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A GENERAL STUDY ON RANDOM INTEGRO-DIFFERENTIAL EQUATIONS OF ARBITRARY ORDER

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Abstract Here the broad study is depending on random integro-differential equations (RIDE) of arbitrary order. The fractional order is in terms of ψ-Hilfer fractional operator. This work reveals the dynamical behaviour such as existence, uniqueness and stability solutions for RIDE involving fractional order. Thus initial value problem (IVP), boundary value problem (BVP), impulsive effect and nonlocal conditions are taken in account to prove the results.

Keywords Random differential equations, fractional derivative, stability.

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1. Introduction

The study of fractional differential equations(FDEs) has emerged as a new branch of applied mathematics, which has been used for construction and various fields of engineering and sciences. We can find numerous applications in viscoelascity, electrochemistry, control, porous media, electromagnetic, etc, see [2, 6, 8, 10]. The domain of FDEs ranges from the theoretical aspects like existence, uniqueness, periodicity, asymptotic behaviour, etc. For the recent studies on FDEs we refer [1, 3, 7, 13, 20, 21].

Since the emergent of fractional calculus then by many fractional derivatives are introduce and developed vastly. There exist many fractional derivative such as Riemann-Liouville, Hadamard, Katugampola, Jumarie, etc. Most recently a fractional derivative with kernel of function is introduced by Vanterler Da C. Sousa and the classical properties with transformation of existing fractional derivative is discussed in [12]. The recent development of ψ-HFD and the theoritical analysis can be seen in [13–17].

Here we study the dynamical behaviour of fractional order integro-differential equations with random variable. Many researchers discussed the randomness of the

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FDEs which arises in uncertainties and complexities. Such deterministic equations are hardly called as Random differential equations (RDEs). The recent development of RDEs of fractional order can be seen in [9, 11, 19]. The main intention of this work is to study existence, uniqueness and stability of solutions for RIDE involving ψ-HFD.

The paper is constructed as follows: In Section 2, we present the main definitions and interesting results. In Section 3, Existence and stability result is established for IVP. In Section 2, stability results for Bvp is discussed. Further dynamical behaviour of impulsive RIDE and nonlocal RIDE are discussed in Section 3 and Section 4 respectively.

2. Preliminaries

Some basic definitions and results are discussed in this section. Let C be the Banach space of all continuous functions $\mathfrak{h} : J \times \Omega \to R$ with the norm

$$\|\mathfrak{g}\|_C = \sup\left\{|\mathfrak{g}(t,\omega)| : t \in J\right\}.$$

We denote the weighted spaces of all continuous functions defined by

$$C_{\gamma,\psi}(J,R) = \left\{\mathfrak{g} : J \to R : (\psi(t) - \psi(0))^{\gamma}\, \mathfrak{g}(t,\omega) \in C\right\},\ 0 \leq \gamma < 1,$$

with the norm

$$\|\mathfrak{g}\|_{C_{\gamma,\psi}} = \sup_{t\in J}\left|(\psi(t) - \psi(0))^{\gamma}\, \mathfrak{g}(t,\omega)\right|.$$

Definition 2.1 ([12]). The left-sided fractional integral of a function $\mathfrak{g}$ with respect to another function ψ on [a, b] is defined by

$$\left(\mathfrak{I}^{\alpha;\psi}\right)\mathfrak{h}(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \psi^{'}(s)\left(\psi(t) - \psi(s)\right)^{\alpha-1}\mathfrak{h}(s)ds,\ t > a. \tag{2.1}$$

Definition 2.2 ([12]). Let $\psi^{'}(t) \neq 0$ $(-\infty \leq a < t < b \leq \infty)$ and $\alpha > 0$, $n \in N$. The Riemann-Liouville fractional derivative of a function $\mathfrak{g}$ with respect to ψ of order α correspondent to the Riemann-Liouville, is defined by

$$\mathfrak{D}^{\alpha;\psi}\mathfrak{h}(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi^{'}(t)}\frac{d}{dt}\right)^n \int_a^t \psi^{'}(s)\left(\psi(t) - \psi(s)\right)^{n-\alpha-1}\mathfrak{h}(s)ds. \tag{2.2}$$

Definition 2.3 ([12]). Let $\alpha > 0$, $n \in N$, $I = [a,b]$ is the interval $(-\infty \leq a < t < b \leq \infty)$, $\mathfrak{g}, \psi \in C^n([a,b],R)$ two functions such that ψ is increasing and $\psi^{'}(t) \neq 0$, for all $t \in I$. The left ψ-Caputo derivative of order α is given by

$$\mathfrak{D}^{\alpha;\psi}\mathfrak{h}(t) = \mathfrak{I}^{n-\alpha;\psi}\left(\frac{1}{\psi^{'}(t)}\frac{d}{dt}\right)^n \mathfrak{h}(t) \tag{2.3}$$

where $n = [\alpha] + 1$ for $\alpha \notin N$ and $\alpha = n$ for $\alpha \in N$.

Definition 2.4 ([12]). The ψ-Hilfer fractional derivative of function f oforder α is given by,

$$\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{h}(t) = \mathfrak{I}^{\beta(1-\alpha);\psi}\left(\frac{1}{\psi^{'}(t)}\frac{d}{dt}\right)\mathfrak{I}^{(1-\beta)(1-\alpha);\psi}\mathfrak{h}(t). \tag{2.4}$$

The ψ-Hilfer fractional derivative as above defined, can be written in the following

$$\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{h}(t) = \mathfrak{I}^{\gamma-\alpha;\psi}\mathfrak{D}^{\gamma;\psi}\mathfrak{h}(t).$$

Lemma 2.1 (Gronwall Lemma [18]). *Suppose $\alpha > 0$, $a(t,\omega)$ is a nonnegative function locally integrable on $J \times \Omega$ (some $T \leq \infty$), and let $g(t,\omega)$ be a nonnegative, nondecreasing continuous function defined on $J \times \Omega$, such that $g(t,\omega) \leq K$ for some constant K. Further let $\mathfrak{h}(t,\omega)$ be a nonnegative locally integrable on $J \times \Omega$ function with*

$$\mathfrak{h}(t,\omega) \leq a(t,\omega) + g(t,\omega)\int_a^t \psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}\mathfrak{h}(s,\omega)ds, \quad (t,\omega) \in J \times \Omega,$$

with some $\alpha > 0$. Then

$$\mathfrak{h}(t,\omega) \leq a(t,\omega) + \int_a^t \left[\sum_{n=1}^{\infty} \frac{(g(t,\omega)\Gamma(\alpha))^n}{\Gamma(n\alpha)}\psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{n\alpha-1}\right] a(s,\omega)ds.$$

Lemma 2.2. *Let $\mathfrak{h} \in PC_{1-\gamma,\psi}$ satisfies the following inequality*

$$|\mathfrak{h}(t,\omega)| \leq c_1 + c_2 \int_0^t \psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}|\mathfrak{h}(s,\omega)|\,ds + \sum_{0<t_k<t} I_k\,|\mathfrak{h}(t_k,\omega)|\,,$$

where c_1 is a nonnegative, continuous and nondecreasing function on J and c_2, I_i are constants. Then

$$|\mathfrak{h}(t,\omega)|$$
$$\leq c_1\left(1 + IE_\alpha(c_2\Gamma(\alpha)(\psi(t)-\psi(0))^\alpha)^k E_\alpha(c_2\Gamma(\alpha)(\psi(t)-\psi(0))^\alpha\right) \ \textit{for}\ t \in (t_k, t_{k+1}],$$

where $I = \sup\{I_k : k = 1, 2, 3, ...,\}$ and $E_\alpha = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$.

Theorem 2.1 ([5], Schauder fixed point theorem). *Let B be closed, convex and nonempty subset of a Banach space E. Let $N : B \to B$ be a continuous mapping such that $N(B)$ is a relatively compact subset of E. Then N has atleast one fixed point in B.*

Theorem 2.2 ([5], Schaefer's Fixed Point Theorem). *Let R be a Banach space and let $\mathfrak{P} : R \to R$ be completely continuous operator. If the set $\{\mathfrak{h} \in R : \mathfrak{h} = \delta\mathfrak{P}\mathfrak{h}$ for some $\delta \in (0,1)\}$ is bounded, then $\mathfrak{P}$ has a fixed point.*

Theorem 2.3 ([5], Krasnoselskii's fixed point theorem). *Let X be a Banach space, let Ω be a bounded closed convex subset of X and let T_1, T_2 be mapping from Ω into X such that $T_1x + T_2y, \in \Omega$ for every pair $x, y \in \Omega$. If T_1 is contraction and T_2 is completely continuous, then the equation $T_1x + T_2x = x$ has a solution on Ω.*

Theorem 2.4 ([5], Banach Fixed Point Theorem). *Suppose Q be a non-empty closed subset of a Banach space E. Then any contraction mapping $\mathfrak{P}$ from Q into itself has a unique fixed point.*

3. Solution of IVP for fractional RIDE

In this section, we consider the IVP for fractional RIDE of the form

$$\begin{cases} \mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{h}(t,\omega) = \mathfrak{g}_\omega\left(t, \mathfrak{h}(t,\omega), \int_0^t k_\omega(t,s,\mathfrak{h}(s,\omega))ds\right), & t \in J := [0,T], \\ \mathfrak{I}^{1-\gamma;\psi}\mathfrak{h}(t,\omega)|_{t=0} = \mu(\omega), \end{cases} \tag{3.1}$$

where $\mathfrak{D}^{\alpha,\beta;\psi}$ is ψ-HFD of orders $\alpha \in (0,1)$ and type $\beta \in [0,1]$, $\mathfrak{h}$ is a random function, ω is the random variable and $\mathfrak{I}^{1-\gamma,\psi}$ is ψ-fractional integral of orders $1-\gamma(\gamma = \alpha+\beta-\alpha\beta)$. Let R be a Banach space, Ω is a probability space and $\mathfrak{g}_\omega : J \times \Omega \times R \times R \to R$ is a given continuous function, $\omega \in \Omega$. For brevity let us take

$$H\mathfrak{h}(t,\omega) = \int_0^t k_\omega(t,s,\mathfrak{h}(s,\omega))ds.$$

We make the following hypotheses to prove our main results.

(H1) There exists a constant $\ell_{\mathfrak{g}}$ such that

$$\begin{aligned}&|\mathfrak{g}_\omega(s,\mathfrak{h}_1(\cdot,\omega),\mathfrak{h}_2(\cdot,\omega)) - \mathfrak{g}_\omega(s,\mathfrak{y}_1(\cdot,\omega),\mathfrak{y}_2(\cdot,\omega))|\\ \le&\ell_{\mathfrak{g}}\left(|\mathfrak{h}_1(\cdot,\omega)-\mathfrak{y}_1(\cdot,\omega)| + |\mathfrak{h}_2(\cdot,\omega)-\mathfrak{y}_2(\cdot,\omega)|\right)\end{aligned}$$

for every $t \in J$ and $\omega \in \Omega$. Set $\tilde{\mathfrak{g}} = \mathfrak{g}(s,0,0)$.

(H2) For all $\mathfrak{h},\mathfrak{y} \in R$, there exists a there exists a constant $\ell_{\mathfrak{h}} > 0$, such that

$$\int_0^t |k_\omega(t,s,\mathfrak{h}) - k_\omega(t,s,\mathfrak{y})| \le \ell_{\mathfrak{h}} |\mathfrak{h}(\cdot,\omega)-\mathfrak{y}(\cdot,\omega)|.$$

Set $\tilde{k} = \int_0^s |k_\omega(s,\tau,0)|\,d\tau$.

(H3) There exists $\lambda_\varphi > 0$ such that for each $t \in J$ and $\omega \in \Omega$, we have

$$\mathfrak{I}^{\alpha;\psi}\varphi(t,\omega) \le \lambda_\varphi\varphi(t,\omega).$$

Lemma 3.1. *A function $\mathfrak{h}$ is the solution of fractional RIDE* (3.1)*, if and only if $\mathfrak{h}$ satisfies the random integral equation*

$$\begin{aligned}\mathfrak{h}(t,\omega) =&\frac{\mu(\omega)}{\Gamma(\gamma)}(\psi(t)-(\psi(0))^{\gamma-1}\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}_\omega(s,\mathfrak{h}(s,\omega),H\mathfrak{h}(s,\omega))ds.\end{aligned} \tag{3.2}$$

Theorem 3.1. *Assume that hypothesis [H1] is satisfied. Then, Eq.* (3.1) *has at least one solution.*

Proof. Consider the operator $\mathfrak{P} : C_{1-\gamma,\psi} \to C_{1-\gamma,\psi}$. Where the equivalent integral Eq.(3.2) which can be written in the operator form

$$\begin{aligned}\mathfrak{P}\mathfrak{h}(t,\omega) =&\frac{\mu(\omega)}{\Gamma(\gamma)}(\psi(t)-\psi(0))^{\gamma-1}\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}_\omega(s,\mathfrak{h}(s,\omega),H\mathfrak{h}(s,\omega))ds.\end{aligned} \tag{3.3}$$

Clearly, the fixed points of the operator $\mathfrak{P}$ is solution of the problem (3.1). For any $\mathfrak{h} \in J \times \Omega$, we have

$$\left|(\mathfrak{P}\mathfrak{h})(t,\omega)\,(\psi(t)-\psi(0))^{1-\gamma}\right|$$

$$
\begin{aligned}
&\leq \frac{|\mu(\omega)|}{\Gamma(\gamma)}+\frac{(\psi(t)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^t\psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{g}_\omega(s,\mathfrak{h}(s,\omega),H\mathfrak{h}(s,\omega))\right|ds\\
&\leq \frac{|\mu(\omega)|}{\Gamma(\gamma)}+\frac{(\psi(t)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^t\psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{g}_\omega(s,\mathfrak{h}(s,\omega),H\mathfrak{h}(s,\omega))\right.\\
&\quad\left.-\mathfrak{g}_\omega(s,0,0)+\mathfrak{g}_\omega(s,0,0)\right|ds\\
&\leq \frac{|\mu(\omega)|}{\Gamma(\gamma)}+\frac{(\psi(t)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^t\psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\left(\ell_\mathfrak{g}\left|\mathfrak{h}(s,\omega)\right|\right.\\
&\quad\left.+\ell_\mathfrak{g}\left|H\mathfrak{h}(s,\omega)\right|+|\tilde{\mathfrak{g}}|\right)ds\\
&\leq \frac{|\mu(\omega)|}{\Gamma(\gamma)}+\frac{(\psi(t)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^t\psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\left(\ell_\mathfrak{g}\left|\mathfrak{h}(s,\omega)\right|\right.\\
&\quad\left.+\ell_\mathfrak{g}\int_0^s|k_\omega(s,\tau,\mathfrak{h}(\tau,\omega))-k_\omega(s,\tau,0)|\,d\tau+\ell_\mathfrak{g}\int_0^s|k_\omega(s,\tau,0)|\,d\tau+|\tilde{\mathfrak{g}}|\right)ds\\
&\leq \frac{|\mu(\omega)|}{\Gamma(\gamma)}+\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}(\psi(T)-\psi(0))^{\alpha}\left(\ell_\mathfrak{g}(1+\ell_\mathfrak{h})r+\ell_\mathfrak{g}\left\|\tilde{k}\right\|_{C_{1-\gamma,\psi}}+\|\tilde{\mathfrak{g}}\|_{C_{1-\gamma,\psi}}\right)\\
&=r.
\end{aligned}
$$

This proves that $\mathfrak{P}$ transforms the ball $B_r=\left\{\mathfrak{h}\in C_{1-\gamma,\psi}:\|\mathfrak{h}\|_{C_{1-\gamma,\psi}}\leq r\right\}$, into itself. We shall show that the operator $\mathfrak{P}:B_r\to B_r$ satisfies all the conditions of Theorem 2.1. The proof will be given in several steps.

Step 1: $\mathfrak{P}$ is continuous.

Let $\mathfrak{h}_n$ be a sequence such that $\mathfrak{h}_n\to\mathfrak{h}$ in $C_{1-\gamma,\psi}$. Then for each $t\in J,\ w\in\Omega$,

$$
\begin{aligned}
&\left|\left(\mathfrak{P}\mathfrak{h}_n(t,\omega)-\mathfrak{P}\mathfrak{h}(t,\omega)\right)(\psi(t)-\psi(0))^{1-\gamma}\right|\\
&\leq \frac{(\psi(t)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^t\psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{g}_\omega(s,\mathfrak{h}_n(s,\omega),H\mathfrak{h}_n(s,\omega))\right.\\
&\quad\left.-\mathfrak{g}_\omega(s,\mathfrak{h}(s,\omega),H\mathfrak{h}(s,\omega))\right|ds\\
&\leq (\psi(t)-\psi(0))^{1-\gamma}\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}(\psi(t)-\psi(0))^{\alpha+\gamma-1}\left\|\mathfrak{g}_\omega(\cdot,\mathfrak{h}_n(\cdot,\omega),H\mathfrak{h}_n(\cdot,\omega))\right.\\
&\quad\left.-\mathfrak{g}_\omega(\cdot,\mathfrak{h}(\cdot,\omega),H\mathfrak{h}(\cdot,\omega))\right\|_{C_{1-\gamma,\psi}}\\
&\leq \frac{B(\gamma,\alpha)}{\Gamma(\alpha)}(\psi(T)-\psi(0))^{\alpha}\left\|\mathfrak{g}_\omega(\cdot,\mathfrak{h}_n(\cdot,\omega),H\mathfrak{h}_n(\cdot,\omega))-\mathfrak{g}(\cdot,\mathfrak{h}(\cdot,\omega),H\mathfrak{h}(\cdot,\omega))\right\|_{C_{1-\gamma,\psi}}
\end{aligned}
$$

since $\mathfrak{g}_\omega$ is continuous, then we have

$$\|\mathfrak{P}\mathfrak{h}_n-\mathfrak{P}\mathfrak{h}\|_{C_{1-\gamma,\psi}}\to 0 \ \text{ as } \ n\to\infty.$$

Step 2: $\mathfrak{P}(B_r)$ is uniformly bounded.

This is clear since $\mathfrak{P}(B_r)\subset B_r$ is bounded.

Step 3: We show that $\mathfrak{P}(B_r)$ is equicontinuous.

Let $t_1,t_2\in J, t_1>t_2$ be a bounded set of $C_{1-\gamma,\psi}$ as in Step 2, and $\mathfrak{h}\in B_r$. Then,

$$
\begin{aligned}
&\left|(\psi(t_1)-\psi(0))^{1-\gamma}\mathfrak{P}\mathfrak{h}(t_1,\omega)-(\psi(t_2)-\psi(0))^{1-\gamma}\mathfrak{P}\mathfrak{h}(t_2,\omega)\right|\\
&\leq\left|\frac{(\psi(t_1)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^{t_1}\psi^{'}(s)(\psi(t_1)-\psi(s))^{\alpha-1}\mathfrak{g}_\omega(s,\mathfrak{h}(s,\omega),H\mathfrak{h}(s,\omega))ds\right.
\end{aligned}
$$

$$-\frac{(\psi(t_2)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^{t_2}\psi^{'}(s)(\psi(t_2)-\psi(s))^{\alpha-1}\mathfrak{g}_\omega(s,\mathfrak{h}(s,\omega),H\mathfrak{h}(s,\omega))ds\Bigg|$$
$$\leq \frac{\|\mathfrak{g}_\omega\|_{C_{1-\gamma,\psi}}}{\Gamma(\alpha)}B(\gamma,\alpha)\left|(\psi(t_1)-\psi(0))^{\alpha}-(\psi(t_2)-\psi(0))^{\alpha}\right|.$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. As a consequence of Step 1-3 together with the Arzela-Ascoli theorem, we can conclude that $\mathfrak{P}$ is continuous and compact. From an application of Schauder's theorem, we deduce that $\mathfrak{P}$ has a fixed point $\mathfrak{h}$ which is a solution of the problem (3.1). □

Lemma 3.2. *Assume that hypothesis [H1] is satisfied. If*

$$\frac{\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})}{\Gamma(\alpha)}(\psi(T)-\psi(0))^{\alpha}B(\gamma,\alpha)<1,$$

then, (3.1) *has unique solution.*

Next, we shall give the definitions g-UHR stable for the problem

$$\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{h}(t,\omega)=\mathfrak{g}_\omega(t,\mathfrak{h}(t,\omega),H\mathfrak{h}(t,\omega)). \tag{3.4}$$

Let $\epsilon > 0$ be a positive real number and $\varphi : J \times \Omega \to R^{+}$ be a continuous function. We consider the following inequalities

$$\left|\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{y}(t,\omega)-\mathfrak{g}_\omega(t,\mathfrak{y}(t,\omega),H\mathfrak{y}(t,\omega))\right|\leq\varphi(t,\omega). \tag{3.5}$$

Definition 3.1. Eq.(3.4) is g-UHR stable with respect to φ if there exists a real number $C_{f,\varphi} > 0$ such that for each solution $\mathfrak{y} : \Omega \to C_{1-\gamma,\psi}$ of the inequality (3.5) there exists a solution $\mathfrak{h} : \Omega \to C_{1-\gamma,\psi}$ of Eq.(3.4) with

$$|\mathfrak{y}(t,\omega)-\mathfrak{h}(t,\omega)|\leq C_{f,\varphi}\varphi(t,\omega).$$

Theorem 3.2. *The hypothesis [H1], [H2] and [H3] holds. Then Eq.*(3.1) *is g-UHR stable.*

Proof. Let $\mathfrak{y}$ be solution of inequality (3.5) and by Lemma 3.2 there exists a unique solution $\mathfrak{h}$ for the problem (3.1). Thus we have

$$\begin{aligned}\mathfrak{h}(t,\omega)=&\frac{\mu(\omega)}{\Gamma(\gamma)}(\psi(t)-\psi(0))^{\gamma-1}\\&+\frac{1}{\Gamma(\alpha)}\int_0^t\psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}_\omega(s,\mathfrak{h}(s,\omega),H\mathfrak{h}(s,\omega))ds.\end{aligned}$$

By differentiating inequality (3.5) for each $t \in J,\ \omega \in \Omega$, we have

$$\begin{aligned}&\Bigg|\mathfrak{y}(t,\omega)-\frac{\mu(\omega)}{\Gamma(\gamma)}(\psi(t)-\psi(0))^{\gamma-1}\\&\quad-\frac{1}{\Gamma(\alpha)}\int_0^t\psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}_\omega(s,\mathfrak{y}(s,\omega),H\mathfrak{y}(s,\omega))ds\Bigg|\leq\lambda_\varphi\varphi(t,\omega).\end{aligned}$$

Hence it follows

$$|\mathfrak{y}(t,\omega)-\mathfrak{h}(t,\omega)|$$

$$
\begin{aligned}
&\leq \Bigg| \mathfrak{y}(t,\omega) - \frac{\mu(\omega)}{\Gamma(\gamma)}(\psi(t)-\psi(0))^{\gamma-1} \\
&\quad - \frac{1}{\Gamma(\alpha)} \int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}_\omega(s,\mathfrak{h}(s,\omega),H\mathfrak{h}(s,\omega))ds \Bigg| \\
&\leq \Bigg| \mathfrak{y}(t,\omega) - \frac{\mu(\omega)}{\Gamma(\gamma)}(\psi(t)-\psi(0))^{\gamma-1} \\
&\quad - \frac{1}{\Gamma(\alpha)} \int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}_\omega(s,\mathfrak{y}(s,\omega),H\mathfrak{y}(s,\omega))ds \Bigg| \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}|\mathfrak{g}_\omega(s,\mathfrak{y}(s,\omega),H\mathfrak{y}(s,\omega))-\mathfrak{g}_\omega(s,\mathfrak{h}(s,\omega),H\mathfrak{h}(s,\omega))|ds \\
&\leq \lambda_\varphi \varphi(t,\omega) + \frac{\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})}{\Gamma(\alpha)} \int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{y}(s,\omega)-\mathfrak{h}(s,\omega)\right| ds \\
&\leq \lambda_\varphi \varphi(t,\omega) + \frac{\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})}{\Gamma(\alpha)} \int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\lambda_\varphi\varphi(s,\omega)ds \\
&:= C_{f,\varphi}\varphi(t,\omega).
\end{aligned}
$$

Thus, Eq.(3.1) is g-UHR stable. □

4. Solution of BVP for fractional RIDE

Consider the fractional RIDE with boundary condition is given by

$$
\begin{cases} \mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{h}(t,\omega) = \mathfrak{g}_\omega(t,\mathfrak{h}(t,\omega),H\mathfrak{h}(t,\omega)), \quad t\in J, \\ a\ \mathfrak{I}^{1-\gamma;\psi}\mathfrak{h}(t,\omega)|_{t=0} + b\ \mathfrak{I}^{1-\gamma;\psi}\mathfrak{h}(t,\omega)|_{t=T} = c. \end{cases} \tag{4.1}
$$

The integral equation of Eq.(4.1) is given by

$$
\begin{aligned}
\mathfrak{h}(t,\omega) = &\left(c - b\mathfrak{I}^{1-\beta+\alpha\beta;\psi}\mathfrak{g}_\omega(T,\mathfrak{h}(T,\omega),H\mathfrak{h}(T,\omega))\right)\frac{(\psi(t)-\psi(0))^{\gamma-1}}{(a+b)\Gamma(\gamma)} \\
&+ \mathfrak{I}^{\alpha;\psi}\mathfrak{g}_\omega(t,\mathfrak{h}(t,\omega),H\mathfrak{h}(t,\omega)),
\end{aligned} \tag{4.2}
$$

where $a,\ b$ and c are some constants.

(H4) There exists a constant such that

$$
|\mathfrak{g}_\omega(\cdot,\mathfrak{h}_1(\cdot,\omega),\mathfrak{h}_2(\cdot,\omega))| \leq m(\cdot,\omega)\left|\mathfrak{h}_1(\cdot,\omega)\right| + n(\cdot,\omega)\left|\mathfrak{h}_2(\cdot,\omega)\right|
$$

for every $t\in J$ and $\omega\in\Omega$. Denote $N(\omega) = \sup n(\cdot,\omega)$ and $M(\omega) = \sup m(\cdot,\omega)$.

(H5) For all $\mathfrak{h}\in R$,

$$
\int_0^t |k_\omega(t,s,\mathfrak{h}(\cdot,\omega))| \leq p(\cdot,\omega)\left|\mathfrak{h}(\cdot,\omega)\right|.
$$

Denote $P(\omega) = \sup p(\cdot,\omega)$

Theorem 4.1. *Assume that [H4] and [H5] are satisfied. Then, Eq.*(4.1) *has at least one solution.*

Proof. Consider the operator $N : C_{1-\gamma,\psi} \to C_{1-\gamma,\psi}$. The equivalent integral Eq.(4.2) which can be written in the operator form

$$N\mathfrak{h}(t,\omega) = \left(c - b\mathfrak{I}^{1-\beta+\alpha\beta;\psi}\mathfrak{g}_\omega(T,\mathfrak{h}(T,\omega),H\mathfrak{h}(T,\omega))\right)\frac{(\psi(t)-\psi(0))^{\gamma-1}}{(a+b)\Gamma(\gamma)} + \mathfrak{I}^{\alpha;\psi}\mathfrak{g}_\omega(t,\mathfrak{h}(t,\omega),H\mathfrak{h}(t,\omega)). \tag{4.3}$$

Claim 1: N is continuous.

Let $\mathfrak{h}_n$ be a sequence such that $\mathfrak{h}_n \to \mathfrak{h}$ in $C_{1-\gamma,\psi}$. Then for each $t \in J$,

$$\begin{aligned}
&\left|(N\mathfrak{h}_n(t,\omega) - N\mathfrak{h}(t,\omega))(\psi(t)-\psi(0))^{1-\gamma}\right| \\
&\leq \left(b\mathfrak{I}^{1-\beta+\alpha\beta;\psi}\left|\mathfrak{g}_\omega(T,\mathfrak{h}_n(T,\omega),H\mathfrak{h}_n(T,\omega)) - \mathfrak{g}_\omega(T,\mathfrak{h}(T,\omega),H\mathfrak{h}(T,\omega))\right|\right)\frac{1}{(a+b)\Gamma(\gamma)} \\
&\quad + (\psi(t)-\psi(0))^{1-\gamma}\mathfrak{I}^{\alpha;\psi}\left|\mathfrak{g}_\omega(t,\mathfrak{h}_n(t,\omega),H\mathfrak{h}_n(t,\omega)) - \mathfrak{g}_\omega(t,\mathfrak{h}(t,\omega),H\mathfrak{h}(t,\omega))\right| \\
&\leq \left(\left(\frac{bB(\gamma,1-\beta+\alpha\beta)}{(a+b)\Gamma(\gamma)\Gamma(1-\beta+\alpha\beta)}(\psi(T)-\psi(0))^{\alpha}\right)\right. \\
&\left. + \frac{B(\gamma,\alpha)}{\Gamma(\alpha)}(\psi(T)-\psi(0))^{\alpha}\right)\left\|\mathfrak{g}_\omega(\cdot,\mathfrak{h}_n(\cdot,\omega),H\mathfrak{h}_n(\cdot,\omega)) - \mathfrak{g}_\omega(\cdot,\mathfrak{h}(\cdot,\omega),H\mathfrak{h}(\cdot,\omega))\right\|_{C_{1-\gamma,\psi}}
\end{aligned}$$

since $\mathfrak{g}_\omega$ is continuous, then we have

$$\|N\mathfrak{h}_n - N\mathfrak{h}\|_{C_{1-\gamma,\psi}} \to 0 \quad \text{as} \quad n \to \infty.$$

Claim 2: N maps bounded sets into bounded sets in $C_{1-\gamma,\psi}$.

Indeed, it is enough to show that for $r > 0$, there exists a positive constant l such that $B_r = \left\{\mathfrak{h} \in C_{1-\gamma,\psi} : \|\mathfrak{h}\|_{C_{1-\gamma,\psi}} \leq r\right\}$,

$$\begin{aligned}
&\left|N\mathfrak{h}(t,\omega)(\psi(t)-\psi(0))^{1-\gamma}\right| \\
&\leq \frac{c}{(a+b)\Gamma(\gamma)} + \frac{b}{(a+b)\Gamma(\gamma)}\mathfrak{I}^{1-\beta+\alpha\beta;\psi}\left|\mathfrak{g}_\omega(T,\mathfrak{h}(T,\omega),H\mathfrak{h}(T,\omega))\right| \\
&\quad + (\psi(t)-\psi(0))^{1-\gamma}\mathfrak{I}^{\alpha;\psi}\mathfrak{g}_\omega\left|(t,\mathfrak{h}(t,\omega),H\mathfrak{h}(t,\omega))\right| \\
&\leq \frac{c}{(a+b)\Gamma(\gamma)} + \frac{b}{(a+b)\Gamma(\gamma)}\mathfrak{I}^{1-\beta+\alpha\beta;\psi}\left(m(T,\omega)\left|\mathfrak{h}(T,\omega)\right| + n(T,\omega)\left|H\mathfrak{h}(T,\omega)\right|\right) \\
&\quad + (\psi(t)-\psi(0))^{1-\gamma}\mathfrak{I}^{\alpha;\psi}\left(m(t,\omega)\left|\mathfrak{h}(t,\omega)\right| + n(t,\omega)\left|H\mathfrak{h}(t,\omega)\right|\right) \\
&\leq \frac{c}{(a+b)\Gamma(\gamma)} + \frac{b}{(a+b)\Gamma(\gamma)}\mathfrak{I}^{1-\beta+\alpha\beta;\psi}\left(m(T,\omega)\left|\mathfrak{h}(T,\omega)\right| + n(T,\omega)p(T,\omega)\left|\mathfrak{h}(T,\omega)\right|\right) \\
&\quad + (\psi(t)-\psi(0))^{1-\gamma}\mathfrak{I}^{\alpha;\psi}\left(m(t,\omega)\left|\mathfrak{h}(t,\omega)\right| + n(t,\omega)p(t,\omega)\left|\mathfrak{h}(t,\omega)\right|\right) \\
&\leq \frac{c}{(a+b)\Gamma(\gamma)} + \left(\frac{bB(\gamma,1-\beta+\alpha\beta)}{(a+b)\Gamma(\gamma)\Gamma(1-\beta+\alpha\beta)}(\psi(T)-\psi(0))^{\alpha}\right. \\
&\quad \left. + \frac{B(\gamma,\alpha)}{\Gamma(\alpha)}(\psi(T)-\psi(0))^{\alpha}\right)\left(M(\omega)\|\mathfrak{h}\|_{C_{1-\gamma,\psi}} + N(\omega)P(\omega)\|\mathfrak{h}\|_{C_{1-\gamma,\psi}}\right) \\
&\leq \frac{c}{(a+b)\Gamma(\gamma)} + \left(\frac{bB(\gamma,1-\beta+\alpha\beta)}{(a+b)\Gamma(\gamma)\Gamma(1-\beta+\alpha\beta)}(\psi(T)-\psi(0))^{\alpha}\right. \\
&\quad \left. + \frac{B(\gamma,\alpha)}{\Gamma(\alpha)}(\psi(T)-\psi(0))^{\alpha}\right)\left(M(\omega) + N(\omega)P(\omega)\right)r
\end{aligned}$$

$:= l.$

Claim 3: N maps bounded sets into equicontinuous set of $C_{1-\gamma,\psi}$.

Let $t_1, t_2 \in J, t_1 > t_2, B_r$ be a bounded set of $C_{1-\gamma,\psi}$ as in Claim 2, and $\mathfrak{h} \in B_r$. Then,

$$\begin{aligned}
&\left|(\psi(t_1)-\psi(0))^{1-\gamma}(N\mathfrak{h}(t_1,\omega)-(\psi(t_2)-\psi(0))^{1-\gamma}N\mathfrak{h}(t_2,\omega))\right| \\
&\leq \left|\frac{(\psi(t_1)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}_\omega(s,\mathfrak{h}(s,\omega),H\mathfrak{h}(s,\omega))ds\right. \\
&\quad \left. + \frac{(\psi(t_1)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}_\omega(s,\mathfrak{h}(s,\omega),H\mathfrak{h}(s,\omega))ds\right|.
\end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. As a consequence of Claim 1 to 3, together with Arzela-Ascoli theorem, we can conclude that $N : C_{1-\gamma,\psi} \to C_{1-\gamma,\psi}$ is continuous and completely continuous.

Claim 4: A priori bounds.

Now it remains to show that the set

$$\eta = \{\mathfrak{h} \in C_{1-\gamma,\psi} : \mathfrak{h} = \delta N\mathfrak{h}, 0 < \delta < 1\}$$

is bounded set.

$$\begin{aligned}
\mathfrak{h}(t,\omega) &= \left(c - b\mathfrak{I}^{1-\beta+\alpha\beta;\psi}\mathfrak{g}_\omega(T,\mathfrak{h}(T,\omega),H\mathfrak{h}(T,\omega))\right)\frac{(\psi(t)-\psi(0))^{\gamma-1}}{(a+b)\Gamma(\gamma)} \\
&\quad + \mathfrak{I}^{\alpha;\psi}\mathfrak{g}_\omega(t,\mathfrak{h}(t,\omega),H\mathfrak{h}(t,\omega)).
\end{aligned}$$

This shows that the set η is bounded. As a consequence of Theorem 2.1, we deduce that N has a fixed point which is a solution of problem (4.1). □

Theorem 4.2. *Assume that hypotheses [H1] and [H2] are fulfilled. If*

$$\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})\left(\left(\frac{bB(\gamma,1-\beta+\alpha\beta)}{(a+b)\Gamma(\gamma)\Gamma(1-\beta+\alpha\beta)}\right)+\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\right)(\psi(T)-\psi(0))^{\alpha} < 1,$$

then, Eq. (4.1) *has unique solution.*

Theorem 4.3. *The hypothesis [H1], [H2] and [H3] holds. Then Eq.*(4.1) *is g-UHR stable.*

Proof. Let $\mathfrak{y}$ be solution of inequality (3.5) and by Theorem 4.2 there $\mathfrak{h}$ is unique solution of the problem

$$\begin{aligned}
\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{h}(t,\omega) &= \mathfrak{g}_\omega(t,\mathfrak{h}(t,\omega),H\mathfrak{h}(t,\omega)), \quad t \in J, \\
a\ \mathfrak{I}^{1-\gamma;\psi}\mathfrak{h}(t,\omega)|_{t=0} &+ b\ \mathfrak{I}^{1-\gamma;\psi}\mathfrak{h}(t,\omega)|_{t=T} = c
\end{aligned}$$

is given by

$$\mathfrak{h}(t,\omega) = A_{\mathfrak{h}} + \frac{1}{\Gamma(\alpha)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}_\omega(s,\mathfrak{h}(s,\omega),H\mathfrak{h}(s,\omega))ds,$$

where

$$A_{\mathfrak{h}} = \left(c - b\mathfrak{I}^{1-\beta+\alpha\beta;\psi}\mathfrak{g}_\omega(T,\mathfrak{h}(T,\omega),H\mathfrak{h}(T,\omega))\right)\frac{(\psi(t)-\psi(0))^{\gamma-1}}{(a+b)\Gamma(\gamma)}.$$

Thus $A_{\mathfrak{h}} = A_{\mathfrak{y}}$.

By differentiating inequality (3.5), we have

$$\left|\mathfrak{y}(t,\omega) - A_{\mathfrak{y}} - \frac{1}{\Gamma(\alpha)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}_{\omega}(s,\mathfrak{y}(s,\omega),H\mathfrak{y}(s,\omega))ds\right|$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\varphi(s,\omega)ds$$
$$\leq \lambda_{\varphi}\varphi(t,\omega).$$

Hence it follows

$$|\mathfrak{y}(t,\omega) - \mathfrak{h}(t,\omega)|$$
$$\leq \left|\mathfrak{y}(t,\omega) - A_{\mathfrak{h}} - \frac{1}{\Gamma(\alpha)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}_{\omega}(s,\mathfrak{h}(s,\omega),H\mathfrak{h}(s,\omega))ds\right|$$
$$\leq \left|\mathfrak{y}(t,\omega) - A_{\mathfrak{h}} - \frac{1}{\Gamma(\alpha)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}_{\omega}(s,\mathfrak{y}(s,\omega),H\mathfrak{y}(s,\omega))ds\right|$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{g}_{\omega}(s,\mathfrak{y}(s,\omega),H\mathfrak{y}(s,\omega)) - \mathfrak{g}_{\omega}(s,\mathfrak{h}(s,\omega),H\mathfrak{h}(s,\omega))\right|ds$$
$$\leq \lambda_{\varphi}\varphi(t,\omega) + \frac{\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})}{\Gamma(\alpha)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{y}(s,\omega)-\mathfrak{h}(s,\omega)\right|ds.$$

By Lemma 2.1, there exists a constant $M^* > 0$ independent of $\lambda_{\varphi}\varphi(t,\omega)$ such that

$$|\mathfrak{y}(t,\omega) - \mathfrak{h}(t,\omega)| \leq M^*\lambda_{\varphi}\varphi(t,\omega) := C_{f,\varphi}\varphi(t,\omega).$$

Thus, Eq.(3.1) is g-UHR stable. □

5. Fractional order RIDE with in abrupt change

In this section, we study the existence, uniqueness and stability of fractional RIDE with impulsive effect is given by

$$\begin{cases} \mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{h}(t,\omega) = \mathfrak{g}_{\omega}(t,\mathfrak{h}(t,\omega),H\mathfrak{h}(t,\omega)), \\ \mathfrak{I}^{1-\gamma;\psi}\mathfrak{h}(t_k,\omega)|_{t=t_k} = I_k\mathfrak{h}(t_k,\omega), \\ \mathfrak{I}^{1-\gamma;\psi}\mathfrak{h}(t,\omega)|_{t=0} = \mathfrak{h}_0, \end{cases} \tag{5.1}$$

where $I_k : R \to R$, and $0 = t_0 < t_1 < ... < t_m < t_{m+1} = T$, $\Delta\mathfrak{I}^{1-\gamma;\psi}\mathfrak{h}(t,\omega)|_{t=t_k} = \mathfrak{I}^{1-\gamma;\psi}\mathfrak{h}(t_k^+,\omega) - \mathfrak{I}^{1-\gamma;\psi}\mathfrak{h}(t_k^-,\omega)$, $\mathfrak{I}^{1-\gamma;\psi}\mathfrak{h}(t_k^+) = \lim_{h\to 0+}\mathfrak{h}(t_k+h,\omega)$ and $\mathfrak{I}^{1-\gamma;\psi}\mathfrak{h}(t_k^-) = \lim_{h\to 0-}\mathfrak{h}(t_k+h,\omega)$ represent the right and left limits of $\mathfrak{h}(t,\omega)$ at $t = t_k$. Consider the weighted space $PC_{\gamma,\psi}(J,R)$ of functions $\mathfrak{g}$ on J defined by

$$PC_{\gamma,\psi}(J,R) = \left\{\mathfrak{g} : J \to R : (\psi(t)-\psi(t_k))^{\gamma}\,\mathfrak{g}(t,\omega) \in PC(J,R)\right\}, 0 \leq \gamma < 1,$$

with the norm

$$\|\mathfrak{g}\|_{PC_{\gamma,\psi}} = \|(\psi(t)-\psi(t_k))^{\gamma}\,\mathfrak{g}(t,\omega)\|_{PC(J,R)} = \max_{tinJ}\left|(\psi(t)-\psi(t_k))^{\gamma}\,\mathfrak{g}(t,\omega)\right|,$$

thus Eq.(5.1) satisfies the integral equation

$$\mathfrak{h}(t,\omega)=\frac{(\psi(t)-\psi(t_k))^{\gamma-1}}{\Gamma(\gamma)}\Bigg[\mathfrak{h}_0+\sum_{0<t_k<t}I_k\mathfrak{h}(t_k,\omega)$$
$$+\sum_{0<t_k<t}\mathfrak{I}^{1-\beta(1-\alpha);\psi}_{t_{k-1}}\mathfrak{g}_\omega(t_k,\mathfrak{h}(t_k,\omega),H\mathfrak{h}(t_k,\omega))\Bigg]+\mathfrak{I}^{\alpha;\psi}_{t_k}\mathfrak{g}_\omega(t,\mathfrak{h}(t,\omega),H\mathfrak{h}(t,\omega)). \quad (5.2)$$

(H6) Let the functions $I_k:R\times\Omega\to R$ are continuous and there exists a constant ℓ_I, such that for all $\mathfrak{h},\overline{\mathfrak{h}}\in R$, $k=1,2,...,m$,

$$\left|I_k(\mathfrak{h}(\cdot,\omega))-I_k(\overline{\mathfrak{h}}(\cdot,\omega))\right|\le\ell_I\left|\mathfrak{h}(\cdot,\omega)-\overline{\mathfrak{h}}(\cdot,\omega)\right|.$$

(H7) For all $\mathfrak{h}\in R$,

$$\left|I_k(\mathfrak{h}(\cdot,\omega))\right|\le i(\omega)\left|\mathfrak{h}(\cdot,\omega)\right|.$$

Set $i(\omega)=\sup I(\omega)$.

Theorem 5.1. *Assume that [H4], [H5] and [H7] are satisfied. Then, Eq.*(5.1) *has at least one solution.*

Proof. Consider the operator $\mathfrak{P}:PC_{1-\gamma,\psi}\to PC_{1-\gamma,\psi}$. The equivalent integral Eq.(5.2) can be written in the operator form

$$\mathfrak{P}\mathfrak{h}(t,\omega)=\frac{(\psi(t)-\psi(t_k))^{\gamma-1}}{\Gamma(\gamma)}\Bigg[\mathfrak{h}_0+\sum_{0<t_k<t}I_k\mathfrak{h}(t_k,\omega)$$
$$+\sum_{0<t_k<t}\mathfrak{I}^{1-\beta(1-\alpha);\psi}_{t_{k-1}}\mathfrak{g}_\omega(t_k,\mathfrak{h}(t_k,\omega),H\mathfrak{h}(t_k,\omega))\Bigg]+\mathfrak{I}^{\alpha;\psi}_{t_k}\mathfrak{g}_\omega(t,\mathfrak{h}(t,\omega),H\mathfrak{h}(t,\omega)). \quad (5.3)$$

Define, $B_r=\left\{\mathfrak{h}\in PC_{1-\gamma,\psi}:\|\mathfrak{h}\|_{PC_{1-\gamma,\psi}}\le r\right\}$.

Claim 1: We check that $\mathfrak{P}(B_r)\subset(B_r)$.

For any $\mathfrak{h}\in PC_{1-\gamma,\psi}$ and $\|\mathfrak{h}\|_{PC_{1-\gamma,\psi}}\le r$, we obtain that

$$\left|\mathfrak{P}\mathfrak{h}(t,\omega)\left(\psi(t)-\psi(t_k)\right)^{1-\gamma}\right|$$
$$\le\frac{1}{\Gamma(\gamma)}\Bigg[\mathfrak{h}_0+\sum_{0<t_k<t}\left|I_k\mathfrak{h}(t_k,\omega)\right|+\sum_{0<t_k<t}\mathfrak{I}^{1-\beta(1-\alpha);\psi}_{t_{k-1}}\left|\mathfrak{g}_\omega(t_k,\mathfrak{h}(t_k,\omega),H\mathfrak{h}(t_k,\omega))\right|\Bigg]$$
$$+\left(\psi(t)-\psi(t_k)\right)^{1-\gamma}\mathfrak{I}^{\alpha;\psi}_{t_k}\left|\mathfrak{g}_\omega(t,\mathfrak{h}(t,\omega),H\mathfrak{h}(t,\omega))\right|$$
$$\le\frac{1}{\Gamma(\gamma)}\Bigg[\mathfrak{h}_0+\sum_{0<t_k<t}i(\omega)\left|\mathfrak{h}(t_k,\omega)\right|+\sum_{0<t_k<t}\mathfrak{I}^{1-\beta(1-\alpha);\psi}_{t_{k-1}}m(\omega)\left|\mathfrak{h}(t_k)\right|+n(\omega)\left|H\mathfrak{h}(t_k)\right|\Bigg]$$
$$+\left(\psi(t)-\psi(t_k)\right)^{1-\gamma}\mathfrak{I}^{\alpha;\psi}_{t_k}m(\omega)\left|\mathfrak{h}(t)\right|+n(\omega)\left|H\mathfrak{h}(t)\right|$$
$$\le\frac{1}{\Gamma(\gamma)}\Bigg[\mathfrak{h}_0+\sum_{0<t_k<t}i(\omega)\left|\mathfrak{h}(t_k,\omega)\right|+\sum_{0<t_k<t}\mathfrak{I}^{1-\beta(1-\alpha);\psi}_{t_{k-1}}m(\omega)\left|\mathfrak{h}(t_k)\right|+n(\omega)p(\omega)\left|\mathfrak{h}(t_k)\right|\Bigg]$$

$$
\begin{aligned}
&+(\psi(t)-\psi(t_k))^{1-\gamma}\,\mathfrak{I}_{t_k}^{\alpha;\psi}\left(m(\omega)\left|\mathfrak{h}(t)\right|+n(\omega)p(\omega)\left|\mathfrak{h}(t)\right|\right)\\
\leq\;&\frac{1}{\Gamma(\gamma)}\Big[\mathfrak{h}_0+mI(\omega)\left(\psi(t)-\psi(t_k)\right)^{\gamma-1}\|\mathfrak{h}\|_{PC_{1-\gamma,\psi}}\\
&+\frac{mB(\gamma,1-\beta(1-\alpha))}{\Gamma(1-\beta(1-\alpha))}\left(M(\omega)+N(\omega)P(\omega)\right)\left(\psi(t)-\psi(t_k)\right)^{\alpha}\|\mathfrak{h}\|_{PC_{1-\gamma,\psi}}\Big]\\
&+(\psi(t)-\psi(t_k))^{1-\gamma}\frac{mB(\gamma,\alpha}{\Gamma(\alpha)}\left(M(\omega)+N(\omega)P(\omega)\right)\left(\psi(t)-\psi(t_k)\right)^{\alpha+\gamma-1}\|\mathfrak{h}\|_{PC_{1-\gamma,\psi}}\\
:=\;&r.
\end{aligned}
$$

Claim 2: The operator $\mathfrak{P}$ is continuous.

Let $\mathfrak{h}_n$ be a sequence such that $\mathfrak{h}_n\to\mathfrak{h}$ in $PC_{1-\gamma,\psi}$. Then

$$
\begin{aligned}
&\left|\left(\mathfrak{P}\mathfrak{h}_n(t,\omega)-\mathfrak{P}\mathfrak{h}(t,\omega)\right)\left(\psi(t)-\psi(t_k)\right)^{1-\gamma}\right|\\
\leq\;&\frac{1}{\Gamma(\gamma)}\Bigg[\sum_{0<t_k<t}\left|I_k(\mathfrak{h}_n(t_k,\omega))-I_k(\mathfrak{h}(t_k,\omega))\right|\\
&+\sum_{0<t_k<t}\mathfrak{I}_{t_{k-1}}^{1-\beta(1-\alpha);\psi}\left|\mathfrak{g}_\omega(t_k,\mathfrak{h}_n(t_k,\omega),H\mathfrak{h}_n(t_k,\omega))-\mathfrak{g}_\omega(t_k,\mathfrak{h}(t_k,\omega),H\mathfrak{h}(t_k,\omega))\right|\Bigg]\\
&+(\psi(t)-\psi(t_k))^{1-\gamma}\,\mathfrak{I}_{t_k}^{\alpha;\psi}\left|\mathfrak{g}_\omega(t,\mathfrak{h}_n(t,\omega),H\mathfrak{h}_n(t,\omega))-\mathfrak{g}_\omega(t,\mathfrak{h}(t,\omega),H\mathfrak{h}(t,\omega))\right|,
\end{aligned}
$$

since $\mathfrak{g}_\omega$ is continuous, then we have

$$\|\mathfrak{P}\mathfrak{h}_n-\mathfrak{P}\mathfrak{h}\|_{PC_{1-\gamma,\psi}}\to 0\quad\text{as}\quad n\to\infty.$$

Claim 3: The operator $\mathfrak{P}(B_r)$ is relatively compact.

Let any $t_1,t_2\in I,t_1>t_2$, one has

$$
\begin{aligned}
&\left|\mathfrak{P}\mathfrak{h}(t_1,\omega)\left(\psi(t_1)-\psi(t_k)\right)^{1-\gamma}-\mathfrak{P}\mathfrak{h}(t_2,\omega)(t_2)\left(\psi(t_2)-\psi(t_k)\right)^{1-\gamma}\right|\\
\leq\;&\frac{1}{\Gamma(\gamma)}\Bigg[\sum_{0<t_k<t_1-t_2}I_k\mathfrak{h}(t_k,\omega)+\sum_{0<t_k<t_1-t_2}\mathfrak{I}_{t_{k-1}}^{1-\beta(1-\alpha)}\mathfrak{g}_\omega(t_k,\mathfrak{h}(t_k,\omega),H\mathfrak{h}(t_k,\omega))\Bigg]\\
&+\|\mathfrak{g}\|_{PC_{1-\gamma,\psi}}\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left|(\psi(t_1)-\psi(t_k))^{\alpha}-(\psi(t_2)-\psi(t_k))^{\alpha}\right|.
\end{aligned}
$$

As $t_1\to t_2$, the right hand side of the above inequality tends to zero. From Claim 1 to 3, together with Arzela-Ascoli theorem, we conclude that $\mathfrak{P}$ completely continuous. □

Theorem 5.2. *Assume that hypotheses [H1], [H2] and [H6] are fulfilled. If*

$$
\Bigg[\frac{m\ell_I\left(\psi(b)-\psi(0)\right)^{\gamma-1}}{\Gamma(\gamma)}
+\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})\left(\frac{mB(\gamma,1-\beta(1-\alpha))}{\Gamma(\gamma)\Gamma(1-\beta(1-\alpha))}+\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\right)\left(\psi(b)-\psi(0)\right)^{\alpha}\Bigg]<1
$$

then, Eq.(5.1) *has unique solution.*

Next, we shall give the definitions of g-UHR stability for Eq.(5.1).

$$\begin{cases} \left|\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{y}(t,\omega)-\mathfrak{g}_\omega(t,\mathfrak{y}(t,\omega),H\mathfrak{y}(t,\omega))\right|\le\varphi(t), \\ \left|\Delta\mathfrak{I}^{1-\gamma;\psi}\mathfrak{y}(t,\omega)|_{t=t_k}-I_k(\mathfrak{y}(t_k))\right|\le\varphi(t). \end{cases} \tag{5.4}$$

Definition 5.1. Eq.(5.1) is g-UHR stable with respect to $\varphi\in PC_{1-\gamma,\psi}$ if there exists a real number $C_{f,\varphi}>0$ such that for each solution $\mathfrak{y}\in PC_{1-\gamma,\psi}$ of the inequality (5.4) there exists a solution $\mathfrak{h}\in PC_{1-\gamma,\psi}$ of Eq.(5.1) with

$$|\mathfrak{y}(t,\omega)-\mathfrak{h}(t,\omega)|\le C_{\mathfrak{g},\varphi}\varphi(t,\omega).$$

Theorem 5.3. *Let hypotheses [H1] - [H3] and [H6] are fullfilled. Then Eq.*(5.1) *is g-UHR stable.*

6. Solution of fractional Nonlocal RIDE

Here we study the existence, uniqueness and stability of nonlocal IVP involving ψ-HFD of the form

$$\begin{cases}\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{h}(t,\omega)=\mathfrak{g}_\omega(t,\mathfrak{h}(t,\omega),H\mathfrak{h}(t,\omega)),\\ \mathfrak{I}^{1-\gamma;\psi}\mathfrak{h}(t,\omega)|_{t=0}=\sum_{i=1}^m c_i\mathfrak{h}(\tau_i,\omega),\quad \tau_i\in J,\end{cases} \tag{6.1}$$

where $\tau_i, i=0,1,...,m$ are prefixed points satisfying $a<\tau_1\le...\le\tau_m<b$ and c_i is real numbers. Here, nonlocal condition $x(0,\omega)=\sum_{i=1}^m c_ix(\tau_i,\omega)$ can be applied in physical problems yields better effect than the initial conditions $x(0,\omega)=x_0$ in [4]. Further (6.1) is equivalent to mixed integral type of the form

$$\mathfrak{h}(t,\omega) = \begin{cases}\frac{T(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\alpha)}\sum_{i=1}^m c_i\int_0^{\tau_i}\psi'(s)\left(\psi(\tau_i)-\psi(s)\right)^{\alpha-1}\mathfrak{g}_\omega(s,\mathfrak{h}(s,\omega),H\mathfrak{h}(s,\omega))ds,\\ +\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}\mathfrak{g}_\omega(s,\mathfrak{h}(s,\omega),H\mathfrak{h}(s,\omega))ds,\end{cases} \tag{6.2}$$

where

$$T=\frac{1}{\Gamma(\gamma)-\sum_{i=1}^m c_i(\psi(\tau_i)-\psi(0))^{\gamma-1}}.$$

Theorem 6.1. *Assume that [H1] and [H2] are satisfied. Then, Eq.*(6.1) *has at least one solution.*

Proof. Consider the operator $P:C_{1-\gamma,\psi}\to C_{1-\gamma,\psi}$, it is well defined and given by

$$P\mathfrak{h}(t,\omega) = \begin{cases}\frac{T(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\alpha)}\sum_{i=1}^m c_i\int_0^{\tau_i}\psi'(s)\left(\psi(\tau_i)-\psi(s)\right)^{\alpha-1}\mathfrak{g}_\omega(s,\mathfrak{h}(s,\omega),H\mathfrak{h}(s,\omega))ds,\\ +\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}\mathfrak{g}_\omega(s,\mathfrak{h}(s,\omega),H\mathfrak{h}(s,\omega))ds.\end{cases} \tag{6.3}$$

Set $\widetilde{\mathfrak{g}}(s) = \mathfrak{g}_\omega(s,0,0)$. Consider the ball $B_r = \left\{\mathfrak{h} \in C_{1-\gamma,\psi} : \|\mathfrak{h}\|_{C_{1-\gamma,\psi}} \leq r\right\}$.

Now we subdivide the operator P into two operator P_1 and P_2 on B_r as follows

$$\begin{aligned} & P_1\mathfrak{h}(t,\omega) \\ &= \frac{T\left(\psi(t)-\psi(0)\right)^{\gamma-1}}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_0^{\tau_i} \psi^{'}(s)\left(\psi(\tau_i)-\psi(s)\right)^{\alpha-1} \mathfrak{g}_\omega(s,\mathfrak{h}(s,\omega),H\mathfrak{h}(s,\omega))ds \end{aligned}$$

and

$$P_2\mathfrak{h}(t,\omega) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1} \mathfrak{g}_\omega(s,\mathfrak{h}(s,\omega),H\mathfrak{h}(s,\omega))ds.$$

The proof is divided into several steps.

Step. 1 $P_1\mathfrak{h} + P_2\mathfrak{y} \in B_r$ for every $\mathfrak{h},\mathfrak{y} \in B_r$.

$$\begin{aligned} & \left|P_1\mathfrak{h}(t,\omega)\left(\psi(t)-\psi(0)\right)^{1-\gamma}\right| \\ &\leq \frac{T}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_0^{\tau_i} \psi^{'}(s)\left(\psi(\tau_i)-\psi(s)\right)^{\alpha-1} \left|\mathfrak{g}_\omega(s,\mathfrak{h}(s,\omega),H\mathfrak{h}(s,\omega))\right| ds \\ &\leq \frac{T}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_0^{\tau_i} \psi^{'}(s)\left(\psi(\tau_i)-\psi(s)\right)^{\alpha-1} \left(\left|\mathfrak{g}_\omega(s,\mathfrak{h}(s,\omega),H\mathfrak{h}(s,\omega)) - \mathfrak{g}_\omega(s,0,0)\right|\right. \\ &\quad \left. + \left|\mathfrak{g}_\omega(s,0,0)\right|\right) ds \\ &\leq \frac{T}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_0^{\tau_i} \psi^{'}(s)\left(\psi(\tau_i)-\psi(s)\right)^{\alpha-1} \left(\ell_{\mathfrak{g}}\left(|\mathfrak{h}(s,\omega)| + |H\mathfrak{h}(s,\omega))|\right) + |\widetilde{\mathfrak{g}}(s)|\right) ds \\ &\leq \frac{T}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_0^{\tau_i} \psi^{'}(s)\left(\psi(\tau_i)-\psi(s)\right)^{\alpha-1} \Bigg(\ell_{\mathfrak{g}}\Bigg(|\mathfrak{h}(s,\omega)| + \ell_{\mathfrak{h}}\,|\mathfrak{h}(s,\omega))| + \left|\tilde{\mathfrak{k}}(s)\right|\Bigg) \\ &\quad + |\tilde{\mathfrak{g}}(s)|\Bigg) ds. \end{aligned}$$

This gives

$$\begin{aligned} \|P_1\mathfrak{h}\|_{C_{1-\gamma,\psi}} \leq \frac{B(\gamma,\alpha)T}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \left(\psi(\tau_i)-\psi(0)\right)^{\alpha+\gamma-1} \Big(\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})\,\|\mathfrak{h}\|_{C_{1-\gamma,\psi}} \\ + \ell_{\mathfrak{g}} \left\|\tilde{\mathfrak{k}}\right\|_{C_{1-\gamma,\psi}} + \|\widetilde{\mathfrak{g}}\|_{C_{1-\gamma,\psi}}\Big). \qquad (6.4) \end{aligned}$$

For operator P_2

$$\begin{aligned} & \left|P_2\mathfrak{h}(t,\omega)\left(\psi(t)-\psi(0)\right)^{1-\gamma}\right| \\ &\leq \frac{\left(\psi(t)-\psi(0)\right)^{1-\gamma}}{\Gamma(\alpha)} \int_0^t \psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1} \left|\mathfrak{g}_\omega(s,\mathfrak{h}(s,\omega),H\mathfrak{h}(s,\omega))\right| ds \\ &\leq \frac{\left(\psi(t)-\psi(0)\right)^{1-\gamma}}{\Gamma(\alpha)} \left(\psi(t)-\psi(0)\right)^{\alpha+\gamma-1} B(\gamma,\alpha) \Big(\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})\,\|\mathfrak{h}\|_{C_{1-\gamma,\psi}} \\ &\quad + \ell_{\mathfrak{g}} \left\|\tilde{\mathfrak{k}}\right\|_{C_{1-\gamma,\psi}} + \|\widetilde{\mathfrak{g}}\|_{C_{1-\gamma,\psi}}\Big). \end{aligned}$$

Thus we obtain

$$\|P_2\mathfrak{h}\|_{1-\gamma} \leq \frac{B(\gamma,\alpha)}{\Gamma(\alpha)} (\psi(t)-\psi(0))^{\alpha} \Big(\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}}) \|\mathfrak{h}\|_{C_{1-\gamma,\psi}} + \ell_{\mathfrak{g}} \left\|\tilde{\mathfrak{k}}\right\|_{C_{1-\gamma,\psi}} + \|\widetilde{\mathfrak{g}}\|_{C_{1-\gamma,\psi}} \Big). \tag{6.5}$$

Linking (6.4) and (6.5), for every $\mathfrak{h}, \mathfrak{y} \in B_r$,

$$\|P_1\mathfrak{h} + P_2\mathfrak{y}\|_{C_{1-\gamma,\psi}} \leq \|P_1\mathfrak{h}\|_{C_{1-\gamma,\psi}} + \|P_2\mathfrak{y}\|_{C_{1-\gamma,\psi}} \leq r.$$

Step. 2 P_1 is a contration mapping.

For any $\mathfrak{h}, \mathfrak{y} \in B_r$

$$\begin{aligned}
&\left|(P_1\mathfrak{h}(t,\omega) - P_1\mathfrak{y}(t,\omega)) (\psi(t)-\psi(0))^{1-\gamma}\right| \\
&\leq \frac{T}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_0^{\tau_i} \psi^{'}(s) (\psi(\tau_i)-\psi(s))^{\alpha-1} |\mathfrak{g}_\omega(s,\mathfrak{h}(s,\omega), H\mathfrak{h}(s,\omega)) \\
&\quad -\mathfrak{g}_\omega(s,\mathfrak{y}(s,\omega), H\mathfrak{y}(s,\omega))| \, ds \\
&\leq \frac{\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})T}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_0^{\tau_i} \psi^{'}(s) (\psi(\tau_i)-\psi(s))^{\alpha-1} |\mathfrak{h}(s,\omega) - \mathfrak{y}(s,\omega)| \, ds \\
&\leq \frac{\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})T}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i (\psi(\tau_i)-\psi(0))^{\alpha+\gamma-1} B(\gamma,\alpha) \|\mathfrak{h}-\mathfrak{y}\|_{C_{1-\gamma,\psi}}.
\end{aligned}$$

This gives

$$\|P_1\mathfrak{h} - P_1\mathfrak{y}\|_{C_{1-\gamma,\psi}} \leq \frac{\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})T}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i (\psi(\tau_i)-\psi(0))^{\alpha+\gamma-1} B(\gamma,\alpha) \|\mathfrak{h}-\mathfrak{y}\|_{C_{1-\gamma,\psi}}.$$

The operator P_1 is contraction.

Step. 3 The operator P_2 is compact and continuous.

According to Step 1, we know that

$$\begin{aligned}
&\|P_2\mathfrak{h}\|_{C_{1-\gamma,\psi}} \\
&\leq \frac{B(\gamma,\alpha)}{\Gamma(\alpha)} (\psi(t)-\psi(0))^{\alpha} \left(\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}}) \|\mathfrak{h}\|_{C_{1-\gamma,\psi}} + \ell_{\mathfrak{g}} \left\|\tilde{\mathfrak{k}}\right\|_{C_{1-\gamma,\psi}} + \|\widetilde{\mathfrak{g}}\|_{C_{1-\gamma,\psi}} \right).
\end{aligned}$$

So operator P_2 is uniformly bounded.

Now we prove the compactness of operator B.

For $0 < t_1 < t_2 < T$, we have

$$\begin{aligned}
&|P_2\mathfrak{h}(t_1,\omega) - P_2\mathfrak{h}(t_2,\omega)| \\
&\leq \Bigg| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \psi^{'}(s) (\psi(t_1)-\psi(s))^{\alpha-1} \mathfrak{g}_\omega(s,\mathfrak{h}(s,\omega), H\mathfrak{h}(s,\omega)) ds \\
&\quad - \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \psi^{'}(s) (\psi(t_2)-\psi(a))^{\alpha-1} \mathfrak{g}_\omega(s,\mathfrak{h}(s,\omega), H\mathfrak{h}(s,\omega)) ds \Bigg| \\
&\leq \|\mathfrak{g}_\omega\|_{C_{1-\gamma,\psi}} B(\gamma,\alpha) \left| (\psi(t_1)-\psi(0))^{\alpha+\gamma-1} - (\psi(t_2)-\psi(0))^{\alpha+\gamma-1} \right|
\end{aligned}$$

tending to zero as $t_1 \to t_2$. Thus P_2 is equicontinuous. Hence, the operator P_2 is compact on B_r by the Arzela-Ascoli Theorem. It follows from Theorem 2.3 that the problem (6.1) has at least one solution. □

Theorem 6.2. *If hypothesis (H1) and the constant*

$$\delta = \frac{\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})B(\gamma,\alpha)}{\Gamma(\alpha)}\left(T\sum_{i=1}^{m} c_i\left(\psi(\tau_i)-\psi(0)\right)^{\alpha+\gamma-1} + \left(\psi(T)-\psi(0)\right)^{\alpha}\right) < 1$$

holds. Then, Eq.(6.1) *has unique solution.*

Next, we shall give the definitions of g-UHR stability for Eq.(5.1).

$$\left|\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{y}(t,\omega) - \mathfrak{g}_{\omega}(t,\mathfrak{y}(t,\omega),H\mathfrak{y}(t,\omega))\right| \le \varphi(t,\omega). \tag{6.6}$$

Definition 6.1. Eq. (6.1) is g-UHR stable with respect to $\varphi \in C_{1-\gamma,\psi}$ if there exists a real number $C_{f,\varphi} > 0$ such that for each solution $\mathfrak{y} \in C_{1-\gamma,\psi}$ of the inequality (6.6) there exists a solution $\mathfrak{h} \in C_{1-\gamma,\psi}$ of Eq. (6.1) with

$$|\mathfrak{y}(t,\omega) - \mathfrak{h}(t,\omega)| \le C_{\mathfrak{g},\varphi}\varphi(t,\omega).$$

Theorem 6.3. *Let hypotheses (H1)–(H3) are fullfilled. Then Eq.*(6.1) *is g-UHR stable.*

Proof. Let $\mathfrak{y}$ be solution of inequality (6.6) and by Theorem 6.2 there $\mathfrak{h}$ is unique solution of equation

$$\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{h}(t,\omega) = \mathfrak{g}_{\omega}(t,\mathfrak{h}(t,\omega),H\mathfrak{h}(t,\omega)),$$
$$\mathfrak{I}^{1-\gamma;\psi}\mathfrak{h}(t,\omega)|_{t=0} = \sum_{i=1}^{m} c_i\mathfrak{h}(\tau_i,\omega), \quad \tau_i \in J$$

is given by

$$\mathfrak{h}(t,\omega) = A_{\mathfrak{h}} + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}\mathfrak{g}_{\omega}(s,\mathfrak{h}(s,\omega),H\mathfrak{h}(s,\omega))ds,$$

where

$$A_{\mathfrak{h}} = \frac{T\left(\psi(t)-\psi(0)\right)^{\gamma-1}}{\Gamma(\alpha)}\sum_{i=1}^{m} c_i\int_0^{\tau_i}\psi'(s)\left(\psi(\tau_i)-\psi(s)\right)^{\alpha-1}\mathfrak{g}_{\omega}(s,\mathfrak{h}(s,\omega),H\mathfrak{h}(s,\omega)).$$

Thus $A_{\mathfrak{h}} = A_{\mathfrak{y}}$.

By differentiating inequality (6.6), we have

$$\left|\mathfrak{y}(t,\omega) - A_{\mathfrak{y}} - \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}_{\omega}(s,\mathfrak{y}(s,\omega),H\mathfrak{y}(s,\omega))ds\right|$$
$$\le \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\varphi(s,\omega)ds$$
$$\le \lambda_{\varphi}\varphi(t,\omega).$$

Hence it follows

$$|\mathfrak{y}(t,\omega) - \mathfrak{h}(t,\omega)|$$

$$
\begin{aligned}
&\leq \left|\mathfrak{y}(t,\omega) - A_{\mathfrak{h}} - \frac{1}{\Gamma(\alpha)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}_\omega(s,\mathfrak{h}(s,\omega),H\mathfrak{h}(s,\omega))ds\right| \\
&\leq \left|\mathfrak{y}(t,\omega) - A_{\mathfrak{h}} - \frac{1}{\Gamma(\alpha)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}_\omega(s,\mathfrak{y}(s,\omega),H\mathfrak{y}(s,\omega))ds\right| \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{g}_\omega(s,\mathfrak{y}(s,\omega),H\mathfrak{y}(s,\omega))\right. \\
&\quad \left. -\mathfrak{g}_\omega(s,\mathfrak{h}(s,\omega),H\mathfrak{h}(s,\omega))\right| ds \\
&\leq \lambda_\varphi\varphi(t,\omega) + \frac{\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})}{\Gamma(\alpha)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{y}(s,\omega)-\mathfrak{h}(s,\omega)\right| ds.
\end{aligned}
$$

By Lemma 2.1, there exists a constant $M^* > 0$ independent of $\lambda_\varphi\varphi(t,\omega)$ such that

$$|\mathfrak{y}(t,\omega) - \mathfrak{h}(t,\omega)| \leq M^*\lambda_\varphi\varphi(t,\omega) := C_{f,\varphi}\varphi(t,\omega).$$

Thus, Eq. (6.1) is g-UHR stable. □

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