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HIGHER ORDER DUALITY FOR A NEW CLASS OF NONCONVEX SEMI-INFINITE MULTIOBJECTIVE FRACTIONAL PROGRAMMING WITH SUPPORT FUNCTIONS

Tadeusz Antczak[1] and Kalpana Shukla[2,†]

Abstract In the paper, a new class of semi-infinite multiobjective fractional programming problems with support functions in the objective and constraint functions is considered. For such vector optimization problems, higher order dual problems in the sense of Mond-Weir and Schaible are defined. Then, various duality results between the considered multiobjective fractional semi-infinite programming problem and its higher order dual problems mentioned above are established under assumptions that the involved functions are higher order $(\Phi, \rho, \sigma^{\alpha})$-type I functions. The results established in the paper generalize several similar results previously established in the literature.

Keywords Semi-infinite multiobjective fractional programming, support function, Mond Weir dual, Schaible type dual, higher order $(\Phi, \rho, \sigma^{\alpha})$-type I functions.

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1. Introduction

In recent years, semi-infinite programming problems have been an active research topic due to their applications in several areas of modern research such as in economics, engineering design, approximation theory, optimal control, physics, robotics, transportation problems, etc.

A semi-infinite programming problem is called a mathematical programming problem with a finite number of variables and infinitely many constraints. Semi-infinite multiobjective fractional programming problems arise when more than one objective function, being a ratio of two functions or several such ratios, is to be optimized over feasible set described by infinite number of constraints. The main reason for interest in semi-infinite multiobjective fractional programming stems from the fact that programming models could better fit in the real problems. There are many works devoted to the study of optimality conditions and duality results for semi-infinite multiobjective programming problems (see, e.g., [6, 9, 10, 22, 27, 28, 30–32, 35, 40–42, 45–52]).

†The corresponding author. Email:bhukalpanabhu@gmail.com(K. Shukla)
1Faculty of Mathematics and Computer Science, University of d, Banacha 22, 90-238, Poland
2Department of Mathematics, Manav Rachna University, Faridabad, India

The higher order duality theory for generalized convex multiobjective optimization problems is a field of the optimization theory which has intensively developed during the last five decades. This is a consequence of the fact that the study of second and higher order duality is significant due to the computational advantage over the first order duality as it provides tighter bounds for the value of objective function when approximations are used because there are more parameters involved. In the last few years, many researchers have studied higher order duality results for various classes of optimization problems (see, e.g., [1–5, 7, 8, 11, 12, 16–18, 20, 21, 23–25, 29, 30, 33, 34, 37–39, 43, 44]).

In this paper, a new class of nondifferentiable nonconvex semi-infinite multiobjective fractional programming problems in which numerators and denominators of the objective functions and, moreover, all constraints contain a term involving the support function of a convex set. For the considered multicriteria optimization problem, we formulate higher order Mond-Weir and Schaible type duals. Then, for the considered nondifferentiable semi-infinite multiobjective programming problem, we prove various higher order duality results in the sense of Mond-Weir and in the sense of Schaible under hypotheses of the concept of higher order $(\Phi, \rho, \sigma^{\alpha})$-type I functions introduced in the paper. Thus, we generalize and extend similar higher order duality results earlier established in the literature to a new class of nondifferentiable nonconvex semi-infinite multiobjective fractional programming problems in which numerator and denominator of the objective functions and all the constraints contain a term involving the support function of a convex set.

2. Prelimanaries

In this section, we provide some definitions and some results that we shall use in the sequel. Let R^n denote the n-dimensional Euclidean space.

The following convention for equalities and inequalities will be used throughout the paper.

For any $x = (x_1, \cdots x_n)^T, y = (y_1, \cdots y_n)^T$ in R^n, we define:

(i) $x = y$ if and only if $x_i = y_i$ for all $i = 1, \cdots, n$;

(ii) $x < y$ if and only if $x_i < y_i$ for all $i = 1, \cdots, n$;

(iii) $x \leqq y$ if and only if $x_i \leqq y_i$ for all $i = 1, \cdots, n$;

(iv) $x \leq y$ if and only if $x \leqq y$ and $x \neq y$.

Let C be a compact convex subset of R^n . The support function of C at $x \in R^n$ is defined by $s(x|C) = \max\left\{x^T c : c \in C\right\}$.

It is well-known that every support function is a sublinear function defined on R^n and, therefore, it is convex, as well as proper and lower semicontinuous.

The support function $s(x|C)$ of a compact convex set $C \subseteq R^n$, being convex and everywhere finite, has a subgradient at every $x \in R^n$ (see, Rockafellar [34]). This means that, at every $x \in R^n$, there exists $\xi \in R^n$ such that

$$s(z|C) \geqq s(x|C) + \xi^T (z - x) \text{ for all } z \in C.$$

The subdifferential of $s(x|C)$ is given by

$$\partial s(x|C) = \left\{\xi \in C : \xi^T x = s(x|C)\right\}.$$

For any set $C \subseteq R^n$, the normal cone to C at any point $x \in C$, denoted by $N_C(x)$, is defined by

$$N_C(x) = \left\{ y \in R^n : y^T (z - x) \leqq 0, \forall z \in C \right\}.$$

If C is a compact convex set, then $y \in N_C(x)$ if and only if $s(y|C) = x^T y$, or equivalently $x \in \partial s(y|C)$.

Now, consider the semi-infinite vector optimization problem defined by

$$\begin{aligned} &\textit{minimize } f(x) \\ &\textit{subject to } \ h(x,y) \leqq 0, y \in Y, \quad (SIVP) \\ &x \in X, \end{aligned}$$

where X is a nonempty open convex subset of R^n, Y is a nonempty compact set of $R^m, f := (f_1, \cdots, f_k) : X \to R^k$ is a differentiable on $X, h : X \times Y \to R$ is such that, for each $y \in Y, h(., y)$ is differentiable on X and, for each $x \in X, h(x, .)$ is continuous on Y. Let A be the set of all feasible solutions of the problem (SIVP), that is, $A = \left\{ x \in X : h(x,y) \leqq 0 \ \forall \ y \in Y \right\}$.

Definition 2.1. A point $\overline{x} \in A$ is a weakly efficient (weak Pareto) solution of (SIVP) if there is no other feasible solution x such that $f(x) < f(\overline{x})$.

Definition 2.2. A point $\overline{x} \in A$ is an efficient (Pareto) solution of (SIVP) if there is no another feasible solution x such that $f(x) \leq f(\overline{x})$.

Motivated by Kaul et al. [21] and Ferrara and Stefanescu [13], we introduce a new concept of generalized convexity for the considered semi-infinite vector optimization problem (SIVP). Namely, we define the notion of higher order $(\Phi, \rho, \sigma^\alpha)$-type I functions. In the following definition, an element of the $(n+1)$-dimensional Euclidean space R^{n+1} is represented as the ordered pair (z, r) with $z \in R^n$ and $r \in R$. Let α be an integer, $\rho = (\rho_1, \cdots, \rho_k) \in R^k, \sigma^\alpha = (\sigma_1, \cdots, \sigma_\alpha) \in R^\alpha$ and Φ be a real-valued function defined on $X \times X \times R^{n+1}$ such that $\Phi(x, u, (0, a)) \geqq 0$ for all $(x, u) \in X \times X$ and any $a \geqq 0$ and, moreover, $\Phi(x, u, .)$ is a convex function. Further, assume that $K := (K_1, \cdots K_k) : R^n \times R^n \to R^k$ and $H : R^n \times Y \times R^n \to R$ be differentiable functions.

Definition 2.3. It is said that the pair (f, h) is higher order $(\Phi, \rho, \sigma^\alpha)$-type I functions at $u \in X$ on X with respect to functions K and H if the inequalities

$$f_i(x) - f_i(u) - K_i(u,p) + p^T \nabla_p K_i(u,p) \geqq \Phi\left(x, u\left(\nabla f_i(u) + \nabla_p K_i(u,p), \rho_i\right)\right), \quad i = 1, \cdots, k,$$

$$-h\left(u, y^j\right) - H\left(u, y^j, q\right) + q^T \nabla_q H\left(u, y^j, q\right) \geqq \Phi\left(x, u, \left(\nabla h\left(u, y^j\right) + \nabla_q H\left(u, y^j, q\right), \sigma_{y^j}\right)\right), \quad j = 1, \cdots, \alpha$$

hold for all $x \in X, p \in R^n, q \in R^n$. If these inequalities are fulfilled for any $u \in X$, then the pair (f, h) is higher order $(\Phi, \rho, \sigma^\alpha)$-type I functions on X with respect to functions K and H.

Definition 2.4. It is said that the pair (f, h) is higher order strictly $(\Phi, \rho, \sigma^\alpha)$-type I functions at $u \in X$ on X with respect to functions K and H if the inequalities

$$f_i(x) - f_i(u) - K_i(u,p) + p^T \nabla_p K_i(u,p) > \Phi\left(x, u\left(\nabla f_i(u) + \nabla_p K_i(u,p), \rho_i\right)\right) \quad i = 1, \cdots, k,$$

$$-h\left(u,y^{j}\right)-H\left(u,y^{j},q\right)+q^{T}\nabla_{q}H\left(u,y^{j},q\right)\geqq\Phi\left(x,u,\left(\nabla h\left(u,y^{j}\right)+\nabla_{q}H\left(u,y^{j},q\right),\sigma_{y^{j}}\right)\right)\qquad j=1,\cdots,\alpha$$

hold for all $x\in X,p\in R^{n},q\in R^{n}$. If these inequalities are fulfilled for any $u\in X$, then the pair (f,h) is higher order strictly $(\Phi,\rho,\sigma^{\alpha})$-type I functions on X with respect to functions K and H.

Remark 2.1. Note that the definition of higher order $(\Phi,\rho,\sigma^{\alpha})$-type I functions generalizes several concept of generalized convexity, earlier introduced to optimization theory. Indeed, if Y is a finite index set and we denote by $h\left(u,y^{j}\right)=h_{j}\left(u\right),H\left(u,y^{j},q\right)=H_{j}\left(u,p\right),j=1,\cdots,\alpha$, we have the following special cases:

(a) If $K_{i}\left(x,u\right)=0,i=1,\cdots,k,H_{j}\left(u,p\right)=0,j=1,\cdots,\alpha,\Phi\left(x,u,\left(\nabla f_{i}\left(u\right)+\nabla_{p}K_{i}\left(u,p\right),\rho_{i}\right)\right)=\left[\eta\left(x,u\right)\right]^{T}\nabla f_{i}\left(u\right),i=1,\cdots,k,\Phi\left(x,u,\left(\nabla h_{j}\left(u\right)+\nabla_{p}H_{j}\left(u,p\right),\rho_{j}\right)\right)=\left[\eta\left(x,u\right)\right]^{T}\nabla h_{j}\left(u\right),j=1,\cdots,\alpha$, then Definition 2.3 reduces to the definition of $V-$ type I functions introduced by Kaul et al. [21].

(b) If f_i and h_j are twice differentiable and, moreover $K_{i}\left(u,p\right)=\frac{1}{2}p^{T}\nabla^{2}f_{i}\left(u\right)p,\Phi\left(x,u,\left(\nabla f_{i}\left(u\right)+\nabla_{p}K_{i}\left(u,p\right),\rho_{i}\right)\right)=\left[\eta\left(x,u\right)^{T}\right]\left(\nabla f_{i}\left(u\right)+\nabla^{2}f_{i}\left(u\right)p\right),i=1,\cdots,k,H_{j}\left(u,p\right)=\frac{1}{2}p\nabla^{2}h_{j}\left(u\right)p,\Phi\left(x,u,\left(\nabla h_{j}\left(u\right)+\nabla_{p}H_{j}\left(u,p\right),\sigma_{j}\right)\right)=\left[\eta\left(x,u\right)^{T}\right]\nabla h_{j}\left(u\right)+\nabla^{2}h_{j}\left(u\right)p,j=1,\cdots,\alpha$, where $\eta:X\times X\rightarrow R^{n}$ is a vector-valued function, then Definition 2.3 reduces to definition of second order type I functions introduced by Mond and Zang [31] (see also Mishra and Rueda [26]).

(c) If $\Phi\left(x,u,\left(\nabla f_{i}\left(u\right)+\nabla_{p}K_{i}\left(u,p\right),\rho_{i}\right)\right)=\left[\eta\left(x,u\right)\right]^{T}\nabla_{p}K_{i}\left(u,p\right),i=1,\cdots,k$, $\Phi\left(x,u,\left(\nabla h_{j}\left(u\right)+\nabla_{p}H_{j}\left(u,p\right),\sigma_{j}\right)\right)=\left[\eta\left(x,u\right)\right]^{T}\nabla_{p}H_{j}\left(u,p\right),j=1,\cdots,\alpha$, where $\eta:X\times X\rightarrow R^{n}$ is a vector-valued function, then Definition 2.3 gives the definition of higher order type I functions with respect to η introduced in the scalar case by Zhang [52](see also Mishra and Rueda [26]).

(d) If f_i and h_j are twice differentiable and, moreover, $K_{i}\left(u,p\right)=\frac{1}{2}p\nabla^{2}f_{i}\left(u\right)$, $\Phi\left(x,u,\left(\nabla f_{i}\left(u\right)+\nabla_{p}K_{i}\left(u,p\right),\rho_{i}\right)\right)=\left[\eta\left(x,u\right)\right]^{T}\left(\nabla f_{i}\left(u\right)+\nabla^{2}f_{i}\left(u\right)p\right)+\rho_{i}\left\Vert\theta\left(x,u\right)\right\Vert^{2},i=1,\cdots,k$. $H_{j}\left(u,q\right)=\frac{1}{2}q\nabla^{2}h_{j}\left(u\right)q$, $\Phi\left(x,u,\left(\nabla h_{j}\left(u\right)+\nabla_{p}H_{j}\left(u,p\right),\sigma_{j}\right)\right)=\left[\eta\left(x,u\right)\right]^{T}\left(\nabla h_{j}\left(u\right)+\nabla^{2}h_{j}\left(u\right)q\right)+\sigma_{j}\left\Vert\theta\left(x,u\right)\right\Vert^{2},j=1\cdots,\alpha$, where $\eta:X\times X\rightarrow R^{n}$ is a vector-valued function, $\theta:X\times X\rightarrow R^{n}$,then Definition 2.3 gives the definition of second order $V-\rho-(\eta,\theta)$-Type I functions defined in the scalar case by Padhan and Nahak [32].

(e) If $\Phi\left(x,u,\left(\nabla f_{i}\left(u\right)+\nabla_{p}K_{i}\left(u,p\right),\rho_{i}\right)\right)=F\left(x,u,\left(\nabla f_{i}\left(u\right)+\nabla_{p}K_{i}\left(u,p\right)\right)\right)+\rho_{i}d^{2}\left(x,u\right),i=1,\cdots,k,\Phi\left(x,u,\left(\nabla h_{j}\left(u\right)+\nabla_{p}H_{j}\left(u,p\right),\sigma_{j}\right)\right)=F\left(x,u,\left(\nabla h_{j}\left(u\right)+\nabla_{p}H_{j}\left(u,p\right)\right)\right)+\sigma_{j}d^{2}\left(x,u\right),j=1,\cdots,\alpha$, where a functional $F:X\times X\times R^{n}\rightarrow R$ is sublinear (in its third argument), $d:X\times X\rightarrow R$, then the definition of higher order $(\Phi,\rho,\sigma^{\alpha})$-type I functions reduced to the definition of higher-order (F,ρ,d) type I functions given by Suneja et al. [37].

(f) If f_i and h_j are twice differentiable and moreover, $K_{i}\left(u,p\right)=\frac{1}{2}p\nabla^{2}f_{i}\left(u\right)$, $\Phi\left(x,u,\left(\nabla f_{i}\left(u\right)+\nabla_{p}K_{i}\left(u,p\right),\rho_{i}\right)\right)=F\left(x,u,\left(\nabla f_{i}\left(u\right)+\nabla^{2}f_{i}\left(u\right)p\right)\right)+\rho_{i}d^{2}\left(x,u\right),i=1,\cdots,k,H_{j}\left(u,q\right)=$

$\frac{1}{2}q\nabla^2 h_j(u)q, \Phi(x,u,(\nabla h_j(u)+\nabla_p H_j(u)p,\sigma_j)) = F\left(x,u,\alpha_j^2\left(\nabla h_j(u)+\nabla^2 h_j(u)q\right)\right)+\sigma_j d^2(x,u), j=1,\cdots,\alpha,$
where $F : X\times X\times R^n \to R$ is sublinear(in its third argument), $d : X\times X \to R$, then Definition 2.3 gives the definition of second order (F,ρ_i,σ_j)-type I functions defined by Srivastava and Govil [36].

(g) If f_i and h_j are twice differentiable and, moreover, $K_i(u,p)=\frac{1}{2}p\nabla^2 f_i(u)$, $\Phi(x,u,(\nabla f_i(u)+\nabla_p K_i(u,p),\rho_i)) = F\left(x,u,\alpha_i^1(x,u)\left(\nabla f_i(u)+\nabla^2 f_i(u)p\right)\right)+\rho_i d^2(x,u), i=1,\cdots,k$, $H_j(u,q)=\frac{1}{2}q\nabla^2 h_j(u)q, \Phi(x,u,(\nabla h_j(u)+\nabla_p H_j(u,p),\sigma_j)) = F\left(x,u,\alpha_j^2(x,u)\left(\nabla h_j(u)+\nabla^2 h_j(u)q\right)\right)+\sigma_j d^2(x,u), j=1,\cdots,\alpha$, where $F : X\times X\times R^n \to R$ is sublinear (in its third argument), $\alpha_i^1 : X\times X \to R_+\{0\}, i=1,\cdots,k, \alpha_j^2 : X\times X\to R_+-\{0\}, j=1,\cdots,\alpha, d : X\times X\to R$ then Definition 2.3 gives the definition of second order (F,α,ρ,σ,d)-type I functions introduced by Hachimi and Aghezzaf [19].

(h) If $\Phi(x,u,(\nabla f_i(u)+\nabla_p K_i(u,p),\rho_i)) = F\left(x,u,\alpha^1(x,u)+\nabla_p K_i(u,p)\right)+\rho_i d^2(x,u), i = 1,\cdots,k, \Phi(x,u,(\nabla h_j(u)+\nabla_p H_j(u,p),\sigma_j)) = F\left(x,u,\alpha^2(x,u)+\nabla_p H_j(u,p)\right)+\sigma_j d^2(x,u), j = 1,\cdots,\alpha$, where a functional $F : X\times X\times R^n\to R$ is sublinear (in its third argument), $\alpha^1,\alpha^2 : X\times X\to R_+-\{0\}, d : X\times X\to R$, then the Definition 2.3 of higher order type I function reduced to the definition of higher-order (F,α,ρ,d) type I functions given by Ahmad et al. [5].

In the paper, consider the nondifferentiable semi-infinite multiobjective fractional programming problem defined by

$$\begin{aligned}&\mathit{minimize}\left(\frac{f_1(x)+s(x|C_1)}{g_1(x)-s(x|D_1)},\cdots,\frac{f_k(x)+s(x|C_k)}{g_k(x)-s(x|D_k)}\right)\\&\mathit{subject\ to}\ h\left(x,y^j\right)+s(x|E_j)\leqq 0,\ y^j\in Y,\qquad (\mathit{SIMFP})\\&x\in X,\end{aligned}$$

where X is a nonempty open convex subset of $R^n, Y\subseteq R^m$ is a nonempty compact set, $f:=(f_1,\cdots,f_k) : X\to R^k, g:=(g_1,\cdots,g_k) : X\to R^k,\ h : X\times Y\to R$ are continuously differentiable such that $f_i(x)+s(x|C_i)\geqq 0,\ g_i(x)-s(x|D_i)>0,\ i=1,\cdots,k$. Let A be the set of all feasible solutions of the problem $(SIMFP)$, that is, $A=\left\{x\in X : h\left(x,y^j\right)+s(x|E_j)\leqq 0\,\forall y^j\in Y\right\}$. For $\overline{x}\in A$, we define the set of active inequality constraints as $Y(\overline{x})=\left\{y^j\in Y : h\left(\overline{x},y^j\right)+s(\overline{x}|E_j)=0\right\}$. Note that the set $Y(\overline{x})$ can be empty. It is obvious that, for each $\overline{x}\in A$, each index $\overline{y}^j\in Y(\overline{x})$ is a global minimizer of the corresponding parameter-depending ($\overline{x}$ is the parameter) problem $\max\left\{h\left(\overline{x},y^j\right)+s(\overline{x}|E_j)\right\}$ s.t. $y^j\in Y$.

Based on the necessary optimality conditions established by Guerra-Vazquez and Ruckmann [14], Kanzi and Nobakhtian [22], Mishra and Jayswal [27] for multiobjective semi-infinite optimization problems and Husain and Z. Jabeen [17], Suneja et al. [37] established for multiobjective fractional programming with the support function, we now give the Karush-Kuhn-Tucker type necessary optimality conditions for the considered nondifferentiable semi-infinite multiobjective fractional programming problem $(SIMFP)$.

Theorem 2.1 (The Karush-Kuhn-Tucker type necessary optimality conditions)**.** *Let $\overline{x}\in A$ be a weakly efficient solution of the considered nondifferentiable semi-infinite multiobjective fractional programming problem $(SIMFP)$ and the suitable* *constraint qualification be fulfilled at* $\overline{x}$*. Then there exist* $\overline{\lambda} = \left(\overline{\lambda}_1, \cdots, \overline{\lambda}_k\right) \in R^k_+, \overline{\lambda} \neq 0$*, an integer* $\overline{\alpha}$ *such that* $0 \leqq \overline{\alpha} \leqq n, \overline{\mu} = \left(\overline{\mu_1}, \cdots, \overline{\mu}_{\overline{\alpha}} \in R^{\alpha}_+\right)$ *and* $y^j \in Y\left(\overline{x}\right)$ *such that*

$$\sum_{i=1}^{k} \overline{\lambda}_i \nabla \left(\frac{f_i(\overline{x}) + z_i^T \overline{x}}{g_i\left(\overline{x}\right) - v_i^T \overline{x}}\right) + \sum_{j=1}^{\overline{\alpha}} \overline{\mu}_j \nabla \left(h\left(\overline{x}, y^j\right) + w_j^T \overline{x}\right) = 0, \tag{2.1}$$

$$\sum_{j=1}^{\overline{\alpha}} \overline{\mu}_j \left(h\left(\overline{x}, y^j\right) + w_j^T \overline{x}\right) = 0, \tag{2.2}$$

$$z_i^T \overline{x} = s\left(\overline{x}|C_i\right), i = 1, \cdots, k, \tag{2.3}$$

$$v_i^T \overline{x} = s\left(\overline{x}|D_i\right), i = 1, \cdots, k, \tag{2.4}$$

$$w_j^T \overline{x} = s\left(\overline{x}|E_j\right), j = 1, \cdots, \overline{\alpha}. \tag{2.5}$$

3. Higher order Mond-Weir duality

In this section, for the primal semi-infinite multiobjective fractional programming problem ($SIMFP$), we define its higher order dual in the sense of Mond-Weir as follows:

$$\begin{gathered}
maximize \left(\frac{f_1\left(u\right) + s\left(u|C_1\right)}{g_1\left(u\right) - s\left(u|D_1\right)}, \cdots, \frac{f_k\left(u\right) + s\left(u|C_k\right)}{g_k\left(u\right) - s\left(u|D_k\right)}\right) \\
subject\ to \sum_{i=1}^{k} \lambda_i \left(\nabla \left(\frac{f_i\left(u\right) + u^T z_i}{g_i\left(u\right) - u^T v_i}\right) + \nabla_p K_i\left(u, p\right)\right) + \\
\sum_{j=1}^{\alpha} \mu_j \left(\nabla \left(h\left(u, y^j\right) + u^T w_j\right) + \nabla_q H\left(u, y^j, q\right)\right) = 0, \\
\sum_{j=1}^{\alpha} \mu_j \left(h\left(u, y^j + u^T w_j\right) + H\left(u, y^j, q\right) - q^T \nabla_q H\left(u, y^j, q\right)\right) \geqq 0, \\
\sum_{i=1}^{k} \lambda_i \left(K_i\left(u, p\right) - p^T \nabla_p K_i\left(u, p\right)\right) \geqq 0, \\
x \in X, \lambda_i \geqq 0, i = 1, \cdots, k, \sum_{i=1}^{k} \lambda_i = 1, 0 \leq \alpha \leq n, \mu_j \geqq 0, j = 1, \cdots, \alpha, y^1, \cdots, y^{\alpha} \in Y.
\end{gathered} \tag{3.1}$$

Let Q denote the set of all feasible solutions of the problem (MWD), that is, the set of $\left(u, \lambda, \mu, \alpha, y^1, \cdots, y^{\alpha}, z, v, w, p, q\right)$ satisfying all constraints of (MWD), where $z = (z_1, \cdots, z_k), v = (v_1, \cdots, v_k), w = (w_1, \cdots, w_{\alpha})$. Further, by U denote the projection of the set Q on X, that is, the set $U = \left\{u \in X : \left(u, \lambda, \mu, \alpha, y^1, \cdots, y^{\alpha}, z, v, w, p, q\right) \in Q\right\}$.

Before we prove various higher order weak duality results in the sense of Mond-Weir, let us define the function $\varphi = (\varphi_1, \cdots, \varphi_k) : X \to R^k$ such that $\varphi_i\left(a\right) = \frac{f_i(a) + (a)^T z_i}{g_i(a) - (a)^T v_i}, i = 1, \cdots, k$ and the function $\psi := (\psi_1, \cdots, \psi_{\alpha}) : X \to R^{\alpha}$ such that $\psi_j\left(a\right) = h_i\left(a, y^j\right) + a^T w_j, j = 1, \cdots, \alpha$.

Theorem 3.1 (Higher order weak duality). *Let* x *and* $(u,\lambda,\mu,\alpha,y^1,\cdots,y^{\alpha},z,v,w,p,q)$ *be any feasible solutions of the problems* ($SIMFP$) *and* (MWD)*, respectively. Fur-*

ther, assume that the pair (φ, ψ) is higher order $(\Phi, \rho, \sigma^{\alpha})$-type I functions at u on $A \cup U$ with respect to K and H. If $\sum_{i=1}^{k} \lambda_i \rho_i + \sum_{j=1}^{\alpha} \mu_j \sigma_{y^j} \geqq 0$, then the inequality

$$\left(\frac{f_1(x) + x^T z_1}{g_1(x) - x^T v_1}, \cdots, \frac{f_k(x) + x^T z_k}{g_k(x) - x^T v_k} \right) < \left(\frac{f_1(u) + u^T z_1}{g_1(u) - u^T v_1}, \cdots, \frac{f_k(u) + u^T z_k}{g_k(u) - u^T v_k} \right). \tag{3.2}$$

Proof. Let x and $(u, \lambda, \mu, \alpha, y^1, \cdots, y^{\alpha}, z, v, w, p, q)$ be any feasible solutions of the problems $(SIMFP)$ and (MWD), respectively. We proceed by contradiction. Suppose, contrary to the result, that the inequalities

$$\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} < \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i}, i = 1, \cdots, k \tag{3.3}$$

hold. From the assumption, the pair (φ, ψ) is higher order $(\Phi, \rho, \sigma^{\alpha})$-type I functions at u on $A \cup U$ with respect to K and H. Hence, by Definition 2.3, (3.3) implies that

$$\Phi\left(x, u, \left(\nabla\left(\frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i}\right) + \nabla_p K_i(u,p), \rho_i\right)\right) + K_i(u,p) - p^T \nabla_p K_i(u,p) < 0, \quad i = 1, \cdots, k. \tag{3.4}$$

Since $\lambda_i \geqq 0, i = 1, \cdots, k$, and $\lambda \neq 0$, (3.4) yields

$$\sum_{i=1}^{k} \left[\Phi\left(x, u, \left(\nabla\left(\frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i}\right) + \nabla_p K_i(u,p), \rho_i\right)\right) + K_i(u,p) - p^T \nabla_p K_i(u,p) \right] < 0.$$

By the third constraint of the problem (MWD), the above inequality gives

$$\sum_{i=1}^{k} \lambda_i \Phi\left(x, u, \left(\nabla\left(\frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i}\right) + \nabla_p K_i(u,p), \rho_i\right)\right) < 0. \tag{3.5}$$

From the second inequality in Definition 2.3, we have

$$\begin{aligned} & - h(u, y^j) - u^T w_j - H(u, y^j, q) + q^T \nabla_q H(u, y^j, q) \\ & \geqq \Phi\left(x, u, \left(\nabla\left(h(u, y^j) + u^T w_j\right) + \nabla_q H(u, y^j, q), \sigma_{y^j}\right)\right), j = 1, \cdots, \alpha. \end{aligned} \tag{3.6}$$

Since $\mu_j \geqq 0$, (3.6) yields

$$\begin{aligned} & - \sum_{j=1}^{\alpha} \mu_j \left(h(u, y^j) + u^T w_j + H(u, y^j, q) - q^T \nabla_q H(u, y^j, q) \right) \\ & \geqq \sum_{j=1}^{\alpha} \Phi\left(x, u, \left(\nabla\left(h(u, y^j) + u^T w_j\right) + \nabla_q H(u, y^j, q), \sigma_y^j\right)\right). \end{aligned} \tag{3.7}$$

Then, by the second constraint of the problem (MWD), (3.7) implies

$$\sum_{j=1}^{\alpha} \mu_j \Phi\left(x, u, \left(\nabla\left(h(u, y^j) + u^T w_j\right) + \nabla_q H(u, y^j, q), \sigma_{y^j}\right)\right) \leqq 0. \tag{3.8}$$

Let us set that

$$\beta_i = \frac{\lambda_i}{\sum_{i=1}^{k} \lambda_i + \sum_{j=1}^{\alpha} \mu_j}, i = 1, \cdots, k, \vartheta_j = \frac{\mu_j}{\sum_{i=1}^{k} \lambda_i + \sum_{j=1}^{\alpha} \mu_j}, \ j = 1, \cdots, \alpha. \tag{3.9}$$

Hence, (3.9) implies that $\beta_i \geqq 0, i = 1, \cdots, k$, and for at least one $i, \beta_i > 0, \vartheta_j \geqq 0, j = 1, \cdots, \alpha$ and, moreover, $\sum_{i=1}^{k} \beta_i + \sum_{j=1}^{\alpha} \vartheta_j = 1$. Using (3.9) together with (3.5) and (3.8), we get

$$\sum_{i=1}^{k} \beta_i \Phi \left(x, u, \left(\nabla \left(\frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) + \nabla_p K_i\left(u, p\right), \rho_i \right) \right)$$
$$+ \sum_{j=1}^{\alpha} \vartheta_j \Phi \left(x, u, \left(\nabla \left(h\left(u, y^j\right) + u^T w_j \right) + \nabla_q H\left(u, y^j, q\right), \sigma_{y^j} \right) \right) < 0. \tag{3.10}$$

Since $\sum_{i=1}^{k} \beta_i + \sum_{j=1}^{\alpha} \vartheta_j = 1$ and Φ is a real-valued convex function defined on $X \times X \times R^{n+1}$, by the definition of convexity, (3.10) implies

$$\Phi \left(x, u, \left(\sum_{i=1}^{k} \beta_i \left[\nabla \left(\frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) + \nabla_p K_i\left(u, p\right) \right] + \sum_{j=1}^{\alpha} \vartheta_j \left[\nabla \left(h\left(u, y^j\right) + u^T w_j \right) \right.\right.\right.$$
$$\left.\left.\left. + \nabla_q H\left(u, y^j, q\right) \right], \sum_{i=1}^{k} \beta_i \rho_i + \sum_{j=1}^{\alpha} \vartheta_j \sigma_{y^j} \right) \right) < 0.$$

Then, by the first constraint of the problem (MWD), the above inequality gives

$$\Phi \left(x, u, \left(0, \sum_{i=1}^{k} \beta_i \rho_i + \sum_{j=1}^{\alpha} \vartheta_j \sigma_{y^j} \right) \right) < 0. \tag{3.11}$$

By assumption $\sum_{i=1}^{p} \lambda_i \rho_i + \sum_{j=1}^{\alpha} \mu_j \rho_{y^j} \geqq 0$. Then, (3.9) implies that

$$\sum_{i=1}^{p} \beta_i \rho_i + \sum_{j=1}^{\alpha} \vartheta_j \sigma_{y^j} \geqq 0. \tag{3.12}$$

Since $\Phi\left(x, u, (0, a)\right) \geqq 0$ for and $a \geqq 0$, (3.12) implies that the inequality

$$\Phi \left(x, u, \left(0, \sum_{i=1}^{k} \beta_i \rho_i + \sum_{j=1}^{\alpha} \vartheta_j \sigma_{y^j} \right) \right) \geqq 0$$

holds, contradicting (3.11). Hence, the proof of this theorem is completed.

Under the stronger assumption imposed on the functions constituting the considered vector optimization problems, the following result can be proved. □

Theorem 3.2 (Higher order weak duality). *Let x and $(u,\lambda,\mu,\alpha,y^1, \cdots ,y^\alpha,z,v,w,p,q)$ be any feasible solutions of the problems $(SIMFP)$ and (MWD), respectively. Further, assume that the pair (φ, ψ) is higher order strictly $(\Phi, \rho, \sigma^\alpha)$-type I functions* *at u on $A \cup U$ with respect to K and H. If $\sum_{i=1}^{p} \lambda_i \rho_i + \sum_{j=1}^{\alpha} \mu_j \rho_{y^j} \geqq 0$, then the inequality*

$$\left(\frac{f_1(x) + x^T z_1}{g_1(x) - x^T v_1}, \cdots, \frac{f_k(x) + x^T z_k}{g_k(x) - x^T v_k}\right) \leq \left(\frac{f_1(u) + x^T z_1}{g_1(u) - u^T v_1}, \cdots, \frac{f_k(u) + u^T z_k}{g_k(u) - u^T v_k}\right)$$

does not hold.

Now, we prove higher order strong duality theorem in the sense of Mond-Weir.

Theorem 3.3 (Higher order strong duality)**.** *Let $\overline{x} \in A$ be a (weakly) efficient solution of the semi-infinite multiobjective fractional programming problem $(SIMFP)$, and the suitable constraint qualification be fulfilled at $\overline{x}$. Then, there exist $\overline{\lambda} = \left(\overline{\lambda}_1, \cdots, \overline{\lambda}_k\right) \in R_+^k, \overline{\lambda} \neq 0$, an integer α such that $0 \leqq \overline{\alpha} \leqq n, \overline{\mu} = \left(\overline{\mu}_1, \cdots, \overline{\mu}_{\overline{\alpha}}\right) \in R_+^{\overline{\alpha}}$ and $\overline{y}^j \in Y(\overline{x})$, such that if $K_i(\overline{x}, 0) = 0, i = 1, \cdots, k, H\left(\overline{x}, 0, \overline{y}^j\right) = 0, j = 1, \cdots, \overline{\alpha}, \left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\alpha}, \overline{y}^1, \cdots, \overline{y}^{\overline{\alpha}}, \overline{z}, \overline{v}, \overline{w}, \overline{p} = 0, \overline{q} = 0\right)$, is a feasible solution in the problem (MWD) and the corresponding values of the objective functions of the problems $(SIMFP)$ and (MWD) are equal. Further, if all hypotheses of the weak duality Theorem (3.1,3.2) are satisfied, then $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\alpha}, \overline{y}^1, \cdots, \overline{y}^{\overline{\alpha}}, \overline{z}, \overline{v}, \overline{w}, \overline{p} = 0, \overline{q} = 0\right)$ is an (weakly) efficient solution of a maximum type for the problem (MWD).*

Proof. Since $\overline{x} \in A$ is a (weakly) efficient solution of the semi-infinite multiobjective fractional programming problem $(SIMFP)$, by Theorem 2.1, there exist $\overline{\lambda} = \left(\overline{\lambda}_1, \cdots, \overline{\lambda}_k\right) \in R_+^k, \overline{\lambda} \neq 0$, an integer $\overline{\alpha}$ such that $0 \leqq \overline{\alpha} \leqq n, \overline{\mu} = \left(\overline{\mu}_1, \cdots, \overline{\mu}_{\overline{\alpha}}\right) \in R_+^{\alpha}$ and $y^j \in Y(\overline{x})$ an integer such that the conditions (2.1)-(2.5) are fulfilled. This means that the solution $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\alpha}, \overline{y}^1, \cdots, \overline{y}^{\overline{\alpha}}, \overline{z}, \overline{v}, \overline{w}, \overline{p} = 0, \overline{q} = 0\right)$ satisfies these conditions. Since $K_i(\overline{x}, 0) = 0, i = 1, \cdots, k, H\left(\overline{x}, 0, \overline{y}^j\right) > 0, j = 1, \cdots, \overline{\alpha}$, by the conditions (2.1)-(2.5), it also follows that the solution $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\alpha}, \overline{y}^1, \cdots, \overline{y}^{\overline{\alpha}}, \overline{z}, \overline{v}, \overline{w}, \overline{p} = 0, \overline{q} = 0\right)$ is feasible for the problem (MWD). We now show that if $\overline{x}$ is a weakly efficient solution of the problem $(SIMFP)$, then $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\alpha}, \overline{y}^1, \cdots, \overline{y}^{\overline{\alpha}}, \overline{z}, \overline{v}, \overline{w}, \overline{p} = 0, \overline{q} = 0\right)$ is a weakly efficient solution of a maximum type for the problem (MWD). By means of contradiction, suppose that $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\alpha}, \overline{y}^1, \cdots, \overline{y}^{\overline{\alpha}}, \overline{z}, \overline{v}, \overline{w}, \overline{p} = 0, \overline{q} = 0\right)$ is not a weakly efficient solution of a maximum type for the problem (MWD). Then, by definition, there does exist $\left(\widetilde{u}, \widetilde{\lambda}, \widetilde{\mu}, \widetilde{\alpha}, \widetilde{y}^1, \cdots, \widetilde{y}^{\widetilde{\alpha}}, \widetilde{z}, \widetilde{v}, \widetilde{w}, \widetilde{p}, \widetilde{q}\right)$ such that the inequalities

$$\frac{f_i(\overline{x}) + \overline{x}^T \widetilde{z}_i}{g_i(\overline{x}) - \overline{x}^T v_i} < \frac{f_i(\widetilde{u}) + \widetilde{u}^T \widetilde{z}_i}{g_i(\widetilde{u}) - \widetilde{u}^T \widetilde{v}_i}, \qquad i = 1, \cdots, k$$

hold which is a contradiction to the higher order weak duality theorem 3.1. The proof of a efficiency of a maximum type for the problem (MWD) is similar. Thus, the proof of this theorem is completed. $\square$

Theorem 3.4 (Higher order strict converse duality)**.** *Let $\overline{x}$ and $(\overline{u}, \overline{\lambda}, \overline{\mu}, \overline{\alpha}, \overline{y}^1, \cdots, \overline{y}^{\overline{\alpha}}, \overline{z}, \overline{v}, \overline{w}, \overline{p}, \overline{q})$ be feasible solutions of the problems $(SIMFP)$ and (MWD), respectively, such that*

$$\frac{f_i(\overline{x}) + \overline{x}^T \overline{z_i}}{g_i(\overline{x}) - \overline{x}^T \overline{v_i}} \leqq \frac{f_i(\overline{u}) + \overline{u}^T \overline{z_i}}{g_i(\overline{u}) - \overline{u}^T \overline{v_i}} - \sum_{j=1}^{\overline{\alpha}} \overline{\mu_j} \left(h\left(\overline{u}, \overline{y}^j\right) + \overline{u}^T \overline{w_j} + H\left(\overline{u}, \overline{y}^j, q\right) - q^T \nabla_q H\left(\overline{u}, \overline{y}^j, q\right)\right). \tag{3.13}$$

Further, let the function $\varphi := (\varphi_1, \cdots, \varphi_k) : X \to R^k$ *be defined by* $\varphi_i(a) = \frac{f_i(a)+a^T\overline{z_i}}{g_i(a)-a^T\overline{v_i}}, i = 1, \cdots, k$, *and the function* $\psi := (\psi_1, \cdots, \psi_k) : X \to R^\alpha$ *be defined by* $\psi_j(a) = h_i\left(a, \overline{y}^j\right) + a^T\overline{w}_j, j = 1, \cdots, \overline{\alpha}$. *Furthermore, assume that the pair* (ϕ, ψ) *is higher order strictly* $\left(\Phi, \rho, \sigma^{\overline{\alpha}}\right)$*-type I at* $\overline{u}$ *on* $A \cup U$ *with respect to* K *and* H. *If* $\sum_{i=1}^{p} \overline{\lambda}_i\rho_i + \sum_{j=1}^{\overline{\alpha}} \overline{\mu}_j\rho_{\overline{y}^j} \geqq 0$, *then* $\overline{x} = \overline{u}$.

Proof. By means of contradiction, suppose that $\overline{x} \neq \overline{u}$ .By the assumption, the pair (φ, ψ) is higher order $\left(\Phi, \rho, \sigma^{\overline{\alpha}}\right)$-type I at $\overline{u}$ on $A \cup U$ with respect to K and H. Then, by Definition (2.3), (3.13) gives that

$$-\sum_{j=1}^{\alpha} \overline{\mu}_j \left(h\left(\overline{u}, \overline{y}^j\right) + \overline{u}^T\overline{w}^j + H\left(\overline{u}, \overline{y}^j, q\right) - q^T\nabla_q H\left(\overline{u}, \overline{y}^j, q\right)\right)$$
$$>\Phi\left(\overline{x}, \overline{u}, \left(\nabla\left(\frac{f_i(\overline{u}) + \overline{u}^T\overline{z_i}}{g_i(\overline{u}) - \overline{u}^T\overline{v_i}}\right) + \nabla_p K_i(\overline{u}, p), \rho_i\right)\right) + K_i(\overline{u}, p) - p^T K_i(\overline{u}, p),$$
$$i = 1, \cdots, k.$$

Then, by the second constraint of the problem (MWD), the above inequality yields

$$\Phi\left(\overline{x}, \overline{u}, \left(\nabla\left(\frac{f_i(\overline{u}) + \overline{u}^T\overline{z_i}}{g_i(\overline{u}) - \overline{u}^T\overline{v_i}}\right) + \nabla_p K_i(\overline{u}, p), \rho_i\right)\right) + K_i(\overline{u}, p) - p^T\nabla_p K_i(\overline{u}, p) < 0,$$
$$i = 1, \cdots, k. \tag{3.14}$$

Since $\overline{\lambda}_i \geqq 0, i = 1, \cdots, k$ and $\overline{\lambda} \neq 0$, (3.14) gives

$$\sum_{i=1}^{k} \overline{\lambda_i} \left[\Phi\left(\overline{x}, \overline{u}, \left(\nabla\left(\frac{f_i(\overline{u}) + \overline{u}^T\overline{z_i}}{g_i(\overline{u}) - \overline{u}^T\overline{v_i}}\right) + \nabla_p K_i(\overline{u}, p), \rho_i\right)\right) + K_i(\overline{u}, p) - p^T\nabla_p(\overline{u}, p)\right] < 0.$$

By the third constraint of the problem (MWD), the above inequality yields

$$\sum_{i=1}^{k} \lambda_i \left(\Phi\left(\overline{x}, \overline{u}, \left(\nabla\left(\frac{f_i(\overline{u}) + \overline{u}^T\overline{z_i}}{g_i(\overline{u}) - \overline{u}^T\overline{v_i}}\right) + \nabla_p K_i(\overline{u}, p), \rho_i\right)\right)\right) < 0. \tag{3.15}$$

Using the second inequality in Definition(2.4),we obtain

$$-h\left(\overline{u}, \overline{y}^j\right) - \overline{u}^T\overline{w}^j - H\left(\overline{u}, \overline{y}^j, q\right) - q^T\nabla_q H\left(\overline{u}, \overline{y}^j, q\right) \geqq$$
$$\Phi\left(\overline{x}, \overline{u}, \left(\nabla\left(h\left(\overline{u}, \overline{y}^j\right) + \overline{u}^T\overline{w}^j\right) + \nabla_q H\left(\overline{u}, \overline{y}^j, q\right), \sigma_{\overline{y}^j}\right)\right), j = 1, \cdots, \alpha. \tag{3.16}$$

Since $\overline{\mu}_j \geqq 0, j = 1, \cdots, \overline{\alpha}$, (3.16) implies

$$-\sum_{j=1}^{\overline{\alpha}} \overline{\mu_j} \left(h\left(\overline{u}, \overline{y}^j\right) + \overline{u}^T\overline{w}^j + H\left(\overline{u}, \overline{y}^j, q\right) - q^T H\left(\overline{u}, \overline{y}^j, q\right)\right) \geqq$$
$$\sum_{j=1}^{\overline{\alpha}} \mu_j \Phi\left(\overline{x}, \overline{u}, \left(\nabla\left(h\left(\overline{u}, \overline{y}^j\right) + \overline{u}^T\overline{w}_j\right) + \nabla_q H\left(\overline{u}, \overline{y}^j, q\right), \sigma_{\overline{y}^j}\right)\right). \tag{3.17}$$

Then, by the second constraint of the problem (MWD), (3.7) implies

$$\sum_{j=1}^{\overline{\alpha}} \mu_j \Phi\left(\overline{x}, \overline{u}, \left(\nabla\left(h\left(\overline{u}, \overline{y}^j\right) + \overline{u}^T\overline{w}_j\right) + \nabla_q H\left(\overline{u}, \overline{y}^j, q\right), \sigma_{\overline{y}^j}\right)\right) \leqq 0. \tag{3.18}$$

Let us define

$$\overline{\beta}_i = \frac{\overline{\lambda}_i}{\sum_{i=1}^{k} \overline{\lambda}_i + \sum_{i=1}^{\overline{\alpha}} \overline{\mu}_j}, i = 1, \cdots, k, \quad \overline{\vartheta}_j = \frac{\overline{\mu}_j}{\sum_{i=1}^{k} \overline{\lambda}_i + \sum_{i=1}^{\overline{\alpha}} \overline{\mu}_j}, j = 1, \cdots, \overline{\alpha}. \tag{3.19}$$

Hence, (3.19) implies that $\overline{\beta}_i \geqq 0, i = 1, \cdots, k,$ and for at least one $i, \beta_i > 0, \overline{\vartheta}_j \geqq 0, j = 1, \cdots, \overline{\alpha}$ and, moreover,$\sum_{i=1}^{k} \overline{\beta}_i + \sum_{j=1}^{\overline{\alpha}} \overline{\vartheta}_j = 1$. Using (3.19) together with (3.15) and (3.18),we get

$$\sum_{i=1}^{k} \overline{\beta}_i \Phi\left(\overline{x}, \overline{u}\left(\nabla\left(\frac{f_i(\overline{u})+\overline{u}^T\overline{z}_i}{g_i(\overline{u})-\overline{u}^T\overline{v}_i}\right)+\nabla_p K_i(\overline{u},p), \rho_i\right)\right)+\sum_{j=1}^{\overline{\alpha}} \overline{\vartheta}_j \Phi\left(\overline{x}, \overline{u}\left(\nabla\left(h(\overline{u},\overline{y}^j)\right.\right.\right.$$
$$\left.\left.\left. + \ \overline{u}^T\overline{w}_j\right) + \nabla_q H(\overline{u},\overline{y}^j,q), \sigma_{y^j}\right)\right) < 0. \tag{3.20}$$

Since $\sum_{i=1}^{k} \overline{\beta}_i + \sum_{j=1}^{\overline{\alpha}} \overline{\vartheta}_j = 1$ and Φ is a real-valued convex function defined on $X \times X \to R^{n+1}$, by the definition of convexity, (3.20) gives

$$\Phi\left(\overline{x}, \overline{u}\left(\sum_{k}^{i=1} \overline{\beta}_i \left[\nabla\left(\frac{f_i(\overline{u}) + \overline{u}^T\overline{z}_i}{g_i(\overline{u}) - \overline{u}^T\overline{v}_i}\right) + \nabla_p K_i(\overline{u},p)\right] + \sum_{j=1}^{\overline{\alpha}} \overline{\vartheta}_j \left[\nabla\left(h(\overline{u},\overline{y}^j)\right.\right.\right.$$
$$\left.\left.\left. + \ \overline{u}^T\overline{w}_j\right) + \nabla_q H(\overline{u},\overline{y}^j,q)\right], \sum_{i=1}^{k} \overline{\beta}_i\rho_i + \sum_{j=1}^{\alpha} \overline{\vartheta}_j\sigma_{\overline{y}^j}\right)\right) < 0.$$

Thus, by the first constraint of the problem (MWD), the above inequality implies

$$\Phi\left(\overline{x}, \overline{u}, \left(0, \sum_{i=1}^{k} \overline{\beta}_i\rho_i + \sum_{j=1}^{\overline{\alpha}} \overline{\vartheta}_j\sigma_{\overline{y}^j}\right)\right) < 0. \tag{3.21}$$

By assumption, $\sum_{i=1}^{k} \overline{\lambda}_i + \sum_{j=1}^{\overline{\alpha}} \overline{\mu}_j > 0$. Hence, (3.19) yields that

$$\sum_{i=1}^{k} \overline{\beta}_i\rho_i + \sum_{j=1}^{\overline{\alpha}} \overline{\vartheta}_j\sigma_{\overline{y}^j} \geqq 0. \tag{3.22}$$

Since $\Phi(\overline{x}, \overline{u}, (0, a)) \geqq 0$ for any $a \geqq 0$, (3.2) implies that the inequality

$$\Phi\left(\overline{x}, \overline{u}, \left(0, \sum_{i=1}^{k} \overline{\beta}_i\rho_i + \sum_{j=1}^{\alpha} \overline{\vartheta}_j\sigma_{\overline{y}^j}\right)\right) \geqq 0$$

holds, contradicting (3.21). Hence, the proof of this theorem is completed. □

4. Higher order Schaible type dual

In this section, for the primal semi-infinite multiobjective fractional programming problem $(SIMFP)$, we formulate its higher order Schaible dual problem as follows

maximize $(\tau_1, \cdots, \tau_k)$

$$\text{s.t.}\nabla\left\{\sum_{i=1}^{k}\lambda_i\left[f_i\left(u\right)+u^Tz_i-\tau_i\left(g_i\left(u\right)-u^Tv_i\right)\right]+\sum_{j=1}^{\alpha}\mu_j\left(\nabla\left(h\left(u,y^j\right)+u^Tw_j\right)\right)\right\}$$

$$+\sum_{i=1}^{k}\lambda_i\left(\nabla_pK_i\left(u,p\right)-\tau_i\nabla_pG_i\left(u,p\right)\right)+\sum_{j=1}^{\alpha}\mu_j\nabla_qH\left(u,y^j,q\right)=0,$$

$$\sum_{i=1}^{k}\lambda_i\left\{\left[f_i\left(u\right)+u^Tz_i-\tau_i\left(g_i\left(u\right)-u^Tv_i\right)\right]+\left(K_i\left(u,p\right)-\tau_iG_i\left(u,p\right)\right)\right.$$
$$\left.-p^T\nabla_p\left(K_i\left(u,p\right)-\tau_iG_i\left(u,p\right)\right)\right\}\geqq 0,$$

$$\sum_{j=1}^{\alpha}\mu_j\nabla\left(h\left(u,y^j\right)+u^Tw_j+H\left(u,y^j,q\right)-q^T\nabla_qH\left(u,y^j,q\right)\right)\geqq 0,$$

$$x\in X,\lambda_i\geqq 0,i=,\cdots,p,\lambda e=1,0\leqq\alpha\leqq n,\mu_j\geqq 0,j=1,\cdots,\alpha,\tau\geqq 0,y^1,\cdots,y^\alpha\in Y.$$

Let S denote the set of all feasible solutions of the problem (SD), that is, the set of $\left(u,\lambda,\mu,\alpha,y^1,\cdots,y^\alpha,z,v,w,p,q\right)$ satisfying all constraints of (SD), where $z=(z_1,\cdots,z_k),v=(v_1,\cdots,v_k),w=(w_1,\cdots,w_\alpha)$. Further, by U denote the projection of the set S on X, that is, the set $U=\left\{u\in X:\left(u,\lambda,\mu,\tau,\alpha,y^1,\cdots,y^\alpha,z,v,w,p,q\right)\in S\right\}$.

Further, let us define the function $\varphi:=(\varphi_1,\cdots,\varphi_{2k}):X\to R^{2k}$ such that $\varphi_i\left(a\right)=f_i\left(a\right)+a^Tz_i,i=1,\cdots,k$ and $\varphi_{k+i}\left(a\right)=-\left(g_i\left(a\right)-a^Tv_i\right),i=1,\cdots,k$, the function $\psi:=(\psi_1,\cdots,\psi_\alpha):X\to R^\alpha$ such that $\psi_j\left(a\right)=h_i\left(a,y^j\right)+a^Tw_j,j=1,\cdots,\alpha$, and, moreover, $K_G=(K_{G_1},\cdots,K_{G_{2k}}):R^n\times R^n\to R^{2k}$, where $K_{G_i}\left(a,p\right)=K_i\left(a,p\right),i=1,\cdots,k,K_{G_{k+i}}\left(a,p\right)=-G_i\left(a,p\right),i=1,\cdots,k$.

Theorem 4.1 (Weak duality). *Let x and $\left(u,\lambda,\mu,\tau,\alpha,y^1,\cdots,y^\alpha,z,v,w,p,q\right)$ be any feasible solutions of the problems $(SIMFP)$ and (SD), respectively. Further, assume that the pair (φ,ψ) is higher order $(\Phi,\rho,\sigma^\alpha)$-type I at u on $A\cup U$ with respect to K_G and H, where $\rho=(\rho_1,\cdots,\rho_{2k})=\left(\rho_{f_1},\cdots,\rho_{f_k},\rho_{g_1,\cdots,\rho_{g_k}}\right)\in R^{2k}$.If $\sum_{i=1}^{k}\lambda_i\rho_{f_i}+\sum_{i=1}^{k}\lambda_i\tau_i\rho_{g_i}+\sum_{j=1}^{\alpha}\mu_j\rho_{y^j}\geqq 0$, then the inequality*

$$\left(\frac{f_1\left(x\right)+x^Tz_1}{g_1\left(x\right)-x^Tv_1},\cdots,\frac{f_k\left(x\right)+x^Tz_k}{g_k\left(x\right)-x^Tv_k}\right)<(\tau_1,\cdots,\tau_k) \tag{4.1}$$

doesnt hold.

Proof. Let x and $\left(u,\lambda,\mu,\tau,\alpha,y^1,\cdots,y^\alpha,z,v,w,p,q\right)$ be any feasible solutions of the problems $(SIMFP)$ and (SD),respectively. We proceed by contradiction. Suppose, contrary to the result, that the inequalities

$$\frac{f_i\left(x\right)+x^Tz_i}{g_i\left(x\right)-x^Tv_i}<\tau_i,i=\cdots,k \tag{4.2}$$

hold. Hence, (4.2) yields

$$f_i\left(x\right)+x^Tz_i-\tau_i(g_i\left(x\right)-x^Tv_i)<0,i=1,\cdots,k. \tag{4.3}$$

Multiplying each inequality (4.3) by $\lambda_i\geqq 0,i=1,\cdots,k,\lambda^Te=1$, and then adding both sides of the resulting inequalities, we get

$$\sum_{i=1}^{k}\lambda_i\left[f_i\left(x\right)+x^Tz_i-\tau_i\left(g_i\left(x\right)-x^Tv_i\right)\right]<0,i=1,\cdots,k. \tag{4.4}$$

From the assumption, the pair (ϕ,φ) is higher order $(\Phi,\rho,\sigma^{\alpha})$-type I at u on $A\cup U$ with respect to K_G and H, where $\rho=(\rho_1,\cdots,\rho_{2k})=(\rho_{f_1},\cdots,\rho_{f_k},\rho_{g_1},\cdots,\rho_{g_k})\in R^{2k}$. Hence, by Definition(2.3), we have that

$$f_i(x)+x^Tz_i-\left(f_i(u)+u^Tz_i\right)-K_i(u,p)+p^T\nabla_pK_i(u,z)\geqq \Phi\left(x,u,\left(\nabla\left(f_i(u)+u^Tz_i\right)+\nabla_pK_i(u,p),\rho_{f_i}\right)\right),i=1,\cdots,k, \tag{4.5}$$

$$-\left(g_i(x)-x^Tv_i\right)+\left(g_i(u)-u^Tv_i\right)+G_i(u,p)-p^T\nabla_pG_i(u,z)\geqq \Phi\left(x,u,\left(-\nabla\left(g_i(u)+u^Tv_i\right)-\nabla_pG_i(u,p),\rho_{g_i}\right)\right),i=1,\cdots,k, \tag{4.6}$$

$$-h\left(u,y^j\right)-H\left(u,y^j,q\right)+q^T\nabla_qH\left(u,y^j,q\right)\geqq \Phi\left(x,u,\left(\nabla h\left(u,y^j\right)+\nabla_qH\left(u,y^j,q\right),\sigma_{y^j}\right)\right),j=1,\cdots,\alpha. \tag{4.7}$$

Multiplying each inequality (4.5) by λ_i, each inequality (4.6) by $\lambda_i\tau_i$ and each inequality (4.7) by μ_j and then adding both sides of the resulting inequalities, we get

$$\sum_{i=1}^{k}\lambda_i\left[f_i(x)+x^Tz_i-\tau_i\left(g_i(x)-x^Tv_i\right)\right]-\sum_{i=1}^{k}\lambda_i\Big\{\left[f_i(u)+u^Tz_i-\tau_i\left(g_i(u)-u^Tv_i\right)\right] + (K_i(u,p)-\tau_iG_i(u,p))-p^T\nabla_p(K_i(u,p)-\tau_iG_i(u,p))\Big\}-\sum_{j=1}^{\alpha}\mu_jh\left(u,y^j\right)+u^Tw_j + H\left(u,y^j,q\right)-q^T\nabla_qH\left(u,y^j,q\right)\geqq\sum_{i=1}^{k}\lambda_i\left[\Phi\left(x,u,\left(\nabla\left(f_i(u)+u^Tz_i\right)+\nabla_pK_i(u,p),\rho_{f_i}\right)\right) +\tau_i\Phi\left(x,u,\left(-\nabla\left(g_i(u)-u^Tv_i\right)\nabla_pG_i(u,p),\rho_{g_i}\right)\right)\right]+\sum_{j=1}^{\alpha}\mu_j\Phi\left(x,u,\left(\nabla h\left(u,y^j\right)+\nabla_qH\left(u,y^j,q\right) ,\sigma_{y^j}\right)\right).$$

By the constraints of the problem (SD), it follows that

$$\sum_{i=1}^{k}\lambda_i\left[f_i(x)+x^Tz_i-\tau_i\left(g_i(x)-x^Tv_i\right)\right]\geqq\sum_{i=1}^{k}\lambda_i\left[\Phi\left(x,u,\left(\nabla\left(f_i(u)+u^Tz_i\right) +\nabla_pK_i(u,p),\rho_{f_i}\right)\right)+\tau_i\Phi\left(x,u,\left(-\nabla\left(g_i(u)-u^Tv_i\right)-\nabla_pG_i(u,p),\rho_{g_i}\right)\right)\right] +\sum_{j=1}^{\alpha}\mu_j\Phi\left(x,u,\left(\nabla h\left(u,y^j\right)+\nabla_qH\left(u,y^j,q\right),\sigma_{y^j}\right)\right). \tag{4.8}$$

Combining (4.4) and (4.8), we get

$$\sum_{i=1}^{k}\lambda_i\left[\Phi\left(x,u,\left(\nabla\left(f_i(u)+u^Tz_i\right)+\nabla_pK_i(u,p),\rho_{f_i}\right)\right) +\tau_i\Phi\left(x,u,\left(-\nabla\left(g_i(u)-u^Tv_i\right)-\nabla_pG_i(u,p),\rho_{g_i}\right)\right)\right] +\sum_{j=1}^{\alpha}\mu_j\Phi\left(x,u,\left(\nabla h\left(u,y^j\right)+\nabla_qH\left(u,y^j,q\right),\sigma_{y^j}\right)\right)<0. \tag{4.9}$$

Let us set that

$$\beta_i = \frac{\lambda_i}{\sum_{i=1}^{k} \lambda_i (1+\tau_i) + \sum_{j=1}^{\alpha} \mu_j}, \ i = 1, \cdots, k,$$
$$\vartheta_j = \frac{\mu_j}{\sum_{i=1}^{k} \lambda_i (1+\tau_i) + \sum_{j=1}^{\alpha} \mu_j}, \ j = 1, \cdots, \alpha. \tag{4.10}$$

Hence $\beta_i \geqq 0, i = 1, \cdots, k$, and for at least one $i, \beta_i > 0, \vartheta_j \geqq 0, j = 1 \cdots, \alpha$, and, moreover, $\sum_{i=1}^{k} \beta_i (1+\tau_i) + \sum_{j=1}^{\alpha} \vartheta_j = 1$. Using (4.10) in (4.9), we obtain

$$\sum_{i=1}^{k} \beta_i \left[\Phi \left(x, u, \left(\nabla \left(f_i(u) + u^T z_i \right) + \nabla_p K_i(u,p), \rho_{f_i} \right) \right) \right.$$
$$\left. + \tau_i \Phi \left(x, u, \left(-\nabla \left(g_i(u) - u^T v_i \right) - \nabla_p G_i(u,p), \rho_{g_i} \right) \right) \right]$$
$$+ \sum_{j=1}^{\alpha} \vartheta_j \Phi \left(x, u, \left(\nabla h \left(u, y^j \right) + \nabla_q H \left(u, y^j, q \right), \sigma_{y^j} \right) \right) < 0. \tag{4.11}$$

Since $\sum_{i=1}^{k} \beta_i (1+\tau_i) + \sum_{j=1}^{\alpha} \vartheta_j = 1$ and Φ is a real-valued convex function defined on $X \times X \times R^{n+1}$, by the definition of convexity, (4.11) implies

$$\Phi \left(x, u, \left(\sum_{i=1}^{k} \beta_i \left[\nabla \left(f_i(u) + u^T z_i \right) - \tau_i \nabla \left(g_i(u) - u^T v_i \right) + \nabla_p K_i(u,p) - \tau_i \nabla_p G_i(u,p) \right] \right. \right.$$
$$\left. \left. + \sum_{j=1}^{\alpha} \vartheta_j \left[\nabla \left(h\left(u, y^j\right) + u^T w_j \right) + \nabla_q H\left(u, y^j, q\right) \right], \sum_{i=1}^{k} \beta_i \rho_{f_i} + \sum_{i=1}^{k} \beta_i \tau_i \rho_{f_{g_i}} + \sum_{j=1}^{\alpha} \vartheta_j \sigma_{y^j} \right) \right) < 0.$$

Then, by the first constraint of the problem (MWD), the above inequality gives

$$\Phi \left(x, u, \left(0, \sum_{i=1}^{k} \beta_i \rho_{f_i} + \sum_{i=1}^{k} \beta_i \tau_i \rho_{f_{g_i}} + \sum_{j=1}^{\alpha} \vartheta_j \sigma_{y^j} \right) \right) < 0. \tag{4.12}$$

By assumption, $\sum_{i=1}^{k} \lambda_i \rho_{f_i} + \sum_{i=1}^{k} \lambda_i \tau_i \rho_{g_i} + \sum_{j=1}^{\alpha} \mu_j \rho_{y^j} \geqq 0$.Then,(4.10) yields that

$$\sum_{i=1}^{k} \beta_i \rho_{f_i} + \sum_{i=1}^{k} \beta_i \tau_i \rho_{f_{g_i}} + \sum_{j=1}^{\alpha} \vartheta_j \sigma_{y^j} \geqq 0. \tag{4.13}$$

Since $\Phi(x, u, (0, a)) \geqq 0$ for any $a \geqq 0$, (4.13) implies that the inequality

$$\Phi \left(x, u, \left(0, \sum_{i=1}^{k} \beta_i \rho_{f_i} + \sum_{i=1}^{k} \beta_i \tau_i \rho_{f_{g_i}} + \sum_{j=1}^{\alpha} \vartheta_j \sigma_{y^j} \right) \right) \geqq 0$$

holds, contradicting (4.12). Hence, the proof of this theorem is completed.

Under the stronger assumption imposed on the functions constituting the considered vector optimization problems, the stronger result can be proved.

Theorem 4.2 (Higher order weak duality). *Let x and $\left(u, \lambda, \mu, \tau, \alpha, y^1, \cdots, y^\alpha, z, v, w, p, q\right)$ be any feasible solutions of the problems $(SIMFP)$ and (SD), respectively. Further, assume that Further, assume that the* *pair (φ,ψ) is higher order strictly $(\Phi,\rho,\sigma^{\alpha})$-type I at u on $A\cup U$ with respect to K_G and H, where $\rho=(\rho_1,\cdots,\rho_{2k})=(\rho_{f_i},\cdots,\rho_{f_k},\rho_{g_i},\cdots,\rho_{g_k})\in R^{2k}$. If $\sum_{i=1}^{k}\lambda_i\rho_{f_i}+\sum_{i=1}^{k}\lambda_i\tau_i\rho_{g_i}+\sum_{j=1}^{\alpha}\mu_j\rho_{y^j}\geqq 0$, then the inequality*

$$\left(\frac{f_1(x)+x^Tz_1}{g_1(x)-x^Tv_1},\cdots,\frac{f_k(x)+x^Tz_k}{g_k(x)-x^Tv_k}\right)\leq(\bar{\tau}_1,\cdots,\bar{\tau}_k)$$

doesn't hold.
Now, we formulate higher order strong duality theorem in the sense of Mond-Weir.

Theorem 4.3 (Higher order strong duality). *Let $\overline{x}\in A$ be a weakly solution (an efficient solution) of the semi-infinite multiobjective fractional programming problem (SIMFP) and the suitable constraint qualification be fulfilled at $\overline{x}$. Then, there exist $\overline{\lambda}=\left(\overline{\lambda_1},\cdots,\overline{\lambda_k}\right)\in R_+^k,\overline{\lambda}\neq 0$, an integer $\overline{\alpha}$. such that $0\leqq\alpha\leqq n,\overline{\mu}=(\overline{\mu}_1,\cdots,\overline{\mu}_{\overline{\alpha}})\in R_+^{\overline{\alpha}}$ and $\overline{y}^j\in Y(\overline{x})$, such that if $K_i(\overline{x},0)=0,G_i(\overline{x},0)=0,i=1,\cdots,k,H\left(\overline{x},0,\overline{y}^j\right)=0,j=1,\cdots,\overline{\alpha},\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{\tau},\overline{\alpha},\overline{y}^1,\cdots,\overline{y}^{\overline{\alpha}},\overline{z},\overline{v},\overline{w},\overline{p}=0,\overline{q}=0\right)$ is a feasible solution in the problem (SD) and the corresponding values of the objective functions of the problems $(SIMFP)$ and (SD) are equal. Further, if all hypotheses of the weak duality theorem (4.1) or (4.2) are satisfied, then $\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{\tau},\overline{\alpha},\overline{y}^1,\cdots,\overline{y}^{\overline{\alpha}},\overline{z},\overline{v},\overline{w},\overline{p}=0,\overline{q}=0\right)$ is a weakly solution (an efficient solution) of a maximum type for the problem (SD).*

Theorem 4.4 (Strict converse duality). *Let $\overline{x}\in A$ and $\left(\overline{u},\overline{\lambda},\overline{\mu},\overline{\tau},\overline{\alpha},\overline{y}^1,\cdots,\overline{y}^{\overline{\alpha}},\right.$ $\left.\overline{z},\overline{v},\overline{w},\overline{p},\overline{q}\right)$ be feasible solutions of the problems $(SIMFP)$ and (SD), respectively, such that*

$$\left(\frac{f_1\overline{(x)}+\overline{x}^T\overline{z}_1}{g_1\overline{(x)}-\overline{x}^T\overline{v}_1}\right),\cdots,\left(\frac{f_k\overline{(x)}+\overline{x}^T\overline{z}_k}{g_k\overline{(x)}-\overline{x}^T\overline{z}_k}\right)=(\overline{\tau}_1,\cdots,\overline{\tau}_k)\,. \tag{4.14}$$

Further, assume that the pair (φ,ψ) is higher order strictly $(\Phi,\rho,\sigma^{\alpha})$-type I at $\overline{u}$ with respect to K_G and H, where $\rho=(\rho_1,\cdots,\rho_{2k})=(\rho_{f_1},\cdots,\rho_{f_k},\rho_{g_1},\cdots,\rho_{g_k})\in R^{2k}$. If $\sum_{i=1}^{k}\overline{\lambda}_i\rho_{f_i}+\sum_{i=1}^{k}\overline{\lambda}_i\overline{\tau}_i\rho_{g_i}+\sum_{j=1}^{\overline{\alpha}}\overline{\mu}_j\sigma_{\overline{y}_j}\geqq 0$, then $\overline{x}=\overline{u}$.

Proof. Let $\overline{x}$ and $\left(\overline{u},\overline{\lambda},\overline{\mu},\overline{\tau},\overline{\alpha},\overline{y}^1,\cdots,\overline{y}^{\overline{\alpha}},\overline{z},\overline{v},\overline{w},\overline{p},\overline{q}\right)$ be feasible solutions of $(SIMFP)$ and (SD), respectively, such that (4.14) is satisfied. We proceed by contradiction. Suppose, contrary to the result, that $\overline{x}=\overline{u}$. By (4.14), it follows that

$$f_i(\overline{x})+\overline{x}^T\overline{z}_i-\overline{\tau}_i\left(g_i(\overline{x})-\overline{x}^T\overline{v}_i\right)=0,\ i=1,\cdots,k. \tag{4.15}$$

Multiplying each inequality (4.15) by $\overline{\lambda}_i\geqq 0,i=1,\cdots,k,\overline{\lambda}^Te=1$, and then adding both sides of the resulting inequalities, we get

$$\sum_{i=1}^{k}\overline{\lambda}_i\left[f_i(\overline{x})+\overline{x}^T\overline{z}_i-\overline{\tau}_i\left(g_i(\overline{x})-\overline{x}^T\overline{v}_i\right)\right]=0. \tag{4.16}$$

By hypotheses, the pair (ϕ,ψ) is higher order $(\Phi,\rho,\sigma^{\alpha})$-type I at u on $A\cup U$ with respect to K_G and H, where $\rho=(\rho_1,\cdots,\rho_{2k})=(\rho_{f_1},\cdots,\rho_{f_k},\rho_{g_1},\cdots,\rho_{g_k})\in R^{2k}$ Hence, by Definition(2.4), we have that

$$f_i(\overline{x})+\overline{x}^T\overline{z}_i-\left(f_i(\overline{u})+\overline{u}^T\overline{z}_i\right)-K_i(u,p)+p^T\nabla_pK_i(\overline{u},z)$$

$$> \Phi\left(\overline{x}, \overline{u}, \left(\nabla\left(f_i\left(\overline{u}\right) + \overline{u}^T \overline{z}_i\right) + \nabla_p K_i\left(\overline{u}, p\right), \rho_{f_i}\right)\right), i = 1, \cdots, k, \tag{4.17}$$

$$\begin{aligned}&-\left(g_i\left(\overline{x}\right) + \overline{x}^T \overline{v}_i\right) + \left(g_i\left(\overline{u}\right) + \overline{u}^T \overline{v}_i\right) + G_i\left(u, p\right) - p^T \nabla_p G_i\left(\overline{u}, z\right)\\ &> \Phi\left(\overline{x}, \overline{u}, \left(-\nabla\left(g_i\left(\overline{u}\right) + \overline{u}^T \overline{v}_i\right) - \nabla_p G_i\left(\overline{u}, p\right), \rho_{g_i}\right)\right), i = 1, \cdots, k,\end{aligned} \tag{4.18}$$

$$\begin{aligned}&-h\left(u, y^j\right) - H\left(\overline{u}, \overline{y}^j, q\right) + q^T \nabla_q H\left(\overline{u}, \overline{y}^j, q\right)\\ &> \Phi\left(\overline{x}, \overline{u}, \left(\nabla h\left(\overline{u}, \overline{y}^j\right) + \nabla_q H\left(\overline{u}, y^j, q\right), \sigma_{\overline{y}^j}\right)\right), j = 1 \cdots, \overline{\alpha}.\end{aligned} \tag{4.19}$$

Multiplying each inequality (4.17) by $\overline{\lambda}_i \geqq 0, i = 1, \cdots, k$, each inequality (4.18) by $\overline{\tau}_i \overline{\lambda}_i \geqq 0, i = 1, \cdots, k$, and each inequality (4.19) by $\overline{\mu}_j \geqq 0, j = 1, \cdots, \overline{\alpha}$, and then adding both sides of the resulting inequalities, we obtain

$$\begin{aligned}&\sum_{i=1}^{k} \overline{\lambda}_i \left[f_i\left(\overline{x}\right) + \overline{x}^T z_i - \overline{\tau}_i\left(g_i\left(\overline{x}\right) - \overline{x}^T v_i\right)\right] - \sum_{i=1}^{k} \overline{\lambda}_i \left\{\left[f_i\left(\overline{u}\right) + \overline{u}^T \overline{z}_i - \overline{\tau}_i\left(g_i\left(\overline{u}\right) - \overline{u}^T \overline{v}_i\right)\right]\right.\\ &\left.+ (K_i\left(\overline{u}, p\right) - \overline{\tau}_i G_i\left(\overline{u}, p\right)) - p^T \nabla_p\left(K_i\left(\overline{u}, p\right) - \overline{\tau}_i G_i\left(\overline{u}, p\right)\right)\right\} - \sum_{j=1}^{\overline{\alpha}} \overline{\mu}_j \left(h\left(\overline{u}, \overline{y}^j\right) + \overline{u}^T \overline{w}_j\right.\\ &\left.+ H\left(\overline{u}, \overline{y}^j, q\right) - q^T \nabla_q H\left(\overline{u}, \overline{y}^j, q\right)\right) > \sum_{i=1}^{k} \overline{\lambda}_i \Phi\left[\overline{x}, \overline{u}, \left(\nabla\left(f_i\left(\overline{u}\right) + \overline{u}^T \overline{z}_i\right) + \nabla_p K_i\left(\overline{u}, p\right), \rho_{f_i}\right)\right]\\ &+ \overline{\tau}_i \Phi\left(\overline{x}, \overline{u}, \left(-\nabla\left(g_i\left(\overline{u}\right) + \overline{u}^T v_i\right) - \nabla_p G_i\left(\overline{u}, p\right), \rho_{g_i}\right)\right) + \sum_{j=1}^{\overline{\alpha}} \overline{\mu}_j \Phi\left(x, u, \left(\nabla h\left(\overline{u}, \overline{y}^j\right)\right.\right.\\ &\left.\left.+ \nabla_q H\left(\overline{u}, \overline{y}^j, q\right), \sigma_{\overline{y}^j}\right)\right).\end{aligned} \tag{4.20}$$

From the constraints of the problem (SD), it follows that

$$\begin{aligned}&\sum_{i=1}^{k} \overline{\lambda}_i \left[f_i\left(\overline{x}\right) + \overline{x}^T \overline{z}_i - \overline{\tau}_i\left(g_i\left(\overline{x}\right) - \overline{x}^T \overline{v}_i\right)\right] \geqq\\ &\sum_{i=1}^{k} \overline{\lambda}_i \left[\Phi\left(\overline{x}, \overline{u}, \left(\nabla\left(f_i\left(\overline{u}\right) + \overline{u}^T \overline{z}_i\right) + \nabla_p K_i\left(\overline{u}, p\right), \rho_{f_i}\right)\right)\right.\\ &\left.+ \tau_i \Phi\left(\overline{x}, \overline{u}, \left(-\nabla\left(g_i\left(\overline{u}\right) - \overline{u}^T \overline{v}_i\right) - \nabla_p G_i\left(\overline{u}, p\right), \rho_{g_i}\right)\right)\right]\\ &+ \sum_{j=1}^{\overline{\alpha}} \overline{\mu}_j \Phi\left(\overline{x}, \overline{u}, \left(\nabla h\left(\overline{u}, y^j\right) + \nabla_q H\left(\overline{u}, y^j, q\right), \sigma_{\overline{y}^j}\right)\right).\end{aligned} \tag{4.21}$$

By (4.16) and (4.21), we have

$$\begin{aligned}&\sum_{i=1}^{k} \overline{\lambda}_i \left[\Phi\left(\overline{x}, \overline{u}, \left(\nabla\left(f_i\left(\overline{u}\right) + \overline{u}^T \overline{z}_i\right) + \nabla_p K_i\left(\overline{u}, p\right), \rho_{f_i}\right)\right) + \overline{\tau}_i \Phi\left(\overline{x}, \overline{u}, \left(-\nabla\left(g_i\left(\overline{u}\right) - \overline{u}^T \overline{v}_i\right)\right.\right.\right.\\ &\left.\left.\left.- \nabla_p G_i\left(\overline{u}, p\right), \rho_{g_i}\right)\right)\right] + \sum_{j=1}^{\overline{\alpha}} \overline{\mu}_j \Phi\left(\overline{x}, \overline{u}, \left(\nabla h\left(\overline{u}, y^j\right) + \nabla_q H\left(\overline{u}, y^j, q\right), \sigma_{\overline{y}^j}\right)\right) < 0\end{aligned} \tag{4.22}$$

Let us set that

$$\overline{\beta}_i = \frac{\overline{\lambda_i}}{\sum_{i=1}^{k} \overline{\lambda_i}\left(1+\overline{\tau_i}\right) + \sum_{j=1}^{\overline{\alpha}} \overline{\mu_j}}, \quad i = 1, \cdots, k,$$

$$\overline{\vartheta_j} = \frac{\overline{\mu}_j}{\sum_{i=1}^{k} \overline{\lambda}_i\left(1+\overline{\tau}_i\right) + \sum_{j=1}^{\overline{\alpha}} \overline{\mu_j}}, \ j = 1, \cdots, \overline{\alpha}. \tag{4.23}$$

Hence $\overline{\beta}_i \geqq 0, i=1, \cdots, k$, and for at least one $i, \overline{\beta}_i > 0$, $\overline{\vartheta}_j \geqq 0$, $j=1, \cdots,$ α and $\sum_{i=1}^{k} \overline{\beta}_i\left(1+\overline{\tau}_i\right) + \sum_{j=1}^{\alpha} \overline{\vartheta}_j = 1$. By (4.22) and (4.23), we get

$$\begin{aligned}
&\sum_{i=1}^{k} \overline{\beta}_i \left[\Phi\left(\overline{x}, \overline{u}, \left(\nabla\left(f_i\left(\overline{u}\right) + \overline{u}^T z_i\right) + \nabla_p K_i\left(\overline{u}, p\right), \rho_{f_i}\right)\right)\right. \\
&\left. + \overline{\tau}_i \Phi\left(\overline{x}, \overline{u}, \left(-\nabla\left(g_i\left(\overline{u}\right) + \overline{u}^T v_i\right) - \nabla_p G_i\left(\overline{u}, p\right), \rho_{g_i}\right)\right)\right] \\
&+ \sum_{j=1}^{\overline{\alpha}} \overline{\vartheta}_j \Phi\left(\overline{x}, \overline{u}, \left(\nabla h\left(\overline{u}, y^j\right) + \nabla_q H\left(\overline{u}, \overline{y}^j, q\right), \sigma_{\overline{y}^j}\right)\right) < 0.
\end{aligned} \tag{4.24}$$

Since $\sum_{i=1}^{k} \overline{\beta}_i\left(1+\overline{\tau}_i\right) + \sum_{j=1}^{\alpha} \overline{\vartheta}_j = 1$ and Φ is a real-valued convex function defined on $X \times X \times R^{n+1}$, by the definition of convexity, (4.24) implies

$$\begin{aligned}
&\Phi\left(\overline{x}, \overline{u}, \left(\sum_{i=1}^{k} \overline{\beta}_i \left[\nabla\left(f_i\left(\overline{u}\right) + \overline{u}^T z_i\right) - \overline{\tau}_i \nabla\left(g_i\left(\overline{u}\right) + \overline{u}^T v_i\right) + \nabla_p K_i\left(\overline{u}, p\right) - \overline{\tau}_i \nabla_p G_i\left(\overline{u}, p\right)\right]\right.\right. \\
&\left.\left. + \sum_{j=1}^{\overline{\alpha}} \overline{\vartheta}_j \left[\nabla\left(h\left(\overline{u}, y^j\right) + \overline{u}^T w_j\right) + \nabla_q H\left(\overline{u}, \overline{y}^j, q\right)\right], \sum_{i=1}^{k} \overline{\beta}_i \rho_{f_i} + \sum_{i=1}^{k} \overline{\beta}_i \tau_i \rho_{f_{g_i}} + \sum_{j=1}^{\overline{\alpha}} \overline{\vartheta}_j \sigma_{y^j}\right)\right) < 0.
\end{aligned}$$

Then, by the first constraint of the problem (MWD), the above inequality gives

$$\Phi\left(\overline{x}, \overline{u}\left(0, \sum_{i=1}^{k} \overline{\beta}_i \rho_{f_i} + \sum_{i=1}^{k} \overline{\beta}_i \overline{\tau}_i \rho_{g_i} - \sum_{j=1}^{\alpha} \overline{\vartheta}_j \sigma_{\overline{y}^j}\right)\right) < 0. \tag{4.25}$$

By assumption, $\sum_{i=1}^{k} \overline{\lambda}_i \rho_{f_i} + \sum_{i=1}^{k} \overline{\lambda}_i \overline{\tau}_i \rho_{g_i} - \sum_{j=1}^{\overline{\alpha}} \overline{\mu}_j \rho_{\overline{y}^j} \geqq 0$. Then, (4.23) yields that

$$\sum_{i=1}^{k} \overline{\beta}_i \rho_{f_i} + \sum_{i=1}^{k} \overline{\beta}_i \overline{\tau}_i \rho_{g_i} - \sum_{j=1}^{\alpha} \overline{\vartheta}_j \sigma_{\overline{y}^j} \geqq 0. \tag{4.26}$$

Since $\Phi\left(\overline{x}, \overline{u}, (0, a)\right) \geqq 0$ for any $a \geqq 0$, (4.26) implies that the inequality

$$\Phi\left(\overline{x}, \overline{u}\left(0, \sum_{i=1}^{k} \overline{\beta}_i \rho_{f_i} + \sum_{i=1}^{k} \overline{\beta}_i \overline{\tau}_i \rho_{g_i} - \sum_{j=1}^{\alpha} \overline{\vartheta}_j \sigma_{\overline{y}^j}\right)\right) \geqq 0$$

holds, contradicting (4.25). Hence, the proof of this theorem is completed. □

References

[1] B. Aghezzaf, *Second order mixed type duality in multiobjective programming problems*, J. Math. Anal. Appl., 2003, 285, 97–106.

[2] I. Ahmad and Z. Husain, *Second order (F, α, ρ, d)-convexity and duality in multiobjective programming*, Inform. Sci., 2006, 176, 3094–3103.

[3] I. Ahmad, *Higher-order duality in nondifferentiable minimax fractional programming involving generalized convexity*, J. Inequal. Appl., 2012, 306.

[4] I. Ahmad, *Unified higher order duality in nondifferentiable multiobjective programming involving cones*, Math. Comp. Model., 2012, 55, 419–425.

[5] I. Ahmad, Z. Husain and S. Sharma, *Higher-order duality in nondifferentiable multiobjective programming*, Numer. Funct. Anal. Opt., 2007, 28, 989–1002.

[6] T. Antczak, *Sufficient optimality conditions for semi-infinite multiobjective fractional programming under $(\Phi, \rho) - V$-invexity and generalized $(\Phi, \rho) - V$-invexity*, Filomat, 2016, 30, 3649–3665.

[7] T. Antczak and V. Singh, *Optimality and duality for minmax fractional programming with support functions under $B - (p, r)$-Type I assumptions*, Math. Comp. Model., 2013, 57, 1083–1100.

[8] C. R. Bector and S. Chandra, *First and second order duality for a class of nondifferentiable fractional programming problems*, J. Inf. Optim. Sci., 1986, 7, 335–348.

[9] G. Caristi, M. Ferrara and A. Stefanescu, *Semi-infinite multiobjective programming with generalized invexity*, Math. Reports, 2010, 12, 217–233.

[10] V. Chankong and Y. Y. Haimes, *Multiobjective Decision Making: Theory and Methodology*, North-Holland, New York, 1983.

[11] X. Chen, *Higher-order symmetric duality in nondifferentiable multiobjective programming problems*, J. Math. Anal. Appl., 2004, 290, 423–435.

[12] R. Dubey and S. K. Gupta, *On duality for a second-order multiobjective fractional programming problem involving type-I functions*, Geo. Math. Journal, 2017. DOI: https://doi.org/10.1515/gmj–2017–0038.

[13] M. Ferrara and M. V. Stefanescu, *Optimality conditions and duality in multiobjective programming with invexity*, Yug. J. Oper. Res., 2008, 8, 153–165.

[14] F. Guerra-Vazquez and J. J. Ruckmann, *On proper efficiency in multiobjective semi-infinite optimization, in: H. Xu, K. L. Teo, Y. Zhang (eds.),* Optimization and Control Techniques and Applications, Springer Proceedings in Mathematics & Statistics 86, Springer-Verlag Berlin Heidelberg, 2014.

[15] R. Gupta and M. Srivastava, *Optimality and duality in multiobjective programming involving support functions*, RAIRO-Oper. Res., 2017, 51, 433–446.

[16] I. Husain and Z. Jabeen, *Second order duality for fractional programming with support functions*, Opsearch, 2004, 41, 121–135.

[17] I. Husain and Z. Jabeen, *On fractional programming containing support functions*, J. Appl. Math. Comput., 2005, 18, 361–376.

[18] Z. Husain, I. Ahmad and S. Sharma, *Second order duality for minimax fractional programming*, Optim. Lett. 2009, 3, 277–286.

[19] M. Hachimi and B. Aghezzaf, *Second order duality in multiobjective programming involving generalized type I functions*, Numer. Func. Annal. Opt., 2005, 25, 725–736.

[20] A. Jayswal, D. Kumar and R. Kumar, *Second order duality for nondifferentiable multiobjective programming problem involving* $(F, \alpha, \rho, d)-V-$*-type I functions*, Optim Lett., 2010, 4, 211–226.

[21] R. N. Kaul, S. K. Suneja and M. K. Srivastava, *Optimality criteria and duality in multiple objective optimization involving generalized invexity*, J. Optimiz. Theory App., 1994, 80, 465–482.

[22] N. Kanzi and S. Nobakhtian, *Optimality conditions for nonsmooth semi-infinite multiobjective programming*, Optim. Lett., 2014, 8, 1517–1528.

[23] M. Kapoor, S. K. Suneja and M. B. Grover, *Higher order optimality and duality in fractional vector optimization over cones*, Tam. J. Math., 2017, 48, 273–287.

[24] O. L. Mangasarian, *Second and higher-order duality in nonlinear programming*, J. Math. Anal. Appl., 1975, 51, 607–620.

[25] S. K. Mishra and N. G. Rueda, *Higher order generalized invexity and duality in non- differentiable mathematical programming*, J. Math. Anal. Appl., 2002, 272, 496–506.

[26] S. K. Mishra and N. Rueda, *Higher-order generalized invexity and duality in mathematical programming*, J. Math. Anal. Appl., 2000, 247, 173–182.

[27] S. K. Mishra and M. Jaiswal, *Optimality conditions and duality for nondifferentiable multi- objective semi-infinite programming*, Viet. J. Math., 2012, 40, 331–343.

[28] S. K. Mishra, M. Jaiswal and L.T.H. An, *Duality for nonsmooth semi-infinite programming problems*, Optim. Lett., 2012, 6, 261–271.

[29] B. Mond and T. Weir, *Generalized convexity and higher order duality*, J. Math. Sci., 1983, 16–18, 74–94.

[30] B. Mond and J. Zhang, *Higher order invexity and duality in mathematical programming, in: J.P. Crouzeix, et al. (Eds.)*, Generalized Convexity, Generalized Monotonicity: Recent Results. Kluwer Academic, 1998, 357–372.

[31] B. Mond and J. Zhang, *Duality for multiobjective programming involving second order V-invex functions, in: B. M. Glover and V. Jeyakumar (Eds.)*, Proceedings of Optimization Mini conference, 1995, 89–100.

[32] S. K. Padhan and C. Nahak, *Second-and higher-order generalized invexity and duality in mathematical programming*, Int. J. Math. Oper. Res., 2013, 5, 170–182.

[33] P. Pankaj and B. C. Joshi, *Higher order duality in multiobjective fractional programming problem with generalized convexity*, Yug. J. Oper. Res., 2017, 27, 249–264.

[34] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, 1970.

[35] Y. Singh, S. K. Mishra and K. K. Lai, *Optimality and duality for nonsmooth semi-infinite multi-objective programming with support functions*, Yug. J. Oper. Res., 2017, 27, 205–218.

[36] M. Srivastava and M. Govil, *Second order duality for multiobjective programming involving* (F, ρ, σ)*-type I functions*, Opsearch, 2000, 37, 316–326.

[37] S. K. Suneja, M. K. Srivastava and M. Bhatia, *Higher order duality in multiobjective fractional programming with support functions*, J. Math. Anal. Appl., 2008, 347, 8–17.

[38] S. K. Suneja, S. Sharma and P. Yadav, *Generalized higher-order cone-convex functions and higher-order duality in vector optimization*, Annal. Oper. Res., 2018, 269, 709–725.

[39] A. K. Tripathy and G. Devi, *Second order multi-objective mixed symmetric duality containing square root term with generalized invex function*, Opsearch, 2013, 50, 260–281.

[40] R. U. Verma, *The sufficient efficiency conditions in semiinfinite multiobjective fractional programming under higher order exponential type hybrid type invexities*, Acta Math. Sci., 2015, 35, 1437–1453.

[41] R. U. Verma, *Semi-Infinite Fractional Programming*, Infosys Science Foundation Series in Mathematical Sciences Springer Nature Singapore Pte Ltd., 2017.

[42] R. U. Verma and G. J. Zalmai, *Hanson-Antczak-type* $(\alpha, \beta, \gamma, \epsilon, \eta, \omega, \rho, \vartheta) - V$*-sonvexities in semi-infinite multiobjective fractional programs for second-order parametric duality models*, Comm. Appl. Non. Anal., 2017, 24, 61–92.

[43] X. Yang, K. L. Teo and X. Yang, *Higher order generalized convexity and duality in non-differentiable multiobjective mathematical programming*, J. Math. Anal. Appl., 2004, 297, 48–55.

[44] L. Yang, L. Yang and T. Liu, *Duality in fractional semi-infinite programming with generalized convexity*, Third International Semi-infinite Programming and Computing, Wuxi, China, 4–6 June 2010, IEEE.

[45] G. J. Zalmai, *Semiinfinite multiobjective fractional programming problems involving Hadamard directionally differentiable functions. Part II: first-order parametric models*, Trans. Math. Prog. Appl., 2013, 1, 1–34.

[46] G. J. Zalmai, *Hanson-Antczak-type generalized* $(\alpha, \beta, \gamma, \xi, \eta, \zeta, \rho, \theta) - V$*-invex functions in semi-infinite multiobjective fractional programming. Part I: Sufficient efficiency conditions*, Adv. Nonlin. Variation. Inequal., 2013, 16, 91–114.

[47] G. J. Zalmai, *Semiinfinite multiobjective fractional programming problems involving Hadamard directionally differentiable functions, part III: First-order parameter-free duality models*, Trans. Math. Prog. Appl., 2014, 2, 31–65.

[48] G. J. Zalmai and Q. Zhang, *Semiinfinite multiobjective fractional programming. Part I: Sufficient efficiency conditions*, J. App. Anal., 2010, 16, 199–224.

[49] G. J. Zalmai and Q. Zhang, *Semiinfinite multiobjective fractional programming. Part II: Duality models*, J. Appl. Anal., 2011, 17, 1–35.

[50] G. J. Zalmai and Q. Zhang, *Global parametric sufficient efficiency conditions for semiinfinite multiobjective fractional programming problems containing generalized* $(\alpha, \eta, \rho) - V$*-invex functions*, Acta Math. Appl. Sinica, 2013, 29, 63–78.

[51] G. J. Zalmai and Q. Zhang, *Parametric duality models for semiinfinite multiobjective fractional programming problems containing generalized* $(\alpha, \eta, \rho) - V$*-invex functions*, Acta Appl. Math. Sinica, English Serie, 2013, 29, 225–240.

[52] Z. Zhang, *Generalized convexity and higher order duality for mathematical programming problems*, Ph.D. thesis, La Trobe University, Australia, 1998.