\begin{aligned}\frac{\mathrm{dS(t)}}{\mathrm{dt}} &= \Lambda_s - \mu_s S(t) - \int_0^\infty \beta(a) f(S(t))h(i(t,a))\mathrm{d}a \\ &\ge \Lambda_s - \left(\mu_s + \hat{\beta} f'(0)h'(0)\left(\frac{\Lambda}{\mu} + \epsilon\right)\right) S(t),\end{aligned}$$

which implies that

$$\liminf_{t\to\infty} S(t) \ge \frac{\Lambda_s}{\mu_s + \hat{\beta} f'(0)h'(0)(\frac{\Lambda}{\mu} + \epsilon)}.$$

Letting ϵ to 0 gives the required result. $\square$

The following result can be proved by similar arguments as those for Proposition 5 of McCluskey ([30]).

Proposition 3.4. *There exist $\mathcal{T}$ and $\epsilon > 0$ such that $e(t,0)$, $i(t,0) > \epsilon$ for all $t \geq \mathcal{T}$.*

4. Asymptotic smothness and global attractor

In order to show the existence of an attractor, it is necessary to obtain the asymptotic smoothness of the semiflow Φ. For this purpose, we need the following two results.

Proposition 4.1. *Let*

$$J(t) = \int_0^\infty \beta(a)h(i(t,a))\mathrm{d}a \text{ and } L(t) = \int_0^\infty \delta(a)e(t,a)\mathrm{d}a,$$

then the functions $J(t)$ and $L(t)$ are Lipschitz continuous.

Proof. We only give the proof of J being Lipschitz continuous as that for L is similar. Let $\tilde{\beta} = \int_0^\infty \beta(a)\mathrm{d}a$. From Assumption 2.2, there exists a positive constant M_h such that $|h(i(t+l,a)) - h(i(t,a))| \leq M_h|i(t+l,a) - i(t,a)|$. Then

$$\begin{aligned}
&|J(t+l) - J(t)| \\
&= \left|\int_0^\infty \beta(a)h(i(t+l,a))\mathrm{d}a - \int_0^\infty \beta(a)h(i(t,a))\mathrm{d}a\right| \\
&= \left|\int_0^\infty \beta(a)(h(i(t+h,a)) - h(i(t,a)))\mathrm{d}a\right| \\
&\leq M_h \int_0^\infty \beta(a)|i(t+h,a) - i(t,a)|\mathrm{d}a \\
&= M_h\left(\int_0^h \beta(a)i(t+h,a)\mathrm{d}a + \int_h^\infty \beta(a)i(t+h,a)\mathrm{d}a\right. \\
&\quad \left. - \int_0^\infty \beta(a)i(t,a)\mathrm{d}a\right).
\end{aligned}$$

Then following the discussion in [30], we have

$$|J(t+h) - J(t)| \leq [M_h(\hat{\beta}C\hat{\gamma} + C\hat{\mu}_i + 2\tilde{\Lambda}_i) + M_hM_\beta C]h.$$

Denoting $\Delta_J = M_h(\hat{\beta}C\hat{\gamma} + C\hat{\mu}_i + 2\tilde{\Lambda}_i) + M_hM_\beta C$ completes the proof. $\square$

The following lemma proposed in [36] provides us with the method to prove the asymptotic smoothness of the semi-flow.

Lemma 4.1 (Theorem 3.2 [36])**.** *The semiflow $\Phi : \mathbb{R}_+ \times \mathcal{X}_+ \to \mathcal{X}_+$ is asymptotically smooth if there are maps Ψ, $\Theta : \mathbb{R}_+ \times \mathcal{X}_+ \to \mathcal{X}_+$ such that $\Phi(t,x) = \Psi(t,x) + \Theta(t,x)$ and the following hold for any bounded closed set $C \subset \mathcal{X}_+$ that is forward invariant under Φ:*

(i) $\lim_{t\to\infty} \mathrm{diam}\Theta(t,C) = 0$;

(ii) *There exists $t_C \geq 0$ such that $\Psi(t, C)$ has compact closure for each $t \geq t_C$.*

In order to verify the second condition of Lemma 4.1, we need the following lemma.

Lemma 4.2 (Theorem B.2 [36])**.** *A set $C \in L^1_+(0, \infty)$ has compact closure if and only if the following conditions hold:*

(i) $\sup_{f\in C} \int_0^\infty f(a)\mathrm{d}a < \infty$;

(ii) $\lim_{r\to\infty} \int_r^\infty f(a)\mathrm{d}a \to 0$ *uniformly in* $f \in C$;

(iii) $\lim_{h\to 0^+} \int_0^\infty |f(a+h) - f(a)|\mathrm{d}a \to 0$ *uniformly in* $f \in C$;

(iv) $\lim_{h\to 0^+} \int_0^h f(a)\mathrm{d}a \to 0$ *uniformly in* $f \in C$.

Based on the above preparations, we are ready to show the asymptotic smoothness.

Theorem 4.1. *The semiflow Φ generated by (2.1) is asymptotically smooth.*

Proof. We first define two maps Ψ and Θ such that $\Phi = \Psi + \Theta$, where

$$\begin{cases} \Psi(t, x_0) = (S(t), \tilde{e}(t, \cdot), \tilde{i}(t, \cdot)), \\ \Theta(t, x_0) = (0, \tilde{\varphi}_e(t, \cdot), \tilde{\varphi}_i(t, \cdot)), \end{cases}$$

where

$$\tilde{e}(t, a) = \begin{cases} f(S(t-a))J(t-a)\Omega(a) & \text{for } 0 \leq a \leq t, \\ 0 & \text{for } 0 \leq t \leq a, \end{cases} \tag{4.1}$$

$$\tilde{i}(t, a) = \begin{cases} L(t-a)\Gamma(a) & \text{for } 0 \leq a \leq t, \\ 0 & \text{for } 0 \leq t \leq a, \end{cases} \tag{4.2}$$

$$\tilde{\varphi}_e(t, a) = \begin{cases} \displaystyle\int_0^a \Lambda_e(\epsilon)\frac{\Omega(a)}{\Omega(\epsilon)}\mathrm{d}\epsilon & \text{for } 0 \leq a \leq t, \\ \displaystyle\varphi_e(a-t)\frac{\Omega(a)}{\Omega(a-t)} + \int_{a-t}^a \Lambda_e(\epsilon)\frac{\Omega(a)}{\Omega(\epsilon)}\mathrm{d}\epsilon & \text{for } 0 \leq t \leq a, \end{cases} \tag{4.3}$$

$$\tilde{\varphi}_i(t, a) = \begin{cases} \displaystyle\int_0^a \Lambda_i(\epsilon)\frac{\Gamma(a)}{\Gamma(\epsilon)}\mathrm{d}\varepsilon & \text{for } 0 \leq a \leq t, \\ \displaystyle\varphi_i(a-t)\frac{\Gamma(a)}{\Gamma(a-t)} + \int_{a-t}^a \Lambda_i(\epsilon)\frac{\Gamma(a)}{\Gamma(\epsilon)}\mathrm{d}\varepsilon & \text{for } 0 \leq t \leq a. \end{cases} \tag{4.4}$$

Let $C \subset \mathcal{X}_+$ be any bounded closed set which is forward invariant under Φ.

We first verify that Θ satisfies condition (i) of Lemma 4.1. For $x_0 \in \Omega$ satisfying $\|x_0\|_\mathcal{X} \leq r$, we have

$$\begin{aligned} &\|\Theta(t, x_0)\|_\mathcal{X} \\ =&|0| + \int_0^\infty |\tilde{\varphi}_e(t, a)|\mathrm{d}a + \int_0^\infty |\tilde{\varphi}_i(t, a)|\mathrm{d}a \end{aligned}$$

$$
\begin{aligned}
&= \int_t^\infty \left|\varphi_e(a-t)\frac{\Omega(a)}{\Omega(a-t)}\right| \mathrm{d}a + \int_t^\infty \left|\varphi_i(a-t)\frac{\Gamma(a)}{\Gamma(a-t)}\right| \mathrm{d}a \\
&= \int_0^\infty \left|\varphi_e(\epsilon)\frac{\Omega(\epsilon+t)}{\Omega(\epsilon)}\right| \mathrm{d}\varepsilon + \int_0^\infty \left|\varphi_i(\epsilon)\frac{\Gamma(\epsilon+t)}{\Gamma(\epsilon)}\right| \mathrm{d}\varepsilon \\
&= \int_0^\infty \left|\varphi_e(\epsilon)e^{-\int_\epsilon^{\epsilon+t}(\mu_e(\tau)+\gamma(\tau))\mathrm{d}\tau}\right| \mathrm{d}\varepsilon + \int_0^\infty \left|\varphi_i(\epsilon)e^{-\int_\epsilon^{\epsilon+t}(\mu_i(\tau)+\delta(\tau))\mathrm{d}\tau}\right| \mathrm{d}\varepsilon \\
&\le e^{-\mu t}\int_0^\infty |\varphi_e(\epsilon)|\mathrm{d}\varepsilon + e^{-\mu t}\int_0^\infty |\varphi_i(\epsilon)|\mathrm{d}\varepsilon \\
&\le e^{-\mu t}\|x_0\|_{\mathcal{X}} \\
&\le e^{-\mu t} r, \qquad t \in \mathbb{R}_+.
\end{aligned}
$$

This shows that $\|\Theta(t,x_0)\|_{\mathcal{X}} \to 0$ as $t \to \infty$, which implies that $\|\Theta(t,x_0)\|_{\mathcal{X}}$ approaches $0 \in \mathcal{Y}$ with uniform exponential speed. This completes the proof of (i) of Lemma 4.1.

Now we prove that Lemma 4.2 holds. By using Proposition 3.2 it is easy to verify conditions (i), (ii) and (iv) of Lemma 4.2 are satisfied since

$$0 \le \tilde{e}(t,a) \le f'(0)h'(0)\bar{\beta}M^2 e^{-\bar{\mu}\bar{\gamma}a}.$$

It remains to show condition (iii). For sufficiently small $u \in (0,t)$, we have

$$
\begin{aligned}
&\int_0^\infty |\tilde{e}(a+u,t) - \tilde{e}(a,t)|\mathrm{d}a \\
&= \int_0^{t-u} |f(S(t-a-u))J(t-a-u)\Omega(a+u) - f(S(t-a))J(t-a)\Omega(a)|\mathrm{d}a \\
&\quad + \int_{t-u}^t |0 - f(S(t-a))J(t-a)\Omega(a)|\mathrm{d}a \\
&\le f'(0)\int_0^{t-u} |S(t-a-u)J(t-a-u)\Omega(a+u) - S(t-a)J(t-a)\Omega(a)|\mathrm{d}a \\
&\quad + f'(0)\int_{t-u}^t |S(t-a)J(t-a)\Omega(a)|\mathrm{d}a \\
&\le f'(0)h'(0)\hat{\beta}C^2 u + f'(0)\int_0^{t-u} S(t-a-u)J(t-a-u)|\Omega(a+u) - \Omega(a)|\mathrm{d}a \\
&\quad + f'(0)\int_0^{t-u} |S(t-a-u)J(t-a-u) - S(t-a)J(t-a)|\Omega(a)\mathrm{d}a \\
&\le f'(0)h'(0)\hat{\beta}C^2 u + \Xi + \Pi,
\end{aligned}
\tag{4.5}
$$

where

$$
\begin{aligned}
\Xi &= f'(0)\hat{\beta}C^2 \int_0^{t-u} |\Omega(a+u) - \Omega(a)|\mathrm{d}a, \\
\Pi &= f'(0)\int_0^{t-u} |S(t-a-u)J(t-a-u) - S(t-a)J(t-a)|\Omega(a)\mathrm{d}a.
\end{aligned}
$$

From (3.1) and (3.2), we have

$$0 \le \int_0^{t-u} |\Omega(a+u) - \Omega(a)|\mathrm{d}a$$

$$
\begin{aligned}
&= \int_0^{t-u} \Omega(a) - \Omega(a+u)\mathrm{d}a \\
&= \int_0^{u} \Omega(a)\mathrm{d}a - \int_{t-u}^{t} \Omega(a)\mathrm{d}a \\
&\le \int_0^{u} \Omega(a)\mathrm{d}a \\
&\le u.
\end{aligned}
$$

Finally, we deal with the last term in (4.5). Since

$$\left|\frac{\mathrm{d}S(t)}{\mathrm{d}t}\right| \le \Lambda_s + \mu_s C + \hat{\beta} f'(0)h'(0)C^2,$$

$S(t)$ is Lipschitz continuous with a Lipschitz coefficient M_S, where $M_s = \Lambda_s + \mu_s C + \hat{\beta} f'(0)h'(0)C^2$. Then

$$
\begin{aligned}
&|S(t-a-u)J(t-a-u) - S(t-a)J(t-a)| \\
\le& |S(t-a-u) - S(t-a)||J(t-a)| + |S(t-a-u)||J(t-a-u)J(t-a)| \\
\le& M_s u h'(0)\hat{\beta} C + \Delta_J C.
\end{aligned}
$$

Substituting the above two equations into (4.5), we have

$$
\begin{aligned}
&\int_0^{\infty} |\tilde{e}(a+u,t) - \tilde{e}(a,t)|\mathrm{d}a \\
\le& f'(0)h'(0)\hat{\beta}C^2 u + f'(0)\hat{\beta}C^2 u + f'(0)M_s u h'(0)\hat{\beta} C + f'(0)\Delta_J C.
\end{aligned}
\tag{4.6}
$$

The constant in (4.6) does not depend on the initial condition x_0. Then Lemma 4.2 holds. Consequently, $\tilde{e}(t,a)$ remains in a pre-compact subset Z_e in $L^1_+(0,\infty)$. Similarly $\tilde{i}(t,a)$ remains in a pre-compact subset Z_i in $L^1_+(0,\infty)$. This completes the proof of asymptotic smoothness. □

Because of the asymptotic smoothness, point dissipativeness, and boundedness of orbits of bounded sets, the following result on the existence of global attractor followes immediately from Theorem 3.4.6 of Hale [14].

Theorem 4.2. *The semiflow $\Phi(t)$ has a global attractor $\mathcal{A}$ in $\mathcal{X}_+$, which attracts all bound subsets of $\mathcal{X}_+$.*

5. The equilibrium and its local stability

Because there is immigration into the second equation or the third equation of (2.1), there is no disease-free equilibrium. An endemic equilibrium $E^* = (S^*, e^*(a), i^*(a))$ satisfies

$$
\begin{cases}
0 = \Lambda_s - \mu_s S^* - \displaystyle\int_0^{\infty} \beta(a) f(S^*) h(i^*(a))\mathrm{d}a, \\
\dfrac{\mathrm{d}e^*(a)}{\mathrm{d}a} = \Lambda_e(a) - (\mu_e(a) + \gamma(a))e^*(a), \\
\dfrac{\mathrm{d}i^*(a)}{\mathrm{d}a} = \Lambda_i(a) - (\mu_i(a) + \delta(a))i^*(a),
\end{cases}
\tag{5.1}
$$

and

$$\begin{cases} e^*(0) = \displaystyle\int_0^\infty \beta(a) f(S^*) h(i^*(a)) \mathrm{d}a, \\ i^*(0) = \displaystyle\int_0^\infty \gamma(a) e^*(a) \mathrm{d}a. \end{cases} \tag{5.2}$$

Solving the last two equations of (5.1) gives

$$\begin{cases} e^*(a) = e^*(0)\Omega(a) + \displaystyle\int_0^a \Lambda_e(\sigma) \frac{\Omega(a)}{\Omega(\sigma)} \mathrm{d}\sigma, \\ i^*(a) = i^*(0)\Gamma(a) + \displaystyle\int_0^a \Lambda_i(\sigma) \frac{\Gamma(a)}{\Gamma(\sigma)} \mathrm{d}\sigma. \end{cases} \tag{5.3}$$

Now, we show the existence of endemic equilibrium to (2.1).

Proposition 5.1. *System (2.1) only has an equilibrium E^*, which is endemic.*

Proof. The first equations of (5.1) and (5.2) give

$$e^*(a) = (\Lambda - \mu_s S^*)\Omega(a) + \int_0^a \Lambda_e(\sigma) \frac{\Omega(a)}{\Omega(\sigma)} \mathrm{d}\sigma. \tag{5.4}$$

Then

$$i^*(0) = (\Lambda - \mu_s S^*) \int_0^\infty \gamma(a)\Omega(a) \mathrm{d}a + \int_0^\infty \int_0^a \gamma(a) \Lambda_e(\sigma) \frac{\Omega(a)}{\Omega(\sigma)} \mathrm{d}\sigma \mathrm{d}a \tag{5.5}$$

and

$$\begin{aligned} i^*(a) &= i^*(0)\Gamma(a) + \int_0^a \Lambda_i(\sigma) \frac{\Gamma(a)}{\Gamma(\sigma)} \mathrm{d}\sigma \\ &= (\Lambda - \mu_s S^*)\Gamma(a) \int_0^\infty \gamma(a)\Omega(a) \mathrm{d}a \\ &\quad + \Gamma(a) \int_0^\infty \int_0^a \gamma(a) \Lambda_e(\sigma) \frac{\Omega(a)}{\Omega(\sigma)} \mathrm{d}\sigma \mathrm{d}a \\ &\quad + \int_0^a \Lambda_i(\sigma) \frac{\Gamma(a)}{\Gamma(\sigma)} \mathrm{d}\sigma. \end{aligned} \tag{5.6}$$

Let

$$P = \int_0^\infty \gamma(a)\Omega(a) \mathrm{d}a, \qquad Q = \int_0^\infty \beta(a)\Gamma(a) \mathrm{d}a,$$

and

$$M = \int_0^\infty \int_0^a \gamma(a) \Lambda_e(\sigma) \frac{\Omega(a)}{\Omega(\sigma)} \mathrm{d}\sigma \mathrm{d}a, \qquad N = \int_0^\infty \int_0^a \Lambda_i(\sigma) \frac{\Gamma(a)}{\Gamma(\sigma)} \mathrm{d}\sigma \mathrm{d}a.$$

Then the first equation of (5.1) gives that S^* is a zero of G in $(0, \frac{\Lambda}{\mu_s})$, where

$$G(S) = \Lambda - \mu_s S - \int_0^\infty \beta(a) f(S) h((\Lambda - \mu_s S)\Gamma(a) P + \Gamma(a) M + N) da.$$

Note that $G(0) = \Lambda > 0$ and $G(\frac{\Lambda}{\mu_s}) < 0$. By the Intermediate Vale Theorem, G has a zero in $(0, \frac{\Lambda}{\mu_s})$. Moreover, since $\frac{\Lambda - \mu_s S}{f(S)}$ is decreasing and $\int_0^\infty \beta(a) h((\Lambda -$

$\mu_s S)\Gamma(a)P + \Gamma(a)M + N)da$ is concave down, we easily see that G only has one zero in $(0, \frac{\Lambda}{\mu_s})$. Denote this unique positive zero of G is $(0, \frac{\Lambda}{\mu_s})$ by S^*. Then we get a unique endemic equilibrium $E^* = (S^*, e^*(a), i^*(a))$ with e^* and i^* being given by (5.4) and (5.6), respectively. □

Next we consider the local stability of E^*.

Theorem 5.1. *The endemic equilibrium E^* is locally asymptotically stable.*

Proof. Introduce the following perturbation variables

$$x_1(t) = S(t) - S^*, \quad x_2(t,a) = e(t,a) - e^*(a), \quad x_3(t,a) = i(t,a) - i^*(a).$$

Linearizing (2.1) at E^* and setting $x_1(t) = x_1^* e^{\lambda t}$, $x_2(t,a) = x_2^*(a)e^{\lambda t}$, and $x_3(t,a) = x_3^*(a)e^{\lambda t}$, we obtain the characteristic equation at E^*, which is

$$\begin{vmatrix} \lambda + \mu_S + \varpi & 0 & \pi \\ \varpi & -1 & \pi \\ 0 & \int_0^\infty \gamma(a)e^{-\lambda a}\Omega(a)\mathrm{d}a & -1 \end{vmatrix} = 0,$$

where $\varpi = \int_0^\infty \beta(a)f'(S^*)h(i^*(a))\mathrm{d}a$ and $\pi = \int_0^\infty \beta(a)f(S^*)h'(i^*(a))e^{-\lambda a}\Gamma(a)\mathrm{d}a$. After expanding, the characteristic equation is

$$\begin{aligned} &\int_0^\infty \beta(a)f'(S^*)h(i^*(a))\mathrm{d}a + \lambda + \mu_s \\ =&(\lambda + \mu_s)\int_0^\infty \beta(a)f(S^*)h'(i^*(a))e^{-\lambda a}\Gamma(a)\mathrm{d}a \int_0^\infty \gamma(a)e^{-\lambda a}\Omega(a)\mathrm{d}a. \end{aligned}$$

By way of contradiction, we assume that it has an eigenvalue λ_0 with $\mathrm{Re}(\lambda_0) \geq 0$. Then

$$\begin{aligned} &\left|(\lambda_0 + \mu_s)\int_0^\infty \beta(a)f(S^*)h'(i^*(a))e^{-\lambda_0 a}\Gamma(a)\mathrm{d}a \int_0^\infty \gamma(a)e^{-\lambda_0 a}\Omega(a)\mathrm{d}a\right| \\ \leq&\left|(\lambda_0 + \mu_s)\int_0^\infty \beta(a)f(S^*)\frac{h(i^*(a))}{i^*(0)\Gamma(a)}e^{-\lambda_0 a}\Gamma(a)\mathrm{d}a \int_0^\infty \gamma(a)e^{-\lambda_0 a}\Omega(a)\mathrm{d}a\right| \\ \leq&\left|(\lambda_0 + \mu_s)\frac{e^*(0)}{i^*(0)}\int_0^\infty \gamma(a)e^{-\lambda_0 a}\Omega(a)\mathrm{d}a\right| \\ \leq&|\lambda_0 + \mu_s| \end{aligned}$$

and

$$\begin{aligned} &\left|\int_0^\infty \beta(a)f'(S^*)h(i^*(a))\mathrm{d}a + \lambda_0 + \mu_s\right| \\ =&\left|\lambda_0 + \frac{\Lambda_s - e^*(0)}{S^*} + \frac{e^*(0)f'(S^*)}{f(S^*)}\right| \\ \geq&\left|\lambda_0 + \frac{(\Lambda_s - e^*(0))f'(S^*)}{f(S^*)} + \frac{e^*(0)f'(S^*)}{f(S^*)}\right| \\ =&\left|\lambda_0 + \frac{\Lambda_s f'(S^*)}{f(S^*)}\right|. \end{aligned}$$

Here we have used (5.1), (5.2), (5.3) and Assumption 2.2 which implies

$$f'(x)x \le f(x) \le f'(0)x.$$

Thus we get

$$\left|\lambda_0 + \frac{\Lambda_s f'(S^*)}{f(S^*)}\right| \le |\lambda_0 + \mu_s|,$$

which is impossible. In fact, we have

$$\begin{aligned}|\lambda_0 + \mu_s|^2 - \left|\lambda_0 + \frac{\Lambda_s f'(S^*)}{f(S^*)}\right|^2 &= \left(2\mathrm{Re}(\lambda_0) + \mu_s + \frac{\Lambda_s f'(S^*)}{f(S^*)}\right)\left(\mu_s - \frac{\Lambda_s f'(S^*)}{f(S^*)}\right)\\ &< 0,\end{aligned}$$

since

$$\begin{aligned}\mu_s - \frac{\Lambda_s f'(S^*)}{f(S^*)} &= \frac{\Lambda_s - e^*(0)}{S^*} - \frac{\Lambda_s f'(S^*)}{f(S^*)}\\ &\le \frac{(\Lambda_s - e^*(0))f'(S^*)}{f(S^*)} - \frac{\Lambda_s f'(S^*)}{f(S^*)}\\ &= \frac{-e^*(0)f'(S^*)}{f(S^*)}\\ &< 0.\end{aligned}$$

This completes the proof. □

6. Global stability of the endemic equilibrium

In this section, we use Lyapunov functional to prove the global stability of the endemic equilibrium E^*. We start with some properties of solutions of (2.1).

Proposition 6.1. *For any solution $(S(t), e(t,\cdot), i(t,\cdot))$ of (2.1), the following identities hold.*

(a) $\int_0^\infty \beta(a) f(S^*) h(i^*(a)) \left(\frac{f(S(t))h(i(t,a))}{f(S^*)h(i^*(a))} - \frac{e(t,0)}{e^*(0)}\right) \mathrm{d}a = 0.$

(b) $\int_0^\infty \gamma(a) e^*(a) \left(\frac{e(t,a)}{e^*(a)} - \frac{i(t,0)}{i^*(0)}\right) \mathrm{d}a = 0.$

(c) $\int_0^\infty \beta(a) f(S^*) h(i^*(a)) \mathrm{d}a - \frac{e^*(0)}{i^*(0)} \int_0^\infty \gamma(a) e^*(a) \mathrm{d}a = 0.$

Proof. First,

$$\begin{aligned}&\int_0^\infty \beta(a) f(S^*) h(i^*(a)) \frac{e(t,0)}{e^*(0)} \mathrm{d}a\\ &= \int_0^\infty \beta(a) f(S^*) h(i^*(a)) \frac{e(t,0)}{\int_0^\infty \beta(a) f(S^*) h(i^*(a)) \mathrm{d}a} \mathrm{d}a\\ &= e(t,0)\\ &= \int_0^\infty \beta(a) f(S(t)) h(i(t,a)) \mathrm{d}a\\ &= \int_0^\infty \beta(a) f(S^*) h(i^*(a)) \frac{f(S(t))h(i(t,a))}{f(S^*)h(i^*(a))} \mathrm{d}a.\end{aligned}$$

This proves (a).

Similarly, we can investigate case (b) and obtain

$$\int_0^\infty \gamma(a)e^*(a)\frac{i(t,0)}{i^*(0)}\mathrm{d}a = i(t,0) = \int_0^\infty \gamma(a)e^*(a)\frac{e(t,a)}{e^*(a)}\mathrm{d}a.$$

Finally, (c) is obvious due to the boundary conditions (2.2). □

Now, we are ready to prove the main result of this paper.

Theorem 6.1. *The endemic equilibrium E^* of (2.1) is globally asymptotically stable.*

Proof. By Theorem 5.1, it suffices to show that E^* is globally attractive. For this purpose, we define a Lyapunov functional as

$$L(t) = L_1(t) + \frac{e^*(0)}{i^*(0)}L_2(t) + L_3(t),$$

where

$$\begin{aligned}
L_1(t) &= S(t) - \int_{S^*}^{S(t)} \frac{f(S^*)}{f(\tau)}\mathrm{d}\tau,\\
L_2(t) &= \int_0^\infty \alpha_e(a)g\left(\frac{e(t,a)}{e^*(a)}\right)\mathrm{d}a,\\
L_3(t) &= \int_0^\infty \alpha_i(a)g\left(\frac{i(t,a)}{i^*(a)}\right)\mathrm{d}a,\\
\alpha_e(a) &= \int_a^\infty \gamma(\theta)e^*(\theta)\mathrm{d}\theta,\\
\alpha_i(a) &= \int_a^\infty \beta(\theta)f(S^*)h(i^*(\theta))\mathrm{d}\theta,\\
g(x) &= x - 1 - \ln x.
\end{aligned}$$

Since $g(x)$ has the global minimum value 0 only at $x = 1$ and $f(\tau)$ is an increasing function, we know that L is nonnegative and $L = 0$ only at the endemic equilibrium E^*.

Now we calculate the derivative of L. Firstly,

$$\begin{aligned}
\frac{\mathrm{d}L_1(t)}{\mathrm{d}t} &= \left(1 - \frac{f(S^*)}{f(S(t))}\right)\frac{\mathrm{d}S(t)}{\mathrm{d}t}\\
&= \left(1 - \frac{f(S^*)}{f(S(t))}\right)(\mu_s S^* - \mu_s S(t))\\
&\quad + \left(1 - \frac{f(S^*)}{f(S(t))}\right)\left(\int_0^\infty \beta(a)f(S^*)h(i^*(a))\mathrm{d}a\right.\\
&\quad \left. - \int_0^\infty \beta(a)f(S(t))h(i(t,a))\mathrm{d}a\right)\\
&= \frac{\mu_s}{f(S(t))}(f(S(t)) - f(S^*))(S^* - S(t))\\
&\quad + \int_0^\infty \beta(a)f(S^*)h(i^*(a))\left(1 - \frac{f(S(t))h(i(t,a))}{f(S^*)h(i^*(a))}\right.
\end{aligned}$$

$$-\frac{f(S^*)}{f(S(t))}+\frac{h(i(t,a))}{h(i^*(a))}\Bigg)\mathrm{d}a.$$

Next, recall that

$$\frac{e(t,a)}{e^*(a)}=\frac{e(t-a,0)\Omega(a)+\Psi(a)\Omega(a)}{e^*(0)\Omega(a)+\Psi(a)\Omega(a)}=\frac{e(t-a,0)+\Psi(a)}{e^*(0)+\Psi(a)},$$

where

$$\Psi(a)=\int_0^a\frac{\Lambda_e(\sigma)}{\Omega(\sigma)}\mathrm{d}\sigma.$$

Using the similar calculation as that in [30], we have

$$\begin{aligned}\frac{\mathrm{d}L_2(t)}{\mathrm{d}t}&=\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty\alpha_e(a)g\left(\frac{e(t-a,0)+\Psi(a)}{e^*(0)+\Psi(a)}\right)\mathrm{d}a\\&=\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^t\alpha_e(t-\tau)g\left(\frac{e(\tau,0)+\Psi(t-\tau)}{e^*(0)+\Psi(t-\tau)}\right)\mathrm{d}\tau\\&=\alpha_e(t-\tau)g\left(\frac{e(\tau,0)+\Psi(t-\tau)}{e^*(0)+\Psi(t-\tau)}\right)\Bigg|_{\tau=t}\\&\quad+\int_{-\infty}^t\alpha_e'(t-\tau)g\left(\frac{e(\tau,0)+\Psi(t-\tau)}{e^*(0)+\Psi(t-\tau)}\right)\mathrm{d}\tau\\&\quad+\textstyle\int_{-\infty}^t\alpha_e(t-\tau)g'\left(\frac{e(\tau,0)+\Psi(t-\tau)}{e^*(0)+\Psi(t-\tau)}\right)\left(\frac{e(\tau,0)+\Psi(t-\tau)}{e^*(0)+\Psi(t-\tau)}\right)'\mathrm{d}\tau.\end{aligned}$$

Note that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{e(\tau,0)+\Psi(t-\tau)}{e^*(0)+\Psi(t-\tau)}\right)\\&=\frac{\Psi'(t-\tau)(e^*(0)+\Psi(t-\tau))-\Psi'(t-\tau)(e(\tau,0)+\Psi(t-\tau))}{(e^*(0)+\Psi(t-\tau))^2}\\&=\Psi'(t-\tau)\frac{e^*(0)+\Psi(t-\tau)-\Psi(t-\tau)-e(\tau,0)}{(e^*(0)+\Psi(t-\tau))^2}\\&=\frac{\Psi'(t-\tau)}{e^*(0)+\Psi(t-\tau)}\left(1-\frac{e(\tau,0)+\Psi(t-\tau)}{e^*(0)+\Psi(t-\tau)}\right)\\&=\frac{\Psi'(t-\tau)}{e^*(0)+\Psi(t-\tau)}\left(1-\frac{e(t,t-\tau)}{e^*(t-\tau)}\right).\end{aligned}$$

Then with the change of variable we get

$$\begin{aligned}\frac{\mathrm{d}L_2(t)}{\mathrm{d}t}&=\alpha_e(0)g\left(\frac{e(t,0)}{e^*(0)}\right)+\int_0^\infty\alpha_e'(a)g\left(\frac{e(t,a)}{e^*(a)}\right)\mathrm{d}a\\&\quad+\int_0^\infty\frac{\alpha_e(a)\Lambda_e(a)}{e^*(a)}\left(1-\frac{e^*(a)}{e(t,a)}\right)\left(1-\frac{e(t,a)}{e^*(a)}\right)\mathrm{d}a.\end{aligned}$$

Here we have used $g'(x)=1-\frac{1}{x}$ and $\frac{\Psi'(a)}{e^*(0)+\Psi(a)}=\frac{\Lambda_e(a)}{e^*(a)}$. Since $\alpha_e'(a)=-\gamma(a)e^*(a)$ and $\alpha_e(0)=\int_0^\infty\gamma(a)e^*(a)da$, we can rewrite $\frac{\mathrm{d}L_2(t)}{\mathrm{d}t}$ as

$$\frac{\mathrm{d}L_2(t)}{\mathrm{d}t}=\int_0^\infty\gamma(a)e^*(a)\left(g\left(\frac{e(t,0)}{e^*(0)}\right)-g\left(\frac{e(t,a)}{e^*(a)}\right)\right)\mathrm{d}a$$

$$-\int_0^\infty \frac{\alpha_e(a)\Lambda_e(a)}{e^*(a)}\frac{(e(t,a)-e^*(a))^2}{e(t,a)e^*(a)}\mathrm{d}a.$$

Similarly,

$$\begin{aligned}\frac{\mathrm{d}L_3(t)}{\mathrm{d}t} = &\int_0^\infty \beta(a)f(S^*)h(i^*(a))\left(g\left(\frac{i(t,0)}{i^*(0)}\right)-g\left(\frac{i(t,a)}{i^*(a)}\right)\right)\mathrm{d}a\\ &-\int_0^\infty \frac{\alpha_i(a)\Lambda_i(a)}{i^*(a)}\frac{(i(t,a)-i^*(a))^2}{i(t,a)i^*(a)}\mathrm{d}a.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\mathrm{d}L(t)}{\mathrm{d}t} = &\frac{\mu_s}{f(S(t))}(f(S(t))-f(S^*))(S^*-S(t))+H\\ &-\frac{e^*(0)}{i^*(0)}\int_0^\infty \frac{\alpha_e(a)\Lambda_e(a)}{e^*(a)}\frac{(e(t,a)-e^*(a))^2}{e(t,a)e^*(a)}\mathrm{d}a\\ &-\int_0^\infty \frac{\alpha_i(a)\Lambda_i(a)}{i^*(a)}\frac{(i(t,a)-i^*(a))^2}{i(t,a)i^*(a)}\mathrm{d}a,\end{aligned} \tag{6.1}$$

where

$$\begin{aligned}H = &\int_0^\infty \beta(a)f(S^*)h(i^*(a))\left(1-\frac{f(S(t))h(i(t,a))}{f(S^*)h(i^*(a))}-\frac{f(S^*)}{f(S(t))}+\frac{h(i(t,a))}{h(i^*(a))}\right)\mathrm{d}a\\ &+\frac{e^*(0)}{i^*(0)}\int_0^\infty \gamma(a)e^*(a)\left(g\left(\frac{e(t,0)}{e^*(0)}\right)-g\left(\frac{e(t,a)}{e^*(a)}\right)\right)\mathrm{d}a\\ &+\int_0^\infty \beta(a)f(S^*)h(i^*(a))\left(g\left(\frac{i(t,0)}{i^*(0)}\right)-g\left(\frac{i(t,a)}{i^*(a)}\right)\right)\mathrm{d}a.\end{aligned}$$

Because of Assumption 2.2, $\frac{\mu_s}{f(S(t))}(f(S(t))-f(S^*))(S^*-S(t))\le 0$. Now we deal with H. By using (i) and (ii) of Proposition 6.1, we have

$$\begin{aligned}H = &\int_0^\infty \beta(a)f(S^*)h(i^*(a))\left(1-\frac{e(t,0)}{e^*(0)}-\frac{f(S^*)}{f(S(t))}+\frac{h(i(t,a))}{h(i^*(a))}\right)\mathrm{d}a\\ &+\int_0^\infty \beta(a)f(S^*)h(i^*(a))\left(\ln\frac{f(S^*)}{f(S(t))}-\ln\frac{f(S^*)}{f(S(t))}\right)\mathrm{d}a\\ &+\frac{e^*(0)}{i^*(0)}\int_0^\infty \gamma(a)e^*(a)\left(\frac{e(t,0)}{e^*(0)}-\ln\frac{e(t,0)}{e^*(0)}-\frac{e(t,a)}{e^*(a)}+\ln\frac{e(t,a)}{e^*(a)}\right)\mathrm{d}a\\ &+\int_0^\infty \beta(a)f(S^*)h(i^*(a))\left(\frac{i(t,0)}{i^*(0)}-\ln\frac{i(t,0)}{i^*(0)}-\frac{i(t,a)}{i^*(a)}+\ln\frac{i(t,a)}{i^*(a)}\right)\mathrm{d}a\\ = &-\int_0^\infty \beta(a)f(S^*)h(i^*(a))g\left(\frac{f(S^*)}{f(S(t))}\right)\mathrm{d}a\\ &+\int_0^\infty \beta(a)f(S^*)h(i^*(a))\left(-\frac{e(t,0)}{e^*(0)}-\ln\frac{f(S^*)}{f(S(t))}+\frac{h(i(t,a))}{h(i^*(a))}\right)\mathrm{d}a\\ &+\frac{e^*(0)}{i^*(0)}\int_0^\infty \gamma(a)e^*(a)\left(\frac{e(t,0)}{e^*(0)}-\ln\frac{e(t,0)}{e^*(0)}-\frac{e(t,a)}{e^*(a)}+\ln\frac{e(t,a)}{e^*(a)}\right)\mathrm{d}a\\ &+\int_0^\infty \beta(a)f(S^*)h(i^*(a))\left(\frac{i(t,0)}{i^*(0)}-\ln\frac{i(t,0)}{i^*(0)}-\frac{i(t,a)}{i^*(a)}+\ln\frac{i(t,a)}{i^*(a)}\right)\mathrm{d}a.\end{aligned}$$

With the help of Proposition 6.1 again, we obtain

$$
\begin{aligned}
H = & -\int_0^\infty \beta(a) f(S^*) h(i^*(a)) g\left(\frac{f(S^*)}{f(S(t))}\right) \mathrm{d}a \\
& + \int_0^\infty \beta(a) f(S^*) h(i^*(a)) \left(1 - \frac{f(S(t))h(i(t,a))e^*(0)}{f(S^*)h(i^*(a))e(t,0)} - \frac{e(t,0)}{e^*(0)}\right) \mathrm{d}a \\
& + \int_0^\infty \beta(a) f(S^*) h(i^*(a)) \left(-\ln \frac{f(S^*)}{f(S(t))} + \frac{h(i(t,a))}{h(i^*(a))}\right) \mathrm{d}a \\
& + \frac{e^*(0)}{i^*(0)} \int_0^\infty \gamma(a) e^*(a) \left(\frac{e(t,0)}{e^*(0)} - \ln \frac{e(t,0)}{e^*(0)} - \frac{e(t,a)}{e^*(a)} + \ln \frac{e(t,a)}{e^*(a)}\right) \mathrm{d}a \\
& + \frac{e^*(0)}{i^*(0)} \int_0^\infty \gamma(a) e^*(a) \left(1 - \frac{e(t,a)i^*(0)}{e^*(a)i(t,0)}\right) \mathrm{d}a \\
& + \int_0^\infty \beta(a) f(S^*) i^*(a) \left(\frac{i(t,0)}{i^*(0)} - \ln \frac{i(t,0)}{i^*(0)} - \frac{i(t,a)}{i^*(a)} + \ln \frac{i(t,a)}{i^*(a)}\right) \mathrm{d}a \\
= & -\int_0^\infty \beta(a) f(S^*) i^*(a) g\left(\frac{f(S^*)}{f(S(t))}\right) \mathrm{d}a \\
& - \int_0^\infty \beta(a) f(S^*) i^*(a) g\left(\frac{f(S(t))h(i(t,a))e^*(0)}{f(S^*)h(i^*(a))e(t,0)}\right) \mathrm{d}a \\
& + \int_0^\infty \beta(a) f(S^*) i^*(a) \left(g\left(\frac{h(i(t,a))}{h(i^*(a))}\right) - g\left(\frac{i(t,a)}{i^*(a)}\right)\right) \mathrm{d}a \\
& - \frac{e^*(0)}{i^*(0)} \int_0^\infty \gamma(a) e^*(a) g\left(\frac{e(t,a)i^*(0)}{e^*(a)i(t,0)}\right) \mathrm{d}a.
\end{aligned}
$$

From [34, Proposition A.1], $\left(g\left(\frac{h(i(t,a))}{h(i^*(a))}\right) - g\left(\frac{i(t,a)}{i^*(a)}\right)\right) \leq 0$. It follows that

$$
\frac{\mathrm{d}L(t)}{\mathrm{d}t} \leq 0.
$$

Since L is bounded on any solution $x(t)$, the omega limit set of $x(t)$ must be contained in $\mathcal{M}$, the largest invariant subset of $\{\frac{\mathrm{d}L}{\mathrm{d}t} = 0\}$. From $\{\frac{\mathrm{d}L}{\mathrm{d}t} = 0\}$, we have $S(t) = S^*$. Take $S(t)$ into system (2.1), gives us $e(t,a) = e^*(a)$ and $i(t,a) = i^*(a)$. It follows that $\mathcal{M} = \{E^*\}$. By the Lyapunov-LaSalle invariance principle, E^* is globally attractive. This completes the proof. $\square$

7. Application and numerical simulations

In this section we apply our results to an example and give some numerical simulations, consider the following model with saturation incidence rate, which have been studied in [29, 49].

$$
\begin{cases}
\dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda_s - \mu_s S(t) - \dfrac{S(t)}{1+\alpha S(t)} \displaystyle\int_0^\infty \beta(a) i(t,a) \mathrm{d}a, \\
\left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a}\right) e(t,a) = \Lambda_e(a) - (\mu_e(a) + \gamma(a)) e(t,a), \\
\left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a}\right) i(t,a) = \Lambda_i(a) - (\mu_i(a) + \delta(a)) i(t,a),
\end{cases}
\tag{7.1}
$$

with the boundary and initial conditions

$$\begin{cases} e(t,0) = \dfrac{S(t)}{1+\alpha S(t)} \displaystyle\int_0^\infty \beta(a) i(t,a) \mathrm{d}a, \\ i(t,0) = \displaystyle\int_0^\infty \gamma(a) e(t,a) \mathrm{d}a, \\ x_0 = (S(0), e(0,\cdot), i(0,\cdot)) = (S_0, e_0, i_0) \in \mathcal{X}_+, \end{cases} \tag{7.2}$$

where α is the saturation constant.

From Assumption (2.2), we can easily check (7.1) with (7.2) is a special case of (2.1) with (2.2) and (2.3). By Theorems (6.1), we obtain the following corollary.

Corollary 7.1. *The endemic equilibrium E^{**} of system (7.1) with (7.2) is globally asymptotically stable.*

Next, to verify the validity of the theoretical results of this paper, we perform numerical simulation to the special case (7.1) with (7.2). Denote $\Lambda_e,\ \Lambda_i,\ \gamma,\ \mu_i$ and δ are the averages of $\Lambda_e(a),\ \Lambda_i(a),\ \gamma(a),\ \mu_i(a)$ and $\delta(a)$, respectively. Since the transmission of tuberculosis can described by SEIR model, we choose the parameters from references of tuberculosis. Initial condition is set as $(4500, 300, 0)$. The parameter values used for simulations are listed in Table 2. We set the maximum age for the upper bound of latent and infection age as 10 years and

$$\beta(a) = \beta\left(1 + \sin\frac{(a-5)\pi}{10}\right), \quad \mu_i(a) = \mu_i\left(1 + \sin\frac{(a-5)\pi}{10}\right),$$

$$\Lambda_e(a) = \Lambda_e\left(1 + \sin\frac{(a-5)\pi}{10}\right), \quad \Lambda_i(a) = \Lambda_i\left(1 + \sin\frac{(a-5)\pi}{10}\right)$$

and

$$\delta(a) = \delta\left(1 + \sin\frac{(a-5)\pi}{10}\right), \quad \text{for} \quad 0 \le a \le 10.$$

From Corollary 7.1, the theoretical result is the endemic equilibrium E^{**} of system (7.1) with (7.2) is globally asymptotically stable. This fact is revealed by Figure 1, it shows an example in which $(S(t), e(t,a), i(t,a))$ converge to the positive steady states $(S^*, e^*(a), i^*(a))$. Furthermore, we show the distribution of $e(t,a)$ and $i(t,a)$ at age $a = 5$, Figure 2 shows that this is a stationary distribution. In Figure 3, we choose different averages value of $\Lambda_i(a)$, the simulation shows the higher of Λ_i, the higher levels of endemic state.

8. Discussion

In this paper, we have investigated the global behavior of an SEIR epidemiological model with infection age and immigration. By constructing suitable Lyapunov functional, we have succeeded in showing the global asymptotic stability of the endemic equilibrium.

Our model is an extended work to that of McCluskey [30], which the model is a special of ours with $f(S(t)) = S(t)$ and $h(i(t,a)) = i(t,a)$. But the mathematical analysis here is much more difficult because of the nonlinear incidence rate. As special examples of the model, we considered the age-structured SEIR models with

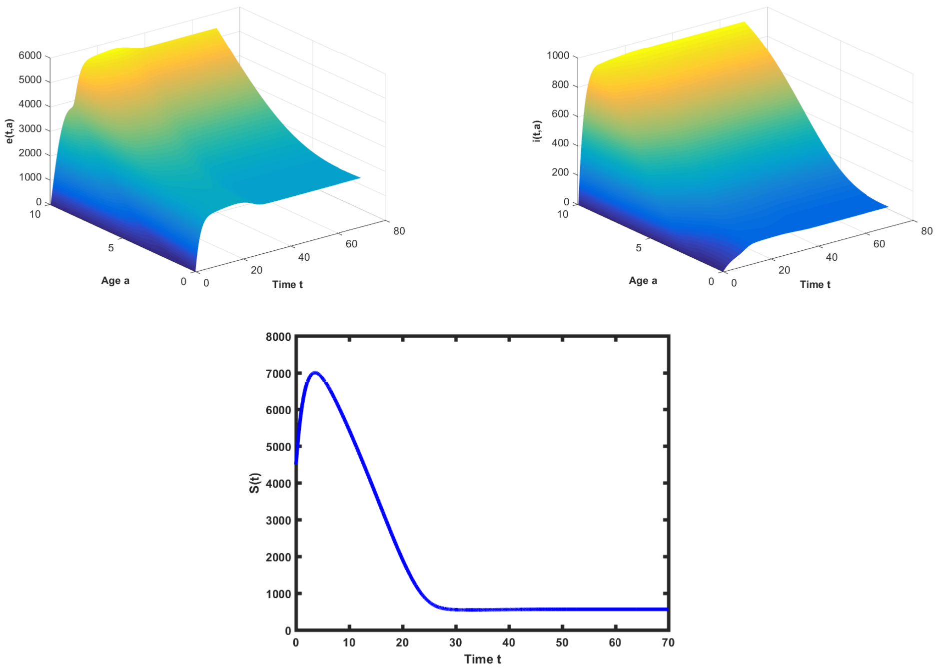

Figure 1. The long time dynamical behaviors (7.1) with (7.2)

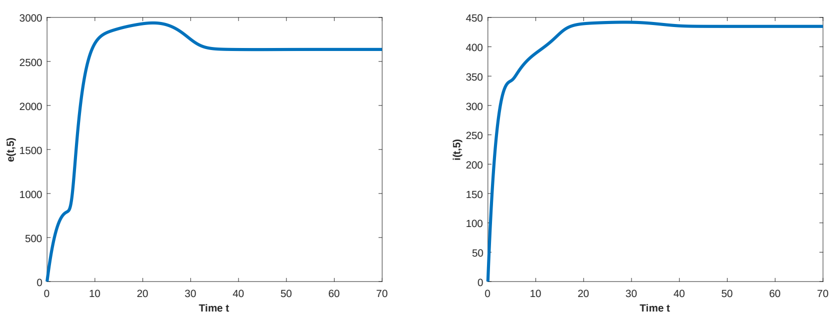

Figure 2. The long time dynamical behaviors at age $a = 5$ of (7.1) with (7.2)

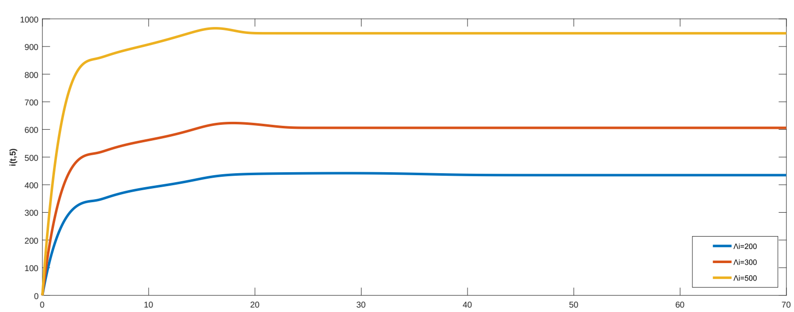

Figure 3. The long time dynamical behaviors of $i(t, a)$ at age $a = 5$ on different Λ_i

Table 2. Parameter values used for simulations of model (7.1).

Parameter	Value	Unit	References
β	0.003	Person $^{-1}$ year $^{-1}$	[47]
μ_S, μ_e	1/70	year $^{-1}$	[6]
γ	0.00368	year $^{-1}$	[3]
μ_i	0.17	year $^{-1}$	[46]
δ	0.01	year $^{-1}$	[38]
Λ_S	2000	Person year^{-1}	Assumed
α	0.008	Person $^{-1}$ year $^{-1}$	Assumed
Λ_e	400	Person year^{-1}	Assumed
Λ_i	200	Person year^{-1}	Assumed

saturation incidence rate which is an application on transmission of tuberculosis, the numerical simulations that come to be consistent with theoretical results.

The results of this paper show that the disease would not die out if there is immigration of exposed and/or infected individuals. These results also provide guidelines for control the spread of infectious diseases, just like border screening. For example, during the 2009 influenza A (H1N1) pandemic in China, there was the isolation of those detected infected individuals from the border screening [42,48].

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