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THE METHOD OF LOWER AND UPPER SOLUTIONS FOR DAMPED ELASTIC SYSTEMS IN BANACH SPACES*

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Abstract In this paper, we are concerned with the initial value problem of a class of damped elastic systems in an order Banach spaces E. By employing the method of lower and upper solutions, we discuss the existence of extremal mild solutions between lower and upper mild solutions for such problem with the associated semigroup is equicontinuous. In addition, two examples are given to illustrate our results.

Keywords Damped elastic systems, extremal mild solution, the method of lower and upper solution, measure of noncompactness.

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1. Introduction

In this article, we use a monotone iterative method in the presence of lower and upper mild solutions to discuss the existence of extremal mild solutions for the semilinear damped elastic systems in an order Banach spaces E:

$$\begin{cases} u''(t) + \rho Bu'(t) + Au(t) = f(t, u(t), Gu(t)), \quad 0 < t < a, \\ u(0) = u_0 \in D(A), \quad u'(0) = u_1 \in E, \end{cases} \tag{1.1}$$

where $A : D(A) \subset E \to E$ and $B : D(B) \subset E \to E$ are densely defined closed (possibly unbounded) linear operators on a complex Banach space E and $f \in C([0, a] \times E \times E, E)$ and

$$Gu(t) = \int_0^t K(t, s)u(s)ds$$

is a Volterra integral operator with integral kernel $K \in C(\nabla, \mathbb{R}^+), \nabla = \{(t, s) : 0 \le s \le t \le a\}$. Throughout this paper, we always assume that

$$K_0 = \sup_{t \in J} \int_0^t K(t, s)ds.$$

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In 1982, Chen and Russell [2] investigated the following linear elastic system described by the second order equation

$$u''(t) + Bu'(t) + Au(t) = 0 \tag{1.2}$$

in a Hilbert space H with inner $(\cdot, \cdot)$, where A (the elastic operator) and B (the damping operator) are positive definite selfadjoint operators in H. They reduced (1.2) to the first order equation in $H \times H$

$$\frac{d}{dt}\begin{pmatrix} A^{\frac{1}{2}}u \\ u' \end{pmatrix} = \begin{pmatrix} 0 & A^{\frac{1}{2}} \\ -A^{\frac{1}{2}} & -B \end{pmatrix}\begin{pmatrix} A^{\frac{1}{2}}u \\ u' \end{pmatrix}.$$

Let $V = D(A^{\frac{1}{2}}), \mathscr{H} = V \times H$ with the naturally induced inner products. Then, (1.2) is equivalent to the first order equation in $\mathscr{H}$

$$\frac{d}{dt}\begin{pmatrix} A^{\frac{1}{2}}u \\ u' \end{pmatrix} = \mathscr{A}_B \begin{pmatrix} A^{\frac{1}{2}}u \\ u' \end{pmatrix},$$

where

$$\mathscr{A}_B = \begin{pmatrix} 0 & I \\ -A & -B \end{pmatrix},$$

$$D(\mathscr{A}_B) = D(A) \times [D(A^{\frac{1}{2}}) \cap D(B)].$$

Chen and Russell [2] conjectured that $\mathscr{A}_B$ is the infinitesimal generator of an analytic semigroup on $\mathscr{H}$ if

$$D(A^{\frac{1}{2}}) \subset D(B)$$

and either of the following two inequalities holds for some $\beta_1, \beta_2 > 0$:

$$\beta_1(A^{\frac{1}{2}}v, v) \leq (Bv, v) \leq \beta_2(A^{\frac{1}{2}}v, v), \ v \in D(A^{\frac{1}{2}});$$

$$\beta_1(Av, v) \leq (B^2v, v) \leq \beta_2(Av, v), \ v \in D(A).$$

The complete proofs of the two conjectures were given by Huang [12, 13]. Then, other sufficient conditions for $\mathscr{A}_B$ or its closure $\overline{\mathscr{A}_B}$ to generate an analytic or differentiable semigroup on $\mathscr{H}$ were discussed in [4, 14–17, 19], by choosing B to be an operator comparable with A^α for $0 < \alpha \leq 1$, based on an explicit matrix representation of the resolvent operator of $\mathscr{A}_B$ or $\overline{\mathscr{A}_B}$.

In [7–9], Fan et al. studied the existence, the asymptotic stability of solutions and the analyticity and exponential stability of associated semigroups for the following elastic system with structural damping given by

$$\begin{cases} u''(t) + \rho Au'(t) + A^2u(t) = f(t, u(t)), \quad 0 < t < a, \\ u(0) = u_0 \in D(A), \quad u'(0) = u_1 \in E, \end{cases} \tag{1.3}$$

where $A : D(A) \subset E \to E$ is a sectorial linear operator on a complex Banach space E and $\rho > 0$ is a constants.

In [22], the authors considered nonlinear evolution equations of second order in Banach spaces

$$\begin{cases} u''(t) + \rho A u'(t) + A^2 u(t) = f(t, u(t), u_t), \quad t \in I = [0, T], \\ u(s) = \varphi(s), \quad s \le 0, \\ u'(0) + h(u) = \psi, \end{cases}$$

where u is the unknown function defined on I and taking values in E, u_t is the history state defined by $u_t : (-\infty, 0] \to E, u_t(s) = u(t+s), t \in I$. By means of the fixed point for condensing maps, they proved the existence and exponential decay of mild solutions.

In [23, 27], the authors discussed the polynomial stability of elastic systems, the discussion is based on the operator semigroups theory and some fixed point theorem. In [6], T. Diagana studied the well-posedness and existence of bounded solutions to the linear elastic systems with damping

$$\begin{cases} u''(t) + \rho B u'(t) + A u(t) = f(t), \quad t > 0, \\ u(0) = u_0 \in D(A), \quad u'(0) = u_1 \in E, \end{cases}$$

where $A : D(A) \subset E \to E$ and $B : D(B) \subset E \to E$ are densely defined closed (possibly unbounded) linear operators on a complex Banach space E and $f : \mathbb{R}_+ \to E$ is a continuous function. But the theory still remains to develop to nonlinear case.

On the other hand, the monotone iterative method based on lower and upper solutions is an effective and flexible mechanism. It yields monotone sequences of lower and upper approximate solutions that converge to the minimal and maximal solutions between the lower and upper solutions. Lately, the monotone iterative method has been extended to evolution equations in ordered Banach spaces by Li [21].

In [10], Fan and Li used a monotone iterative technique in the presence of lower and upper solutions to discuss the existence of extremal mild solutions and positive mild solutions to the initial value problem of second order semilinear evolution equations in ordered Banach space E

$$\begin{cases} u''(t) + \rho A u'(t) + A^2 u(t) = f(t, u(t)), \quad 0 < t < a, \\ u(0) = u_0 \in D(A), \quad u'(0) = u_1 \in E, \end{cases} \tag{1.4}$$

where $\rho \ge 2$ is a constant, $A : D(A) \subset E \to E$ is a closed linear operator and $-A$ generates a C_0-semigroup $T(t)(t \ge 0)$ on E, $f \in C(J, E), J = [0, a], a > 0$ is a constant.

However, motivated by the above works, ideas and methods based on the paper [6], in this paper, we give the expression of the solution of problem (1.1), which is different from the expression given in article [6]. Moreover, we obtain the existence of the minimal and maximal mild solution, and the mild solutions between the minimal and maximal mild solution of the problem (1.1) through the monotone iterative and measure of noncompactness. Our results presented in this paper is differential from [10]. First of all, the equations (1.1) we consider is different the equations (1.4) from [10]. Our research is extensive, which contains the equations (1.4). When $B = A^{\frac{1}{2}}$, equations (1.1) is transformed into equations (1.4); then our results improve and generalize many classical results [6–10].

The paper is organized as follows: In Section 2, we introduce some notations and recall some basic known results. In Section 3 we present the existence of extremal mild solutions for damped elastic systems (1.1) in order Banach space. In Section 4, we give an example to illustrate our results.

2. Preliminaries

Let E be an ordered complex Banach space with the norm $\|\cdot\|$ and partial order $\leq$, whose positive cone $P = \{x \in E : x \geq 0\}$ is normal with normal constant N. For any constant $a > 0$, denote $J = [0, a]$. Let $C(J, E)$ denote the Banach space of all continuous E-value functions on interval J with the norm $\|u\| = \max_{t\in J} \|u(t)\|$. Evidently, $C(J, E)$ is also an ordered Banach space induced by the convex cone $P' = \{u \in E | u(t) \geq 0, t \in J\}$, which is also a normal cone. The notations $D(L)$ stand for the domain of L.

Let $\alpha(\cdot)$ denote the Kuratowski measure of noncompactness of the bounded set. For the details of the definition and properties of the measure of noncompactness, see [1, 5]. For any $B \subset C(J, E)$ and $t \in J$, set $B(t) = \{u(t) : u \in B\} \subset E$. If B is bounded in $C(J, E)$, then $B(t)$ is bounded in E, and $\alpha(B(t)) \leq \alpha(B)$.

Now we introduce some basic definitions and properties about Kuratowski measure of noncompactness that will be used in sequel.

Lemma 2.1 ([20]). *Let E be a Banach space, and let $D \subset E$ be bounded. Then there exists a countable set $D_0 \subset D$, such that $\alpha(D) \leq 2\alpha(D_0)$.*

Lemma 2.2 ([28]). *Let E be a Banach space, and let $D = \{u_n\} \subset C([b_1, b_2], E)$ be a bounded and countable set for constants $-\infty < b_1 < b_2 < +\infty$. Then $\alpha(D(t))$ is Lebesgue integral on $[b_1, b_2]$, and*

$$\alpha\Big(\Big\{ \int_{b_1}^{b_2} u_n(t)dt : n \in \mathbb{N}\Big\}\Big) \leq 2 \int_{b_1}^{b_2} \alpha(D(t))dt.$$

Lemma 2.3 ([21]). *Let E be a Banach space, and let $D \subset C([b_1, b_2], E)$ be bounded and equicontinuous. Then $\alpha(D(t))$ is continuous on $[b_1, b_2]$, and*

$$\alpha(D) = \max_{t\in[b_1,b_2]} \alpha(D(t)).$$

Definition 2.1 ([18]). Let E be a Banach space, and let S be a nonempty subset of E. A continuous mapping $Q : S \to E$ is called to be strict α-set-contraction operator if there existence a constant $0 \leq k < 1$ such that, for every bounded set $\Omega \subset S$,

$$\alpha(Q(\Omega)) \leq k\alpha(\Omega).$$

Lemma 2.4 ([11]). *Let P be a normal cone of the ordered Banach space E and $v_0, w_0 \in E$ with $v_0 \leq w_0$. Suppose that $Q : [v_0, w_0] \to E$ is a nondecreasing strict α-set-contraction operator such that $v_0 \leq Qv_0$ and $Qw_0 \leq w_0$. The Q has a minimal fixed point $\underline{u}$ and a maximal fixed point $\overline{u}$ in $[v_0, w_0]$; Moreover, $v_n \to \underline{u}$ and $w_n \to \overline{u}$, where $v_n = Qv_{n-1}$ and $w_n = Qw_{n-1} (n = 1, 2, \dots)$ which satisfy*

$$v_0 \leq v_1 \leq \cdots \leq v_n \leq \cdots \leq \underline{u} \leq \overline{u} \leq \cdots \leq w_n \leq \cdots \leq w_1 \leq w_0.$$

Lemma 2.5 (Sadovskii's fixed point theorem)**.** *Let E be a Banach space and Ω_0 be a nonempty bounded convex closed set in E. If $Q : \Omega_0 \to \Omega_0$ is a condensing mapping, then Q has a fixed point in Ω_0.*

Lemma 2.6 ([26])**.** *Assume $f \in C(J, E)$ and that A is the infinitesimal generator of C_0-semigroup $(T(t))_{t\geq 0}$. Then the inhomogeneous Cauchy problem*

$$\begin{cases} u'(t) = Au(t) + f(t), & t \in J, \\ u(0) = u_0 \in D(A) \end{cases} \tag{2.1}$$

has a mild solution u given by

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds, \quad t \in J.$$

Thoughts and methods based on paper [9]. We consider the following linear damped elastic system

$$\begin{cases} u''(t) + \rho Bu'(t) + Au(t) = h(t), & t \in J, \\ u(0) = u_0 \in D(A), \quad u'(0) = u_1 \in E, \end{cases} \tag{2.2}$$

where $A : D(A) \subset E \to E$ and $B : D(B) \subset E \to E$ are densely defined closed (possibly unbounded) linear operators on a complex Banach space E and $h : J \to E$.

For the second order evolution equation

$$u''(t) + \rho Bu'(t) + Au(t) = h(t), \tag{2.3}$$

it was rewritten as

$$\Big(\frac{d}{dt} + E_1(\rho)\Big)\Big(\frac{d}{dt} + E_2(\rho)\Big)u = h(t), \ t > 0. \tag{2.4}$$

That is,

$$\frac{d^2u}{dt^2} + \Big(E_1(\rho) + E_2(\rho)\Big)\frac{du}{dt} + E_1(\rho)E_2(\rho)u = h(t). \tag{2.5}$$

It follows from (2.3) and (2.5) that

$$E_1(\rho) + E_2(\rho) = \rho B, \qquad E_1(\rho)E_2(\rho) = A. \tag{2.6}$$

By (2.6), we have

(i) if $C(\rho) = \rho^2B^2 - 4A = L^2(\rho) > 0$, then

$$\begin{aligned} E_1(\rho) &= \frac{\rho B - \sqrt{\rho^2B^2 - 4A}}{2} = \frac{\rho B - L(\rho)}{2}, \\ E_2(\rho) &= \frac{\rho B + \sqrt{\rho^2B^2 - 4A}}{2} = \frac{\rho B + L(\rho)}{2}, \end{aligned} \tag{2.7}$$

(ii) if $C(\rho) = \rho^2B^2 - 4A = L^2(\rho) = 0$, then

$$E_1(\rho) = E_2(\rho) = \frac{\rho B}{2}.$$

(iii) if $C(\rho) = \rho^2 B^2 - 4A = -L^2(\rho) < 0$, then

$$E_1(\rho) = \frac{\rho B - \sqrt{\rho^2 B^2 - 4A}}{2} = \frac{\rho B - iL(\rho)}{2},$$
$$E_2(\rho) = \frac{\rho B + \sqrt{\rho^2 B^2 - 4A}}{2} = \frac{\rho B + iL(\rho)}{2}, \tag{2.8}$$

Remark 2.1. In order to study the existence to Eq.(1.1), we will make use of the above linear operator which links both A and B: $C(\rho) = \rho^2 B^2 - 4A = L^2(\rho)$ with $D(C(\rho)) = D(B^2) \cap D(A)$. In the following discussion, we will focus on the following cases: $C(\rho) = L^2(\rho) > 0$ and $C(\rho) = L^2(\rho) = 0$, for densely closed linear operator $L(\rho) : D(L(\rho)) \subset E \to E$. Obviously, $C(\rho) = 0$ corresponds to the case studied in papers [7, 8].

Lemma 2.7. *Assume that there exists a densely defined closed linear operator $L(\rho)$: $D(L(\rho)) \subset E \to E$ such that $u_0 \in D(L(\rho)) \cap D(B)$ and $C(\rho) = \rho^2 B^2 - 4A = L^2(\rho)$ and $BL(\rho) = L(\rho)B$. Let $h \in C(J, E)$, $-E_1(\rho)$ and $-E_2(\rho)$ are respectively the infinitesimal generators of C_0-semigroups $T_1(t)(t \geq 0)$ and $T_2(t)(t \geq 0)$. Then Eq.(2.2) has a unique solution given by*

$$\begin{aligned} u(t) =& T_2(t)u_0 + \int_0^t T_2(t-s)T_1(s)(u_1 + E_2(\rho)u_0)ds \\ &+ \int_0^t \int_0^s T_2(t-s)T_1(s-\tau)h(\tau)d\tau ds, \end{aligned}$$

where $E_1(\rho)$, $E_2(\rho)$ were defined in (2.7).

Proof. Let

$$\frac{du}{dt} + E_2(\rho)u = v(t), \quad t \in J,$$

which means

$$v_0 := v(0) = u_1 + E_2(\rho)u_0.$$

So we reduce the linear elastic system (2.2) to the following two abstract Cauchy problems in Banach space E:

$$\begin{cases} \frac{dv}{dt} + E_1(\rho)v = h(t), \quad t \in J, \\ v(0) = v_0, \end{cases} \tag{2.9}$$

and

$$\begin{cases} \frac{du}{dt} + E_2(\rho)u = v(t), \quad t \in J, \\ u(0) = u_0. \end{cases} \tag{2.10}$$

It is clear that (2.9) and (2.10) are linear inhomogeneous initial value problems for $-E_1(\rho)$ and $-E_2(\rho)$ respectively. Thus, by operator semigroups theory [26],$-E_1(\rho)$ and $-E_2(\rho)$ are infinitesimal generators of C_0-semigroups, which implies initial value problems (2.9) and (2.10) are well-posed.

Thus using Lemma 2.6, if $h \in C(J, E)$, the Eq. (2.9) has a mild solution v given by

$$v(t) = T_1(t)v_0 + \int_0^t T_1(t-s)h(s)ds. \tag{2.11}$$

Similarly, if $v \in C(J, E)$, then the mild solution of the Eq. (2.10) expressed by

$$u(t) = T_2(t)u_0 + \int_0^t T_2(t-s)v(s)ds. \tag{2.12}$$

Substituting (2.11) into (2.12), we get

$$\begin{aligned} u(t) =& T_2(t)u_0 + \int_0^t T_2(t-s)T_1(s)(u_1 + E_2(\rho)u_0)ds \\ &+ \int_0^t \int_0^s T_2(t-s)T_1(s-\tau)h(\tau)d\tau ds. \end{aligned}$$

□

Based on the above discussion, motivated by the definition of mild solutions in [9], we give the definition of mild solution of the problem (1.1) as follows.

Definition 2.2. Let $f \in C(J \times E \times E, E)$, $-E_1(\rho)$ and $-E_2(\rho)$ are respectively the infinitesimal generators of C_0-semigroups $T_1(t)(t \geq 0)$ and $T_2(t)(t \geq 0)$. A function $u : J \to E$ is said to be a mild solution of the problem (1.1) if $u(0) = u_0, u'(0) = u_1$ and

$$\begin{aligned} u(t) =& T_2(t)u_0 + \int_0^t T_2(t-s)T_1(s)(u_1 + E_2(\rho)u_0)ds \\ &+ \int_0^t \int_0^s T_2(t-s)T_1(s-\tau)f(\tau, u(\tau), Gu(\tau))d\tau ds, \end{aligned}$$

where $E_1(\rho)$, $E_2(\rho)$ were defined in (2.7).

Remark 2.2. In the case $C(\rho) = -L^2(\rho)$, the expression of mild solution for the problem (1.1) and the conclusion of Theorem 3.1 are correct and meaningful in complex Banach spaces. For more detail to see [6].

Definition 2.3. If a function $v_0 \in C^2(J, E) \cap C(J, E)$ satisfy

$$\begin{aligned} v_0''(t) + \rho B v_0'(t) + A v_0(t) \leq f(t, v_0(t), Gv_0(t)), \quad t \in J, \\ v_0(0) \leq v_0, \quad v_0'(0) \leq v_1, \end{aligned} \tag{2.13}$$

we call it a lower solution of the problem (1.1); if all the inequalities of (2.13) are reversed, we call it an upper solution of the problem (1.1).

Definition 2.4. If a function $\mu \in C(J, E)$ satisfy

$$\begin{aligned} \mu(t) \leq& T_2(t)\mu_0 + \int_0^t T_2(t-s)T_1(s)(\mu_1 + E_2(\rho)\mu_0)ds \\ &+ \int_0^t \int_0^s T_2(t-s)T_1(s-\tau)f(\tau, \mu(\tau), G\mu(\tau))d\tau ds, \end{aligned} \tag{2.14}$$

we call it a lower mild solution of the problem (1.1); if the inequalities of (2.14) are reversed, we call it an upper mild solution of the problem (1.1), where $E_1(\rho)$, $E_2(\rho)$ were defined in (2.7).

Definition 2.5. A C_0-semigroup $T(t)(t \geq 0)$ in E is said to be equicontinuous if $T(t)$ is continuous by operator norm for every $t > 0$.

Definition 2.6. A C_0-semigroup $T(t)(t \geq 0)$ in E is called to be positive, if order inequality $T(t)x \geq \theta$ holds for each $x \geq \theta, x \in E$ and $t \geq 0$.

3. Main results

For $v, w \in C(J, E)$ with $v \le w$, we use $[v, w]$ to denote the order interval $\{u \in C(J, E) | v \le u \le w\}$ in $C(J, E)$, and $[v(t), w(t)]$ to denote the order interval $\{u \in E | v(t) \le u(t) \le w(t), t \in J\}$ in E. Let $\mathcal{L}(E)$ be the Banach space of all linear and bounded operators on E. Since $T_1(t)(t \ge 0)$ and $T_2(t)(t \ge 0)$ are C_0-semigroup on E, then there exist $M_1 \ge 1$ and $M_2 \ge 1$ such that

$$M_1 = \sup_{t \in J} \|T_1(t)\|_{\mathcal{L}(E)}, \qquad M_2 = \sup_{t \in J} \|T_2(t)\|_{\mathcal{L}(E)}.$$

For the convenience of discussion, we define the mapping $Q : C(J, E) \to C(J, E)$ by

$$\begin{aligned} Q(u)(t) =& T_2(t)u_0 + \int_0^t T_2(t-s)T_1(s)(u_1 + E_2(\rho)u_0)ds \\ &+ \int_0^t \int_0^s T_2(t-s)T_1(s-\tau)f(\tau, u(\tau), Gu(\tau))d\tau ds. \end{aligned} \tag{3.1}$$

Theorem 3.1. *Let E be an ordered Banach space, whose positive cone P is normal, there exists a densely defined closed linear operator $L(\rho) : D(L(\rho)) \subset E \to E$ such that $u_0 \in D(L(\rho)) \cap D(B)$ and $C(\rho) = \rho^2 B^2 - 4A = L^2(\rho)$, $BL(\rho) = L(\rho)B$ and $-E_1(\rho)$ and $-E_2(\rho)$ generate positive and equicontinuous C_0-semigroups $T_1(t)(t \ge 0)$ and $T_2(t)(t \ge 0)$ respectively, $f \in C(J \times E \times E, E)$. If the problem (1.1) has a lower mild solution $v_0 \in (J, E)$ and an upper mild solution $w_0 \in (J, E)$ with $v_0 \le w_0$. Suppose that the following conditions are satisfied:*

(H1)

$$f(t, u_2, v_2) - f(t, u_1, v_1) \ge \theta,$$

for any $t \in J$, and $v_0(t) \le u_1 \le u_2 \le w_0(t), Gv_0(t) \le v_1 \le v_2 \le Gw_0(t)$.

(H2) There exists a constant $0 < L_1 < \frac{1}{4M_1 M_2 a^2 (1+2K_0)}$, such that

$$\alpha(\{f(t, u_n, v_n)\}) \le L_1(\alpha(\{u_n\} + \alpha(\{v_n\}),$$

for $\forall t \in J$, and equicontinuous countable and increasing or decreasing monotone set $\{u_n\} \subset [v_0(t), w_0(t)], \{v_n\} \subset [Gv_0(t), Gw_0(t)]$.

Then for every $u_1 \in E$, the problem (1.1) has a minimal mild solution $\underline{u}$ and a maximal mild solution $\overline{u}$ in $[v_0, w_0]$; moreover,

$$v_n(t) \to \underline{u}(t), \qquad w_n(t) \to \overline{u}(t), \quad (n \to +\infty) \text{ uniformly for } t \in J,$$

where $v_n(t) = Qv_{n-1}(t), w_n(t) = Qw_{n-1}(t)$ which satisfy

$$v_0(t) \le v_1(t) \le \cdots \le v_n(t) \le \cdots \le \underline{u}(t) \le \overline{u}(t) \le \cdots \le w_n(t) \le \cdots \le w_1(t) \le w_0(t), \ \forall t \in J.$$

Proof. Define the mapping $Q : [v_0, w_0] \to C(J, E)$ is given by (3.1). By Definition 2.2, it is obvious that the mild solution of the problem (1.1) is equivalent to the fixed point of Q.

First, we prove that Q is continuous in $C(J, E)$. To this end, let $u_n \in C(J, E)$ be a sequence such that $u_n \to u$ in $C(J, E)$. By the continuity of nonlinear term f with respect to the second variable, for each $s \in J$, we have

$$f(s, u_n(s), Gu_n(s)) \to f(s, u(s), Gu(s)), \quad n \to \infty, \tag{3.2}$$

that is for all $\epsilon > 0$, there exists N, when $n > N$, we have

$$\|f(s, u_n(s), Gu_n(s)) - f(s, u(s), Gu(s))\| \leq \epsilon. \tag{3.3}$$

Now, we have

$$\begin{aligned}\|Q(u_n)(t) - Q(u)(t)\| &\leq M_1 M_2 \int_0^t \int_0^s (\|f(\tau, u_n(\tau), Gu_n(\tau)) - f(\tau, u(\tau), Gu(\tau))\| \\ &\leq M_1 M_2 a^2 \|f(\tau, u_n(\tau), G_n(\tau)) - f(\tau, u(\tau), Gu(\tau))\|.\end{aligned}$$

So, when $n > N$, we have

$$\|Q(u_n) - Q(u)\| \leq M_1 M_2 a^2 \epsilon,$$

which means that $Q : [v_0, w_0] \to [v_0, w_0]$ is continuous.

Next, we show $Q : [v_0, w_0] \to [v_0, w_0]$ is a increase operator, and $v_0 \leq Q(v_0)$, $Q(w_0) \leq w_0$. In fact, for $\forall t \in J, v_0(t) \leq u_1(t) \leq u_2(t) \leq w_0$, from the assumptions (H1), we have

$$f(t, u_1(t), Gu_1(t)) \leq f(t, u_2(t), Gu_2(t)).$$

By the positivity of operators $T_1(t)$ and $T_2(t)$, thus

$$\begin{aligned}&\int_0^t \int_0^s T_2(t-s) T_1(s-\tau) f(\tau, u_1(\tau), Gu_1(\tau)) d\tau ds \\ \leq &\int_0^t \int_0^s T_2(t-s) T_1(s-\tau) f(\tau, u_2(\tau), Gu_2(\tau)) d\tau ds.\end{aligned}$$

Hence from (3.1) we see that $Q(u_1) \leq Q(u_2)$, which means that Q is a increase operator.

By the definition of lower mild solution and upper mild solution, we can conclude that $v_0 \leq Q(v_0)$ and $Q(w_0) \leq w_0$, respectively. So, $Q : [v_0, w_0] \to [v_0, w_0]$ is continuous monotone operator.

In the following, we demonstrate that the operator $Q : [v_0, w_0] \to [v_0, w_0]$ is equicontinuous. For any $u \in [v_0, w_0]$ and $0 < t' < t'' \leq a$, we obtain that

$$\begin{aligned}&\|(Q_2 u)(t'') - Q_2 u)(t')\| \leq \|T_2(t'') u_0 - T_2(t') u_0\| \\ &+ \Big\| \int_0^{t'} \Big[T_2(t''-s) - T_2(t'-s)\Big] T_1(s)(u_1 + E_2(\rho) u_0) ds \Big\| \\ &+ \Big\| \int_{t'}^{t''} T_2(t''-s) T_1(s)(u_1 + E_2(\rho) u_0) ds \Big\| + \Big\| \int_0^{t'} \int_0^s [T_2(t''-s) - T_2(t'-s)] \\ &\times T_1(s-\tau) f(\tau, u(\tau), Gu(\tau)) d\tau ds \Big\| \\ &+ \Big\| \int_{t'}^{t''} \int_0^s T_2(t''-s) \times T_1(s-\tau) f(\tau, u(\tau), Gu(\tau)) d\tau ds \Big\| \\ :=& I_1 + I_2 + I_3 + I_4 + I_5,\end{aligned}$$

where

$$I_1 = \|T_2(t'')u_0 - T_2(t')u_0\|,$$
$$I_2 = \Big\| \int_0^{t'} \Big[T_2(t''-s) - T_2(t'-s)\Big] T_1(s)(u_1 + E_2(\rho)u_0)ds \Big\|,$$
$$I_3 = \Big\| \int_{t'}^{t''} T_2(t''-s)T_1(s)(u_1 + E_2(\rho)u_0)ds \Big\|,$$
$$I_4 = \Big\| \int_0^{t'} \int_0^{s} [T_2(t''-s) - T_2(t'-s)] \times T_1(s-\tau)f(\tau, u(\tau), Gu(\tau))d\tau ds \Big\|$$
$$I_5 = \Big\| \int_{t'}^{t''} \int_0^{s} T_2(t''-s) \times T_1(s-\tau)f(\tau, u(\tau), Gu(\tau))d\tau ds \Big\|.$$

In fact, we only need to check I_1, I_2, I_3, I_4 and I_5 tend to 0 independently of $u \in [v_0, w_0]$ when $t'' - t' \to 0$.

Since $T_1(t)(t \geq 0)$ is a equicontinuous C_0 semigroup, thus, $T_1(t)u_0$ is uniformly continuous on J and thus $\lim_{t'' \to t'} I_1 = 0$.

Since $T_2(t)(t \geq 0)$ is a equicontinuous C_0 semigroup, for I_2, we have

$$\begin{aligned} I_2 &\leq \int_0^{t'} \Big\|T_2(t''-s) - T_2(t'-s)\Big\|_{\mathcal{L}(E)} \times \|T_1(s)\|_{\mathcal{L}(E)} \|u_1 + E_2(\rho)u_0\| ds \\ &\leq M_1 \|u_1 + E_2(\rho)u_0\| \int_0^{t'} \Big\|T_2(t''-s) - T_2(t'-s)\Big\|_{\mathcal{L}(E)} ds, \end{aligned}$$

for $t \in J$ allows us to conclude that $\lim_{t'' \to t'} I_2 = 0$.

By the normality of the cone P, there exists $\overline{M} > 0$ such that

$$\|f(t, u(t), Gu(t))\| \leq \overline{M}, \quad u \in [v_0, w_0].$$

For I_4, we have

$$\begin{aligned} I_4 \leq& \int_0^{t'} \int_0^{s} \Big\|T_2(t''-s) - T_2(t'-s)\Big\|_{\mathcal{L}(E)} \times \|T_1(s-\tau)\|_{\mathcal{L}(E)} \|f(\tau, u(\tau), Gu(\tau))\| d\tau ds \\ \leq& M_1 a \overline{M} \times \int_0^{t'} \Big\|T_2(t''-s) - T_2(t'-s)\Big\|_{\mathcal{L}(E)} ds. \end{aligned}$$

Consequent, $\lim_{t'' \to t'} I_4 = 0$.

For I_3, I_5, we have

$$I_3 \leq M_1 M_2 \|u_1 + E_2(\rho)u_0\| \cdot |t'' - t'|,$$
$$I_5 \leq M_1 M_2 \overline{M} |t'' - t'|.$$

Hence, $\lim_{t'' \to t'} I_3 = \lim_{t'' \to t'} I_5 = 0$.

As a result, $\|(Qu)(t'') - (Qu)(t')\|$ tends to 0 independently of $u \in \Omega_R$ as $t'' - t' \to 0$, which means that $Q : [v_0, w_0] \to [v_0, w_0]$ is equicontinuous.

Now, we show that the operator Q is a α-set-contractive. For any bounded $D \subset [v_0, w_0]$, $Q(D)$ is bounded and equicontinuous. Therefore, by Lemma 2.1, we know that there exists a countable set $D_0 = \{u_n\} \subset D$, such that

$$\alpha(Q(D)) \leq 2\alpha(Q(D_0)). \tag{3.4}$$

Since $Q(D_0) \subset Q(D)$ is bounded and equicontinuous, we know from Lemma 2.3 that

$$\alpha(Q(D_0)) = \max_{t\in J} \alpha(Q(D_0)(t)). \tag{3.5}$$

For $t \in J$, by Lemma 2.2, we get

$$\alpha(G(D_0)(t)) = \alpha\Big(\Big\{ \int_0^t K(t,s)u_n(s)ds : n \in \mathbb{N}\Big\}\Big) \leq 2K_0\alpha(D_0).$$

For every $t \in J$, by Lemma 2.2, the assumption (H3) and (3.2), we have

$$\begin{aligned}
\alpha(Q(D_0)(t)) = \alpha\Big(\Big\{&T_2(t)u_0 + \int_0^t T_2(t-s)T_1(s)(u_1 + E_2(\rho)u_0)ds \\
&+ \int_0^t \int_0^s T_2(t-s)T_1(s-\tau)f(\tau, u_n(\tau), Gu_n(\tau))d\tau ds\Big\}\Big) \\
&\leq 2M_1M_2a \int_0^t \alpha(\{f(\tau, u_n(\tau), Gu_n(\tau))\})d\tau \\
&\leq 2M_1M_2a \int_0^t L_1[\alpha(D_0(s)) + \alpha(G(D_0)(s))]ds \\
&\leq 2M_1M_2a^2(L_1 + 2L_1K_0)\alpha(D).
\end{aligned} \tag{3.6}$$

Therefore, from (3.4) and (3.6) we know that

$$\alpha(Q(D)) \leq \gamma\alpha(D).$$

where $\gamma = 4M_1M_2a^2(L_1 + 2L_1K_0) < 1$.

Therefore, $Q : [v_0, w_0] \to [v_0, w_0]$ is strict set contraction operator. Hence, our conclusion follows from Lemma 2.4. □

If we replace the assumptions (H2) by the following assumptions:

$(H3)$ There exist a constant $L_1 > 0$ such that for all $t \in J$,

$$\alpha(\{f(t, u_n, v_n)\}) \leq L_1(\alpha(\{u_n\}) + \alpha(\{v_n\})),$$

and increasing or decreasing sequences $\{u_n\} \subset [v_0(t), w_0(t)]$, $\{v_n\} \subset [Gv_0(t), Gw_0(t)]$.

Theorem 3.2. *Let E be an ordered Banach space, whose positive cone P is normal, there exists a densely defined closed linear operator $L(\rho) : D(L(\rho)) \subset E \to E$ such that $u_0 \in D(L(\rho)) \cap D(B)$ and $C(\rho) = \rho^2B^2 - 4A = L^2(\rho)$, $BL(\rho) = L(\rho)B$ and $-E_1(\rho)$ and $-E_2(\rho)$ generate positive and equicontinuous C_0-semigroups $T_1(t)(t \geq 0)$ and $T_2(t)(t \geq 0)$ respectively, $f \in C(J \times E \times E, E)$. If the problem (1.1) has a lower mild solution $v_0 \in (J, E)$ and an upper mild solution $w_0 \in (J, E)$ with $v_0 \leq w_0$. Suppose that the conditions (H1) and (H3) are satisfied, then for every $u_1 \in E$, the problem (1.1) has minimal and maximal mild solutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0 respectively.*

Proof. By the proof of Theorem 3.1, we known that the operator $Q : [v_0, w_0] \to [v_0, w_0]$ is continuous increase operator.

Now, we define two sequences $\{v_n\}$ and $\{w_n\}$ in $[v_0, w_0]$ by the iterative scheme

$$v_n = Q(v_{n-1}), \quad w_n = Q(w_{n-1}), \quad n = 1, 2, \ldots. \tag{3.7}$$

Then from the monotonicity of Q, it follows that

$$v_0 \le v_1 \le v_2 \le \cdots \le v_n \le \cdots \le w_n \le \cdots \le w_2 \le w_1 \le w_0. \tag{3.8}$$

In what follows we prove that $\{v_n\}$ and $\{w_n\}$ are convergent in J.

For convenience, let $B = \{v_n : n \in \mathbb{N}\}$ and $B_0 = \{v_{n-1} : n \in \mathbb{N}\}$. Then $B = Q(B_0)$. From $B_0 = B \cup \{v_0\}$ it follow that $\alpha(B_0(t)) = \alpha(B(t))$ for $t \in J$. Let $\varphi(t) := \alpha(B(t)), t \in J$.

For $t \in J$, by Lemma 2.2, we get

$$\begin{aligned}\int_0^t \alpha(G(B_0)(t)) &= \int_0^t \alpha\Big(\Big\{\int_0^t K(t,s)v_{n-1}(s)ds : n \in \mathbb{N}\Big\}\Big)\\ &\le 2K_0 \int_0^t \alpha(B_0(s))ds\\ &= 2K_0 \int_0^t \varphi(s)ds,\end{aligned}$$

therefore

$$\int_0^t \alpha(G(B_0)(s))ds \le 2K_0 \int_0^t \varphi(s)ds.$$

Thus, by Lemma 2.2, the assumption (H3) and (3.2), we have

$$\begin{aligned}\varphi(t) &= \alpha(B(t)) = \alpha(Q(B_0)(t))\\ &= \alpha\Big(\Big\{T_2(t)u_0 + \int_0^t T_2(t-s)T_1(s)(u_1 + E_2(\rho)u_0)ds\\ &\quad + \int_0^t \int_0^s T_2(t-s)T_1(s-\tau)f(\tau, v_{n-1}(\tau), Gv_{n-1}(\tau))d\tau ds\Big\}\Big)\\ &\le 2M_1M_2a \int_0^t \alpha\Big(\Big\{f(\tau, v_{n-1}(\tau), Gv_{n-1}(\tau))\Big\}\Big)d\tau\\ &\le 2M_1M_2L_1a \int_0^t (\alpha(B_0(s)) + \alpha(G(B_0)(s)))ds\\ &\le 4M_1M_2L_1a(1+2K_0) \int_0^t \varphi(s)ds.\end{aligned}$$

Hence by the Gronwall's inequality, $\varphi(t) = 0$, a.e. $t \in J$. So $\int_0^t \varphi(s)ds \equiv 0$, by the above inequality, $\varphi(t) \le 0$, combing this with the property of noncompactness, $\varphi(t) \equiv 0, t \in J$.

Hence, for any $t \in J, \{v_n(t)\}$ is precompact, and $\{v_n(t)\}, \{w_n(t)\}$ has a convergent subsequence. Combing this with the monotonicity (3.8), we easily prove that $\{v_n(t)\}$ itself is convergent, i.e., $\lim_{n\to\infty} v_n(t) = \underline{u}(t)$, $t \in J$. Similarly, $\lim_{n\to\infty} w_n(t) = \overline{u}(t)$, $t \in J$.

It follows from (3.7) and the Lebesgue dominated convergence theorem that and

$$\underline{u} = Q\underline{u}, \overline{u} = Q\overline{u}.$$

Combing this with monotonicity (3.8), we see that $v_0 \le \underline{u} \le \overline{u} \le w_0$. By the monotonicity of Q, it is easy to see that $\underline{u}$ and $\overline{u}$ are the minimal and maximal fixed points of Q in $[v_0, w_0]$. Therefore, $\underline{u}$ and $\overline{u}$ are the minimal and maximal mild solutions of the problem (1.1) in $[v_0, w_0]$, and $\underline{u}$ and $\overline{u}$ can be obtained by the iterative scheme (3.7) starting from v_0 and w_0, respectively. □

Now, we discuss the existence of the mild solution to the problem (1.1) between the minimal and maximal mild solutions $\underline{u}$ and $\overline{u}$. If we replace the assumptions (H3) by the following assumptions:

$(H4)$ There exists a constant $L_1 > 0$ such that

$$\alpha(f, D_1, D_2) \leq L_1(\alpha(D_1) + \alpha(D_2)),$$

for any $t \in J$, where $D_1 = \{v_n\} \subset [v_0(t), w_0(t)]$ and $D_2 = \{v_n\} \subset [Gv_0(t), Gw_0(t)]$.

We will have the following existence result.

Theorem 3.3. *Let E be an ordered Banach space, whose positive cone P is normal, there exists a densely defined closed linear operator $L(\rho) : D(L(\rho)) \subset E \to E$ such that $u_0 \in D(L(\rho)) \cap D(B)$ and $C(\rho) = \rho^2 B^2 - 4A = L^2(\rho)$, $BL(\rho) = L(\rho)B$ and $-E_1(\rho)$ and $-E_2(\rho)$ generate positive equicontinuous C_0-semigroups $T_1(t)(t \geq 0)$ and $T_2(t)(t \geq 0)$ respectively, $f \in C(J \times E \times E, E)$. If the problem (1.1) has a lower mild solution $v_0 \in (J, E)$ and an upper mild solution $w_0 \in (J, E)$ with $v_0 \leq w_0$ such that assumptions (H1) and $(H4)$ hold, the problem (1.1) has a minimal mild solution $\underline{u}$ and a maximal mild solution $\overline{u}$ between v_0 and w_0, and at least has one mild solution between $\underline{u}$ and $\overline{u}$.*

Proof. We can easily see that $(H4) \Rightarrow$ (H3). Hence, by Theorem 3.2, the problem (1.1) has a minimal mild solution $\underline{u}$ and a maximal mild solution $\overline{u}$ between v_0 and w_0. Next, we prove the existence of the mild solution of the equation (1.1) between $\underline{u}$ and $\overline{u}$. Clearly, $Q : [v_0, w_0] \to [v_0, w_0]$ is continuous and the mild solution of the problem (1.1) is equivalent to the fixed point of operator Q . For any bounded $D \subset [v_0, w_0]$, by Lemma 2.1, we know that there exists a countable set $D_0 = \{u_n\} \subset D$, such that

$$\alpha(Q(D)) \leq 2\alpha(Q(D_0)). \tag{3.9}$$

Since $Q(D_0) \subset Q(D)$ is bounded and equicontinuous, we know from Lemma 2.3 that

$$\alpha(Q(D_0)) = \max_{t \in J} \alpha(Q(D_0)(t)). \tag{3.10}$$

For every $t \in J$, by Lemma 2.2, the assumption (H4) and (3.9), we have

$$\begin{aligned}\alpha(Q(D_0)(t)) = \alpha\Big(\Big\{&T_2(t)u_0 + \int_0^t T_2(t-s)T_1(s)(u_1 + E_2(\rho)u_0)ds \\ &+ \int_0^t \int_0^s T_2(t-s)T_1(s-\tau)[f(\tau, u_n(\tau), Gu_n(\tau))]d\tau ds\Big\}\Big) \\ &\leq 2M_1M_2a \int_0^t \alpha(\{f(\tau, u_n(\tau), Gu_n(\tau))\})d\tau \\ &\leq 2M_1M_2a \int_0^t L_1(\alpha(D_0(s)) + \alpha(G(D_0)(s)))ds \\ &\leq 2M_1M_2a^2(L_1 + 2L_1K_0)\alpha(D).\end{aligned} \tag{3.11}$$

Therefore, from (3.9) and (3.11) we know that

$$\alpha(Q(D)) \leq 4M_1M_2a^2(L_1 + 2L_1K_0)\alpha(D).$$

(i) $[2M_1M_2a^2(L_1+2L_1K_0)]<1$, then the operator $Q:[v_0,w_0]\to[v_0,w_0]$ is a condensing mapping. It follows from Lemma 2.5 that Q has at least one fixed point u in $[v_0,w_0]$, so u is the mild solution of the problem (1.1) in $[v_0,w_0]$.

(ii) If $[2M_1M_2a^2(L_1+2L_1K_0)]\geq 1$. Divide $J=[0,a]$ into n equal parts, let $\Delta_n: 0=t'_0<t'_1<\cdots<t'_n=a$, such that

$$[4M_1M_2(L_1+2L_1K_0)\|\Delta_n\|^2]<1. \tag{3.12}$$

By (i) and (3.12), the problem (1.1) has mild solution $u_1(t)$ in $[0,t'_1]$; Again by (i) and (3.10), if Eq.(1.1) with $u(t'_1)=u_1(t'_1)$ as initial value, then it has mild solution $u_2(t)$ in $[t'_1,t'_2]$ and satisfies $u_2(t'_1)=u_1(t'_1)$. Thus, the mild solution of the equation continuously extend from $[0,t'_1]$ to $[0,t'_2]$; Continuing such a process, the mild solution of the equation can be continuously extended to J. So, we obtain a mild solution $u\in C(J,E)$ of the problem (1.1), which satisfies $u(t)=u_i(t), t'_{i-1}\leq t\leq t'_i, i=1,2,\ldots,n$.

Finally, since $u=Qu, v_0\leq u\leq w_0$, by the monotonicity of Q

$$v_1=Q(v_0)\leq Q(u)\leq Q(w_0)=w_1.$$

Similarly, $v_2\leq u\leq w_2$, in general, $v_n\leq u\leq w_n$, letting $n\to\infty$, we get $\underline{u}\leq u\leq\overline{u}$. Therefore, the problem (1.1) at least has one mild solution between $\underline{u}$ and $\overline{u}$. □

Remark 3.1. The analytic semigroup and differentiable semigroup are equicontinuous semigroup [26]. In the application of partial differential equations, such as parabolic and strongly damped wave equations, the corresponding solution semigroup are analytic semigroup. Therefore, Theorem 3.2 and Theorem 3.3 have some broad applicability.

4. Examples

Let $\Omega\subset\mathbb{R}^N$ be an open bounded set with sufficiently smooth boundary $\partial\Omega$ and let $E=L^p(\Omega)$. Then E is a Banach space equipped with the L^p-norm $\|\cdot\|_p$.

Example 4.1. Let $p=2$. we consider the following damping elastic system

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2}-2\gamma\Delta\frac{\partial u(t,x)}{\partial t}+\Delta^2 u(t,x)=\frac{1}{10}\sin u(t,x)+\frac{1}{5}\int_0^t(t-s)u(s,x)ds, & (t,x)\in J\times\Omega,\\ \Delta u(t,x)=u(t,x)=0, \quad (t,x)\in J\times\partial\Omega, & \\ u(0,x)=u_0(x), \quad \frac{\partial}{\partial t}u(0,x)=u_1(x), \quad x\in\Omega, & \end{cases} \tag{4.1}$$

where $\gamma=\rho\geq 1$ is constant, Δ stands for the Laplace operator in the space variable x, $J=[0,1]$, we defined the linear operators A and B in E by

$$Au=\Delta^2u, \quad u\in D(A)=D(\Delta^2)=\{u\in H^4(\Omega):\Delta u=u=0 \text{ on } \partial\Omega\},$$

$$Bu=-2\Delta u, \quad u\in D(B)=H_0^1(\Omega)\cap H_0^2(\Omega).$$

Clearly, $C(\rho)=\rho^2B^2-4A=4\Delta^2(\rho^2-1)=L^2$, where $L=2\Delta(\rho^2-1)^{\frac{1}{2}}$. It is clear that $BL=LB$. Further,

$$E_1(\rho)=-(\rho+(\rho^2-1)^{\frac{1}{2}})\Delta=-\sigma_1\Delta, \quad E_2(\rho)=-(\rho-(\rho^2-1)^{\frac{1}{2}})\Delta=-\sigma_2\Delta, \tag{4.2}$$

where $\sigma_1 = (\rho + (\rho^2 - 1)^{\frac{1}{2}}), \sigma_2 = (\rho - (\rho^2 - 1)^{\frac{1}{2}})$, $E_1(\rho)$ and $E_2(\rho)$ are invertible bounded linear operator on $L^2(\Omega)$ for all $\rho > 0$.

Since Δ generates an operator semigroup $\{T(t)\}_{t\geq 0}$ which is compact, analytic, and uniformly bounded. By the maximum principle, we can find that $\{T(t)\}_{t\geq 0}$ is a positive semigroup in E (see [21, 24, 26]). Furthermore, for any $\rho \geq 1$, (4.2), yield $\sigma_1 > 0$, $\sigma_2 > 0$. Thus, by operator semigroups theory [26], $-E_1(\rho) = \sigma_1\Delta$ and $-E_2(\rho) = \sigma_2\Delta$ are the infinitesimal generator of equicontinuous C_0-semigroup $T_1(t)_{t\geq 0}$ and $T_2(t)_{t\geq 0}$ on $L^2(\Omega)$, respectively. It follow that

$$T_1(t) = T(\sigma_1 t), \quad T_2(t) = T(\sigma_2 t), \quad t \geq 0,$$

which is exponential stable, i.e,

$$\|T_1(t)\| \leq e^{-\lambda_1\sigma_1 t}, \quad \|T_2(t)\| \leq e^{-\lambda_1\sigma_2 t}$$

with λ_1 being the first eigenvalue of Δ.

Let $u(t) = u(t, \cdot)$, $f(t, u(t), Gu(t)) = \frac{1}{10}\sin u(t, \cdot) + \frac{1}{5}\int_0^t (t-s)u(s, \cdot)ds$, then the problem (4.1) can be reformulated as the following abstract second order evolution equation in E

$$\begin{cases} u''(t) + \rho Bu'(t) + Au(t) = f(t, u(t), Gu(t)), & t \in J, \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \tag{4.3}$$

In order to solve the problem (4.1), we also need the following assumptions:

(1) $u_0 \in D(L) \cap D(B),\ u_1 \in L^2(\Omega)$.

(2) The partial derivative $f'_u(t, x, u)$ is continuous.

Theorem 4.1. *If the assumptions (1) and (2) are satisfied, then Problem (4.1) has a mild solution $u \in C(J, L^2(\Omega))$.*

Proof. Since $f(t, x, u(t, x), Gu(t, x)) = \frac{1}{10}\sin u(t, x) + \frac{1}{5}\int_0^t (t-s)u(s, x)ds$ is continuous on $[0, 1] \times [0, +\infty) \times E \times E$ and satisfying

$$|f'_u(t, x, u)| = \frac{1}{10}|\cos u(t, x)| + \frac{1}{10}|u(t, x)|, \quad (t, x, u) \in [0, 1] \times [0, +\infty) \times E; \tag{4.4}$$

$$f(t, x, 0, 0) = \sin 0 + 0 = 0, \quad (t, x) \in [0, 1] \times [0, +\infty).$$

From (4.4), for $u, v \in E$, we have

$$\|f(t, x, u, v)\| \leq \frac{1}{10}\|u\| + \frac{1}{10}\|v\|, \quad (t, x) \in [0, 1] \times [0, +\infty),$$

$$\alpha(f(t, D_1, D_2)) \leq \frac{1}{10}\alpha(D_1) + \frac{1}{10}\alpha(D_2), \quad t \in J,$$

$$K_0 = \sup_{t\in J}\int_0^t (t-s)ds = \frac{1}{2}.$$

Now take $M_1 = M_2 = 1$, we calculate

$$0 < L_1 < \frac{1}{4a^2 M_1 M_2(1 + 2K_0)} = \frac{1}{8}.$$

From all the assumptions, it is easily seen that the conditions in Theorem 3.1 are satisfied. Hence, by Theorem 3.1, the problem (4.1) has a mild solution $u \in C(J, E)$, which means u is a mild solution for the problem (1.1). $\square$

Example 4.2. Let $p \in [2, \infty)$. Consider the following damping elastic system

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + 2\rho\Delta \frac{\partial u(t,x)}{\partial t} + \Delta u(t,x) = f(t,x,u(t,x),Gu(t,x)), & (t,x) \in J \times \Omega, \\ u(t,x) = 0, \quad (t,x) \in J \times \partial\Omega, \\ u(0,x) = u_0(x), \quad \frac{\partial}{\partial t} u(0,x) = u_1(x), \quad x \in \Omega, \end{cases} \tag{4.5}$$

where $\rho > 0$ is constant, the function $f : J \times \Omega \times E \times E \to E$ is continuous, and Δ stands for the Laplace operator in the space variable x, $J = [0, 1]$, we defined the linear operators A and B in E by

$$Au = \Delta u, \quad u \in D(A) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega),$$

$$Bu = 2\Delta u, \quad u \in D(B) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega).$$

Clearly, $C(\rho) = \rho^2 B^2 - 4A = 4(\rho^2\Delta^2 - \Delta) = L^2$, where $L = 2(\rho^2\Delta^2 - \Delta)^{\frac{1}{2}}$. It is clear that $BL = LB$. Further,

$$E_1(\rho) = R_1(\rho)\Delta, \qquad E_2(\rho) = R_2(\rho)\Delta,$$

where $R_1(\rho) = [\rho I - (\rho^2 I + (-\Delta)^{-1})^{\frac{1}{2}}]$ and $R_2(\rho) = [\rho I + (\rho^2 I + (-\Delta)^{-1})^{\frac{1}{2}}]$ are invertible bounded linear operator on $L^p(\Omega)$ for all $\rho > 0$.

Moreover,

$$-R_1^{-1}(\rho)(-E_1(\rho)) = -R_2^{-1}(\rho)(-E_2(\rho)) = \Delta$$

generates an operator semigroup $\{T(t)\}_{t\geq 0}$ which is compact, analytic, and uniformly bounded. By the maximum principle, we can find that $\{T(t)\}_{t\geq 0}$ is a positive semigroup on $L^p(\Omega)$ (see [21, 24, 26]), and $-R_1^{-1}(\rho), -R_2^{-1}(\rho)$ are invertible. Thus, by operator semigroups theory [26], $-E_1(\rho) = -R_1(\rho)\Delta$ and $-E_2(\rho) = -R_2(\rho)\Delta$ generate positive C_0-semigroups $T_1(t)(t \geq 0)$ and $T_2(t)(t \geq 0)$ on $L^p(\Omega)$.

Let $u(t) = u(t, \cdot)$, $f(t, u(t), Gu(t)) = f(t, \cdot, u(t, \cdot), Gu(t, \cdot))$, then the problem (4.5) can be reformulated as the equations (1.1).

Theorem 4.2. *If the following conditions*

(F1) Let $u_0 \in D(L) \cap D(B), u_1 \in L^p(\Omega)$, $u_0(x), u_1(x) \geq 0, x \in \Omega, f(t,x,0,0) \geq 0$ *and there exists a function* $w = w(t,x) \in C(J \times \Omega)$ *such that*

$$\begin{cases} \frac{\partial^2 w(t,x)}{\partial t^2} + 2\rho\Delta \frac{\partial w(t,x)}{\partial t} + \Delta w(t,x) \geq f(t,x,w(t,x),Gw(t,x)), & (t,x) \in J \times \Omega, \\ w(t,x) = 0, \quad (t,x) \in J \times \partial\Omega, \\ w(0,x) \geq w_0(x), \quad \frac{\partial}{\partial t} w(0,x) \geq w_1(x), \quad x \in \Omega. \end{cases}$$

(F2) There exits a constant $M > 0$ *such that*

$$f(t,x,u_2,v_2) - f(t,x,u_1,v_1) \geq -M(u_2 - u_1)$$

for any $t \in J$, *and* $0 \leq u_1 \leq u_2 \leq w(t,x), 0 \leq v_1 \leq v_2 \leq Gw(t,x)$.

(F3) There exists a constant $L > 0$ *such that*

$$\alpha(\{f(t,u_n,v_n)\}) \leq L(\alpha(\{u_n\}) + \alpha(\{v_n\})),$$

for $\forall t \in J$, *and increasing or decreasing monotonic sequences* $\{u_n\} \subset [v_0(t), w_0(t)]$ *and* $\{v_n\} \subset [Gv_0(t), Gw_0(t)]$.

Then the problem (4.5) has minimal and maximal mild solutions between 0 *and* $w(x,t)$*, which can be obtained by a monotone iterative procedure starting from* 0 *and* $w(t)$*, respectively.*

Proof. Assumption (F1) implies that $v_0 \equiv 0$ and $w_0 \equiv w(x,t)$ are lower and upper solutions of the problem (4.5), respectively, and from (F1) and (F2), it is easy to verify that all conditions (H1) are satisfied under the constant $M_1 = M_2 = 1$. So our conclusion follows from Theorem 3.2. □

5. Conclusions

This paper investigates the existence of the extremal mild solutions for damped elastic systems in Banach spaces. By introducing a new concept of lower and upper mild solutions, we construct a new monotone iterative method for damped elastic systems and obtain the existence of extremal mild solutions between lower and upper mild solutions for the problem under the situation that the associated semigroup is equicontinuous. Here, we do not need the associated semigroup is compact. Our results presented in this paper improve and generalize many classical results [7–9]. For future work will be focused on investigate the asymptotic stability of solutions, and the analyticity and exponential stability of associated semigroup for damping elastic system in Banach spaces.

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