Journal of Applied Analysis and Computation
Volume 9, Number 4, August 2019, 1443–1453
Website:http://jaac.ijournal.cn
DOI:10.11948/2156-907X.20180276

ITERATIVE METHOD FOR A CLASS OF FOURTH-ORDER P-LAPLACIAN BEAM EQUATION*

Zhanbing Bai[1,†], Zengji Du[2] and Shuo Zhang[3]

Abstract This paper considers the existence of the solutions for a class of fourth-order p-Laplacian. The boundary value problem considered can describe the tiny deformation of an elastic beam. By using a novel efficient iteration method, the existence and uniqueness result of solution for the problem is obtained. An example is given to illustrate the main results.

Keywords Fourth-order p-Laplacian, uniqueness, iterative method.

MSC(2010) 34B15, 34B18.

1. Introduction

This paper studies a class of fourth-order p-Laplace boundary value problem.

$$[\phi_p(u''(t))]'' = f(t, u(t), u''(t)), \quad t \in (0, 1), \tag{1.1}$$

$$u(0) = 0, \ u(1) = au(\xi), \ u''(0) = 0, \ \phi_p(u''(1)) = b\phi_p(u''(\zeta)), \tag{1.2}$$

where $\phi_p(t) = |t|^{p-2} \cdot t, \ p > 1, \ 0 < \xi, \zeta < 1, \ 0 \le a < 1/\xi, \ 0 \le b < 1/\zeta, \ 1/p+1/q = 1, \ p, q \ge 0$, function $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous.

Problems of the above form occur in beam theory [8], for example,

(1). a beam with small deformations (also called geometric linearity);

(2). a beam of a material which satisfies a nonlinear power-like stress-strain law;

(3). a beam with two-sided links (for example, springs) which satisfies a nonlinear power-like elasticity law.

Many scholars have investigated the boundary value problem (1.1), (1.2) via fixed point theorem, nonlinear alternative on cone, upper and lower solutions, or coincidence degree theorem [1]– [40]. The best known setting is the boundary value problem when $p = 2, a = b = 0$.

[†]the corresponding author. Email address:zhanbingbai@163.com (Z. Bai)
[1]School of Mathematics and System Science, Shandong University of Science and Technology, Qingdao 266590, China
[2]School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou 221116, China
[3]School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China
[*]The authors were supported by National Natural Science Foundation of China (11571207), National Science Foundation of Shandong (ZR2018MA011) and the Taishan Scholar project.

This problem arises when one describes deformations of an elastic beam. Usually both ends are simply supported, or one end is simply supported and the other end is clamped by sliding clamps. Also vanishing moments and shear forces at the beam ends are frequently included in the boundary conditions [18]. One derivation of this fourth order equation plus the two-point boundary conditions occurs when the method of lines is used in the description over regions of certain partial differential equations describing the detection of an elastic beam. Ma etc [25] investigated the following problem in 1997,

$$u^{(4)}(t) = f(t, u(t), u''(t)), \quad 0 < t < 1, \tag{1.3}$$

$$u(0) = u(1) = u''(0) = u''(1) = 0. \tag{1.4}$$

For this kind of problem, the method of upper and lower solutions is effective. They obtained the upper and lower solutions via some new maximum principles and constructed monotone sequences to get the existence of the positive solutions. The maximum principles of fourth-order linear operator play a very important role in their proofs. However, when $p \neq 2$, the operator $[\phi_p(u'')]''$ is a nonlinear operator, the maximum principle and the Fredholm alternative cannot be applied.

For $p \neq 2, a \neq 0, b \neq 0$, Bai etc [6] investigated the problem:

$$[\phi_p(u''(t))]'' = f(t, u(t), u''(t)), t \in (0, 1), \tag{1.5}$$

$$u(0) = 0,\ u(1) = au(\xi),\ u''(0) = 0,\ u''(1) = bu''(\zeta), \tag{1.6}$$

where $\phi_p(t) = |t|^{p-2} \cdot t,\ p > 1,\ 0 < \xi, \zeta < 1,\ 0 \le a < \frac{1}{\xi},\ 0 \le b < \min\{\frac{1}{\zeta},\ \phi_q(\frac{1}{\zeta})\}$, function $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. The existence of iterative solutions for problem (1.5), (1.6) without any growth restriction on the nonlinear term f is obtained. The solution is between a lower solution β and an upper solution α. However, the construction of upper and lower solutions is difficult and complicate. For most boundary value problems, we even can't get suitable upper and lower solutions. We pointed here that there was a small mistake in [6] that the boundary condition $u''(1) = bu''(\zeta)$ should be $\phi_p(u''(1)) = b\phi_p(u''(\zeta))$.

More recently, Bachouche etc [2] proved existence of positive solutions to a more general fourth-order semipositone φ-Laplacian boundary value problem. The nonlinearity may have time singularity and change sign. Existence results are proved using the Krasnoselskii and the Leggett-Williams fixed point theorems.

In 2017, Dang etc [13] proposed a novel efficient method for boundary value problem (1.3), (1.4) and obtained the existence of positive solutions. They reduced the problem to two second-order operator equations. Under some easily verified conditions on this function in a specified bounded domain he proved the contraction of the operator. This guarantees the existence and uniqueness of a solution of the problem. This idea also be used for other beam equation [14, 32].

Motivated by the mentioned excellent works, in this paper, we use the method due to Dang to obtain the existence of the positive solution. Differently from [3, 10, 25], we must consider influence of the index p. Specifically, we define a more general operator A to consider the problem (1.1), (1.2). Some preliminaries are presented in section 2. The main results will be given in section 3. An example is presented in section 4.

2. Preliminaries

Suppose that $E = C[0,1]$ is a real Banach space with the maximum norm $\|u\| = \max_{t\in[0,1]} |u(t)|$. Given $\eta \in C^2[0,1]$, consider the following boundary value problem

$$[\phi_p(u''(t))]'' = f(t, \eta(t), \eta''(t)), \quad t \in (0,1), \tag{2.1}$$
$$u(0) = 0,\ u(1) = au(\xi),\ u''(0) = 0,\ \phi_p(u''(1)) = b\phi_p(u''(\zeta)), \tag{2.2}$$

where $0 \le a < 1/\xi, 0 \le b < 1/\zeta$.

Lemma 2.1. *Boundary value problem* (2.1), (2.2) *has a unique solution* $u(t)$ *and*

$$u(t) = (T\eta)(t) = (HF(\eta))(t),$$

where

$$(F\eta)(s) = \int_0^1 G_1(s,\tau) f(\tau, \eta(\tau), \eta''(\tau))d\tau,$$
$$(Hy)(t) = \int_0^1 G_2(t,s)\phi_q(y(s))ds,$$

$$G_1(t,s) = \begin{cases} s \in [0,\zeta] : \begin{cases} \frac{t}{1-b\zeta}[(1-s) - b(\zeta - s)], & t \le s; \\ \frac{s}{1-b\zeta}[(1-t) - b(\zeta - t)], & s \le t, \end{cases} \\ s \in [\zeta,1] : \begin{cases} \frac{1}{1-b\zeta} t(1-s), & t \le s; \\ \frac{1}{1-b\zeta}[s(1-t) + b\zeta(t-s)], & s \le t, \end{cases} \end{cases}$$

$$G_2(t,s) = \begin{cases} s \in [0,\xi] : \begin{cases} \frac{t}{1-a\xi}[(1-s) - a(\xi - s)], & t \le s; \\ \frac{s}{1-a\xi}[(1-t) - a(\xi - t)], & s \le t, \end{cases} \\ s \in [\xi,1] : \begin{cases} \frac{1}{1-a\xi} t(1-s), & t \le s; \\ \frac{1}{1-a\xi}[s(1-t) + a\xi(t-s)], & s \le t. \end{cases} \end{cases}$$

Moreover, $G_1(t,s) \ge 0, G_2(t,s) \ge 0$, *for* $t, s \in [0,1]$.

Proof. Let $y = \phi_p(u'')$, then boundary value problem (2.1), (2.2) changed as

$$y''(t) = f(t, \eta(t), \eta''(t)), \quad t \in (0,1), \tag{2.3}$$
$$u''(t) = \phi_q(y(t)), \tag{2.4}$$
$$y(0) = 0, \quad y(1) = by(\zeta), \tag{2.5}$$
$$u(0) = 0, \quad u(1) = au(\xi). \tag{2.6}$$

With the condition $0 \le a < 1/\xi, 0 \le b < 1/\zeta$, the Green function for the second order three-point boundary value problem was given in [17]. We denote the Green function of (2.3), (2.5) with $G_1(t,s)$, and the Green function of (2.4), (2.6) with $G_2(t,s)$. The positivity of the Green functions are a special case of [5]. We refer the readers to [5,17] for details. A computation show that for $p > 1$, there is $\phi_q = \phi_p^{-1}$. In fact, let

$$s = \phi_p(t) = |t|^{p-2} \cdot t$$

$$= \begin{cases} t^{p-1}, & \text{for } t \geq 0; \\ -|t|^{p-1}, & \text{for } t < 0, \end{cases}$$

then

$$t = \begin{cases} s^{\frac{1}{p-1}}, & \text{for } s \geq 0; \\ -|s|^{\frac{1}{p-1}}, & \text{for } s < 0. \end{cases}$$
$$= \begin{cases} s^{q-1}, & \text{for } s \geq 0; \\ -|s|^{q-1}, & \text{for } s < 0. \end{cases}$$
$$= |s|^{q-2} \cdot s = \phi_q(s).$$

To sum up, the proof is complete. □

Lemma 2.2. *Given $\varphi \in C[0,1]$, let*

$$v(t) = \phi_q\left(\int_0^1 G_1(t,s)\varphi(s)ds\right), \qquad u(t) = \int_0^1 G_2(t,\tau)v(\tau)d\tau,$$
$$M_1 = \max_{0\leq t\leq 1}\int_0^1 G_1(t,s)ds, \qquad M_2 = \max_{0\leq t\leq 1}\int_0^1 G_2(t,s)ds.$$

Then

$$\|v\| \leq M_1^{q-1}\|\varphi\|^{q-1}, \quad \|u\| \leq M_2 M_1^{q-1}\|\varphi\|^{q-1}.$$

where $p, q > 0,\ 1/p + 1/q = 1,\ \|\cdot\|$ is the maximum norm.

Proof. Noticed that $q - 1 = 1/(p-1)$ and ϕ_q is increasing, so

$$v(t) = \phi_q\left(\int_0^1 G_1(t,s)\varphi(s)ds\right) \leq \phi_q\left(\int_0^1 G_1(t,s)||\varphi||ds\right)$$
$$\leq \phi_q\left(\int_0^1 G_1(t,s)ds\|\varphi\|\right) \leq M_1^{q-1}||\varphi||^{q-1}.$$

Thus, $\|v\| \leq M_1^{q-1}\|\varphi\|^{q-1}$. Similarly, $\|u\| \leq M_2 M_1^{q-1}\|\varphi\|^{q-1}$. The proof is completed. □

Lemma 2.3 (Lemma 2.2, [33])**.** *The following relations hold:*
(1) If $1 < q \leq 2$, then for all $u, v \in \mathbb{R}$,

$$|\phi_q(u+v) - \phi_q(u)| \leq 2^{2-q}|v|^{q-1},$$

(2) If $q > 2$, then for all $u, v \in \mathbb{R}$,

$$|\phi_q(u+v) - \phi_q(u)| \leq (q-1)(|u| + |v|)^{q-2}|v|.$$

3. Main results

In this section, we use the Banach fixed point theorem to obtain the existence of positive solution of boundary value problem (1.1), (1.2).

Given $M > 0$, denote by the set D_M as

$$D_M = \left\{(t, u, v) \in \mathbb{R}^3 \mid 0 \le t \le 1,\ |u| \le M_2 M_1^{q-1} M^{q-1},\ |v| \le M_1^{q-1} M^{q-1}\right\}.$$

Denote by $B[O, M]$ a closed ball centred at O with the radius M in the space of continuous functions $C[0, 1]$.

Theorem 3.1. *Suppose that $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, $1 < p \le 2$ and there exist three numbers $M, L_1, L_2 \ge 0$ such that*

(i) $|f(t, u, v)| \le M$, *for* $(t, u, v) \in D_M$*;*

(ii) $|f(t, u_2, v_2) - f(t, u_1, v_1)| \le L_1|u_2 - u_1| + L_2|v_2 - v_1|$, *for* $(t, u_i, v_i) \in D_M,\ i = 1, 2$*;*

(iii) $k := (q - 1)(3M)^{q-2} M_1^{q-1}(L_1 M_2 + L_2) < 1$.

Then the boundary value problem (1.1), (1.2) *has a unique solution* $u(t) \in C[0, 1]$ *such that*

$$|u(t)| \le M_2 M_1^{q-1} M^{q-1}, \qquad |u''(t)| \le M_1^{q-1} M^{q-1}.$$

Proof. Firstly, define an operator $A : C[0, 1] \to C[0, 1]$ by

$$(A\varphi)(t) = f\left(t, \int_0^1 G_2(t, \tau)\phi_q\left(\int_0^1 G_1(\tau, s)\varphi(s)ds\right)d\tau\ , \right.$$
$$\left. \phi_q\left(\int_0^1 G_1(t, s)\varphi(s)ds\right)\right).$$

By the continuity of $G_1(t, s), G_2(t, s)$ and $f(t, u, v)$, it is easy to check that A is a continuous operator. With Lemma 2.1, it is clear that if $\varphi(t)$ is a fixed point of the operator A, then

$$u(t) = \int_0^1 G_2(t, \tau)\phi_q\left(\int_0^1 G_1(\tau, s)\varphi(s)ds\right)d\tau$$

is a solution of the problem (1.1), (1.2). On the contrary, if problem (1.1), (1.2) has a solution $u(t)$, then $\varphi(t) = [\phi_p(u''(t))]''$ is a fixed point of the operator A.

Secondly, we show that the operator A maps $B[O, M]$ into itself. For any $\varphi(t) \in B[O, M]$, by Lemma 2.2, we have

$$|u(t)| \le M_2 M_1^{q-1} M^{q-1}, \quad |v(t)| \le M_1^{q-1} M^{q-1}.$$

Consequently, for any $t \in [0, 1]$, we have $(t, u(t), v(t)) \in D_M$. Meanwhile, from assumption (i) of Theorem 3.1, we can conclude that

$$|(A\varphi)(t)| = |f(t, u(t), v(t))| \le M, \qquad t \in [0, 1],$$

thus $(A\varphi)(t) \in B[O, M]$. So the operator A maps $B[O, M]$ into itself.

Thirdly, we show that the operator $A : B[O, M] \to B[O, M]$ is a contraction mapping. Since $B[O, M]$ is a subspace of $C([0, 1], \|\cdot\|)$, so $B[O, M]$ is a complete distance space. By using the assumption (ii) of Theorem 3.1, Lemma 2.2, and (2) of Lemma 2.3, for $\varphi_1(t), \varphi_2(t) \in B[O, M]$, there is

$$|(A\varphi_2)(t) - (A\varphi_1)(t)|$$

$$
\begin{aligned}
&= |f(t,u_2(t),v_2(t)) - f(t,u_1(t),v_1(t))| \\
&\le L_1|u_2(t)-u_1(t)| + L_2|v_2(t)-v_1(t)| \\
&\le L_1 \left|\int_0^1 G_2(t,\tau)\left[\phi_q(\int_0^1 G_1(\tau,s)\varphi_2(s)ds) - \phi_q(\int_0^1 G_1(\tau,s)\varphi_1(s)ds)\right] d\tau\right| \\
&\quad + L_2 \left|\phi_q\left(\int_0^1 G_1(t,s)\varphi_2(s)ds\right) - \phi_q\left(\int_0^1 G_1(t,s)\varphi_1(s)ds\right)\right| \\
&\le L_1 \left|\int_0^1 G_2(t,\tau)\left[(q-1)\left(\left|\int_0^1 G_1(\tau,s)\varphi_1(s)ds\right|\right.\right.\right. \\
&\quad + \left.\left.\left.\left|\int_0^1 G_1(\tau,s)(\varphi_2(s)-\varphi_1(s))ds\right|\right)^{q-2} \cdot \left|\int_0^1 G_1(\tau,s)(\varphi_2(s)-\varphi_1(s))ds\right|\right] d\tau\right| \\
&\quad + L_2(q-1)\left(\left|\int_0^1 G_1(\tau,s)\varphi_1(s)ds\right| + \left|\int_0^1 G_1(\tau,s)(\varphi_2(s)-\varphi_1(s))ds\right|\right)^{q-2} \\
&\quad \cdot \left|\int_0^1 G_1(\tau,s)(\varphi_2(s)-\varphi_1(s))ds\right| \\
&\le L_1M_2(q-1)(3MM_1)^{q-2}M_1\|\varphi_2-\varphi_1\| \\
&\quad + L_2(q-1)(3MM_1)^{q-2}M_1\|\varphi_2-\varphi_1\| \\
&\le (q-1)(3M)^{q-2}M_1^{q-1}(L_1M_2+L_2)\|\varphi_2-\varphi_1\| \\
&= k\|\varphi_2-\varphi_1\|.
\end{aligned}
$$

Hence,

$$\|(A\varphi_2)-(A\varphi_1)\| \le k\|\varphi_2-\varphi_1\|, \quad 0<k<1.$$

Thus, the operator $A : B[O,M] \to B[O,M]$ is a contraction mapping and it has a unique fixed point in $B[O,M]$. According to Lemma 2.1, Lemma 2.2, we can obtain that the boundary value problem (1.1), (1.2) has a unique solution $u(t) \in C[0,1]$ such that

$$|u(t)| \le M_2M_1^{q-1}M^{q-1}, \qquad |u''(t)| \le M_1^{q-1}M^{q-1}.$$

The proof is complete. □

Now we consider the following iterative process:

(1). $u_1(t) = v_1(t) = 0$,

For $n = 2,3,4,\cdots$, let

(2). $\varphi_n(t) = f(t,u_{n-1}(t),v_{n-1}(t))$,

(3). $v_n(t) = \phi_q\left(\int_0^1 G_1(t,s)\varphi_n(s)ds\right)$,

(4). $u_n(t) = \int_0^1 G_2(t,s)v_n(s)ds$.

By using the Banach contracting mapping principle, the sequence $\{\varphi_n(t)\}$ converges with the rate of geometric progression to the fixed-point of the operator A, denote it as $\varphi^*(t)$. And there holds the estimation

$$\|\varphi_n-\varphi^*\| \le \frac{k^n}{1-k}\|\varphi_1-\varphi_0\|. \tag{3.1}$$

Therefore, we can obtain a iterative sequence solution $\{u_n(t)\}$ of the problem (1.1), (1.1)

$$u_n(t) = \int_0^1 G_2(t,\tau)\phi_q\left(\int_0^1 G_1(\tau,s)\varphi_n(s)ds\right)d\tau,$$

which converges to the unique solution u^* of the problem (1.1), (1.1).

Remark 3.1. For $p > 2$, to use the similar method, one has to apply (1) of Lemma 2.3. In this situation, the nonlinearity f should satisfy

$$|f(t, u_2, v_2) - f(t, u_1, v_1)| \leq L_1|u_2 - u_1|^{p-1} + L_2|v_2 - v_1|^{p-1},$$

for $(t, u_i, v_i) \in D_M, i = 1, 2$. However, it is well know that under the condition that the exponent of the Hölder $\alpha > 1$ the function is constant. So, the problem is not solved yet.

4. An Example

Example 4.1. Consider the following boundary value problem:

$$[\phi_p(u''(t))]'' = -3u^2(t)u''(t) + 3u(t) - 4u''(t) + 5\sin(\pi t), \tag{4.1}$$

$$u(0) = u(1) = u''(0) = u''(1) = 0. \tag{4.2}$$

Let $f(t, u, v) = -3u^2v + 3u - 4v + 5\sin(\pi t)$. Firstly, we choose $p = 2$.

From the definitions of M_1, M_2, we can calculate that $M_1 = M_2 = 1/8$. Then, we need choose a suitable $M > 0$ such that all conditions of Theorem 3.1 are satisfied. Clearly, for $(t, u, v) \in D_M$, there is

$$\begin{aligned} |f(t, u, v)| &\leq | - 3u^2v + 3u - 4v + 5\sin(\pi t)| \\ &\leq 3\left(\frac{M}{64}\right)^2 \frac{M}{8} + 3\frac{M}{64} + 4\frac{M}{8} + 5\sin(\pi t) \leq M \end{aligned}$$

as soon as $11.4 < M < 64.02$. Thus, choose $M = 12$, the condition (i) of Theorem 3.1 hold in D_M.

In the other hand, for $(t, u, v) \in D_M$,

$$|f_u| = | - 6uv + 3| \leq 6\frac{M}{64}\frac{M}{8} + 3 \leq 3.5,$$

$$|f_v| = | - 3u^2 - 4| \leq 3\left(\frac{M}{64}\right)^2 + 4 \leq 4.5.$$

So we choose $L_1 = 3.5, \quad L_2 = 4.5$, the condition (ii) of Theorem 3.1 is satisfied.

Moreover,

$$k = M_1(L_1M_2 + L_2) \approx 0.6172 < 1,$$

the condition (iii) of Theorem 3.1 is satisfied. Hence, the problem (4.1), (4.2) has a uniqueness solution (see Figure 1 and Table 1 for the iterative process).

Table 1. The numerical approximation of the solution (4.1), (4.2) for $p = 2$

	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
2	0	0.0031	0.0188	0.0446	0.0685	0.0781	0.0685	0.0446	0.0188	0.0031	0
3	0	0.0026	0.0131	0.0273	0.0384	0.0425	0.0384	0.0273	0.0131	0.0026	0
4	0	0.0027	0.0149	0.0340	0.0517	0.0588	0.0517	0.0340	0.0149	0.0027	0
5	0	0.0027	0.0143	0.0314	0.0459	0.0514	0.0459	0.0314	0.0143	0.0027	0
6	0	0.0027	0.0145	0.0324	0.0484	0.0548	0.0484	0.0324	0.0145	0.0027	0

We can find that the iterative method is very effective.

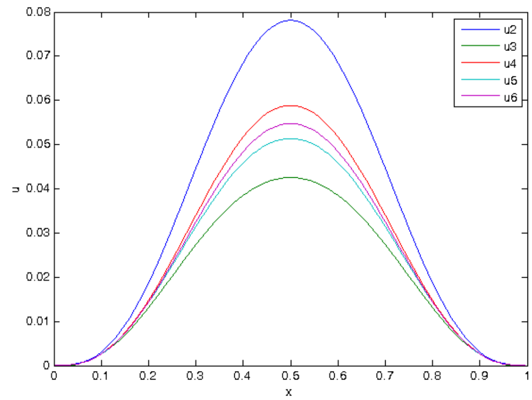

Figure 1. The approximation of the solution (4.1), (4.2) for $p = 2$

Secondly, we consider the case that $1 < p < 2$. We choose $p = 3/2$.

In this case, the conditions $(i), (ii)$ of Theorem 3.1 are satisfied clearly. We only need notice that

$$k = (q-1)(3M)^{q-2} M_1^{q-1}(L_1 M_2 + L_2) = \frac{1}{2} 9^{-\frac{1}{2}} (\frac{1}{8})^{\frac{1}{2}} (\frac{3.5}{8} + 4.5) \approx 0.291 < 1,$$

the condition (iii) of Theorem 3.1 is satisfied, too. Hence, the problem (4.1), (4.2) has a uniqueness solution (see Figure 2 and Table 2 for the iterative process).

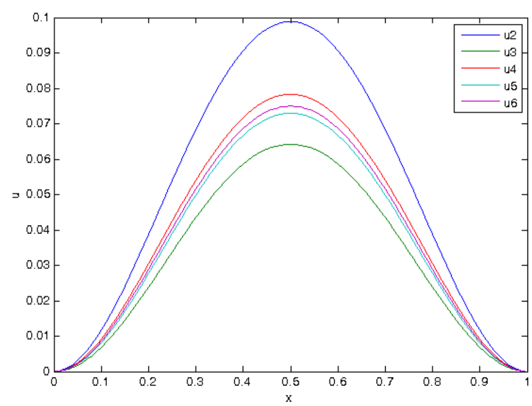

Figure 2. The approximation of the solution (4.1), (4.2) for $p = 3/2$

Table 2. The numerical approximation of the solution (4.1), (4.2) for $p = 3/2$

	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
2	0	0.0119	0.0388	0.0684	0.0906	0.0988	0.0906	0.0684	0.0388	0.0119	0
3	0	0.0069	0.0239	0.0435	0.0586	0.0642	0.0586	0.0435	0.0239	0.0069	0
4	0	0.0093	0.0305	0.0540	0.0717	0.0782	0.0717	0.0540	0.0305	0.0093	0
5	0	0.0082	0.0278	0.0498	0.0667	0.0729	0.0667	0.0498	0.0278	0.0082	0
6	0	0.0087	0.0289	0.0515	0.0686	0.0750	0.0686	0.0515	0.0289	0.0087	0

Acknowledgements. The authors are grateful to the anonymous referees for their useful suggestions which improve the contents of this article.

References

[1] P. Agarwal, H. Lü and D. ORegan, *Positive solutions for the boundary value problem* $(|u''|^{p-2}u'')'' - \lambda q(t)f(u(t)) = 0$, Mem. Differential Equations Math. Phys., 2003, 28, 33–44.

[2] K. Bachouche, A. Benmezai and S. Djebali, *Positive solutions to semi-positone fourth-order* φ*-Laplacian BVPs*, Positivity, 2017, 21, 193–212.

[3] Z. Bai, *Positive solutions of some nonlocal fourth-order boundary value problem*, Appl. Math. Comput., 2010, 215, 4191–4197.

[4] Z. Bai, Y. Chen, H. Lian and S. Sun, *On the existence of blow up solutions for a class of fractional differential equations*, Fract. Calc. Appl. Anal., 2014, 17(4), 1175–1187.

[5] Z. Bai, W. Ge and Y. Wang, *Multiplicity results for some second-order four-point boundary-value problems*, Nonlinear Anal.-Theor., 2005, 60, 491–500.

[6] Z. Bai, B. Huang and W. Ge, *The iterative solutions for some fourth-order p-Laplace equation boundary value problem*, Appl. Math. Lett., 2006, 19, 8–14.

[7] Z. Bai and Y. Zhang, *Solvability of fractional three-point boundary value problems with nonlinear growth*, Appl. Math. Comput., 2011, 218(5), 1719–1725.

[8] F. Bernis, *Compactness of the support in convex and non-convex fourth order elasticity problems*, Nonlinear Anal.-Theor., 1982, 6, 1221–1243.

[9] C. Chen, H. Song and H. Yang, *Liouville type theorems for stable solutions of p-Laplace equation in* R^n, Nonlinear Anal.-Theor., 2017, 160, 44–52.

[10] Y. Cui and J. Sun, *Existence of multiple positive solutions for fourth-order boundary value problems in Banach spaces*, Bound. Value Probl., 2012. DOI: 10.1186/1687-2770-2012-107.

[11] Y. Cui and J. Sun, *A generalization of Mahadevan's version of the Krein-Rutman theorem and applications to p-Laplacian boundary value problems*, Abstr. Appl. Anal., 2012. DOI: 10.1155/2012/305279.

[12] Y. Cui and Y. Zou, *Existence and uniqueness theorems for fourth-order singular boundary value problems*, Comp. Math. Appl., 2009, 58, 1449–1456.

[13] Q. A. Dang, Q. L. Dang and N. Quy, *A novel efficient method for nonlinear boundary value problems*, Numerical Algorithms, 2017, 76, 427–439.

[14] Q.A Dang and N. Quy, *Existence results and iterative method for solving the cantilever beam equation with fully nonlinear term*, Nonlinear Anal.-Real., 2017, 36, 56–68.

[15] X. Dong, Z. Bai and S. Zhang, *Positive solutions to boundary value problems of p-Laplacian with fractional derivative*, Bound. Value Probl., 2017. DOI: 10.1186/s13661-016-0735-z.

[16] M. Feng, P. Li and S. Sun, *Symmetric positive solutions for fourth-order n-dimensional m-Laplace systems*, Bound. Value Probl., 2018. DOI: 10.1186/s13661-018-0981-3.

[17] Y. Guo and W. Ge, *Positive solutions for three-point boundary value problems with dependence on the first order derivative*, J. Math. Anal. Appl., 2004, 290, 291–301.

[18] C. Gupta, *Existence and uniqueness results for the bending of an elastic beam equation at resonance*, J. Math. Anal. Appl., 1998, 135, 208–225.

[19] M. Guo, C. Fu, Y. Zhang, J. Liu and H. Yang, *Study of ion-acoustic solitary waves in a magnetized plasma using the three-dimensional time-space fractional schamel-KdV equation*, Complexity, 2018. DOI: 10.1155/2018/6852548.

[20] H. Li and J. Sun, *Positive solutions of superlinear semipositone nonlinear boundary value problems*, Comp. Math. Appl., 2011, 61, 2806–2815.

[21] Y. Li, *Existence of nontrivial solutions for unilaterally asymptotically linear three-point boundary value problems*, Abstr. Appl. Anal., 2014. DOI: 10.1155/2014/263042.

[22] Y. Li, *A monotone iterative technique for solving the bending elastic beam equations*, Appl. Math. Comput., 2010, 217, 2200–2208.

[23] Y. Li, *Existence of positive solutions for the cantilever beam equations with fully nonlinear terms*, Nonlinear Anal.-Real., 2016, 27, 221–237.

[24] H. Lian, D. Wang, Z. Bai and R. Agarwal, *Periodic and subharmonic solutions for a class of second-order p-Laplacian Hamiltonian systems*, Bound. Value Probl., 2014. DOI: 10.1186/s13661-014-0260-x.

[25] R. Ma, J. Zhang and S. Fu, *The method of lower and upper solutions for fourth-order two-point boundary value problem*, J. Math. Anal. Appl., 1997, 215, 415–422.

[26] Y. Pang and Z. Bai, *Upper and lower solution method for a fourth-order four-point boundary value problem on time scales*, Appl. Math. Comput., 2009, 215, 6, 2243–2247.

[27] K. Sheng, W. Zhang, Z. Bai, *Positive solutions to fractional boundary value problems with p-Laplacian on time scales*, Bound. Value Probl., 2018. DOI: 10.1186/s13661-018-0990-2.

[28] Q. Song and Z. Bai, *Positive solutions of fractional differential equations involving the Riemann-Stieltjes integral boundary condition*, Adv. Differ. Equ-NY, 2018. DOI: 10.1186/s13662-018-1633-8.

[29] Y. Tian, S. Sun and Z. Bai, *Positive solutions of fractional differential equations with p-Laplacian*, J. Funct. Space, 2017. DOI: 10.1155/2017/3187492.

[30] Y. Tian, Y. Wei and S. Sun, *Multiplicity for fractional differential equations with p-Laplacian*, Bound. Value Probl., 2018. DOI: 10.1186/s13661-018-1049-0.

[31] Z. Wang, Y. Xie, J. Lu and Y. Li, *Stability and bifurcation of a delayed generalized fractional-order prey-predator model with interspecific competition*, Appl. Math. Comput., 2019, 347, 360–369.

[32] Y. Wei, Q. Song and Z. Bai, *Existence and iterative method for some fourth order nonlinear boundary value problems*, Appl. Math. Lett., 2019, 87, 101–107.

[33] P. Yan, *Nonresonance for one-dimensional p-Laplacian with regular restoring*, J. Math. Anal. Appl., 2003, 285, 141–154.

[34] J. Zhang, G. Zhang and H. Li, *Positive solutions of second-order problem with dependence on derivative in nonlinearity under Stieltjes integral boundary condition*, Electron. J. Qual. Theo., 2018. DOI: 10.14232/ejqtde.2018.1.4.

[35] X. Zhang and Y. Cui, *Positive solutions for fourth-order singular p-Laplacian differential equations with integral boundary conditions*, Bound. Value Probl., 2010. DOI: 10.1155/2010/862079.

[36] X. Zhang, L. Liu, Y. Wu and Y. Cui, *Entire blow-up solutions for a quasilinear p-Laplacian Schrödinger equation with a non-square diffusion term*, Appl. Math. Lett., 2017, 74, 85–93.

[37] Y. Zhang, *Existence results for a coupled system of nonlinear fractional multipoint boundary value problems at resonance*, J. Inequal. Appl. 2018. DOI: 10.1186/s13660-018-1792-x.

[38] Y. Zou, *On the existence of positive solutions for a fourth-order boundary value problem*, J. Funct. Space, 2017. DOI: 10.1155/2017/4946198.

[39] Y. Zou and Y. Cui, *Uniqueness result for the cantilever beam equation with fully nonlinear term*, J. Nonlinear. Sci. App., 2017, 10, 4734–4740.

[40] Y. Zou and G. He, *Fixed point theorem for systems of nonlinear operator equations and applications to (p_1, p_2)-Laplacian system*, Mediterr. J. Math., 2018. DOI: 10.1007/s00009-018-1119-7.