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LANDWEBER ITERATIVE METHOD FOR AN INVERSE SOURCE PROBLEM OF TIME-FRACTIONAL DIFFUSION-WAVE EQUATION ON SPHERICALLY SYMMETRIC DOMAIN*

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Abstract In this paper, an inverse source problem of time-fractional diffusion-wave equation on spherically symmetric domain is considered. In general, this problem is ill-posed. Landweber iterative method is used to solve this inverse source problem. The error estimates between the regularization solution and the exact solution are derived by an a-priori and an a-posteriori regularization parameters choice rules. The numerical examples are presented to verify the efficiency and accuracy of the proposed methods.

Keywords Time-fractional diffusion-wave equation, identifying the unknown source, Landweber iterative method, parameter choice rule.

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1. Introduction

In this work, we focus on an inverse source problem for the time-fractional diffusion-wave equation on spherically symmetric domain as follows:

$$\begin{cases} D_t^\alpha u(r,t) - \frac{2}{r}u_r(r,t) - u_{rr}(r,t) = f(r), & 0<r<r_0, \quad 0<t<T, \quad 1<\alpha<2, \\ u(r_0,t)=0, & 0\le t\le T, \\ u(r,0)=\varphi(r), & 0\le r\le r_0, \\ u_t(r,0)=\psi(r), & 0\le r\le r_0, \\ \lim_{r\to 0} u(r,t) \ \ \text{bounded}, & 0<t<T, \\ u(r,T)=g(r), & 0\le r\le r_0, \end{cases} \tag{1.1}$$

where the time-fractional derivative D_t^α is the Caputo fractional derivative with respect to t, r_0 is the radius, $g(r)\in L^2[0,r_0;r^2]$ is given, $f(r)$ is unknown source. The Caputo fractional derivative of order $\alpha(1<\alpha<2)$ defined by [13]

$$D_t^\alpha u(x,t) = \frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{\partial^2 u(x,s)}{\partial s^2}\frac{ds}{(t-s)^{\alpha-1}}, \qquad 1<\alpha<2. \tag{1.2}$$

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We use the final time to identify the unknown source $f(r)$. Physically, $g(r)$ can be measured, there will be measurement errors, and we assume the function $g^\delta(r) \in L^2[0, r_0; r^2]$ is the measurable data which satisfies

$$\|g^\delta(r) - g(r)\|_{L^2[0,r_0;r^2]} \le \delta, \tag{1.3}$$

where the positive constant $\delta > 0$ represents a bound on measurement error. And throughout this paper, $L^2[0, r_0; r^2]$ denotes the Hilbert space of Lebesgue measurable function f with weight r^2 in $[0, r_0]$. $(\cdot, \cdot)$ and $\|\cdot\|$ denote inner product and norm on $L^2[0, r_0; r^2]$, respectively, i.e.,

$$\|f\| = (\int_0^{r_0} r^2 |f(r)|^2 dr)^{\frac{1}{2}}, \quad (f, g) = \int_0^{r_0} r^2 f(r) g(r) dr. \tag{1.4}$$

For the inverse problems of diffusion equation on spherically symmetric domain, Cheng et al. used different regularization methods to deal with it. For example, in [2], the authors used the modified Tikhonov regularization method to deal with inverse heat conduction problem. In [3], the authors used the Tikhonov type's regularization method and the Fourier regularization method to deal with the same problem as [2]. In [4], the modified quasi-boundary value method is used to deal with a radially symmetric inverse heat conduction problem. From [2–4], when the authors considered the high dimensional inverse heat conduction problem, the regularization parameter is a priori choice, which depends on priori bound, but the a priori bound is difficult to obtain in practical application. In [23], the authors used the quasi-boundary value method to identify the initial value of heat equation on a columnar symmetric domain. But, the equation is integer order, not fractional order. However, time-fractional diffusion equation has received much attention recently, due to many applications in various areas of engineering. The mathematical theory and associated numerical method for the anomalous diffusion equation have often been discussed, see [6–8, 12]. The inverse source problem about fractional diffusion equation attracted many authors and its physical background can be found in [17]. Yang et al [24, 25] studied an inverse source problem in a time-fractional diffusion equation by a mollification regularization method and quasi-reversibility regularization method. An inversion source problem of time-fractional diffusion equation is studied by a truncation method [35], a Tikhonov regularization method and a simplified Tikhonov regularization method [18], a generalized Tikhonov regularization method [19], a modified quasi-boundary value method [20], Landweber iterative method [26, 27] and quasi-reversibility regularization method [21]. In these references, about source term identification for the time-fractional diffusion equation, they only studied one dimensional situations. At present, the research on high dimensional unknown source identification problem is very difficult. Choulli and Yarmamoto studied two dimensional unknown source identification problem with practical application background in [5] and obtained the conditional stability estimate and the uniqueness of solution, but the authors did not give error estimates. In [28], the authors used the quasi-boundary regularization method to identify the initial value of time-fractional diffusion equation on spherically symmetric domain. And there are few research results about the inverse problem for the time-fractional diffusion-wave equation on spherically symmetric domain. In [9], the authors used the Landweber method and the conjugate gradient method to identify the space-dependent force. In [14], the authors solved the backward problem and identified the unknown source $p(t)$ for the time-fractional diffusion-wave equation. In [15], the author identified the time-dependent source for a time-fractional wave equation in a bound domain and obtained the weak solution of existence, uniqueness and regularity. In [10], the authors gave the solution of existence, uniqueness for identifying the time-dependent source for the time-fractional diffusion-wave equation. The Landweber iterative method and Fourier truncation method are very useful method for solving the ill-posed problem and has been studied for solving various types of inverse problems [11,16,22,29–34]. In this paper, we solve an inverse source problem of time-fractional diffusion-wave equation on spherically symmetric domain by the Landweber iterative method.

The remainder of this paper is composed of five sections. Some preliminary results are presented in Section 2. Section 3 develops the Landweber iterative regularization method and gives convergence estimates under an a-priori regularization parameter choice rule and an a-posteriori regularization parameter choice rule, respectively. Section 4 provides some numerical examples to illustrate the efficiency of our method. Finally, Section 5 gives a simple conclusion.

2. Preliminaries

In this section, we give preliminary results which are very useful for our main conclusion.

Definition 2.1 ([13]). The generalized Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\quad z\in\mathbb{C}, \tag{2.1}$$

where $\alpha>0$ and $\beta\in\mathbb{R}$ are arbitrary constants.

Lemma 2.1 ([13]). *Let $\lambda>0$, then we have*

$$\int_0^{\infty}e^{-pt}t^{\gamma k+\beta-1}{E_{\gamma,\beta}}^{(k)}(\pm at^{\gamma})dt=\frac{k!p^{\gamma-\beta}}{(p^{\gamma}\mp a)^{k+1}},\quad Re(p)>\|a\|^{\frac{1}{\gamma}}, \tag{2.2}$$

where ${E_{\gamma,\beta}}^{(k)}(y):=\frac{d^k}{dy^k}E_{\gamma,\beta}(y)$.

Lemma 2.2 ([13]). *Suppose $\alpha<2$, $\beta\in\mathbb{R}$, $\frac{\pi\alpha}{2}<\mu<min\{\pi,\pi\alpha\}$ and $\mu\le|arg(z)|\le\pi$. Then there exists a constant $C_1>0$ such that*

$$|E_{\alpha,\beta}(z)|\le\frac{C_1}{1+|z|}. \tag{2.3}$$

Lemma 2.3 ([1]). *For $1<\alpha<2$, $\beta\in\mathbb{R}$ and $\eta>0$, we have*

$$E_{\alpha,\beta}(-\eta)=\frac{1}{\Gamma(\beta-\alpha)\eta}+O(\frac{1}{\eta^2}),\ \eta\to\infty. \tag{2.4}$$

Lemma 2.4. *For $\lambda_n=(\frac{n\pi}{r_0})^2, n=1,2,...,$ there exists positive constants C_2, C_3 depending on α,T,C_1 such that*

$$\frac{C_2}{n^2T^{\alpha}}\le|E_{\alpha,1+\alpha}(-\lambda_nT^{\alpha})|\le\frac{C_3}{n^2T^{\alpha}}, \tag{2.5}$$

where $C_2=\frac{r_0^2}{\pi^2}$, $C_3=\frac{r_0^2C_1}{\pi^2}$.

Proof. By Lemma 2.2, we know

$$|E_{\alpha,\alpha+1}(-\lambda_n T^\alpha)| \le \frac{C_1}{1+\lambda_n T^\alpha} = \frac{C_1}{1+(\frac{n\pi}{r_0})^2 T^\alpha} \le \frac{C_1}{(\frac{n\pi}{r_0})^2 T^\alpha} = \frac{C_3}{n^2 T^\alpha}. \tag{2.6}$$

Using Lemma 2.3, we have

$$|E_{\alpha,\alpha+1}(-\lambda_n T^\alpha)| \ge \frac{1}{\Gamma(1)(\frac{n\pi}{r_0})^2 T^\alpha} = \frac{r_0^2}{\pi^2 n^2 T^\alpha} = \frac{C_2}{n^2 T^\alpha}. \tag{2.7}$$

Then the proof is completed. □

Now, we will need the solution of the direct problem (1.1). Applying the separation of variables and Laplace transform of Mittag-Leffler function, we can get the solution of problem (1.1) as follows:

$$\begin{aligned} u(r,t) = \sum_{n=1}^{\infty} [t^\alpha E_{\alpha,\alpha+1}((\frac{n\pi}{r_0})^2 t^\alpha)(f(r),\omega_n(r)) + E_{\alpha,1}(-(\frac{n\pi}{r_0})^2 t^\alpha)(\varphi(r),\omega_n(r)) \\ + tE_{\alpha,2}(-(\frac{n\pi}{r_0})^2 t^\alpha)(\psi(r),\omega_n(r))]\omega_n(r), \end{aligned} \tag{2.8}$$

where

$$\omega_n(r) := \frac{\sqrt{2}n\pi}{\sqrt[3]{r_0^3}} \frac{\sin(\frac{n\pi r}{r_0})}{\frac{n\pi r}{r_0}}, \quad n = 1, 2, \cdots,$$

is a standard orthogonal system with weight r^2 in the $[0, r_0]$ and it is complete in the class of square integrable functions on $[0, r_0]$. Now let $f_n = (f(r), \omega_n(r))$, $\varphi_n = (\varphi(r), \omega_n(r))$, $\psi_n = (\psi(r), \omega_n(r))$ and $g_n = (g(r), \omega_n(r))$. f_n is the Fourier coefficient of $f(r)$, which is defined as follows

$$f_n = \int_0^{r_0} r^2 f(r)\omega_n(r)dr. \tag{2.9}$$

Using $g(r) = u(r,T)$, we have

$$\begin{aligned} g(r) = \sum_{n=1}^{\infty} [T^\alpha E_{\alpha,\alpha+1}((\frac{n\pi}{r_0})^2 T^\alpha) f_n + E_{\alpha,1}(-(\frac{n\pi}{r_0})^2 T^\alpha)\varphi_n \\ + TE_{\alpha,2}(-(\frac{n\pi}{r_0})^2 T^\alpha)\psi_n]\omega_n(r), \end{aligned} \tag{2.10}$$

then

$$\begin{aligned} g_n = (g(r),\omega_n(r)) = T^\alpha E_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2 T^\alpha) f_n + E_{\alpha,1}(-(\frac{n\pi}{r_0})^2 T^\alpha)\varphi_n \\ + TE_{\alpha,2}(-(\frac{n\pi}{r_0})^2 T^\alpha)\psi_n. \end{aligned} \tag{2.11}$$

From (2.11), we can get

$$\begin{aligned} f_n &= \frac{g_n - E_{\alpha,1}(-(\frac{n\pi}{r_0})^2 T^\alpha)\varphi_n - TE_{\alpha,2}(-(\frac{n\pi}{r_0})^2 T^\alpha)\psi_n}{T^\alpha E_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2 T^\alpha)} \\ &= \frac{h_n}{T^\alpha E_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2 T^\alpha)}, \end{aligned} \tag{2.12}$$

where $h_n := g_n - E_{\alpha,1}(-(\frac{n\pi}{r_0})^2T^\alpha)\varphi_n - TE_{\alpha,2}(-(\frac{n\pi}{r_0})^2T^\alpha)\psi_n$.

So

$$f(r) = \sum_{n=1}^{\infty} \frac{h_n}{T^\alpha E_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha)}\omega_n(r). \tag{2.13}$$

To find $f(r)$, we just need to solve the following integral equation

$$(Kf)(r) = h(r), \quad 0 \le r \le r_0. \tag{2.14}$$

It is easily to see that K is a self adjoint operator. And its eigenvalue and eigenvector are

$$k_n = T^\alpha E_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha)$$

and $\omega_n(r)$, respectively, $n = 0, 1, 2, \cdots$. From [35], we know $K : L^2[0, r_0; r^2] \to L^2[0, r_0; r^2]$ is compact operator, thus this problem is ill-posed. Therefore we use the Landweber iterative regularization method to recover it.

In order to obtain the error estimate, we assume that $f(r)$ satisfies the following priori bound condition:

$$\|f(\cdot)\|_{H_p} \le E, \quad p > 0, \quad E > 0, \tag{2.15}$$

and we define $\|f(\cdot)\|_p$ as follows:

$$\|f(\cdot)\|_{H_p} := \|\sum_{n=1}^{\infty}(1+n^2)^{\frac{p}{2}}(f(\cdot), \omega_n(\cdot))\|. \tag{2.16}$$

Theorem 2.1. *Let $f(r) \in H^p$ satisfy a priori bound condition*

$$\|f\|_{H_p} \le E, \; p > 0, \tag{2.17}$$

then we have

$$\|f\| \le C_4 E^{\frac{2}{p+2}} \|h\|^{\frac{p}{p+2}}, p > 0, \tag{2.18}$$

where $C_4 := (\frac{1}{C_2})^{\frac{p}{p+2}}$ is a constant depending on α, T, p.

Proof. From (2.12) and *Hölder* inequality, we have

$$\begin{aligned} \|f\|^2 &= \sum_{n=1}^{\infty} f_n^2 = \sum_{n=1}^{\infty} \frac{h_n^2}{|T^\alpha E_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha)|^2} \\ &= \sum_{n=1}^{\infty} \frac{h_n^{\frac{4}{p+2}}}{|T^\alpha E_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha)|^2} h_n^{\frac{2p}{p+2}} \\ &\le (\sum_{n=1}^{\infty} \frac{h_n^2}{|T^\alpha E_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha)|^{p+2}})^{\frac{2}{p+2}} (\sum_{n=1}^{\infty} h_n^2)^{\frac{p}{p+2}}. \end{aligned} \tag{2.19}$$

Applying (2.5), we get

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{h_n^2}{|T^\alpha E_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha)|^{p+2}} &\le \sum_{n=1}^{\infty} \frac{h_n^2}{|T^\alpha E_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha)|^2}(\frac{n^2}{C_2})^p \\ &\le \sum_{n=1}^{\infty} f_n^2(1+n^2)^p C_2^{-p} = \|f\|_p^2 C_2^{-p}. \end{aligned} \tag{2.20}$$

Combining (2.19) and (2.20), we obtain

$$\|f\|^2 \le C_2^{-\frac{2p}{p+2}} \|f\|_p^{\frac{4}{p+2}} \|h\|^{\frac{2p}{p+2}}.$$

So far, this Theorem is proved. □

3. Regularization method and convergence rates

In this section, we propose the Landweber iterative regularization method to solve this inverse source problem of time-fractional diffusion-wave equation on spherically symmetric domain and give two error estimates under an a-priori regularization parameter choice rule and an a-posteriori parameter choice rule.

We can use the Landweber iterative regularization method to obtain the regularization solution $f^{m,\delta}(r)$ for (1.1). We use operator equation $f = (I - aK^*K)f + aK^*h$ instead of equation $Kf = h$ for some $a > 0$ to get the following iterative format:

$$f^{0,\delta}(r) = 0, \quad f^{m,\delta}(r) = (I - aK^*K)f^{m-1,\delta}(r) + aK^*h^\delta(r), \quad m = 1, 2, 3, \cdots, \tag{3.1}$$

where m is the iterative step number, which is also selected as regularization parameter. a is called the relaxation factor and satisfies $0 < a < \frac{1}{\|K\|^2}$. For K is a self-adjoint operator, we denote operator $R_m : L^2[0, r_0; r^2] \to L^2[0, r_0; r^2]$ as follows

$$R_m = a \sum_{k=0}^{m-1} (I - aK^*K)^k K, \quad m = 1, 2, 3, \cdots,$$

then we obtain

$$f^{m,\delta}(r) = R_m h^\delta(r) = a \sum_{k=0}^{m-1} (I - aK^2)^k K h^\delta(r). \tag{3.2}$$

Using (3.1) and the singular values k_n of K, we get

$$f^{m,\delta}(r) = \sum_{n=1}^{\infty} \frac{1 - (1 - aT^{2\alpha} E^2_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2 T^\alpha))^m}{T^\alpha E_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2 T^\alpha)} h_n^\delta \omega_n(r), \tag{3.3}$$

where $h_n^\delta := g_n^\delta - E_{\alpha,1}(-(\frac{n\pi}{r_0})^2 T^\alpha)\varphi_n - T E_{\alpha,2}(-(\frac{n\pi}{r_0})^2 T^\alpha)\psi_n$.

Because $k_n = T^\alpha E_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2 T^\alpha)$ is singular value of K and $0 < a < \frac{1}{\|K\|^2}$, we can easily see $0 < aT^{2\alpha} E^2_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2 T^\alpha) < 1$.

In the following, we give two convergence estimates for $\|f^{m,\delta}(r) - f(r)\|$ under an a-priori and an a-posteriori choice rule for the regularization parameter.

3.1. The error estimate with a priori parameter choice

Theorem 3.1. *Assume the a priori condition* (1.3) *and* (2.15) *hold, let* $f(r)$ *given by* (2.13) *be the exact solution of problem* (1.1), $f^{m,\delta}(r)$ *given by* (3.3) *be the regularization solution. Choose the regularization parameter* $m = [\gamma]$, *where*

$$\gamma = (\frac{E}{\delta})^{\frac{4}{p+2}}, \tag{3.4}$$

then we can obtain the following error estimate

$$\|f^{m,\delta}(r) - f(r)\| \le C_5 E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}, \tag{3.5}$$

where $[\gamma]$ *denotes the largest integer less than or equal to* b *and* $C_5 = \sqrt{a} + (\frac{p}{aC_2^2})^{\frac{p}{4}}$ *is positive constant.*

Proof. By the triangle inequality, we get

$$\|f^{m,\delta}(r) - f(r)\| \le \|f^{m,\delta}(r) - f^m(r)\| + \|f^m(r) - f(r)\|. \tag{3.6}$$

On the one hand, from (1.3) and (3.3), we have

$$\begin{aligned}\|f^{m,\delta}(r) - f^m(r)\|^2 &= \sum_{n=1}^{\infty} \frac{(1-(1-aT^{2\alpha}E^2_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha))^m)^2}{T^{2\alpha}E^2_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha)}(g_n^\delta - g_n)^2 \\ &\le \sup_{n\in N} D(n)^2\delta^2,\end{aligned} \tag{3.7}$$

where

$$D(n) := \frac{1-(1-aT^{2\alpha}E^2_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha))^m}{T^\alpha E_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha)}.$$

Because $0 < x < 1$, we have $x \le \sqrt{x}$, then

$$1-(1-aT^{2\alpha}E^2_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha))^m \le \sqrt{1-(1-aT^{2\alpha}E^2_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha))^m}. \tag{3.8}$$

Using Bernoulli inequality, we have

$$\sqrt{1-(1-aT^{2\alpha}E^2_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha))^m} \le \sqrt{am}T^\alpha E_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha). \tag{3.9}$$

Combining (3.8) and (3.9), we get

$$1-(1-aT^{2\alpha}E^2_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha))^m \le \sqrt{am}T^\alpha E_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha), \tag{3.10}$$

then

$$D(n) \le \sqrt{am}, \tag{3.11}$$

so

$$\|f^{m,\delta}(r) - f^m(r)\| \le \sqrt{am}\delta. \tag{3.12}$$

On the other hand, from (2.13) and (2.15), we can obtain

$$\begin{aligned}\|f^m(r) - f(r)\|^2 &= \sum_{n=1}^{\infty}(\frac{(1-aT^{2\alpha}E^2_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha))^{2m}}{T^{2\alpha}E^2_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha)}h_n^2 \\ &= \sum_{n=1}^{\infty}(1-aT^{2\alpha}E^2_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha))^{2m}(1+n^2)^{-p}(f_n^2(1+n^2)^p) \\ &\le \sup_{n\in N} M(n)^2E^2,\end{aligned} \tag{3.13}$$

where

$$M(n) := (1 - aT^{2\alpha}E^2_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha))^m(1+n^2)^{-\frac{p}{2}}.$$

Using (2.5), we get

$$M(n) \le (1 - \frac{aC_2^2}{n^4})^m(n^2)^{-\frac{p}{2}}. \tag{3.14}$$

Let

$$F(x) := (1 - \frac{aC_2^2}{x^2})^m x^{-\frac{p}{2}}, \quad x = n^2. \tag{3.15}$$

Suppose x_0 satisfy $F^{'}(x_0) = 0$, then we easily get

$$x_0 = (\frac{aC_2^2(4m+p)}{p})^{\frac{1}{2}},$$

thus we have

$$\begin{aligned} F(x) \le F(x_0) &= (1 - \frac{p}{4m+p})^m(\frac{aC_2^2(4m+p)}{p})^{-\frac{p}{4}} \\ &\le (\frac{p}{(m+1)aC_2^2})^{\frac{p}{4}}, \end{aligned} \tag{3.16}$$

i.e.,

$$M(n) \le (\frac{p}{aC_2^2})^{\frac{p}{4}}(m+1)^{-\frac{p}{4}}. \tag{3.17}$$

So

$$\|f^m(r) - f(r)\| \le (\frac{p}{aC_2^2})^{\frac{p}{4}}(m+1)^{-\frac{p}{4}}E. \tag{3.18}$$

Combining (3.12) and (3.18), we choose $m = [\gamma]$ and we get

$$\|f^{m,\delta}(r) - f(r)\| \le C_5 E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}, \tag{3.19}$$

where $C_5 = \sqrt{a} + (\frac{p}{aC_2^2})^{\frac{p}{4}}$.

This Theorem is proved. □

3.2. The error estimate with a posteriori parameter choice

Assume $\tau > 1$ be given a fixed constant. Stop the algorithm at the first occurrence of $m = m(\delta) \in \mathbb{N}_0$ with

$$\|Kf^{m,\delta}(r) - h^\delta(r)\| \le \tau\delta, \tag{3.20}$$

where $\|h^\delta\| \le \tau\delta$.

Lemma 3.1. *Let $\rho(m) = \|Kf^{m,\delta}(r) - h^\delta(r)\|$, then we have the following results:*
(a) $\rho(m)$ is a continuous function;
(b) $\lim_{m\to 0}\rho(m) = \|h^\delta\|$;
(c) $\lim_{m\to+\infty} = 0$;
(d) $\rho(m)$ is a strictly decreasing function, for any $m \in (0, +\infty)$.

Lemma 3.2. *Let* (3.20) *hold, so the regularization parameter $m = m(\delta) \in \mathbb{N}_0$ satisfies*

$$m \le C_3^{\frac{4}{p+2}}(\frac{p+2}{2aC_2^2})(\frac{E}{(\tau-1)\delta})^{\frac{4}{p+2}}. \tag{3.21}$$

Proof. From (3.2) and (3.3), we have

$$R_m h = \sum_{n=1}^{\infty} \frac{1-(1-aT^{2\alpha}E^2_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha))^m}{T^\alpha E_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha)} h_n\omega_n(r). \tag{3.22}$$

Thus

$$\|KR_mh-h\|^2 = \sum_{n=1}^{\infty}(1-aT^{2\alpha}E^2_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha))^{2m}h_n^2. \tag{3.23}$$

Because $|1-aT^{2\alpha}E^2_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha)|<1$, we obtain $\|KR_{m-1}-I\|\le 1$. Using (3.20), we obtain

$$\begin{aligned}\|KR_{m-1}h-h\| &\ge \|KR_{m-1}h^\delta-h^\delta\| - \|(KR_{m-1}-I)(h-h^\delta)\|\\ &\ge \tau\delta - \|KR_{m-1}-I\|\delta\\ &\ge (\tau-1)\delta.\end{aligned}$$

On the other hand, using (2.15), we obtain

$$\begin{aligned}\|KR_{m-1}h-h\| &= \|\sum_{n=1}^{\infty}(1-(1-aT^{2\alpha}E^2_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha))^{m-1})h_n\omega_n - \sum_{n=1}^{\infty}h_n\omega_n\|\\ &= \sum_{n=1}^{\infty}(1-aT^{2\alpha}E^2_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha))^{m-1}|h_n|\\ &= \sum_{n=1}^{\infty}(1-aT^{2\alpha}E^2_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha))^{m-1}\\ &\quad\cdot|T^\alpha E_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha)||f_n(1+n^2)^{\frac{p}{2}}|(1+n^2)^{-\frac{p}{2}}\\ &= \sup_{n\in N} H(n)E,\end{aligned}$$

where

$$H(n) := (1-aT^{2\alpha}E^2_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha))^{m-1}|T^\alpha E_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha)|(1+n^2)^{-\frac{p}{2}},$$

so

$$(\tau-1)\delta \le H(n)E. \tag{3.24}$$

Using (2.5), we have

$$H(n) \le (1-a\frac{C_2^2}{n^4})^{m-1}(n^2)^{-\frac{p}{2}-1}C_3. \tag{3.25}$$

Let

$$G(x) := (1-a\frac{C_2^2}{x^2})^{m-1}x^{-\frac{p}{2}-1}C_3, \quad x=n^2. \tag{3.26}$$

Suppose x^* satisfies $G^{'}(x^*)=0$, then we easily get

$$x^* = (\frac{aC_2^2(4m+p-2)}{p+2})^{\frac{1}{2}},$$

so

$$G(x) \le G(x^*) = (1 - \frac{p+2}{4m+p-2})^{m-1}(\frac{aC_2^2(4m+p-2)}{p+2})^{-\frac{p+2}{4}}C_3 \\ \le C_3(\frac{p+2}{2maC_2^2})^{\frac{p+2}{4}}. \tag{3.27}$$

Thus we obtain

$$H(n) \le C_3(\frac{p+2}{2aC_2^2})^{\frac{p+2}{4}} m^{-\frac{p+2}{4}}. \tag{3.28}$$

Using (3.24) and (3.28), we get

$$(\tau-1)\delta \le C_3(\frac{p+2}{2aC_2^2})^{\frac{p+2}{4}} m^{-\frac{p+2}{4}} E. \tag{3.29}$$

Thus

$$m \le C_3^{\frac{4}{p+2}}(\frac{p+2}{2aC_2^2})(\frac{E}{(\tau-1)\delta})^{\frac{4}{p+2}}.$$

□

Theorem 3.2. *Assume the a priori condition* (1.3) *and* (2.15) *hold, let* $f(r)$ *given by* (2.13) *be the exact solution of problem* (1.1), $f^{m,\delta}(r)$ *given by* (3.3) *be the regularization solution. Regularization parameter* m *is given by* (4.1). *Then we have the following error estimate*

$$\|f^{m,\delta}(r) - f(r)\| \le (C_5(\tau+1)^{\frac{p}{p+2}} + C_6)E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}, \tag{3.30}$$

where $C_6 = (\frac{p+2}{2C_2^2})^{\frac{1}{2}}(\frac{C_3}{\tau-1})^{\frac{2}{p+2}}$.

Proof. By the triangle inequality, we get

$$\|f^{m,\delta}(r) - f^m(r)\| \le \|f^{m,\delta}(r) - f^m(r)\| + \|f^m(r) - f(r)\|. \tag{3.31}$$

Using (3.12) and Lemma 3.2., we get

$$\|f^{m,\delta}(r) - f^m(r)\| \le \sqrt{am}\delta \le C_6 E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}, \tag{3.32}$$

where $C_6 := (\frac{p+2}{2C_2^2})^{\frac{1}{2}}(\frac{C_3}{\tau-1})^{\frac{2}{p+2}}$. For the second part of the right side of (3.31), we know

$$K(f^m(r) - f(r)) = \sum_{n=1}^{\infty} -(1 - aT^{2\alpha}E_{\alpha,\alpha+1}^2(-(\frac{n\pi}{r_0})^2 T^\alpha))^m h_n \omega_n(r) \\ = \sum_{n=1}^{\infty} -(1 - aT^{2\alpha}E_{\alpha,\alpha+1}^2(-(\frac{n\pi}{r_0})^2 T^\alpha))^m (h_n - h_n^\delta)\omega_n(r) \\ + \sum_{n=1}^{\infty} -(1 - aT^{2\alpha}E_{\alpha,\alpha+1}^2(-(\frac{n\pi}{r_0})^2 T^\alpha))^m h_n^\delta \omega_n(r).$$

Using (1.3) and (3.20), we have

$$\|K(f^m(r) - f(r))\| \le (\tau+1)\delta. \tag{3.33}$$

Due to

$$
\begin{aligned}
&\|f^m(r) - f(r)\|_{H_p}\\
&= (\sum_{n=1}^{\infty}(1 - aT^{2\alpha}E^2_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha))^{2m}\frac{h_n^2}{(T^\alpha E_{\alpha,\alpha+1}(-\lambda_n T^\alpha))^2}(1+n^2)^p)^{\frac{1}{2}}\\
&\le (\sum_{n=1}^{\infty}(1+n^2)^p\frac{h_n^2}{(T^\alpha E_{\alpha,\alpha+1}(-(\frac{n\pi}{r_0})^2T^\alpha))^2})^{\frac{1}{2}}\\
&\le E.
\end{aligned}
$$

Using Theorem 2.1., we have

$$
\|f^m(r) - f(r)\| \le C_4(\tau+1)^{\frac{p}{p+2}}E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}. \tag{3.34}
$$

Therefore

$$
\|f^{m,\delta}(r) - f(r)\| \le (C_4(\tau+1)^{\frac{p}{p+2}} + C_6)E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}.
$$

□

4. Numerical results

In this section, we present numerical examples to illustrate the efficiency and accuracy of the proposed method.

Since the analytic solution of problem (1.1) is difficult to obtain, we construct the final data $g(r)$ by solving the forward problem with given data $f(r)$, $\varphi(r)$ and $\psi(r)$ by a finite difference method. The noisy data are generated by adding a random perturbation, i.e., $g^\delta = g + \varepsilon g \cdot (2rand(size(g) - 1))$. The corresponding noise level is calculated by $\delta = \varepsilon\|g\|$.

Let $r_0 = \pi$, $T = 1$ and consider the following time-fractional diffusion-wave equation

$$
\begin{cases}
D_t^\alpha u(r,t) - \frac{2}{r}u_r(r,t) - u_{rr}(r,t) = f(r), & 0<r<\pi,\quad 0<t<T,\quad 1<\alpha<2,\\
u(r,0) = \varphi(r), & 0 \le r \le \pi,\\
u_t(r,0) = \psi(r), & 0 \le r \le \pi,\\
\lim_{r\to 0} u(r,t) \ \ \text{bounded}, & 0<t<T,\\
u(\pi,t) = 0, & 0 \le t \le T,\\
u(r,T) = g(r), & 0 \le r \le \pi.
\end{cases} \tag{4.1}
$$

Time and space of grid step size are $\Delta t = \frac{T}{N}$ and $\Delta r = \frac{\pi}{M}$, respectively. The grid points on the time interval $[0,T]$ is $t_n = n\Delta t$ $(n = 0,1,\ldots,N)$ and $r_i = i\Delta r$ $(i = 0,1,\ldots,M)$ is grid points on the space interval $[0,r_0]$. The approximate values of each grid points u is denoted by $u_i^n = (r_i,t_n)$.

The time-fractional derivative is given in [1] as follows:

$$
D_t^\alpha u(r_i,t_n) = \frac{(\Delta t)^{1-\alpha}}{\Gamma(3-\alpha)}[b_0\frac{1}{\Delta t}(u_i^n - u_i^{n-1}) - \sum_{j=1}^{n-1}(b_{n-j-1} - b_{n-j})\frac{1}{\Delta t}(u_i^j - u_i^{j-1}) - b_{n-1}\psi(x_i)], \tag{4.2}
$$

where $b_k = (k+1)^{2-\alpha} - (k)^{2-\alpha}$, $k = 0,1,2,\cdots$.

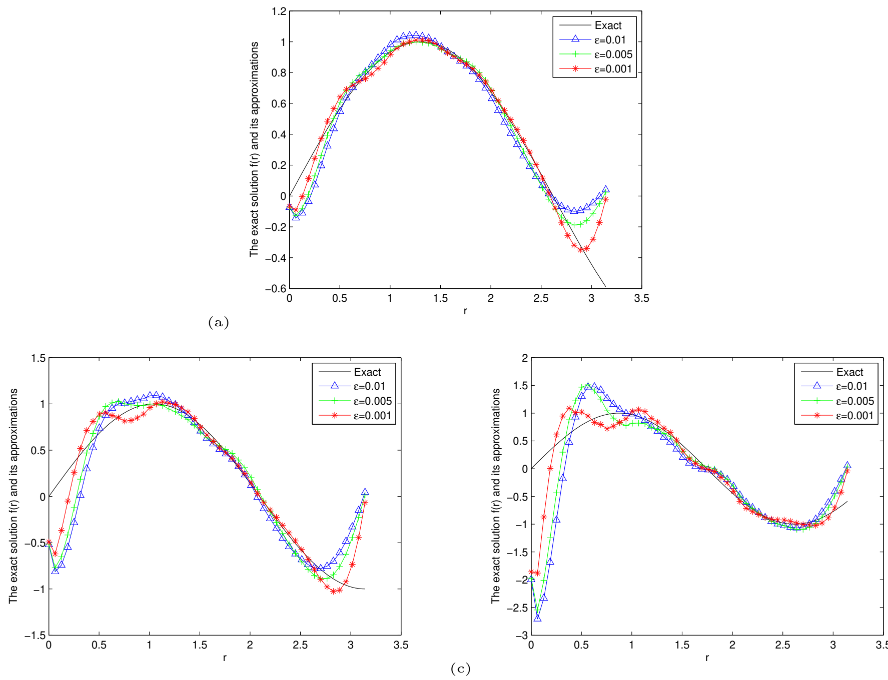

Figure 1. The exact solution and regularization solutions by using the a posteriori parameter choice rule for Example 4.1. (a)$\alpha = 1.2$, (b)$\alpha = 1.5$, (c)$\alpha = 1.8$.

We approximate the space derivatives by

$$u_r(r_i, t_n) \approx \frac{u_{i+1}^n - u_i^n}{\Delta r}, \tag{4.3}$$

$$u_{rr}(r_i, t_n) \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta r)^2}. \tag{4.4}$$

Example 4.1. Choose

$$f(r) = \sin(\alpha r),$$
$$\varphi(r) = 2\alpha \sin r,$$
$$\psi(r) = \sin r.$$

Figure 1 shows the comparisons between the exact solution and its regularized solution for various noise levels $\varepsilon = 0.01, 0.005, 0.001$ in the case of $\alpha = 1.2, 1.5, 1.8$. The iterative step $m = 352, 753, 4670$ for $\alpha = 1.2$, $m = 526, 1042, 6471$ for $\alpha = 1.5$ and $m = 1062, 2306, 19286$ for $\alpha = 1.8$.

Example 4.2. Choose

$$f(r) = \alpha r \cos r,$$
$$\varphi(r) = 4 \cos r,$$
$$\psi(r) = 0.$$

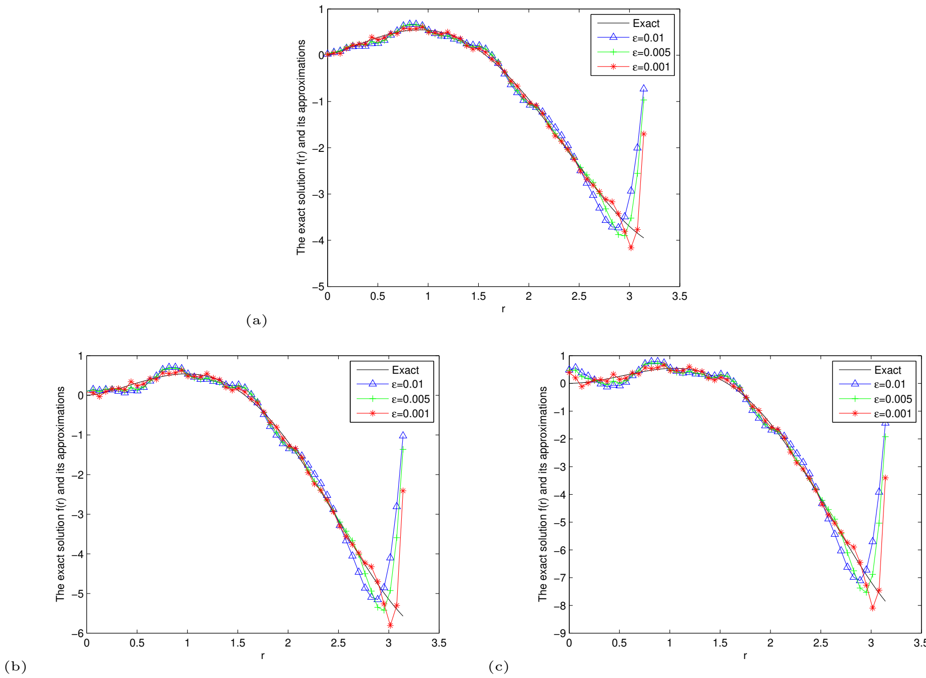

Figure 2. The exact solution and regularization solutions by using the a posteriori parameter choice rule for Example 4.2. (a)$\alpha = 1.2$, (b)$\alpha = 1.5$, (c)$\alpha = 1.8$.

Figure 2 shows the comparisons between the exact solution and its regularized solution for various noise levels $\varepsilon = 0.01, 0.05, 0.001$ in the case of $\alpha = 1.2, 1.5, 1.8$. The iterative step $m = 7518, 22361, 229455$ for $\alpha = 1.2$, $m = 7816, 23233, 238255$ for $\alpha = 1.5$ and $m = 8029, 24013, 244837$ for $\alpha = 1.8$.

According to above two examples, we can find that the smaller ε and α, the fitting effect between the exact solution and regularized solution is better. Meanwhile, numerical examples verify that the Landweber iterative method is efficient and accurate.

5. Conclusion

An inverse problem of the time-fractional diffusion-wave equation on spherically symmetric domain is considered. Based on the conditional stability, we propose a Landweber iterative regularization method to deal with it and derive the a priori and a posteriori convergence estimates. In addition, numerical examples verify that the Landweber iterative regularization method is efficient and accurate.

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