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PROPERTIES OF EIGENVALUES AND SPECTRAL SINGULARITIES FOR IMPULSIVE QUADRATIC PENCIL OF DIFFERENCE OPERATORS

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Abstract In this paper, we investigate the spectral analysis of impulsive quadratic pencil of difference operators. We first present a boundary value problem consisting one interior impulsive point on the whole axis corresponding to the above mentioned operator. After introducing the solutions of impulsive quadratic pencil of difference equation, we obtain the asymptotic equation of the function related to the Wronskian of these solutions to be helpful for further works, then we determine resolvent operator and continuous spectrum. Finally, we provide sufficient conditions guarenteeing finiteness of eigenvalues and spectral singularities by means of uniqueness theorems of analytic functions. The main aim of this paper is demonstrating the impulsive quadratic pencil of difference operator is of finite number of eigenvalues and spectral singularities with finite multiplicities which is an uninvestigated problem proposed in the literature.

Keywords Asymptotic, eigenvalues, impulsive conditions, quadratic pencil of difference equations, resolvent operator, spectral singularities, spectrum.

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1. Introduction

Researchers often encounter some discontinuities or degenerations investigating the mathematical simulations of various physical or chemical phenomena. These phenomena involve short-term perturbations which act abruptly. Over the years, studying the mathematical and physical models of these phenomena has been an indispensable requirement for the scientists. Therefore, the equations involving impulsive effects, often called "impulsive equations", have been object of several books [17, 24, 25]. Recently, theory of impulsive equations has received significant attention on spectral theory and boundary value problems. For instance, spectral properties of Sturm–Liouville, difference, Dirac and different kinds of operators have been investigated with impulsive effects, in other words, transmission effects [5, 11, 19, 21, 22, 26, 27].

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Considering the progress of improvements on this theory, in this paper, we introduce a second-order impulsive quadratic pencil of difference operator on the whole axis with one interior impulsive point. Before presenting our problem, let us shortly give an overview on the existing literature about the spectral theory of differential, difference and quadratic pencil of difference equations without any discontinuity.

The spectral theory of differential equations or Sturm–Liouville equations was introduced by Naimark [23] and then intensively studied by other authors. For detailed information, we refer to [12,18,20] and the references quoted therein. Many results concerning differential equations carry over quite easily to the difference equations.

As is well known, the second-order difference equation

$$\triangle\left(a_{n-1} \triangle y_{n-1}\right)+\left(q_{n}-\lambda\right) y_{n}=0, \quad n \in \mathbb{Z} \tag{1.1}$$

turns into

$$a_{n-1} y_{n-1}+b_{n} y_{n}+a_{n} y_{n+1}=\lambda y_{n}, \quad n \in \mathbb{Z} \tag{1.2}$$

in the event that $b_n = q_n - a_{n-1} - a_n$, where $\triangle$ denotes the forward difference operator, $\{a_n\}_{n\in\mathbb{Z}}$ and $\{q_n\}_{n\in\mathbb{Z}}$ are real or complex sequences. (1.2) gives the discrete analogue of the Sturm–Liouville equation

$$-y''+q(x) y=\lambda^{2} y, \quad x \in \mathbb{R}. \tag{1.3}$$

Over the years, the equation (1.2) has been the subject of investigations due to the wide applicability of difference equations in various areas. For the studies on the spectral and scattering theory of difference equations, we refer to papers [2,3,8,14,15]. Among these papers, the sequences $\{a_n\}$ and $\{b_n\}$ have to satisfy the condition

$$\sum_{n \in \mathbb{Z}}|n|\left(\left|1-a_{n}\right|+\left|b_{n}\right|\right)<\infty, \tag{1.4}$$

in which $n \in \mathbb{Z}$ and $a_n > 0$. The spectral theory of (1.2) and (1.3) is well developed, even the matrix and q-cases are studied [4,6,10].

Despite the fact that the dependence on the spectral parameter λ is linear in (1.1), researchers also studied the equation

$$\triangle\left(a_{n-1} \triangle y_{n-1}\right)+\left(q_{n}+2 \lambda p_{n}+\lambda^{2}\right) y_{n}=0, \quad n \in \mathbb{Z}, \tag{1.5}$$

where this dependence is nonlinear. We should note that, in (1.5), $a_n \neq 0$ for all $n \in \mathbb{Z}$, the sequences $\{a_n\}_{n\in\mathbb{Z}}$, $\{q_n\}_{n\in\mathbb{Z}}$ and $\{p_n\}_{n\in\mathbb{Z}}$ are complex-valued satisfying

$$\sum_{n \in \mathbb{Z}}|n|\left(\left|1-a_{n}\right|+\left|p_{n}\right|+\left|q_{n}\right|\right)<\infty. \tag{1.6}$$

It is worth mentioning here that (1.5) is likewise discrete analogue of quadratic pencil of Schrödinger equation

$$-y''+\left[q(x)+2 \lambda p(x)-\lambda^{2}\right] y=0, \quad x \in \mathbb{R}, \tag{1.7}$$

which is intensively studied in [7,9,16].

In [1], the spectral properties of non–selfadjoint quadratic pencil of Schrödinger type difference operators corresponding to (1.5) has been investigated. This paper gives us opportunity to learn the structure of eigenvalues and spectral singularities of these type operators with general conditions on the whole real line. But there is no any study about spectral analysis of quadratic pencil of difference operators with impulsive conditions in literature. Therefore, motivated by [1] and because of the requirement of literature, in this paper, we handle an impulsive quadratic pencil of difference operator so that we can compare the results to see the effects of a discontinuity at one interior point $n=0$.

This paper aims to investigate the quadratic pencil of difference equation

$$\triangle\left(a_{n-1} \triangle y_{n-1}\right)+\left(q_n+2 \mu p_n+\mu^2\right) y_n=0, \quad n \in \mathbb{Z} \backslash\{-1,0,1\} \tag{1.8}$$

together with the impulsive condition

$$\begin{pmatrix} y_1 \\ \triangle y_1 \end{pmatrix}=\mathbf{T}\begin{pmatrix} y_{-1} \\ \nabla y_{-1} \end{pmatrix}, \quad \mathbf{T}=\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}, \tag{1.9}$$

where $\mu:=2 \cos \frac{z}{2}$ is a spectral parameter, $\det \mathbf{T}>0$, and $\{\alpha_i\}_{i=1,2,3,4}$ are all complex numbers. In (1.9), $\triangle$ denotes the forward difference operator and ∇ denotes the backward difference operator, i.e.

$$\begin{aligned} \triangle y_n &:= y_{n+1}-y_n, \\ \nabla y_n &:= y_n-y_{n-1}. \end{aligned}$$

Throughout this paper, we assume that $\{a_n\}_{n \in \mathbb{Z}}$, $\{q_n\}_{n \in \mathbb{Z}}$, $\{p_n\}_{n \in \mathbb{Z}}$ are complex sequences satisfying (1.6) and $a_n \neq 0$ for all $n \in \mathbb{Z}$. Clearly, (1.9) is called the impulsive condition or point interaction for (1.8), $n=0$ is the single impulsive point for the impulsive boundary value problem (IBVP) (1.8)–(1.9) and $\mathbf{T}$ is called the transfer matrix which is used to continue the solutions from negatif integers to positive integers.

The set up of this paper is summarized as follows: In Section 2, we present some basic concepts concerning the notations and solutions of quadratic pencil of difference equations without an impulsive point to use our further works. Then, we obtain the representations of the solutions of IBVP (1.8)–(1.9). Section 3 discusses the continuous spectrum and resolvent operator of this problem so that we can define the sets of eigenvalues and spectral singularities by means of poles of the resolvent operator. Main theorems and results of this paper are given in last section. So, Section 4 includes sufficient conditions guarenteeing the finiteness of eigenvalues and spectral singularities.

2. Statement of the Problem

Let us introduce the Hilbert space $\ell_2(\mathbb{Z})$ consisting of all complex-valued sequences $y:=\{y_n\}_{n \in \mathbb{Z}}$ with the inner product

$$<y, z>=\sum_{n \in \mathbb{Z}}\left(y_n, z_n\right), \quad \{y_n\}_{n \in \mathbb{Z}},\{z_n\}_{n \in \mathbb{Z}} \in \mathbb{C}$$

such that

$$||y||_{\ell_2}^2 := \sum_{n\in\mathbb{Z}} |y_n|^2 < \infty.$$

We will denote the operator $\mathcal{L}_\mu$ in $\ell_2(\mathbb{Z})$ generated by the IBVP (1.8)–(1.9).

In order to set the theory for our impulsive operator $\mathcal{L}_\mu$, we need some preliminaries. For convenience, we will use the same notations and representations with the reference [1]. The bounded solutions of (1.5) which are represented by

$$f_n^+(z) = \rho_n^+ e^{inz}\left(1 + \sum_{m=1}^{\infty} K_{nm}^+ e^{im\dfrac{z}{2}}\right), \quad n \in \mathbb{Z}, \tag{2.1}$$

and

$$f_n^-(z) = \rho_n^- e^{-inz}\left(1 + \sum_{-\infty}^{m=-1} K_{nm}^- e^{-im\dfrac{z}{2}}\right), \quad n \in \mathbb{Z} \tag{2.2}$$

for $z \in \overline{\mathbb{C}}_+ := \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$, where

$$\rho_n^+ = \left\{\prod_{r=n}^{\infty}(-a_r)\right\}^{-1} \quad \text{and} \quad \rho_n^- = \left\{\prod_{-\infty}^{r=n-1}(-a_r)\right\}^{-1} \tag{2.3}$$

are said to be Jost solutions of (1.5). As is seen, the coefficients $\rho_n^\pm$ and the kernels $K_{nm}^\pm$ are uniquely expressed in terms of $\{a_n\}_{n\in\mathbb{Z}}$, $\{p_n\}_{n\in\mathbb{Z}}$ and $\{q_n\}_{n\in\mathbb{Z}}$. It will be useful to keep in mind that the condition (1.6) assures the convergences of the products in (2.3) and the kernels $K_{nm}^\pm$ [1]. Moreover, we will also need the inequalities for the kernels $K_{nm}^\pm$ obtained as

$$\left|K_{nm}^+\right| \leq c_1 \sum_{r=n+[|\frac{m}{2}|]}^{\infty} \left(|1-a_r| + |p_r| + |q_r|\right), \tag{2.4}$$

$$\left|K_{nm}^-\right| \leq c_2 \sum_{-\infty}^{r=n+[|\frac{m}{2}|]+1} \left(|1-a_r| + |p_r| + |q_r|\right), \tag{2.5}$$

where $[|\dfrac{m}{2}|]$ denotes the integer part of $\dfrac{m}{2}$ and c_1, c_2 are positive constants. In other words, as consequences of (2.4) and (2.5), the solutions f_n^+ and f_n^- satisfying the asymptotic equations

$$f_n^\pm(z) = \exp\left(\pm inz\right)\left[1 + o(1)\right], \quad z \in \overline{\mathbb{C}}_+, \quad n \to \pm\infty, \tag{2.6}$$

$$f_n^\pm(z) = \rho_n^\pm \exp\left(\pm inz\right)\left[1 + o(1)\right], \quad n \in \mathbb{Z}, \quad z = \xi + i\tau, \quad \tau \to \infty, \tag{2.7}$$

are called Jost solutions of (1.5). Note that, $f^+(z) := \{f_n^+(z)\}_{n\in\mathbb{Z}}$ and $f^-(z) := \{f_n^-(z)\}_{n\in\mathbb{Z}}$ are both analytic with respect to z in $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and continuous in $\overline{\mathbb{C}}_+$.

On the other hand, $g^\pm(z) := \{g_n^\pm(z)\}_{n\in\mathbb{Z}}$ are also solutions of (1.5) for $z \in \overline{\mathbb{C}}_- := \{z \in \mathbb{C} : \operatorname{Im} z \leq 0\}$ satisfying the asymptotic equations

$$g_n^\pm(z) = \exp\left(\mp inz\right)\left[1 + o(1)\right], \quad z \in \overline{\mathbb{C}}_-, \quad n \to \pm\infty. \tag{2.8}$$

Similarly, $g_n^\pm$ are analytic with respect to z in $\mathbb{C}_-$, continuous in $\overline{\mathbb{C}}_-$. It is obvious that for z in $\mathbb{C}_-$, $g_n^\pm(z) = f_n^\pm(-z)$ holds.

Definition 2.1. Wronskian of any two solutions $u = \{u_n\}_{n\in\mathbb{Z}}$ and $v = \{v_n\}_{n\in\mathbb{Z}}$ of (1.5) or (1.8) is defined as

$$W[u, v] := a_n \left[u_n v_{n+1} - u_{n+1} v_n\right]. \tag{2.9}$$

Lemma 2.1. *The pairs* $\{f_n^+(z)\}_{n\in\mathbb{Z}}$, $\{f_n^+(-z)\}_{n\in\mathbb{Z}}$ *and* $\{f_n^-(z)\}_{n\in\mathbb{Z}}$, $\{f_n^-(-z)\}_{n\in\mathbb{Z}}$ *form two fundamental systems of solutions of* (1.8) *for* $z \in \mathbb{R}^* := \mathbb{R}\backslash\{z : z = k\pi, k \in \mathbb{Z}\}$.

Proof. It can be easily calculated from (2.9) that

$$W[f^\pm(z), f^\pm(-z)] = \mp 2i \sin z, \quad z \in \mathbb{R}. \tag{2.10}$$

Since $W[f^\pm(z), f^\pm(-z)] \neq 0$ for all $z \in \mathbb{R}^*$, we prove that the pairs $\{f_n^+(z)\}_{n\in\mathbb{Z}}$, $\{f_n^+(-z)\}_{n\in\mathbb{Z}}$ and $\{f_n^-(z)\}_{n\in\mathbb{Z}}$, $\{f_n^-(-z)\}_{n\in\mathbb{Z}}$ are linearly independent. This completes the proof. □

Now, we are ready to continue with the impulsive quadratic pencil of difference equation. We first seek the solutions of (1.8)–(1.9) and express two of them as

$$F_n^+(z) = \begin{cases} \beta_1(z) f_n^-(z) + \beta_2(z) f_n^-(-z), & n \in \mathbb{Z}^-, \\ f_n^+(z), & n \in \mathbb{Z}^+, \end{cases} \tag{2.11}$$

and

$$F_n^-(z) = \begin{cases} f_n^-(z), & n \in \mathbb{Z}^-, \\ \beta_3(z) f_n^+(z) + \beta_4(z) f_n^+(-z), & n \in \mathbb{Z}^+, \end{cases} \tag{2.12}$$

for $\mu = 2\cos\dfrac{z}{2}$, $z \in \mathbb{R}^*$, $\{\beta_i\}_{i=1,2,3,4}$ are arbitrary coefficients depending on z. Using the impulsive condition (1.9), we get uniquely

$$\begin{aligned}\beta_1(z) = \frac{a_{-2}}{2i \sin z \det \mathbf{T}} &\left\{\alpha_3 f_1^+(z) f_{-1}^-(-z) + \alpha_4 f_1^+(z) \bigtriangledown f_{-1}^-(-z)\right. \\ &\left.-\alpha_1 f_{-1}^-(-z) \bigtriangleup f_1^+(z) - \alpha_2 \bigtriangleup f_1^+(z) \bigtriangledown f_{-1}^-(-z)\right\},\end{aligned} \tag{2.13}$$

$$\begin{aligned}\beta_2(z) = \frac{a_{-2}}{2i \sin z \det \mathbf{T}} &\left\{\alpha_1 f_{-1}^-(z) \bigtriangleup f_1^+(z) + \alpha_2 \bigtriangleup f_1^+(z) \bigtriangledown f_{-1}^-(z)\right. \\ &\left.-\alpha_3 f_{-1}^-(z)\ f_1^+(z) - \alpha_4 f_1^+(z) \bigtriangledown f_{-1}^-(z)\right\},\end{aligned} \tag{2.14}$$

$$\begin{aligned}\beta_3(z) = -\frac{a_1}{2i \sin z} &\left\{\alpha_1 f_{-1}^-(z) \bigtriangleup f_1^+(-z) + \alpha_2 \bigtriangleup f_1^+(-z) \bigtriangledown f_{-1}^-(z)\right. \\ &\left.-\alpha_3 f_{-1}^-(z) f_1^+(-z) - \alpha_4 f_1^+(-z) \bigtriangledown f_{-1}^-(z)\right\},\end{aligned} \tag{2.15}$$

$$\begin{aligned}\beta_4(z) = -\frac{a_1}{2i \sin z} &\left\{\alpha_3 f_1^+(z) f_{-1}^-(z) + \alpha_4 f_1^+(z) \bigtriangledown f_{-1}^-(z)\right. \\ &\left.-\alpha_1 f_{-1}^-(z) \bigtriangleup f_1^+(z) - \alpha_2 \bigtriangleup f_1^+(z) \bigtriangledown f_{-1}^-(z)\right\}\end{aligned} \tag{2.16}$$

for all $z \in \mathbb{R}^*$.

Corollary 2.1. *Using* (2.9)-(2.16), *we easily conclude that*

$$G(z) := W[F_n^+(z), F_n^-(z)] = \begin{cases} -2i \sin z \beta_2(z), & n \in \mathbb{Z}^-, \\ \\ -2i \sin z \beta_2(z) \dfrac{a_1}{a_{-2}} \det \boldsymbol{T}, & n \in \mathbb{Z}^+, \end{cases}$$

holds for all $z \in \mathbb{R}^*$.

Remark 2.1. (2.14) and Corollary 2.1 imply that the function β_2 has an analytic continuation from the real axis to the open upper half-plane $\mathbb{C}_+$. Hence, following the same idea of [1], we obtain the analyticity of G in $\mathbb{C}_+$ and continuity in $\overline{\mathbb{C}}_+$, then we get that G is a 4π-periodic function.

Theorem 2.1. *Under the condition* (1.6)*, the function* β_2 *has asymptotic equations for all* $n \in \mathbb{Z}$ *and* $\operatorname{Im} z \to \infty$.

(i) Assume $\sum\limits_{k=1}^{4} \alpha_k \neq 0$. *Then the asymptotic equation*

$$\beta_2(z)e^{-3iz} = \frac{a_{-2}}{\det \boldsymbol{T}} \rho_{-1}^- \rho_1^+ \sum_{k=1}^{4} \alpha_k \left[1 + o(1)\right]$$

is satisfied.

(ii) Assume $\sum\limits_{k=1}^{4} \alpha_k = 0$ *and* $a_{-2} = a_1$. *Then the asymptotic equations*

$$\beta_2(z)e^{-4iz} = -\frac{a_{-2}}{\det \boldsymbol{T}} \left[(\alpha_1+\alpha_2)\rho_{-1}^- \rho_2^+ + (\alpha_2+\alpha_4)\rho_1^+ \rho_{-2}^-\right] \left[1+o(1)\right], \ \ \alpha_2 \neq \alpha_3,$$

$$\beta_2(z)e^{-5iz} = \frac{a_{-2}}{\det \boldsymbol{T}} \alpha_2 \rho_2^+ \rho_{-2}^- \left[1 + o(1)\right], \quad \alpha_2 = \alpha_3$$

are satisfied.

Proof. Equation (2.14) can be rewritten as

$$\begin{aligned}\beta_2(z) = {} & \frac{a_{-2}}{2i \sin z \det \mathbf{T}} \left[(\alpha_1+\alpha_2) f_{-1}^-(z) f_2^+(z) - (\alpha_1+\alpha_2+\alpha_3+\alpha_4) f_{-1}^-(z) f_1^+(z) \right. \\ & \left. -\alpha_2 f_2^+(z)\ f_{-2}^-(z) + (\alpha_2+\alpha_4) f_1^+(z) f_{-2}^-(z)\right]. \end{aligned} \tag{2.17}$$

Thus, from (2.7), we have the asymptotic equation for (2.17).

$$\begin{aligned}\beta_2(z) = {} & -\frac{a_{-2}}{2i \sin z \det \mathbf{T}} e^{2iz} \left\{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\rho_{-1}^- \rho_1^+ - e^{iz}(\alpha_1+\alpha_2)\rho_{-1}^- \rho_2^+ \right. \\ & \left. -e^{iz}(\alpha_2+\alpha_4)\rho_1^+ \rho_{-2}^- + e^{2iz}\alpha_2 \rho_2^+ \rho_{-2}^-\right\} \left[1 + o(1)\right], \quad \operatorname{Im} z \to \infty. \end{aligned} \tag{2.18}$$

This implies two cases:

(i) If $\sum\limits_{k=1}^{4} \alpha_k \neq 0$, then the proof is clear.

(ii) If $\sum\limits_{k=1}^{4} \alpha_k = 0$ and $a_{-2} = a_1$, then (2.18) turns into

$$\beta_2(z) = \frac{a_{-2}}{\det \mathbf{T}} e^{3iz} \left[(\alpha_1+\alpha_2)\rho_{-1}^- \rho_2^+ + (\alpha_2+\alpha_4)\rho_1^+ \rho_{-2}^- - e^{iz}\alpha_2 \rho_2^+ \rho_{-2}^-\right] \left[1+o(1)\right].$$

It is obvious that the asymptotic depends on the subcase:
Considering $(\alpha_1+\alpha_2)\rho_{-1}^- \rho_2^+ + (\alpha_2+\alpha_4)\rho_1^+ \rho_{-2}^- = 0$, namely, $\alpha_2 = \alpha_3$ yields

$$\beta_2(z)e^{-5iz} = \frac{a_{-2}}{\det \mathbf{T}} \alpha_2 \rho_2^+ \rho_{-2}^- \left[1 + o(1)\right],$$

and also considering $\alpha_2 \neq \alpha_3$ yields

$$\beta_2(z)e^{-4iz} = -\frac{a_{-2}}{\det \mathbf{T}}\left[(\alpha_1+\alpha_2)\rho_{-1}^-\rho_2^+ + (\alpha_2+\alpha_4)\rho_1^+\rho_{-2}^-\right][1+o(1)]$$

for $\operatorname{Im} z \to \infty$, where $\rho_1^+ = \left\{\prod_{r=1}^{\infty}(-a_r)\right\}^{-1}$, $\rho_{-1}^- = \left\{\prod_{-\infty}^{r=-2}(-a_r)\right\}^{-1}$,
$\rho_{-2}^- = \left\{\prod_{-\infty}^{r=-3}(-a_r)\right\}^{-1}$ and $\rho_2^+ = \left\{\prod_{r=2}^{\infty}(-a_r)\right\}^{-1}$.

Hence, the proof is completed. □

3. Resolvent Operator and Continuous Spectrum of $\mathcal{L}_\mu$

In this section, we first define the semi-strips

$$P^+ := \{z \in \mathbb{C} : z = x+iy, y>0, -\pi \le x \le 3\pi\}$$

and

$$P_0 := (-\pi, 3\pi) \setminus \{0, \pi, 2\pi\}.$$

Then, we can find two unbounded solutions of the impulsive boundary value problem (1.8)–(1.9) as

$$U_n^+(z) = \begin{cases} \widehat{\beta_1}(z) f_n^-(z) + \beta_2(z)\widehat{f_n}^-(z), & n \in \mathbb{Z}^-, \\ f_n^+(z), & n \in \mathbb{Z}^+, \end{cases} \tag{3.1}$$

and

$$V_n^-(z) = \begin{cases} f_n^-(z), & n \in \mathbb{Z}^-, \\ \widehat{\beta_3}(z) f_n^+(z) + \beta_4(z)\widehat{f_n}^+(z), & n \in \mathbb{Z}^+, \end{cases} \tag{3.2}$$

for $\mu = 2\cos\frac{z}{2}$, $z \in P^+$, where $\left\{\widehat{f^{\pm}}_n(z)\right\}_{n\in\mathbb{Z}}$ satisfy the asymptotic equations

$$\widehat{f^{\pm}}_n(z) = e^{\mp inz}[1+o(1)], \quad z \in \overline{\mathbb{C}}_+, \quad n \to \pm\infty. \tag{3.3}$$

Taking into consideration U_n^+ and V_n^-, we get that the $z-$depending coefficients β_2 and β_4 are expressed as in (2.14) and (2.16), respectively, while $\widehat{\beta_1}$ and $\widehat{\beta_3}$ are obtained as

$$\begin{aligned}\widehat{\beta_1}(z) = \frac{a_{-2}}{2i\sin z \det \mathbf{T}} \Big\{ & \alpha_3 f_1^+(z)\widehat{f_{-1}^-}(z) + \alpha_4 f_1^+(z) \bigtriangledown \widehat{f_{-1}^-}(z) \\ & -\alpha_1 \widehat{f_{-1}^-}(z) \bigtriangleup f_1^+(z) - \alpha_2 \bigtriangleup f_1^+(z) \bigtriangledown \widehat{f_{-1}^-}(z) \Big\}\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\widehat{\beta_3}(z) = -\frac{a_1}{2i\sin z} \Big\{ & \alpha_1 f_{-1}^-(z) \bigtriangleup \widehat{f_1^+}(z) + \alpha_2 \bigtriangleup \widehat{f_1^+}(z) \bigtriangledown f_{-1}^-(z) \\ & -\alpha_3 f_{-1}^-(z)\widehat{f_1^+}(z) - \alpha_4 \widehat{f_1^+}(z) \bigtriangledown f_{-1}^-(z) \Big\},\end{aligned} \tag{3.5}$$

respectively.

By the help of these unbounded solutions, we can construct the resolvent operator of $\mathcal{L}_\mu$. Note that, the set of μ values for which ${\mathcal{L}_\mu}^{-1}$ exists and bounded on $\ell_2(\mathbb{Z})$ is called the resolvent set of $\mathcal{L}_\mu$. If μ belongs to the resolvent set, then the resolvent operator $\mathcal{R}_\mu(\mathcal{L}_\mu)$ of $\mathcal{L}_\mu$ exists.

Theorem 3.1. *The resolvent operator of $\mathcal{L}_\mu$ has the representation*

$$(\mathcal{R}_\mu(\mathcal{L}_\mu)\varphi)_n := \sum_{m\in\mathbb{Z}} \mathcal{G}_{n,m}(z)\varphi(m), \quad \varphi := \{\varphi_m\} \in \ell_2(\mathbb{Z}),$$

where

$$\mathcal{G}_{n,m}(z) = \begin{cases} -\dfrac{U_n^+(z)V_m^-(z)}{W[U^+,V^-](z)}, & m = n-1, n-2, \dots \\ -\dfrac{V_n^-(z)U_m^+(z)}{W[U^+,V^-](z)}, & m = n, n+1, \dots \end{cases}$$

is the Green function for $z \in P^+$ and $m, n \neq \{0\}$. Moreover, for $z \in P^+$, the Wronskian of $U^+(z)$ and $V^-(z)$ can be calculated independently of n as follows:

$$W[U^+(z), V^-(z)] = \begin{cases} -2i\sin z\beta_2(z), & n \in \mathbb{Z}^-, \\ -2i\sin z\dfrac{a_1}{a_{-2}} \det \boldsymbol{T}\beta_2(z), & n \in \mathbb{Z}^+. \end{cases} \tag{3.6}$$

Due to Theorem 3.1, it is not difficult to obtain the continuous spectrum of $\mathcal{L}_\mu$. For this reason, let us firstly give the next lemma and its proof.

Lemma 3.1. *For all $\epsilon > 0$, there exists a positive number c_ϵ such that*

$$||\mathcal{R}_\mu(\mathcal{L}_\mu)|| \geq \frac{c_\epsilon}{|W[U^+,V^-](z)|\sqrt{1-e^{-2\operatorname{Im} z}}}$$

holds for $\mu = 2\cos\frac{z}{2}$, $z \in \mathbb{C}_+$ and $\operatorname{Im} z > \epsilon$.

Proof. Assume $\epsilon > 0$, $z \in \mathbb{C}_+$ and $\operatorname{Im} z > \epsilon$. Then, let us define the function $\mathcal{H}^{m_0}$ by

$$\mathcal{H}_n^{m_0}(z) := \begin{cases} \overline{V_n^-(z)}, & n = m_0-1, m_0-2, \dots, \\ 0, & n = m_0, m_0+1, \dots, \end{cases}$$

for $n \neq 0$. Obviously, $\mathcal{H}_n^{m_0}(z) \in \ell_2(\mathbb{Z})$ and its proof is straightforward from (3.2) and (2.7). Then, we can write

$$\begin{aligned} \mathcal{R}_\mu(\mathcal{L}_\mu)\mathcal{H}_n^{m_0}(z) &= \sum_{-\infty}^{m=m_0-1} \mathcal{G}_{n,m}(z)\overline{V_m^-(z)} \\ &= \frac{U_n^+(z)}{W[U^+,V^-](z)} ||\mathcal{H}_n^{m_0}||^2 \end{aligned} \tag{3.7}$$

for $m_0 < n$ and $n \neq 0$. Now, in order to get an inequality for $|U_n^+(z)|$, for $n \in \mathbb{Z}$, we consider (3.1) and obtain

$$\widehat{\beta_1}(z)f_n^-(z) + \beta_2(z)\widehat{f_n}^-(z) = \frac{a_{-2}}{2i\sin z\det\mathbf{T}}\left[\sum_{k=1}^{4}\alpha_k\rho_n^-\rho_1^+\rho_{-1}^-\left(e^{-inz} - e^{iz(2+n)}\right)\right.$$

$$
\begin{aligned}
&+(\alpha_1+\alpha_2)\rho_n^-\rho_2^+\rho_{-1}^-\left(e^{iz(3+n)}-e^{iz(1-n)}\right)\\
&+(\alpha_2+\alpha_4)\rho_n^-\rho_1^+\rho_{-2}^-\left(e^{iz(3+n)}-e^{-iz(1+n)}\right)\\
&+\alpha_2\rho_n^-\rho_2^+\rho_{-2}^-\left(e^{-inz}-e^{iz(4+n)}\right)\Big]
\end{aligned}
$$

for $n \in \mathbb{Z}^-$. Without loss of generality, similar to Theorem 2.1, we can obtain an asymptotic equation for U_n^+ by

$$
U_n^+(z)=\begin{cases} -\dfrac{a_{-2}}{2i\sin z\det \mathbf{T}}\sum\limits_{k=1}^{4}\alpha_k\rho_n^-\rho_1^+\rho_{-1}^-\left[e^{iz(2+n)}+o(1)\right], & n\in\mathbb{Z}^-, \quad \operatorname{Im} z\to\infty,\\ e^{inz}+o(1), & n\in\mathbb{Z}^+, \quad \operatorname{Im} z\to\infty. \end{cases}
$$

So, assuming $\operatorname{Im} z > \epsilon$, and choosing $m_0 = m_0(\epsilon)$ sufficiently large so that $m_0 < n$, last asymptotic equation implies the inequality

$$
\left|U_n^+(z)\right| > \frac{1}{2}e^{-n\operatorname{Im} z}.
$$

Thus, we arrive at

$$
\left|\left|U_n^+(z)\right|\right|^2 \geq \frac{e^{-2m_0\operatorname{Im} z}}{4(1-e^{-2\operatorname{Im} z})}. \tag{3.8}
$$

Substituting (3.8) in (3.7) after computing the norm of resolvent operator yields

$$
||\mathcal{R}_\mu(\mathcal{L}_\mu)||^2 \geq \frac{c_\epsilon^2}{\left|W\left[U^+,V^-\right](z)\right|^2\left(1-e^{-2\operatorname{Im} z}\right)},
$$

where

$$
c_\epsilon = \frac{||\mathcal{H}_n^{m_0}||^2}{2e^{m_0\operatorname{Im} z}}.
$$

□

Theorem 3.2. $\sigma_c(\mathcal{L}_\mu) = [-2,2]$, *where* $\sigma_c(\mathcal{L}_\mu)$ *denotes the continuous spectrum of the operator* $\mathcal{L}_\mu$.

Proof. In order to prove $\sigma_c(\mathcal{L}_\mu) \subset [-2,2]$, let $\mu_0 = 2\cos\frac{z_0}{2} \in \sigma_c(\mathcal{L}_\mu)$. So, we can write $||\mathcal{R}_\mu(\mathcal{L}_\mu)|| \to \infty$ as $\mu \to \mu_0$ as a requirement of definition of continuous spectrum. Then, by the help of Lemma 3.1, we conclude that $W\left[U_n^+,V_n^-\right](z) \to 0$ as $z \to z_0$. It is clear to see the continuity of the function $W\left[U_n^+,V_n^-\right]$ from (3.6), so that $W\left[U_n^+,V_n^-\right](z) \to W\left[U_n^+,V_n^-\right](z_0)$ as $z \to z_0$. But, since μ_0 belongs to continuous spectrum, $W\left[U_n^+,V_n^-\right](z_0)$ cannot be equal to zero. This yields $\sin z \to 0$, i.e., $\operatorname{Im} z \to 0$ as $z \to z_0$. Thus, we get $\operatorname{Im} z_0 = 0$, i.e., $\mu_0 = 2\cos\frac{z_0}{2} \in [-2,2]$.

Conversely, in order to prove $[-2,2] \subset \sigma_c(\mathcal{L}_\mu)$, let an arbitrary $\mu_0 = 2\cos\frac{z}{2} \in [-2,2]$. This means there is a corresponding z_0 such that $\operatorname{Im} z_0 \to 0$. Hence, by Lemma 3.1, we find $||\mathcal{R}_\mu(\mathcal{L}_\mu)|| \to \infty$ for all $\mu = 2\cos\frac{z}{2} \in [-2,2]$. To complete the proof, it only remains to show that the range $\mathcal{R}_\mu(\mathcal{L}_\mu)$ of values of the operator $\mathcal{L}_\mu$ is dense in the space $\ell_2(\mathbb{Z})$. Therefore, we need to prove that the orthogonal complement of the set $\mathcal{R}_\mu(\mathcal{L}_\mu)$ consists of the zero element only. Indeed, this orthogonal complement coincides with the space of solutions $y \in \ell_2(\mathbb{Z})$ of equation $\mathcal{L}_\mu^* y = 0$, where $\mathcal{L}_\mu^*$ denotes the adjoint operator of $\mathcal{L}_\mu$. Consequently, the proof is completed as desired because of the fact that the equation $\mathcal{L}_\mu^* y = 0$ has no eigenvalue on the real line. □

4. Eigenvalues and Spectral Singularities

This section includes the main results for the spectral theory of impulsive quadratic pencil of difference operators. In spite of mixed calculations, we proved that the finiteness of eigenvalues and spectral singularities of the operator can be still guaranteed in the event that the corresponding equation has an impulsive point. In this respect, this papers differs from [1] and the others. In this section, we will first define the sets of eigenvalues and spectral singularities in terms of the poles of the resolvent operator obtained in Section 3.

Theorem 3.1 and (3.6) point us that in order to investigate the quantitative properties of impulsive boundary value problem (1.8)–(1.9), it is necessary to get the quantitative properties of zeros of the function β_2. Therefore, we can introduce the sets of eigenvalues and spectral singularities of the operator $\mathcal{L}_\mu$ by

$$\sigma_d(\mathcal{L}_\mu) = \left\{\mu = 2\cos\frac{z}{2}, z \in P^+, \beta_2(z) = 0\right\} \tag{4.1}$$

and

$$\sigma_{ss}(\mathcal{L}_\mu) = \left\{\mu = 2\cos\frac{z}{2}, z \in P_0, \beta_2(z) = 0\right\}, \tag{4.2}$$

respectively.

Now, let N_1 and N_2 denote the sets of all zeros of the function β_2 in P^+ and P_0, respectively. This evidently implies

$$N_1 := \left\{z : z \in P^+, \beta_2(z) = 0\right\}, \tag{4.3}$$

$$N_2 := \{z : z \in P_0, \beta_2(z) = 0\}. \tag{4.4}$$

Lemma 4.1. *(i) The set N_1 is bounded, has at most countably many elements and its limit points can lie only in $[-\pi, 3\pi]$.*

(ii) The set N_2 is compact and its linear Lebesgue measure is zero.

Proof.

(i) Since $\det \mathbf{T} > 0$, Theorem 2.1 proves the boundedness of the sets N_1 and N_2. Furthermore, it follows from (2.14) that the function β_2 is analytic in P^+ and 4π-periodic, then the limit points of the zeros of β_2 in P^+ can only lie in $[-\pi, 3\pi]$.

(ii) Since N_2 is a bounded subset of real numbers, in order to prove the compactness of N_2, we need to prove the closeness of N_2. Boundary uniqueness theorems of analytic functions give us that N_2 is a closed set and Privalov Theorem [13] proves that its linear Lebesgue measure is zero.

□

It is not difficult to see that the sets of eigenvalues and spectral singularities can be rewritten according to (4.3) and (4.4) as follows

$$\sigma_d(\mathcal{L}_\mu) := \left\{\mu : \mu = 2\cos\frac{z}{2}, z \in N_1\right\} \tag{4.5}$$

and

$$\sigma_{ss}(\mathcal{L}_\mu) := \left\{\mu : \mu = 2\cos\frac{z}{2}, z \in N_2\right\}, \tag{4.6}$$

respectively.

The following theorem is a direct consequence of (4.5), (4.6) and Lemma 4.1.

Theorem 4.1. *Assume* (1.6)*. Then we have the results.*

(i) The set of eigenvalues of $\mathcal{L}_\mu$ is bounded and countable, its limit points can lie only in $[-2,2]$,

(ii) The set of spectral singularities of $\mathcal{L}_\mu$ is compact and its linear Lebesgue measure is zero.

Now, we proceed by assuming the extra condition

$$\sup_{n\in Z}\left\{e^{\epsilon|n|}\left(|1-a_n|+|p_n|+|q_n|\right)\right\}<\infty, \quad \epsilon>0 \tag{4.7}$$

on the sequences $\{a_n\}$, $\{p_n\}$ and $\{q_n\}$ to assure the finiteness of the sets of eigenvalues and spectral singularities. Before giving the related theorem, we briefly recall some basic facts.

Definition 4.1. The convolution of the sequences $\{u_n\}$ and $\{v_n\}$ is defined by

$$u_n * v_n := \sum_{n\in\mathbb{Z}} u_n v_{n-m}, \tag{4.8}$$

where "$*$" denotes the convolution operation.

Lemma 4.2. *The equality*

$$\sum_{n\in\mathbb{Z}}\left(u_n * v_n\right)e^{i\lambda n}=\sum_{n\in\mathbb{Z}}u_n e^{i\lambda n}\sum_{n\in\mathbb{Z}}v_n e^{i\lambda n} \tag{4.9}$$

holds for all $\lambda\in\mathbb{C}$.

Definition 4.2. The multiplicity of the corresponding eigenvalue or spectral singularity of the operator $\mathcal{L}_\mu$ is called the multiplicity of a zero of the function β_2 in $P^+\cup P_0$.

Theorem 4.2. *If the condition* (4.7) *holds for some $\epsilon>0$, then the operator $\mathcal{L}_\mu$ has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity.*

Proof. Under the condition (4.7), we get from (2.4) and (2.5) that the kernels satisfy

$$\left|K_{j,m}^{+}\right|\le\tilde{c}_1 e^{-\frac{\epsilon}{4}m}, \quad j=1,2;\quad m\in\mathbb{N}, \tag{4.10}$$

$$\left|K_{j,m}^{-}\right|\le\tilde{c}_2 e^{\frac{\epsilon}{4}m}, \quad j=-1,-2;\quad m=-1,-2,\ldots, \tag{4.11}$$

where $\tilde{c}_1,\tilde{c}_2$ are arbitrary constants. Afterwards, we obtain by (4.8)-(4.11) that

$$\left|K_{j,-m}^{-}*K_{2,m}^{+}\right|,\left|K_{j,-m}^{-}*K_{1,m}^{+}\right|\le\tilde{c}_3 e^{-\frac{\epsilon}{4}m}, \quad j=-1,-2;\quad m\in\mathbb{N} \tag{4.12}$$

holds. From (2.17), these inequalities imply the representation for $\beta_2(z)$.

$$\beta_2(z)=\frac{a_{-2}}{2i\sin z\det\mathbf{T}}\left\{(\alpha_1+\alpha_2)\,\rho_{-1}^{-}\rho_2^{+}e^{3iz}\left(1+\sum_{m=1}^{\infty}K_{2m}^{+}e^{im\frac{z}{2}}\right)\left(1+\sum_{-\infty}^{m=-1}K_{-1m}^{-}e^{-im\frac{z}{2}}\right)\right.$$

$$-\left(\alpha_1+\alpha_2+\alpha_3+\alpha_4\right)\rho_{-1}^{-}\rho_1^{+}e^{2iz}\left(1+\sum_{m=1}^{\infty}K_{1m}^{+}e^{im\frac{z}{2}}\right)\left(1+\sum_{-\infty}^{m=-1}K_{-1m}^{-}e^{-im\frac{z}{2}}\right)$$
$$-\alpha_2\rho_{-2}^{-}\rho_2^{+}e^{4iz}\left(1+\sum_{m=1}^{\infty}K_{2m}^{+}e^{im\frac{z}{2}}\right)\left(1+\sum_{-\infty}^{m=-1}K_{-2m}^{-}e^{-im\frac{z}{2}}\right)\Bigg\}$$
$$+\left(\alpha_2+\alpha_4\right)\rho_{-2}^{-}\rho_1^{+}e^{3iz}\left(1+\sum_{m=1}^{\infty}K_{1m}^{+}e^{im\frac{z}{2}}\right)\left(1+\sum_{-\infty}^{m=-1}K_{-2m}^{-}e^{-im\frac{z}{2}}\right)\Bigg\}.$$

By (4.10)–(4.12) and Lemma 4.2, the last representation demonstrates that the function β_2 has an analytic continuation to the half plane $\operatorname{Im} z > -\frac{\epsilon}{4}$. Thus, the limit points of all zeros of the function β_2 in P^+ can not lie in $[-\pi, 3\pi]$. So, we obtain that the bounded sets $\sigma_d(\mathcal{L}_\mu)$ and $\sigma_{ss}(\mathcal{L}_\mu)$ have no limit poins by Theorem 4.2, namely, these sets have a finite number of elements. Analyticity of β_2 in $\operatorname{Im} z > -\frac{\epsilon}{4}$ proves that all zeros of β_2 in $P^+ \cup P_0$ have finite multiplicity. As a direct result of this corollary, all eigenvalues and spectral singularities of $\mathcal{L}_\mu$ have a finite multiplicity under the condition (4.7). □

Henceforth, we will suppose the following condition which is weaker than (4.7). Let us assume that for some $\epsilon > 0$ and $\frac{1}{2} \leq \delta < 1$,

$$\sup_{n\in Z}\left\{e^{\epsilon|n|^{\delta}}\left(|1-a_n|+|p_n|+|q_n|\right)\right\}<\infty, \quad \epsilon>0 \tag{4.13}$$

holds. Under the condition (4.13), the function β_2 is still analytic in $\mathbb{C}_+$ and has infinitely many derivatives by (2.4), (2.5).

In order to investigate the finiteness of eigenvalues and spectral singularities under this condition, we need following notations. Let us denote the sets of all limit points of N_1 and N_2 by N_3 and N_4, respectively and the set of all zeros of β_2 with infinite multiplicity in $P^+ \cup P_0$ by N_5.

Lemma 4.3. *(i)* $N_3 \subset N_2$, $N_4 \subset N_2$, $N_5 \subset N_2$, $N_3 \subset N_5$, $N_4 \subset N_5$,

(ii) $\mu(N_3) = \mu(N_4) = \mu(N_5) = 0$.

Proof. Proof of the lemma is obvious from the boundary uniqueness theorems of analytic functions in [13]. □

For the sake of simplicity, let us define

$$A(z) := \frac{\beta_2(z)2i\sin z\det\mathbf{T}}{a_{-2}} \tag{4.14}$$

which is also analytic in $\mathbb{C}_+$ and infinitely differentiable on the real axis.

In order to give our main result, we need two lemmas.

Lemma 4.4. *Under the condition* (4.13)*, we get that inequality*

$$\left|A^{(k)}(z)\right| \leq H_k \tag{4.15}$$

holds for $z \in P^+$ *and k=0,1,…, where*

$$H_k \leq \widetilde{C}8^k B b^k k! k^{k\frac{1-\delta}{\delta}},$$

$\widetilde{C}, B, b$ *are positive constants depending on* ϵ *and* δ.

Proof. From (2.17) , we can regulate $A(z)$ as

$$\begin{aligned}\left|A^{(k)}(z)\right| \leq \Bigg[&\left|(\alpha_1+\alpha_2)\rho_2^+\rho_{-1}^-\right|\left(3^k+\sum_{m=1}^{\infty}\left(3+\frac{m}{2}\right)^k\left|K_{2,m}^+\right|\right.\\
&\left.+\sum_{-\infty}^{m=-1}\left(3-\frac{m}{2}\right)^k\left|K_{-1,m}^-\right|+\sum_{m=1}^{\infty}\left(\frac{3+m}{2}\right)^k\left|K_{-1,-m}^-*K_{2,m}^+\right|\right)\\
&+\left|(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\rho_1^+\rho_{-1}^-\right|\left(2^k+\sum_{m=1}^{\infty}\left(2+\frac{m}{2}\right)^k\left|K_{1,m}^+\right|\right.\\
&\left.+\sum_{-\infty}^{m=-1}\left(2-\frac{m}{2}\right)^k\left|K_{-1,m}^-\right|+\sum_{m=1}^{\infty}\left(1+\frac{m}{2}\right)^k\left|K_{-1,-m}^-*K_{1,m}^+\right|\right) \quad (4.16)\\
&+\left|\alpha_2\rho_2^+\rho_{-2}^-\right|\left(4^k+\sum_{m=1}^{\infty}\left(4+\frac{m}{2}\right)^k\left|K_{2,m}^+\right|\right.\\
&\left.+\sum_{-\infty}^{m=-1}\left(4-\frac{m}{2}\right)^k\left|K_{-2,m}^-\right|+\sum_{m=1}^{\infty}\left(2+\frac{m}{2}\right)^k\left|K_{-2,-m}^-*K_{2,m}^+\right|\right)\\
&+\left|(\alpha_2+\alpha_4)\rho_1^+\rho_{-2}^-\right|\left(3^k+\sum_{m=1}^{\infty}\left(3+\frac{m}{2}\right)^k\left|K_{1,m}^+\right|\right.\\
&\left.+\sum_{-\infty}^{m=-1}\left(3-\frac{m}{2}\right)^k\left|K_{-2,m}^-\right|+\sum_{m=1}^{\infty}\left(\frac{3+m}{2}\right)^k\left|K_{-2,-m}^-*K_{2,m}^+\right|\right)\Bigg].\end{aligned}$$

Furthermore, in accordance with (2.4), (2.5) and (4.13), we find

$$\left|K_{j,m}^+\right| \leq \tilde{c}_4 e^{-\frac{\epsilon}{2}\left|\frac{m}{2}\right|^\delta}, \quad j=1,2; \quad m\in\mathbb{N}, \tag{4.17}$$

$$\left|K_{j,m}^-\right| \leq \tilde{c}_5 e^{-\frac{\epsilon}{2}\left|\frac{m}{2}\right|^\delta}, \quad j=-1,-2; \quad m=-1,-2,\ldots, \tag{4.18}$$

where $\tilde{c}_4, \tilde{c}_5$ are arbitrary constants. Afterwards, it follows from Lemma 4.2 that

$$\left|K_{j,-m}^-*K_{2,m}^+\right|, \left|K_{j,-m}^-*K_{1,m}^+\right| \leq \tilde{c}_6 e^{-\frac{\epsilon}{2}\left|\frac{m}{2}\right|^\delta}, \quad j=-1,-2; \quad m\in\mathbb{N} \tag{4.19}$$

holds. After some direct calculations, we arrive at

$$\left|A^{(k)}(z)\right| \leq B\tilde{C}8^k\sum_{m=1}^{\infty} m^k e^{-\frac{\epsilon}{2}\left|\frac{m}{2}\right|^\delta}, \tag{4.20}$$

where

$$\begin{aligned}B:=\Big\{&\left|(\alpha_1+\alpha_2)\rho_2^+\rho_{-1}^-\right|+\left|(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\rho_1^+\rho_{-1}^-\right|\\
&+\left|\alpha_2\rho_2^+\rho_{-2}^-\right|+\left|(\alpha_2+\alpha_4)\rho_1^+\rho_{-2}^-\right|\Big\}.\end{aligned}$$

Moreover, if we define

$$D_k := \sum_{m=1}^{\infty} m^k e^{-\frac{\epsilon}{2}(m/2)^\delta},$$

by the help of Gamma function, we estimate

$$D_k \leq \int_0^\infty t^k e^{-\frac{\epsilon}{2}(t/2)^\delta} dt = \frac{2^{k+1+\frac{k+1}{\delta}}}{\delta \epsilon^{\frac{k+1}{\delta}}} \left(\frac{k+1}{\delta}-1\right) \Gamma \left(\frac{k+1}{\delta}-1\right).$$

After that, using the inequalities $1+\dfrac{1}{k} \leq e$ and $k^k \leq e^k k!$ for $k \in \mathbb{N}$, we get

$$D_k \leq b^k k! k^{k\frac{1-\delta}{\delta}}$$

which gives the proof of the lemma. □

Lemma 4.5. *Assume that the 4π-periodic function h is analytic in the open upper half-plane, all of its derivatives are continuous in the closed upper half-plane and*

$$\sup_{z \in P} \left| h^{(k)}(z) \right| \leq H_k, \quad k \in \mathbb{N} \cup \{0\}.$$

The set $M \subset [-\pi, 3\pi]$ with linear Lebesgue measure zero is the set of all zeros of the function h with infinite multiplicity in P. If

$$\int_0^w \ln T(s) d\mu(M_s) = -\infty,$$

where

$$T(s) = \inf_k \frac{H_k s^k}{k!}, \quad k \in \mathbb{N} \cup \{0\}$$

and $\mu(M_s)$ is the linear Lebesgue measure of $s-$neighbourhood of M, $w \in (0, 4\pi)$ is an arbitrary constant, then $h \equiv 0$ [8].

Theorem 4.3. *Under the condition* (4.13)*, $N_5 = \emptyset$.*

Proof. Since the function A is not equal to zero identically, according to Lemma 4.5, we obtain

$$\int_0^w \ln T(s) d\mu(N_5, s) > -\infty, \tag{4.21}$$

where

$$T(s) = \inf_k \frac{H_k s^k}{k!}, \quad k \in \mathbb{N} \cup \{0\},$$

$\mu(N_5, s)$ denotes the Lebesgue measure of s-neighbourhood of N_5 and H_k is defined by Lemma 4.4. By Lemma 4.4, we calculate

$$T(s) \leq B \exp \left\{ -\frac{1-\delta}{\delta} e^{-1} b^{-\frac{\delta}{1-\delta}} s^{-\frac{\delta}{1-\delta}} \right\}.$$

Hence, we see from (4.21) that

$$\int_0^w s^{-\frac{\delta}{1-\delta}} d\mu(N_5, s) \leq -\int_0^w \ln T(s) d\mu(N_5, s) < \infty.$$

Since $\frac{\delta}{1-\delta} \geq 1$, the integral on the left handside is convergent for arbitrary s if and only if $\mu(N_5, s) = 0$, i.e., $N_5 = \emptyset$. □

Since $N_5 = \emptyset$, then it is easy to arrive at the theorem.

Theorem 4.4. *Assume* (4.13)*. Then the operator $\mathcal{L}_\mu$ has a finite number of eigenvalues and spectral singularities and each of them is of finite multiplicity.*

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