

INFINITELY MANY SOLUTIONS FOR A ZERO MASS SCHRÖDINGER-POISSON-SLATER PROBLEM WITH CRITICAL GROWTH*

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Abstract In this paper, we are concerned with the following Schrödinger-Poisson-Slater problem with critical growth:

$$-\Delta u + (u^2 \star \frac{1}{|4\pi x|})u = \mu k(x)|u|^{p-2}u + |u|^4u \text{ in } \mathbb{R}^3.$$

We use a measure representation concentration-compactness principle of Lions to prove that the $(PS)_c$ condition holds locally. Via a truncation technique and Krasnoselskii genus theory, we further obtain infinitely many solutions for $\mu \in (0, \mu^*)$ with some $\mu^* > 0$.

Keywords Schrödinger-Poisson-Slater problem, Zero mass, critical growth, concentration-compactness principle.

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1. Introduction

In this paper we investigate the multiplicity of solutions for the following version of Schrödinger-Poisson-Slater problem with critical growth:

$$-\Delta u + (u^2 \star \frac{1}{|4\pi x|})u = \mu k(x)|u|^{p-2}u + |u|^4u, \quad \text{in } \mathbb{R}^3, \quad (1.1)$$

where $\mu > 0$, $1 < p < 2$ and $k(x) \in L^{\frac{6}{6-p}}(\mathbb{R}^3)$.

The problem (1.1) arises in the study of standing wave solutions for the nonlocal nonlinear Schrödinger equation

$$i \frac{\partial \psi}{\partial t} = -\Delta \psi + V(x)\psi + \lambda(\psi^2 \star \frac{1}{|4\pi x|})\psi - \mu|\psi|^{p-2}\psi, \quad (\psi, t) \in \mathbb{R}^3 \times \mathbb{R} \quad (1.2)$$

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and its stationary counterpart

$$-\Delta u + V(x)u + (u^2 \star \frac{1}{|4\pi x|})u = \mu|u|^{p-2}u \text{ in } \mathbb{R}^3. \tag{1.3}$$

Equation (1.3) appears in the physical as an approximation of the Hartree-Fock model of a quantum many body system of electrons under the presence of the external potential $V(x)$, u^2 denotes the density of electrons in the original many-body system, $(u^2 \star \frac{1}{|4\pi x|})u$ represents the Coulombic repulsion between the electrons, $|u|^{p-2}u$ was introduced by Slater and μ is called Slater constant. For more information on these models and their deduction, see [7, 9, 15, 21].

In the literature there are many papers on the equations for (1.3) by using variational methods since it was introduced in [7]. Recently, a lot of attention has been focused on the study of the existence of solutions, sign-changing solutions, ground states, radial and semiclassical states, see [1, 3, 4, 6, 10, 11, 13, 17, 23, 27, 29–33] and the references therein.

When $V(x) = 0$, equation (1.3) becomes as the following static case

$$-\Delta u + (u^2 \star \frac{1}{|4\pi x|})u = \mu|u|^{p-2}u \text{ in } \mathbb{R}^3, \tag{1.4}$$

which is called “zero mass” problem (see [8]). Compared with problem (1.3), there has been only few works in recent years on the existence of solutions of systems like (1.4). We refer to [12, 22, 24, 25]. More precisely, a limit profile was studied in [25]. For the case of $p \geq 2$, we see [12] for the existence of ground and bound states. We emphasize that (1.4) presents an interesting competition between local and nonlocal nonlinearities, which yields to some non expected situations, as has been shown in the literature [25]. Moreover, due to the absence of a phase term, the standard Sobolev space $H^1(\mathbb{R}^3)$ is not a good framework in the study of (1.4). In [25], a new variational framework was established and the so-called Coulomb-Sobolev function space was introduced:

$$E = \left\{ u \in D^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} dx dy < +\infty \right\},$$

where the double integral expression is called Coulomb energy of the wave. It was shown in [25] that E is a uniformly convex separable Banach space, that $E \subset L^q(\mathbb{R}^N)$ for every $q \in [3, 6]$. Recently, a more general Coulomb-Sobolev space was established in [22] and a family of optimal interpolation inequalities and the existence of ground state solutions was studied for a general static Schrödinger-Poisson-Slater problem. Recently, in [19], the existence of positive solutions for the following equation with critical growth was obtained via a novel perturbation approach in the case of $2 < p < 5$ and truncation technique (see [2]) for $\frac{11}{7} < p < 2$,

$$-\Delta u + (u^2 \star \frac{1}{|4\pi x|})u = \mu|u|^{p-1}u + |u|^4u \text{ in } \mathbb{R}^3. \tag{1.5}$$

To our best knowledge, there is no result about the infinitely many solutions for equation (1.1) in the literature. Motivated by the above and the idea of [2, 5, 16–19, 28], the aim of this paper is to study the existence of infinitely many solutions for Schrödinger-Poisson-Slater equations with critical nonlinearity in \mathbb{R}^3 .

The main difficulty is to show the $(PS)_c$ condition holds, because the embedding $E \hookrightarrow L^q(\mathbb{R}^N)$ is not compact for $q \in [3, 6]$. By applying a measure representation concentration-compactness principle of Lions [20] and more delicate analysis, we prove that $(PS)_c$ condition holds locally. To obtain the infinitely many solutions, we use a new version of the symmetric mountain-pass lemma due to Kajikiya [14]. However, we have to consider a truncation because the functional J is not bounded below.

Our main result reads as follows:

Theorem 1.1. *Suppose that $\Omega := \{x \in \mathbb{R}^3 : k(x) > 0\}$ is an open subset and $0 < |\Omega| < +\infty, 1 < p < 2$. Then there exists $\mu^* > 0$ such that equation (1.1) has a sequence of solutions $\{u_n\}$ with $J(u_n) \leq 0, J(u_n) \rightarrow 0$ and $u_n \rightarrow 0$ in E as $n \rightarrow \infty$ for $\mu \in (0, \mu^*)$.*

The remainder of this paper is organized as follows. Some preliminaries are presented in Section 2. In Section 3, we complete the proof of Theorem 1.1.

2. Preliminaries

Define $\phi_u = \frac{1}{4\pi|x|} \star u^2$, then $u \in E$ if and only if both u and ϕ_u belong to $D^{1,2}(\mathbb{R}^3)$. In such case, problem (1.1) can be rewritten as a system in the following form:

$$\begin{cases} -\Delta u + \phi u = \mu k(x)|u|^{p-2}u + |u|^4u, & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2, & \text{on } \mathbb{R}^3. \end{cases} \quad (2.1)$$

Moreover,

$$\int_{\mathbb{R}^3} |\nabla \phi_u|^2 dx = \int_{\mathbb{R}^3} \phi_u u^2 dx = \int_{\mathbb{R}^6} \frac{u^2(x)u^2(y)}{4\pi|x-y|} dx dy.$$

Define the norm $\|\cdot\| : E \rightarrow \mathbb{R}^+ \cup \{0\}$ for the space E as follows

$$\|u\| := \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx + \left(\int_{\mathbb{R}^6} \frac{u^2(x)u^2(y)}{4\pi|x-y|} dx dy \right)^{1/2} \right)^{1/2}.$$

For the space E , some properties have been proved in [12, 25].

Lemma 2.1 (See [25]). *$(\|\cdot\|, E)$ is a uniformly convex Banach space. Moreover, $C_0^\infty(\mathbb{R}^3)$ is dense in E . $E \hookrightarrow L^q(\mathbb{R}^3)$ continuously for $q \in [3, 6]$ and $E \hookrightarrow L^q(\Omega)$ compactly for $q \in [1, 6)$ with bounded $\Omega \subset \mathbb{R}^3$.*

Define $M : E \rightarrow \mathbb{R}$ as

$$M(u) := \int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^6} \frac{u^2(x)u^2(y)}{4\pi|x-y|} dx dy.$$

We can easily obtain for any $u \in E$,

$$\frac{1}{2}\|u\|^4 \leq M(u) \leq \|u\|^2, \text{ if either } \|u\| \leq 1 \text{ or } M(u) \leq 1. \quad (2.2)$$

The following estimate will be of use.

Lemma 2.2 (See [12]). *There exists $C > 0$ such that*

$$\|u\|_{p+1}^{p+1} \leq CM(u)^{\frac{2p-1}{3}}$$

for $u \in E$ with $p \in [2, 5]$.

We consider the energy functional $J : E \rightarrow \mathbb{R}$ defined by

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx - \frac{\mu}{p} \int_{\mathbb{R}^3} k(x) |u|^p dx - \frac{1}{6} \int_{\mathbb{R}^3} u^6 dx.$$

Owing to $k(x) \in L^{\frac{6}{6-p}}(\mathbb{R}^3)$ and Lemma 2.1, J belongs to $C^1(E, \mathbb{R})$. It is well known that a critical point of J is a weak solution of problem (1.1). To get compactness of the (PS) sequence in E , we recall the well-known concentration-compactness principle due to P. Lions [20](see Lemmas I.1- I.2 of [20] for the details of proof).

Lemma 2.3 (See [20]). *Let $\{u_n\}$ be a sequence weakly converging to u in $D^{1,2}(\mathbb{R}^3)$. Then, up to subsequence,*

(A1) $|\nabla u_n|^2$ weakly converges in $\mathbf{M}(\mathbb{R}^3)$ to a nonnegative measure $\tilde{\mu}$,

(A2) $|u_n|^6$ weakly converges in $\mathbf{M}(\mathbb{R}^3)$ to a nonnegative measure ν ,

and there exist an at most countable index set K , a family $\{x_j : j \in K\}$ of distinct points of \mathbb{R}^3 , and families $\{\nu_j : j \in K\}, \{\tilde{\mu}_j : j \in K\}$ of positive numbers such that

$$\tilde{\mu} \geq |\nabla u|^2 dx + \sum_{j \in K} \tilde{\mu}_j \delta_{x_j}, \quad \nu = |u|^6 dx + \sum_{j \in K} \nu_j \delta_{x_j}$$

and for all $j \in K, S\nu_j^{1/3} \leq \tilde{\mu}_j$, where δ_{x_j} is the Dirac measure at point x_j .

To study the concentration at infinity of the sequence we recall the following quantities.

Lemma 2.4 (See [20]). *Let $\{u_n\}$ be a sequence weakly converging to u in $D^{1,2}(\mathbb{R}^3)$ and define*

$$\nu_\infty = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} |u_n|^6 dx, \quad \tilde{\mu}_\infty = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} |\nabla u_n|^2 dx.$$

Then the quantities ν_∞ and $\tilde{\mu}_\infty$ are well defined and satisfy

$$\nu_\infty + \int_{\mathbb{R}^3} d\nu = \limsup_{n \rightarrow \infty} \int_{|x| > R} |u_n|^6 dx, \quad \tilde{\mu}_\infty + \int_{\mathbb{R}^3} d\tilde{\mu} = \limsup_{n \rightarrow \infty} \int_{|x| > R} |\nabla u_n|^2 dx,$$

and $S\nu_\infty^{1/3} \leq \tilde{\mu}_\infty$, where $\tilde{\mu}$ and ν defined in Lemma 2.3.

Lemma 2.5. *Suppose that $1 < p < 2$ holds, and then any $(PS)_c$ sequence $\{u_n\}$ of J is bounded in E .*

Proof. Let $\{u_n\}$ be a sequence in E such that

$$c + o(1) = J(u_n) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx - \frac{\mu}{p} \int_{\mathbb{R}^3} k(x) |u_n|^p dx - \frac{1}{6} \int_{\mathbb{R}^3} u_n^6 dx, \tag{2.3}$$

and for all $v \in E$

$$\langle J'(u_n), v \rangle = \int_{\mathbb{R}^3} \nabla u_n \nabla v dx + \int_{\mathbb{R}^3} \phi_{u_n} u_n v dx - \mu \int_{\mathbb{R}^3} k(x) |u_n|^{p-1} v dx - \int_{\mathbb{R}^3} u_n^5 v dx = o(\|v\|). \tag{2.4}$$

Combining (2.3) with (2.4), we have

$$\begin{aligned} & c + o(\|u_n\|) \\ &= J(u_n) - \frac{1}{6} J'(u_n) u_n \\ &= \frac{1}{3} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{1}{12} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx - \left(\frac{\mu}{p} - \frac{\mu}{6}\right) \int_{\mathbb{R}^3} k(x) |u_n|^p dx \\ &\geq \frac{1}{3} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{1}{12} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx - \left(\frac{\mu}{p} - \frac{\mu}{6}\right) \left(\int_{\mathbb{R}^3} |k(x)|^{\frac{6}{6-p}} dx\right)^{\frac{6-p}{6}} \left(\int_{\mathbb{R}^3} |u_n|^6 dx\right)^{\frac{p}{6}} \\ &\geq \frac{1}{12} M(u_n) - \left(\frac{\mu}{p} - \frac{\mu}{6}\right) C \|k\|_{\frac{6}{6-p}} M(u_n)^{p/2}, \end{aligned} \tag{2.5}$$

which implies that $\{u_n\}$ is bounded in E due to $1 < p < 2$. □

Lemma 2.6. *Let $1 < p < 2$ and $c < 0$. Then there exists $\mu^* > 0$ such that for $\mu \in (0, \mu^*)$, J satisfies $(PS)_c$ condition.*

Proof. By Lemma 2.5, any $(PS)_c$ sequence $\{u_n\}$ is bounded in E . We may assume that there exists $u \in E$ such that $u_n \rightharpoonup u$ in E as $n \rightarrow \infty$. we have $J'(u) = 0$. It follows from Lemma 2.3 that $|\nabla u_n|^2 dx$ converges in the *weak**-sense of measure to a measure $\tilde{\mu}$ and $|u_n|^6 dx$ converges in the *weak**-sense of measure to a measure ν . Furthermore, we obtain an at most countable index set K and sequences $\{x_i\} \subset \mathbb{R}^3$ and families $\{\tilde{\mu}_i, \nu_i : i \in K\}$ of positive numbers such that

$$\tilde{\mu} \geq |\nabla u|^2 dx + \sum_{j \in K} \tilde{\mu}_j \delta_{x_j}, \quad \nu = |u|^6 dx + \sum_{j \in K} \nu_j \delta_{x_j}$$

and for all $j \in K, S\nu_j^{1/3} \leq \tilde{\mu}_j$.

Now for any $\epsilon > 0$, we define $\chi_\epsilon(x) := \bar{\chi}_\epsilon(x - x_i)$, where $\bar{\chi}_\epsilon \in C_0^\infty(\mathbb{R}^3, [0, 1])$ is such that $\bar{\chi}_\epsilon \equiv 1$ on $B_\epsilon(0)$, $\bar{\chi}_\epsilon \equiv 0$ on $\mathbb{R}^3 \setminus B_{2\epsilon}(0)$ and $|\nabla \bar{\chi}_\epsilon| \in [0, \frac{2}{\epsilon}]$. Now we divide the proof into four steps.

Step 1: For any $i \in K, \nu_i = \tilde{\mu}_i$.

It is clear that the sequence $\{\chi_\epsilon u_n\}$ is bounded, then we have $J'(u_n)(\chi_\epsilon u_n) \rightarrow 0$ as $n \rightarrow \infty$. Thus,

$$\int_{\mathbb{R}^3} u_n \nabla u_n \nabla \chi_\epsilon dx = \int_{\mathbb{R}^3} (-|\nabla u_n|^2 - \phi_{u_n} u_n^2 + \mu k(x) |u_n|^p + u_n^6) \chi_\epsilon dx + o(1). \tag{2.6}$$

Using the Hölder inequality, we obtain the following limit expression:

$$\begin{aligned} \left| \int_{\mathbb{R}^3} u_n \nabla u_n \nabla \chi_\epsilon dx \right| &\leq \left(\int_{B_{2\epsilon}(x_i)} |u_n \nabla u_n|^{3/2} dx \right)^{\frac{2}{3}} \left(\int_{B_{2\epsilon}(x_i)} |\nabla \chi_\epsilon|^3 dx \right)^{\frac{1}{3}} \\ &\leq C \left(\int_{B_{2\epsilon}(x_i)} u_n^6 dx \right)^{\frac{1}{6}} \rightarrow 0 \end{aligned} \tag{2.7}$$

as $\epsilon \rightarrow 0$. Moreover, since $u_n \rightarrow u$ in $L^s_{loc}(\mathbb{R}^3)$ for all $1 < s < 6$ and χ_ϵ has compact support, it follows from that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} u_n \nabla u_n \nabla \chi_\epsilon dx = - \int_{\mathbb{R}^3} (|\nabla u|^2 + \phi_u u^2 - \mu k(x)|u|^p - u^6) \chi_\epsilon dx + \int_{\mathbb{R}^3} \chi_\epsilon d\nu - \int_{\mathbb{R}^3} \chi_\epsilon d\tilde{\mu}. \tag{2.8}$$

Letting $\epsilon \rightarrow 0$ in (2.8), we conclude $\nu_i = \tilde{\mu}_i$.

Step 2: $\nu_\infty \geq \tilde{\mu}_\infty$.

For any $R > 0$, let $\psi_R : \mathbb{R}^3 \rightarrow [0, 1]$ be a smooth function satisfying $\psi_R \equiv 0$ if $|x| \leq R$ and $\psi_R \equiv 1$ if $|x| \geq 2R$, and $|\nabla \psi_R| \leq 2/R$. It is easy to see that $\{u_n \psi_R\}$ is bounded in E and $J'(u_n)(u_n \psi_R) \rightarrow 0$ as $n \rightarrow \infty$, which implies

$$\int_{\mathbb{R}^3} \nabla u_n \nabla (u_n \psi_R) dx + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \psi_R dx = \mu \int_{\mathbb{R}^3} k(x)|u_n|^p \psi_R dx + \int_{\mathbb{R}^3} u_n^6 \psi_R dx. \tag{2.9}$$

Similar to (2.8), by Lemma 2.4, we have

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \left(\mu \int_{\mathbb{R}^3} k(x)|u_n|^p \psi_R dx + \int_{\mathbb{R}^3} u_n^6 \psi_R dx \right) = \nu_\infty. \tag{2.10}$$

On the other hand, we also have

$$\begin{aligned} \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} \nabla u_n \nabla (u_n \psi_R) dx &= \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} (|\nabla u_n|^2 \psi_R + u_n \nabla u_n \nabla \psi_R) dx \\ &= \tilde{\mu}_\infty + \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} u_n \nabla u_n \nabla \psi_R dx \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \int_{\mathbb{R}^3} u_n \nabla u_n \nabla \psi_R dx \right| &\leq \left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \right)^{1/2} \left(\int_{\mathbb{R}^3} |\nabla \psi_R|^2 dx \right)^{1/4} \|u_n\|_4 \\ &\leq \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{C}{R^{3/4}} \left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \right)^{1/2} \|u_n\|_4 = 0. \end{aligned} \tag{2.12}$$

Combining (2.9)-(2.12), we have $\nu_\infty \geq \tilde{\mu}_\infty$.

Step 3: $\nu_i = 0$ for any $i \in K$ and $\nu_\infty = 0$.

Suppose that there exists $i_0 \in K$ such that $\nu_{i_0} > 0$ or $\nu_\infty > 0$. Using Lemmas 2.3-2.4, we have

$$S^3 \nu_{i_0} \leq \tilde{\mu}_{i_0}^3 = \nu_{i_0}^3, \quad S^3 \nu_\infty \leq \tilde{\mu}_\infty^3 = \nu_\infty^3,$$

which implies

$$\nu_{i_0} \geq S^{3/2}, \quad \nu_\infty \geq S^{3/2}. \tag{2.13}$$

For $c < 0$, we have

$$\begin{aligned} 0 > c &= J(u_n) - \frac{1}{6} J'(u_n) u_n \\ &= \frac{1}{3} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{1}{12} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx - \left(\frac{1}{p} - \frac{1}{6} \right) \mu \int_{\mathbb{R}^3} k(x)|u_n|^p dx \\ &\geq \frac{1}{12} M(u_n) - \frac{(6-p)\mu C}{6p} \|k\|_{\frac{6}{6-p}} M(u_n)^{p/2}. \end{aligned} \tag{2.14}$$

This implies that

$$M(u_n) \leq C\mu^{\frac{2}{2-p}}.$$

If $\nu_{i_0} > 0$, then we obtain

$$\begin{aligned} 0 > c &= J(u_n) - \frac{1}{6} J'(u_n)u_n \\ &= \frac{1}{3} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{1}{12} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx - \left(\frac{1}{p} - \frac{1}{6}\right) \mu \int_{\mathbb{R}^3} k(x) |u_n|^p dx \\ &\geq \frac{1}{3} \int_{\mathbb{R}^3} |\nabla u_n|^2 \chi_\epsilon dx - \frac{(6-p)\mu C}{6p} \|k\|_{\frac{6}{6-p}} M(u_n)^{p/2} \\ &\geq \frac{1}{3} \tilde{\mu}_{i_0} - \frac{(6-p)\mu C}{6p} \|k\|_{\frac{6}{6-p}} C^{p/2} \mu^{\frac{p}{2-p}} \\ &\geq \frac{1}{3} \nu_{i_0} - \frac{(6-p)\mu C}{6p} \|k\|_{\frac{6}{6-p}} C^{p/2} \mu^{\frac{p}{2-p}} \\ &\geq \frac{1}{3} S^{3/2} - \frac{(6-p)\mu C}{6p} \|k\|_{\frac{6}{6-p}} C^{p/2} \mu^{\frac{p}{2-p}}. \end{aligned} \tag{2.15}$$

However, we can choose $\mu^* > 0$ small enough such that for every $\mu \in (0, \mu^*)$, the last term on the right-hand side above is greater than zero, which is a contradiction. Similarly, if $\nu_\infty > 0$, we can get

$$\begin{aligned} 0 > c &= J(u_n) - \frac{1}{4} J'(u_n)u_n \\ &\geq \frac{1}{4} \int_{\mathbb{R}^3} |\nabla u_n|^2 \psi_R dx + \frac{1}{12} \int_{\mathbb{R}^3} |u_n|^6 \psi_R dx - \frac{(4-p)\mu C}{4p} \|k\|_{\frac{6}{6-p}} M(u_n)^{p/2} \\ &\geq \frac{1}{12} \nu_\infty - \frac{(4-p)\mu C}{4p} \|k\|_{\frac{6}{6-p}} C^{p/2} \mu^{\frac{p}{2-p}} \\ &\geq \frac{1}{12} S^{3/2} - \frac{(6-p)\mu C}{6p} \|k\|_{\frac{6}{6-p}} C^{p/2} \mu^{\frac{p}{2-p}}. \end{aligned} \tag{2.16}$$

So we can choose $\mu^* > 0$ so small such that for every $\mu \in (0, \mu^*)$, the last term on the right-hand side above is larger than zero, which is also contradiction. Therefore, $\nu_i = 0$ for any $i \in K$ and $\nu_\infty = 0$.

Step 4: $u_n \rightarrow u$ in E .

By the conclusions of Step 3 and Lemmas 2.3-2.4, we obtain

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} |u_n|^6 dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} |u|^6 dx.$$

Using the Fatou Lemma, the following holds immediately

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} |u_n - u|^6 dx = 0.$$

Using the fact that $J'(u) = 0$, we have

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} \phi_u u^2 dx &\leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx \\ &\leq \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} k(x) |u_n|^p dx + \int_{\mathbb{R}^3} u_n^6 dx \end{aligned}$$

$$\begin{aligned} &\leq \int_{\mathbb{R}^3} k(x)|u|^p dx + \int_{\mathbb{R}^3} u^6 dx \\ &= \int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} \phi_u u^2 dx. \end{aligned} \tag{2.17}$$

Hence, we immediately get $u_n \rightarrow u$ in E . □

3. Proof of the main result

In this section, we prove the existence of infinitely many solutions to (1.1) which tend to zero. Let X be a Banach space and denote

$$\Sigma := \{A \subset X \setminus \{0\} : A \text{ is closed in } X \text{ and symmetric with respect to the origin}\}.$$

For $A \in \Sigma$, we define genus $\gamma(A)$ as

$$\gamma(A) := \inf\{m \in \mathbb{N} : \exists \phi \in C(a, \mathbb{R}^m \setminus \{0\}), -\phi(x) = \phi(-x)\}.$$

If there is no mapping ϕ as above for any $m \in \mathbb{N}$, then $\gamma(A) = +\infty$. Let Σ_k denote the family of closed symmetric subsets A of X such that $0 \neq A$ and $\gamma(A) \geq k$. We now list some properties of the genus(see [14, 26]).

Proposition 3.1. *Let A, B be closed symmetric subsets of X which do not contain the origin. Then the following holds.*

- (1) *If there exists an odd continuous mapping from A to B , then $\gamma(A) \leq \gamma(B)$;*
- (2) *If there is an odd homeomorphism from A to B , then $\gamma(A) = \gamma(B)$;*
- (3) *If $\gamma(B) < \infty$, then $\gamma(\overline{A \setminus B}) \geq \gamma(A) - \gamma(B)$;*
- (4) *Then n -dimensional sphere S^n has a genus of $n + 1$ by the Borsuk-Ulam Theorem;*
- (5) *If A is compact, then $\gamma(A) < +\infty$ and there exists $\delta > 0$ such that $U_\delta(A) \in \Sigma$ and $\gamma(U_\delta(A)) = \gamma(A)$, where $U_\delta(A) = \{x \in X : \|x - A\| \leq \delta\}$.*

The following version of the symmetric mountain-pass lemma is due to Kajikiya [14].

Proposition 3.2. *Let X be an infinite-dimensional Banach space and $J \in C^1(X, \mathbb{R})$ and suppose the following conditions holds.*

- (1) *$J(u)$ is even, bounded from below, $J(0) = 0$ and $J(u)$ satisfies the local (PS) condition, i.e. for some $C > 0$, in the case when every sequence $\{u_n\}$ in X satisfying $\lim_{n \rightarrow \infty} J(u_n) = c < C$ and $J'(u_n) \rightarrow 0$ has a convergent subsequence;*
- (2) *For each $k \in \mathbb{N}$, there exists an $A_k \in \Sigma_k$ such that $\sup_{u \in A_k} J(u) < 0$.*

Then either (J_1) or (J_2) below holds.

- (J_1) *There exists a sequence $\{u_k\}$ such that $J'(u_k) = 0$, $J(u_k) < 0$ and $u_k \rightarrow 0$ in X as $k \rightarrow \infty$;*
- (J_2) *There exist two sequences $\{u_k\}, \{v_k\}$ such that $J'(u_k) = 0, J(u_k) < 0$ and $u_k \neq 0, u_k \rightarrow 0$ in X as $k \rightarrow \infty$, $J'(v_k) = 0, J(v_k) < 0, J(v_k) \rightarrow 0$, and $\{v_k\}$ converges to a non-zero limit.*

In the proof of our main result, we need that J is bounded below, which contradicts with the critical growth term. To overcome this difficulty, we introduce a truncation in functional J . It follows from Lemma 2.2 that

$$\begin{aligned} J(u) &= \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx - \frac{\mu}{p} \int_{\mathbb{R}^3} k(x) |u|^p dx - \frac{1}{6} \int_{\mathbb{R}^3} u^6 dx \\ &\geq \frac{1}{4} M(u) - \frac{\mu C}{p} \|k\|_{\frac{6}{6-p}} M(u)^{\frac{p}{2}} - \frac{C}{6} M(u)^3. \end{aligned} \quad (3.1)$$

Let

$$g(t) = \frac{1}{4} t - \frac{\mu C}{p} \|k\|_{\frac{6}{6-p}} t^{\frac{p}{2}} - \frac{C}{6} t^3,$$

then $J(u) \geq g(M(u))$. Note that there exists

$$\mu^{**} > 0 \quad (3.2)$$

such that if $\mu \in (0, \mu^{**})$, then there exist $R_1, R_2 > 0$ such that $g(t) \geq 0$ over interval $I = [R_1, R_2]$, where $g(R_1) = g(R_2) = 0$. Moreover, $g(t) < 0$ over interval $(0, R_1)$ and $(R_2, +\infty)$. Now we consider the following truncation of J :

$$\bar{J}(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx - \frac{\mu}{p} \int_{\mathbb{R}^3} k(x) |u|^p dx - \frac{1}{6} \varphi(M(u)) \int_{\mathbb{R}^3} u^6 dx, \quad (3.3)$$

where $\varphi \in C_0^\infty(\mathbb{R}^+, [0, 1])$, such that $\varphi(t) \equiv 1$ if $t \in [0, R_1]$ and $\varphi(t) \equiv 0$ if $t \in [R_2, +\infty)$. Hence, $\bar{J}(u) \geq g_1(M(u))$, where

$$g_1(t) = \frac{1}{4} t - \frac{\mu C}{p} \|k\|_{\frac{6}{6-p}} t^{\frac{p}{2}} - \frac{C}{6} \varphi(t) t^3.$$

Note that $g_1(t) < 0$ over interval $(0, R_1)$ and $g_1(t) > 0$ over $(R_1, +\infty)$.

The proof of Theorem 1.1 requires some lemmas.

Lemma 3.1. *There exists $\mu^* > 0$ such that if any $0 < \mu < \mu^*$, then the following holds:*

- (i) *If $\bar{J}(u) < 0$, then $M(u) < R_1$ and $J(u) = \bar{J}(u)$.*
- (ii) *\bar{J} satisfies $(PS)_c$ condition for $c < 0$.*

Proof. Consider μ^* and μ^{**} given, respectively, by Lemma 2.6 and (3.2), we choose μ^* sufficiently small, such that $\mu^* < \min\{\mu^*, \mu^{**}\}$. Conclusion (i) follows immediately from the definition of $\bar{J}(u)$.

(ii) Assume that $\bar{J}(u_n) \rightarrow c < 0$ and $\bar{J}'(u_n) \rightarrow 0$ as $n \rightarrow \infty$. By conclusion (i), we have $M(u_n) < R_1$ and $J(u_n) \rightarrow c < 0$ and $J'(u_n) \rightarrow 0$. By Lemma 2.6, there exists $u \in E$ such that $u_n \rightarrow u$ in E . \square

Remark 3.1. Denote $K_c = \{u \in E : \bar{J}'(u) = 0, \bar{J}(u) = c\}$. If μ is as in (ii) above, then K_c is compact for $c < 0$.

Lemma 3.2. *For any $m \in \mathbb{N}$, there is $\varepsilon_m < 0$ such that $\gamma(\bar{J}^{\varepsilon_m}) \geq m$.*

Proof. Denote by $E_0(\Omega)$ the closure of $C_0^\infty(\Omega)$. We extend functions in $E_0(\Omega)$ by 0 outside Ω . Let X_m be a m -dimensional subspace of $E_0(\Omega)$. For any $u \in X_m, u \neq 0$, define $u = r_m w$ with $w \in X_m$ and $\|w\| = 1$ and $r_m > 0$. From the assumptions of

$k(x)$, it is easy to see that, for every $w \in X_m$ with $\|w\| = 1$, there exists $d_m > 0$ such that

$$\int_{\Omega} k(x)|w|^p dx \geq d_m.$$

Take $0 < r_m < \min\{R_1, 1\}$. Since all the norms are equivalent and $0 < |\Omega| < +\infty$, by (2.2) we have

$$\begin{aligned} \bar{J}(u) = J(u) &= \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx - \frac{\mu}{p} \int_{\mathbb{R}^3} k(x)|u|^p dx - \frac{1}{6} \int_{\mathbb{R}^3} u^6 dx \\ &\leq \frac{1}{2} \|u\|^2 - \frac{\mu}{p} \int_{\Omega} k(x)|u|^p dx - \frac{1}{6} \int_{\Omega} u^6 dx \\ &\leq \frac{1}{2} r_m^2 - \frac{\mu}{p} C r_m^p - \frac{1}{6} C r_m^6 := \varepsilon_m. \end{aligned} \tag{3.4}$$

Hence, we can choose r_m so small that $\bar{J}(u) < \varepsilon_m < 0$. Let

$$S_{r_m} = \{u \in E : \|u\| = r_m\}.$$

Then $S_{r_m} \cap X_m \subset \gamma(\bar{J}^{\varepsilon_m})$. Then by Proposition 3.1, we have $\gamma(\bar{J}^{\varepsilon_m}) \geq \gamma(S_{r_m} \cap X_m) \geq m$. □

Therefore, we can denote $\Gamma_m = \{A \in \Sigma : \gamma(A) \geq m\}$ and define

$$c_m := \inf_{A \in \Gamma_m} \sup_{u \in A} \bar{J}(u), \tag{3.5}$$

then

$$-\infty < c_m \leq \varepsilon_m < 0, \quad m \in \mathbb{N}, \tag{3.6}$$

because $\bar{J}^{\varepsilon_m} \in \Gamma_m$ and \bar{J} is bounded from below.

Lemma 3.3. *Let μ be as in Lemma 3.1, Then, all c_m given by (3.5) are critical values of \bar{J} and $c_m \rightarrow 0$ as $m \rightarrow \infty$.*

Proof. It is clear that $c_m \leq c_{m+1}$. By (3.6), we have $c_m < 0$ for any fixed $m \in \mathbb{N}$. Hence $c_m \rightarrow \bar{c} \leq 0$. Moreover, since $(PS)_c$ is satisfied, it follows from a standard argument as in [26] that all c_m given by (3.5) are critical values of \bar{J} . We claim $\bar{c} = 0$. If $\bar{c} < 0$, then by Remark 3.4, we get that $K_{\bar{c}}$ is compact and $K_{\bar{c}} \in \Sigma$. By Proposition 3.1, we have $m_0 : \gamma(K_{\bar{c}}) < +\infty$, which implies that there exists $\delta > 0$ such that $\gamma(K_{\bar{c}}) = \gamma(N_{\delta}(K_{\bar{c}})) = m_0$. By the deformation lemma, there exist $\epsilon > 0$ ($\bar{c} + \epsilon < 0$) and an odd homeomorphism η such that

$$\eta(\bar{J}^{\bar{c}+\epsilon} \setminus N_{\delta}(K_{\bar{c}})) \subset \bar{J}^{\bar{c}-\epsilon}. \tag{3.7}$$

Since c_m is increasing and converges to \bar{c} , there exists $m \in \mathbb{N}$ such that $c_m > \bar{c} - \epsilon$ and $c_{m+m_0} \leq \bar{c}$. There exists $A \in \Gamma_{m+m_0}$ such that $\sup_{u \in A} \bar{J}(u) < \bar{c} + \epsilon$. By proposition 3.1, we have

$$\gamma(A \setminus \bar{N}_{\delta}(K_{\bar{c}})) \geq \gamma(A) - \gamma(N_{\delta}(K_{\bar{c}})) \geq m. \tag{3.8}$$

Thus,

$$\eta(A \setminus \bar{N}_{\delta}(K_{\bar{c}})) \in \Gamma_m.$$

Consequently,

$$\sup_{u \in \eta(A \setminus \bar{N}_{\delta}(K_{\bar{c}}))} \bar{J}(u) \geq c_m > \bar{c} - \epsilon. \tag{3.9}$$

On the other hand, by (3.7) and (3.8), we have

$$\eta(A \setminus N_\delta(K_{\bar{\epsilon}})) \subset \eta(\bar{J}^{\bar{c}+\epsilon} \setminus N_\delta(K_{\bar{\epsilon}})) \subset \bar{J}^{\bar{c}-\epsilon}, \quad (3.10)$$

which contradicts with (3.9). Hence $c_m \rightarrow 0$ as $m \rightarrow \infty$. \square

Now, we conclude with the proof of our main result.

Proof of Theorem 1.1. By the Lemma 3.1, there is $\mu^* > 0$ such that $J(u) = \bar{J}(u)$ if $0 < \mu < \mu^*$ and $\bar{J}(u) < 0$. Then by Lemma 3.1, Lemma 3.2, Lemma 3.3 and (3.6), we can see that all the assumptions of Proposition 3.2 are satisfied for \bar{J} . Therefore, the conclusions of Theorem 1.1 are obtained. \square

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