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ON SOLVABILITY OF SINGULAR INTEGRAL-DIFFERENTIAL EQUATIONS WITH CONVOLUTION

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Abstract In this paper, we study a class of singular integral-different equations of convolution type with Cauchy kernel. By means of the classical boundary value theory, of the theory of Fourier analysis, and of the principle of analytic continuation, we transform the equations into the Riemann-Hilbert problems with discontinuous coefficients and obtain the general solutions and conditions of solvability in class $\{0\}$. Thus, the result in this paper generalizes the classical theory of integral equations and boundary value problems.

Keywords Singular integral-differential equations, Riemann-Hilbert problems; integral operators, Cauchy kernel, convolution type.

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1. Introduction

There were rather complete investigations on the singular integral equations (SIEs) and Riemann-Hilbert problems (R-HPs) [4,10,13,14,26]. Litvinchuk [11,12] studied singular integral-differential equations, in which the class of differentiable functions was extended to the class of a Hölder continuous function, next the singular integral-differential equation which the coefficients contain a discontinuity point of the first kind was also studied. Li and Ren [15–20]proposed a general method to solve SIEs with a mixture of convolution kernel and Cauchy kernel, in which the convolution kernel has discontinuous property, that is to transform this kind of integral equations to a R-HP by using Fourier transform. In this paper, we set up and discuss one class of singular integral-differential equations of convolution type with Cauchy kernel. By using Fourier transform, we transform this class of equations into a R-HP, and obtain the general solution and conditions of solvability in class $\{0\}$. Simultaneously, other classes of singular integral-different equations of convolution type can be also solved by the method of this paper, such as equation of Wiener-Hopf type, equations of dual type and so on.

2. Preliminaries

In this section we present some definitions and lemmas.

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Definition 2.1. If $F(x) \in H \cap L^2(\mathbb{R})$, we say that $F(x) \in \{\{0\}\}$, where H, $L^2(\mathbb{R})$ are the Hölder continuous function space and the Lebesgue space, respectively.

Definition 2.2. The Fourier transform $\mathbb{F}$ of a function $f(t)$ is defined by

$$\mathbb{F}[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{ixt} dt, \tag{2.1}$$

we denote $\mathbb{F}[f(t)] = F(x)$.

And the inverse Fourier transform $\mathbb{F}^{-1}$ of a function $F(x)$ is denoted by

$$\mathbb{F}^{-1}[F(x)] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} F(x) e^{-ixt} dx, \tag{2.2}$$

we also denote $\mathbb{F}^{-1}[F(x)] = f(t)$, where two integrals appeared in (2.1) and (2.2) exist.

Definition 2.3. Let $F(x) = \mathbb{F}f(t)$. If $F(x) \in \{\{0\}\}$, we say that $f(t) \in \{0\}$.

Definition 2.4. The operators P and Q are defined as follows

$$P(f(t)) = f(-t), \quad Q(f(t)) = f(t)\mathrm{sgn}t, \quad t \in \mathbb{R};$$

when $f(t) \in \{0\}$, we define the operator T of Cauchy principal value integral

$$Tf = \frac{1}{\pi i} \int_{\mathbb{R}} \frac{f(\tau)}{\tau - t} d\tau, \quad t \in \mathbb{R}.$$

It is easy to see that

$$\mathbb{F}[Pf(t)] = PF(x), \quad \mathbb{F}^{-1}[PF(x)] = Pf(t).$$

The following lemmas 2.1-2.4 are important to our results.

Lemma 2.1. *Assume that a function $f(t)$ as well as its derivatives $f^{(j)}(t)$ $(1 \leq j \leq n)$ belong to $\{0\}$. Then*

$$\mathbb{F}[f_{\pm}^{(j)}(t)] = (-ix)^j \mathbb{F}[f_{\pm}(t)] - \frac{1}{\sqrt{2\pi}} \sum_{m=0}^{j-1} (-ix)^m f^{(j-m-1)}(0) \ (1 \leq j \leq n), \tag{2.3}$$

where

$$f_+(t) = \begin{cases} f(t), & t \geq 0, \\ 0, & t < 0; \end{cases} \quad f_-(t) = \begin{cases} 0, & t \geq 0, \\ -f(t), & t < 0. \end{cases}$$

Proof. By induction on j. For $j = 1$ we have

$$\begin{aligned} \mathbb{F}[f_+^{'}(t)] &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f_+^{'}(t) e^{ixt} dt = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}+} f^{'}(t) e^{ixt} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}+} e^{ixt} df(t) = \frac{1}{\sqrt{2\pi}} [f(t) e^{ixt}]|_0^{+\infty} - \frac{1}{\sqrt{2\pi}} ix \int_{\mathbb{R}+} f(t) e^{ixt} dt \\ &= -\frac{1}{\sqrt{2\pi}} f(0) + (-ix)\mathbb{F}[f_+(t)]. \end{aligned} \tag{2.4}$$

Suppose that (2.3) is true for $j = k$, that is,

$$\mathbb{F}[f_+^{(k)}(t)] = (-ix)^k \mathbb{F}[f_+(t)] - \frac{1}{\sqrt{2\pi}} \sum_{m=0}^{k-1} (-ix)^m f^{(k-m-1)}(0). \tag{2.5}$$

For $j = k+1$ we find

$$\mathbb{F}[f_+^{(k+1)}(t)] = (-ix)^{k+1} \mathbb{F}[f_+(t)] - \frac{1}{\sqrt{2\pi}} \sum_{m=0}^{k} (-ix)^m f^{(k-m)}(0). \tag{2.6}$$

Therefore, the case $f_+(t)$ is proved.

Similarly, we can prove the case $f_-(t)$. □

Lemma 2.2. *Let $f(t) \in \{0\}$, we have $\mathbb{F}(Tf(t)) = -QF(x)$.*

Proof. Since

$$\begin{aligned} \mathbb{F}[Tf(t)] &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} [\frac{1}{\pi i} \int_{\mathbb{R}} \frac{f(\tau)}{\tau - t} d\tau] e^{ixt} dt \\ &= -\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} [\frac{1}{\pi i} \int_{\mathbb{R}} \frac{e^{ixt}}{t-\tau} dt] f(\tau) d\tau, \end{aligned} \tag{2.7}$$

by the extended residue theorem [10], we have

$$\frac{1}{\pi i} \int_{\mathbb{R}} \frac{e^{ixt}}{t-\tau} dt = \begin{cases} e^{ix\tau}, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -e^{ix\tau}, & \text{if } x < 0. \end{cases} \tag{2.8}$$

Substituting (2.8) into (2.7), we obtain

$$\mathbb{F}[Tf(t)] = -\mathrm{sgn} x \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{ixt} dt = -\mathrm{sgn} x F(x) = -QF(x). \tag{2.9}$$

□

Lemma 2.3 (See [29])**.** *If $f(t) \in \{0\}$ and $\mathbb{F}f(0) = 0$, then $Tf(t) \in \{0\}$.*

Lemma 2.4. *Let $f^{(j)}(t) \in \{0\} (0 \le j \le n)$ and $\mathbb{F}f(0) = 0$, we have $\mathbb{F}[Tf^{(j)}(t)] = -(-ix)^j QF(x)$, where*

$$Tf^{(j)}(t) = \frac{1}{\pi i} \int_{\mathbb{R}} \frac{f^{(j)}(\tau)}{\tau - t} d\tau.$$

Proof. By Lemma 2.1, we obtain

$$\mathbb{F}[f^{(j)}(t)] = \mathbb{F}[f_+^{(j)}(t)] - \mathbb{F}[f_-^{(j)}(t)] = (-ix)^j \mathbb{F}[f_+(t)] - (-ix)^j \mathbb{F}[f_-(t)] = (-ix)^j F(x). \tag{2.10}$$

Again using Lemma 2.2, we have $\mathbb{F}[Tf^{(j)}(t)] = -(-ix)^j QF(x)$. □

For two functions $f(t)$ and $g(t)$, if we use the notation of convolution

$$f * g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t-\tau) g(\tau) d\tau,$$

then it is well known that

$$\mathbb{F}(f * g) = \mathbb{F}f \cdot \mathbb{F}g = FG,$$

where F, G are the Fourier transforms of f, g, respectively.

3. Presentation of the problem

Some practical problems, such as atomic diffusion theory, heat conduction, transport and nuclear collision, and mathematics physics research, are closed related to singular integral-differential equations and integral differential equations with convolution kernel [7, 21]. In fact, the above-mentioned problems can often attribute to finding the solution for the following generalized singular integral-differential equations of convolution type with Cauchy kernel.

Let us solve the following equation

$$\begin{aligned}\sum_{j=0}^{n}\{a_j f^{(j)}(t) + \frac{b_j}{\pi i}\int_{\mathbb{R}}\frac{f^{(j)}(\tau)}{\tau - t}d\tau + \frac{c_j}{\sqrt{2\pi}}\int_{\mathbb{R}^+} k_j(t-\tau)f^{(j)}(\tau)d\tau \\ + \frac{d_j}{\sqrt{2\pi}}\int_{\mathbb{R}^-} h_j(t-\tau)f^{(j)}(\tau)d\tau\} = g(t), \quad t\in\mathbb{R},\end{aligned} \tag{3.1}$$

where a_j, b_j, c_j, d_j $(0 \le j \le n)$ are real constants, and $\sum_{j=0}^{n}|b_j| \neq 0$. The known functions $k_j(t), h_j(t), g(t) \in \{0\}(0 \le j \le n)$, and the unknown function $f(t)$ as well as its derivatives $f^{(j)}(t)(1 \le j \le n)$ belong to $\{0\}$.

The presentation and the solving method of Eq.(3.1) rich a theory of singular integral equation. It is mentioned that the methods of solution for other singular integral-differential equations are still effective. Hence, (3.1) has important meaning not only in application but also in the theory of resolving the equation itself.

In order to solve Eq.(3.1), we may write it as

$$\begin{aligned}\sum_{j=0}^{n}\{a_j f^{(j)}(t) + \frac{b_j}{\pi i}\int_{\mathbb{R}}\frac{f^{(j)}(\tau)}{\tau - t}d\tau + \frac{c_j}{\sqrt{2\pi}}\int_{\mathbb{R}} k_j(t-\tau)f_+^{(j)}(\tau)d\tau \\ - \frac{d_j}{\sqrt{2\pi}}\int_{\mathbb{R}} h_j(t-\tau)f_-^{(j)}(\tau)d\tau\} = g(t), \quad t\in\mathbb{R}.\end{aligned} \tag{3.2}$$

Take the Fourier transforms in both sides of (3.2), by using lemmas 2.1-2.4, we can transform (3.2) to the following Riemann-Hilbert problem:

$$F^+(x) = B(x)F^-(x) + D(x), \quad x\in\mathbb{R}, \tag{3.3}$$

where

$$B(x) = \frac{\sum\limits_{j=0}^{n}[a_j - b_j\mathrm{sgn}x + d_j H_j(x)](-ix)^j}{\sum\limits_{j=0}^{n}[a_j - b_j\mathrm{sgn}x + c_j K_j(x)](-ix)^j},$$

$$D(x) = \frac{G(x) + \frac{1}{\sqrt{2\pi}}\sum\limits_{j=1}^{n}\{[\sum\limits_{m=0}^{j-1}(-ix)^m A_{j,m}](c_j K_j(x) - d_j H_j(x))\}}{\sum\limits_{j=0}^{n}[a_j - b_j\mathrm{sgn}x + c_j K_j(x)](-ix)^j},$$

$$F^{\pm}(x) = \mathbb{F}[f_{\pm}(t)],\ G(x) = \mathbb{F}[g(t)],\ K_j(x) = \mathbb{F}[k_j(t)],\ H_j(x) = \mathbb{F}[h_j(t)],\ 0\le j\le n,$$

and $A_{j,m} = f^{(j-m-1)}(0)(0 \le m \le j-1, 1 \le j \le n)$ are undetermined constants. (3.3) is a R-HP with node $x = 0$ on an infinite straight line, and it can be directly solved by the classical method [6, 27]. In this paper, we shall take another method to solve R-HP (3.3).

We make the following linear transform

$$z = \frac{\zeta}{i(\zeta + i)}. \tag{3.4}$$

(3.4) maps the real axis X in plane Z onto a circle $C : |\zeta + \frac{i}{2}| = \frac{1}{2}$ in plane ζ, which surrounds an interior region S^+ and an exterior region S^-, and maps the upper half-plane Im$z > 0$ and the lower half-plane Im$z < 0$ onto S^+ and S^- respectively. Again let

$$F(z) = \Psi(\zeta),\ G(z) = W(\zeta),\ K_j(z) = B_{1,j}(\zeta),\ H_j(z) = B_{2,j}(\zeta),\ 0 \le j \le n.$$

Then (3.3) is readily reduced to the following R-HP in plane ζ

$$\Psi^+(\tau) = M(\tau)\Psi^-(\tau) + N(\tau), \quad \tau \in C, \tag{3.5}$$

where

$$N(\tau) = \frac{W(\tau) + \frac{1}{\sqrt{2\pi}} \sum\limits_{j=1}^{n} \{[\sum\limits_{m=0}^{j-1} (-\frac{\tau}{\tau+i})^m A_{j,m}](c_j B_{1,j}(\tau) - d_j B_{2,j}(\tau))\}}{\sum\limits_{j=0}^{n} [a_j - b_j \delta(\tau) + c_j B_{1,j}(\tau)](-\frac{\tau}{\tau+i})^j},$$

$$M(\tau) = \frac{\sum\limits_{j=0}^{n} [a_j - b_j \delta(\tau) + d_j B_{2,j}(\tau)](-\frac{\tau}{\tau+i})^j}{\sum\limits_{j=0}^{n} [a_j - b_j \delta(\tau) + c_j B_{1,j}(\tau)](-\frac{\tau}{\tau+i})^j},$$

$$\delta(\tau) = \begin{cases} 1, & \tau \in C_1; \\ -1, & \tau \in C_2, \end{cases}$$

here C_1, C_2 are the left half circles and the right half circles of C, respectively.

Note that the solutions of (3.2), (3.3), and (3.5) are equivalent to each other. On the solutions of R-HP (3.5), we will consider the two cases: the normal type and the non-normal type.

4. The solving method of (3.5)

4.1. The normal type case of (3.5)

If $\sum\limits_{j=0}^{n} [a_j - b_j \delta(\tau) + c_j B_{1,j}(\tau)](-\frac{\tau}{\tau+i})^j \neq 0$ or $\sum\limits_{j=0}^{n} [a_j - b_j \delta(\tau) + d_j B_{2,j}(\tau)](-\frac{\tau}{\tau+i})^j \neq 0$ $(\tau \in C)$, then $\tau = 0$ is a discontinuous point of $M(\tau)$ and $N(\tau)$, therefore it is also node of (3.5), in this case, we call (3.5) the R-HP of normal type with node $\tau = 0$. Let

$$\gamma = \alpha + i\beta = \frac{1}{2\pi i}\{\log M(-0) - \log M(+0)\} \tag{4.1}$$

(the definitions of $M(\pm 0)$ see [15,22]). Because $\log M(\tau)$ has infinite number branches, we may take a continuous branch of $\log M(\tau)$ such as $\log M(-i)=0$, and again take the integer μ such as

$$0 \leq \alpha' = \alpha - \mu < 1, \quad \lambda = \gamma - \mu = \alpha' + i\beta, \tag{4.2}$$

and call μ the index of (3.5). We can define the following piece-wise function

$$X(\zeta) = \begin{cases} e^{\Gamma(\zeta)}, & \zeta \in S^+; \\ (\zeta + \frac{i}{2})^{-\mu} e^{\Gamma(\zeta)}, & \zeta \in S^-, \end{cases} \tag{4.3}$$

where

$$\Gamma(\zeta) = \frac{1}{2\pi i} \int_C \frac{\log M(\tau)}{\tau - \zeta} d\tau, \quad \zeta \notin C, \tag{4.4}$$

therefore, we obtain

$$X^+(\tau) = M(\tau) X^-(\tau), \quad \tau \in C. \tag{4.5}$$

Next we shall solve the R-HP (3.5). Since $\Psi(\zeta)$ is bounded at $\zeta = \infty$, therefore, (3.5) has a solution in class R_0. The homogeneous problem of (3.5) is denoted by

$$\Psi^+(\tau) = M(\tau)\Psi^-(\tau), \quad \tau \in C. \tag{4.6}$$

From (4.5) and (4.6), we have

$$\frac{\Psi^+(\tau)}{X^+(\tau)} = \frac{\Psi^-(\tau)}{X^-(\tau)}. \tag{4.7}$$

Consider the function

$$D(\zeta) = \frac{\Psi(\zeta)}{X(\zeta)},$$

by (4.7), we known that, $D(\zeta)$ is analytic on the complex plane and has a pole-point ∞ with the order μ. By generalized Liouville theorem [10], we can get a general solution of (4.6)

$$\Psi(\zeta) = X(\zeta) P_\mu(\zeta), \tag{4.8}$$

and $P_\mu(\zeta) = e_0 + e_1\zeta + \ldots + e_\mu \zeta^\mu (\mu \geq 0)$ is a polynomial of degree μ with complex coefficients; when $\mu < 0$, $P_\mu(\zeta) \equiv 0$, that is, (4.6) only has zero solution.

In order to solve (3.5), we define the following Cauchy principal value integral with the kernel density function $\frac{N(\tau)}{X^+(\tau)}$, that is,

$$U(\zeta) = \frac{1}{2\pi i} \int_C \frac{N(\tau)}{X^+(\tau)(\tau - \zeta)} d\tau. \tag{4.9}$$

By applying Plemelj's formula [26], we easily prove that the following (4.10) is a special solution of (3.5)

$$U_*(\zeta) = \frac{X(\zeta)}{2\pi i} \int_C \frac{N(\tau)}{X^+(\tau)(\tau - \zeta)} d\tau. \tag{4.10}$$

According to the theory of linear algebra, we obtain a general solution of (3.5) as follows

$$\Psi(\zeta) = U_*(\zeta) + X(\zeta) P_\mu(\zeta), \tag{4.11}$$

that is,

$$\Psi(\zeta) = X(\zeta)[\frac{1}{2\pi i}\int_C \frac{N(\tau)}{X^+(\tau)(\tau-\zeta)}d\tau + P_\mu(\zeta)], \tag{4.12}$$

where $P_\mu(z)$ is as the above. But, when $\mu < 0$, $P_\mu(z) \equiv 0$, and the following $-\mu-1$ solvable conditions

$$\int_C \frac{N(\tau)\tau^j}{X^+(\tau)}d\tau = 0, \quad 0 \le j \le -\mu-2 \tag{4.13}$$

are also satisfied. Therefore, when $\mu < 0$, (3.5) has only solution and its solution is still (4.12) (with $P_\mu(z) \equiv 0$).

Now we consider the case of the solution at $\tau = 0$.

By lemma 2.3, we have $F(x) \in \{\{0\}\}$ and $F(0) = 0$. If $\tau = 0$ is an ordinary node, then $0 < \alpha < 1$ and $\lambda \ne 0$. Because $\Psi(\zeta)$ is continuous at $\tau = 0$, and $F(0) = 0$, we may obtain $\Psi(0) = 0$, thus we have the following solvable condition

$$W(0) + \frac{1}{\sqrt{2\pi}}\sum_{j=1}^{n} A_{j,0}[c_jB_{1,j}(0) - d_jB_{2,j}(0)] = 0. \tag{4.14}$$

If $\tau = 0$ is a special node, then $\alpha = 0$ and $\lambda = i\beta_0$. When $\beta_0 \ne 0$, (4.14) and the following condition must be fulfilled

$$c + \frac{1}{2\pi i}\int_C \frac{N(\tau)}{X^+(\tau)\tau}d\tau = 0, \tag{4.15}$$

where c is a constant term of $P_\mu(\zeta)$. When $\beta_0 = 0$, we have $\lambda = 0$, hence $\Psi(0) = 0$ if and only if (4.14) holds. So the necessary condition of the existence of the solution for (3.5) is (4.14). Once (4.14) and (4.15) are fulfilled, then $F(\zeta) \in H$ near $\tau = 0$ and therefore $\Psi^\pm(\zeta)$ are continuous at $\tau = 0$.

Now we can formulate the main results about the solutions of Eq.(3.1) in the following form.

Theorem 4.1. *Under the case of normal type, the necessary condition of the existence of the solution for Eq.(3.1) is (4.14). Assume that (4.14) is fulfilled.*

(1) Let $\tau = 0$ be an ordinary node. When $\mu \ge -1$, (3.1) is solvable, and its solution is given by $f(t) = \mathbb{F}^{-1}[F(x)]$, where $F(x) = F^+(x) - F^-(x)$; when $\mu < -1$, and (4.13) satisfies, (3.1) has only solution (4.12) (with $P_\mu(z) \equiv 0$). It follows from $F(x) \in \{\{0\}\}$ that $f(t) \in \{0\}$.

(2) Let $\tau = 0$ be a special node. When $\mu \ge -1$, and (4.14), (4.15) hold, (3.1) has a solution; when $\mu < -1$ and (4.13) satisfies, (3.1) has a solution and its solution is the same as the case in (1).

4.2. The non-normal type case of (3.5)

Assume that $N(\tau)$ has some zero-points and pole-points on C, then R-HP (3.5) is called the non-normal type case. Let

$$\sum_{j=0}^{n}[a_j - b_j\delta(\tau) + c_jB_{1,j}(\tau)](-\frac{\tau}{\tau+i})^j$$

and

$$\sum_{j=0}^{n}[a_j - b_j\delta(\tau) + d_jB_{2,j}(\tau)](-\frac{\tau}{\tau+i})^j$$

have common and the same order zero-points $a_1, a_2, \cdots, a_q$, with the orders γ_1, $\gamma_2, \cdots, \gamma_q$ respectively on C;

$$\sum_{j=0}^{n}[a_j - b_j\delta(\tau) + c_jB_{1,j}(\tau)](-\frac{\tau}{\tau+i})^j$$

has some zero-points $b_1, b_2, \cdots, b_s$ with the orders $\alpha_1, \alpha_2, \cdots, \alpha_s$ respectively on C;

$$\sum_{j=0}^{n}[a_j - b_j\delta(\tau) + d_jB_{2,j}(\tau)](-\frac{\tau}{\tau+i})^j$$

has zero-points $c_1, c_2, \cdots, c_l$ with the orders $\beta_1, \beta_2, \cdots, \beta_l$ respectively on C. Here $\alpha_j, \beta_j, \gamma_j$ are positive integers. Again let

$$v_1(\tau) = \prod_{j=1}^{s}(\tau - b_j)^{\alpha_j}, \quad v_2(\tau) = \prod_{j=1}^{l}(\tau - c_j)^{\beta_j},$$

$$\sum_{j=1}^{s}\alpha_j = N_1, \quad \sum_{j=1}^{l}\beta_j = N_2, \quad \sum_{j=1}^{q}\gamma_j = N_3.$$

Thus, (3.5) is reduced to the following form

$$\Psi^+(\tau) = \frac{v_2(\tau)}{v_1(\tau)}M_0(\tau)\Psi^-(\tau) + M(\tau), \quad \tau \in C. \tag{4.16}$$

It follows from $F(x) \in \{\{0\}\}$ that $\Psi(\tau)$ is bounded on C. In order that $\Psi(\tau)$ satisfies the conditions at $a_j (1 \le j \le q)$, the following equalities must be fulfilled

$$\{W(\tau) + \frac{1}{\sqrt{2\pi}}\sum_{j=1}^{n}[\sum_{m=0}^{j-1}(-1)^m(\frac{\tau}{\tau+i})^m A_{j,m}](c_jB_{1,j}(\tau) - d_jB_{2,j}(\tau))\}^{(k)}|_{\tau=a_j} = 0 \tag{4.17}$$

for any $k = 0, 1, \cdots, \gamma_j - 1; j = 1, 2, \cdots, q$. In order that computation of (4.17) is effective, $W(\tau), B_{1,j}(\tau), B_{2,j}(\tau)$ $(0 \le j \le n)$ must exist derivatives until order $\gamma_j - 1$ on the neighborhood of a_j, and all order derivatives satisfy Hölder conditions.

We discuss only the case $a_j + b_j \neq 0, a_j - b_j \neq 0$ $(0 \le j \le n)$ in this paper. On other cases, such as $a_j + b_j = 0, a_j - b_j \neq 0$ $(0 \le j \le n)$; $a_j + b_j \neq 0, a_j - b_j = 0$ $(0 \le j \le n)$ can be discussed similarly. Under satisfying the above conditions, (4.16) is a non-normal type R-HP with discontinuous coefficients. We first discuss the following homogeneous problem

$$\Psi^+(\tau) = \frac{v_2(\tau)}{v_1(\tau)}M_0(\tau)\Psi^-(\tau), \quad \tau \in C. \tag{4.18}$$

By means of the method of solution in [5,12,23,24], and applying the extended Liouville theory and the principle of analytic continuation [8,28], it is easy to obtain the general solution of (4.18):

$$\Psi_*(\zeta) = \begin{cases} X(\zeta)v_2(\zeta)P_{\mu-N_1}(\zeta), & \zeta \in S^+; \\ X(\zeta)v_1(\zeta)P_{\mu-N_1}(\zeta), & \zeta \in S^-, \end{cases} \tag{4.19}$$

where $X(\zeta)$ also takes the from of (4.3), but $\Gamma(\zeta)$ is replaced by the following equality

$$\Gamma(\zeta)=\frac{1}{2\pi i}\int_C\frac{\log M_0(\tau)}{\tau-\zeta}d\tau,\quad \zeta\notin C. \tag{4.20}$$

In the following, we shall discuss the non-homogeneous R-HP (4.16). Define the following piece-wise function

$$\varphi(\zeta)=\frac{1}{2\pi i}\int_C\frac{v_1(\tau)N(\tau)}{X^+(\tau)(\tau-\zeta)}d\tau,\ \zeta\notin C. \tag{4.21}$$

It follows from $F(x)\in\{\{0\}\}$ that $\Psi(\zeta)$ is bounded on C, hence $\Psi(\zeta)$ has no singularity at $\tau=b_j,c_j$. In order to solve (4.16), we need to consider a following Hermite interpolation polynomial

$$\Omega_\rho(\zeta)=e_0\zeta^\rho+e_1\zeta^{\rho-1}+\cdots+e_\rho\quad(\rho=N_1+N_2-1),$$

where $\Omega_\rho(\zeta)$ has some zero-points of orders α_j,β_j at b_j,c_j respectively, and e_j are constants. By means of the above Hermite interpolation polynomial $\Omega_\rho(\zeta)$, we can define the following function

$$Y(\zeta)=\begin{cases}\frac{X(\zeta)(\Omega_\rho(\zeta)-\varphi(\zeta))}{v_1(\zeta)}, & \zeta\in S^+;\\ \frac{X(\zeta)(\Omega_\rho(\zeta)-\varphi(\zeta))}{v_2(\zeta)}, & \zeta\in S^-.\end{cases} \tag{4.22}$$

Since $Y(\zeta)$ is bounded at b_j,c_r $(j=1,2,\cdots,s;r=1,2,\cdots,l)$, therefore, the following conditions of solvability

$$\begin{aligned}&\int_C\frac{v_1(\tau)N(\tau)}{X^+(\tau)(\tau-b_j)^p}d\tau=0,\quad j=1,2,\cdots,s;\ p=0,1,2,\cdots,\alpha_j,\\ &\int_C\frac{v_1(\tau)N(\tau)}{X^+(\tau)(\tau-c_r)^q}d\tau=0,\quad r=1,2,\cdots,l;\ p=0,1,2,\cdots,\beta_r\end{aligned} \tag{4.23}$$

are fulfilled. Under the solvable conditions (4.17) and (4.23), by applying Plemelj's formula, we can verify that (4.22) is a special solution of (4.16). According to the theory of linear algebra, we obtain a general solution of (4.16)

$$\Psi(\zeta)=Y(\zeta)+\Psi_*(\zeta). \tag{4.24}$$

By taking the boundary values to (4.24), we have the following explicit expressions for R-HP (4.16)

$$\begin{aligned}\Psi(\zeta)&=\frac{1}{2}N(\zeta)+\frac{X^+(\zeta)(\Omega_\rho(\zeta)-\varphi(\zeta))}{v_1(\zeta)}+X^+(\zeta)v_2(\zeta)P_{\mu-N_1}(\zeta),\qquad \zeta\in S^+;\\ \Psi(\zeta)&=-\frac{1}{2}\frac{N(\zeta)}{M_0(\zeta)}+\frac{X^-(\zeta)(\Omega_\rho(\zeta)-\varphi(\zeta))}{v_2(\zeta)}+X^-(\zeta)v_1(\zeta)P_{\mu-N_1}(\zeta),\quad \zeta\in S^-.\end{aligned} \tag{4.25}$$

Next, we shall discuss the property of the solution (4.25) at $\tau=0$. If $\tau=0$ is an ordinary node, similar to the above discussion, the solvable condition (4.14) is fulfilled. If $\tau=0$ is a special node, (4.14) is also fulfilled, and the constant term c of $P_\mu(\zeta)$ should take the value

$$c=\frac{\Omega_\rho(0)-\varphi(0)}{v_1(0)v_2(0)}. \tag{4.26}$$

Finally, we come to consider the behavior of the solution at $\tau = \infty$. By (4.22), we have the following results.

(1) When $N_1 - \mu - 1 > 0$, $Y(\zeta)$ has a pole point ∞ with the order $N_1 - \mu - 1$. In order that $\Psi(\zeta)$ is bounded at ∞, one must have

$$e_0 = e_1 = \cdots = e_{N_1-\mu-2} = 0, \tag{4.27}$$

and $\Omega_\rho(\zeta)$ should be a polynomial with the degree $\rho-(N_1-\mu-1)$, where $\rho-(N_1-\mu-1) = N_2+\mu$.

(2) When $N_2+\mu \le -1$, we have $\Omega_\rho(\zeta) \equiv 0$, and when $N_2+\mu < -1$, since $\Psi(\zeta)$ is bounded at ∞, the following conditions of solvability are satisfied

$$\int_C \frac{v_1(\tau)N(\tau)\tau^{k-1}}{X^+(\tau)} d\tau = 0, \quad k = 0, 1, \cdots, -N_2-\mu+1. \tag{4.28}$$

In conclusion, we obtain the following theorem.

Theorem 4.2. *Under conditions $a_j \pm b_j \ne 0$ $(0 \le j \le n)$, the necessary conditions of solvability for (3.1) are (4.14) and (4.17). Assume that both (4.14) and (4.17) are fulfilled.*

(1) Let $\tau = 0$ be an ordinary node. If $\mu - N_1 \ge 0$, (3.1) is solvable and has $\mu - N_1$ linearly independent solutions; if $\mu - N_1 = -1$, (3.1) has only solution; if $\mu - N_1 < -1$, the solvable condition of Eq.(3.1) is (4.27), and if $N_2+\mu < -1$, one require also that (4.28) holds, then (3.1) has just only solution, and its solution is given by (4.24), and when $\mu - N_1 \le -1$, $P_{\mu-N_1}(\zeta) \equiv 0$.

(2) Let $\tau = 0$ be a special node, then (4.14) and (4.26) should be fulfilled. In this case, (3.1) has a solution, and its solution is also given by (4.24).

(3) In the case $\tau = \infty$, the conditions of solvability for Eq. (3.1) are (4.27) and (4.28).

Under all of the above conditions, the solution of (3.1) is given by $f(t) = \mathbb{F}^{-1}[F(x)]$, where $F(x) = F^+(x) - F^-(x)$, $F^\pm(x)$ are given by (4.24). It is easy to prove that $f(t) \in \{0\}$.

Remark 4.1. From $f^{(j)}(t) \in \{0\}$ it follows that $(F^\pm(x))^{(j)} \in \{\{0\}\}(j = 0, 1, \cdots, n)$. In the following, we give the undetermined constants $A_{j,m}$ $(0 \le m \le j-1, 1 \le j \le n)$, note that $\{A_{j,m}\} = \{A_{j,0}\} = \{f(0), f'(0), \cdots, f^{(n-1)}(0)\}$. In the neighborhood of $z = \infty$, we can make the expansion in series for $F^+(x)$, and we only take former finite number terms, that is,

$$\begin{aligned} &F^+(x) = a_1x^{-1} + O(|\, x \,|^{-1}), \\ &F^+(x) = a_1x^{-1} + a_2x^{-2} + O(|\, x \,|^{-2}), \cdots, \\ &F^+(x) = a_1x^{-1} + \cdots + a_nx^{-n} + O(|\, x \,|^{-n}). \end{aligned} \tag{4.29}$$

Because $A_{j,0} = \sqrt{2\pi} i a_j$ $(j = 1, 2, \cdots, n)$, we may obtain $A_{j,0}$ $(j = 1, 2, \cdots, n)$ by solving Eqs.(4.29). Moreover, $A_{j,0}$ $(j = 1, 2, \cdots, n)$ can be determined in the following way.

Owing to

$$\lim_{x\to\infty} \{(-ix)^j F^+(x) - \frac{1}{\sqrt{2\pi}} \sum_{m=0}^{j-1} (-ix)^m A_{j,m}\} = 0, \ j = 1, 2, \cdots, n, \tag{4.30}$$

by solving Eqs. (4.30) we can obtain the undetermined constants $A_{j,0}(j=1,2,\cdots,n)$.

In this paper, we have solved a class of singular integral-different equations of convolution type with Cauchy kernel. Indeed, it is possible to studying the above mentioned equations in Clifford analysis, which is similar to that in [1–3, 9, 25]. Further discussion is omitted here.

5. Conclusions

In this paper, we study the singular integral-different equations of convolution type with Cauchy kernel. This class of equations have important applications in practical problems, such as elastic mechanics, heat conduction, and electrostatics. Hence, the study about Eq. (3.1) has important meaning not only in application but also in the theory of resolving the equation itself. Many problems, such as piezoelectric material, voltage magnetic materials and functional gradient materials, can often attribute the problem to finding solutions for this classes of equations. Hence, the result in this paper improves some results in Refs. [1–7], which provides theoretical basis for solving relative physics problems. Here, our method is different from the ones for the classical R-HPs, and it is novel and effective. Thus, this paper generalizes the theory of classical R-HPs and SIEs.

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