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BEYOND SUMUDU TRANSFORM AND NATURAL TRANSFORM: $\mathbb{J}$-TRANSFORM PROPERTIES AND APPLICATIONS*

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Abstract In this paper, we introduce an efficient integral transform called the $\mathbb{J}$-transform which is a modification of the well-known Sumudu transform and the Natural transform for solving differential equations with real applications in applied physical sciences and engineering. The $\mathbb{J}$-transform is more advantageous than both the Sumudu transform and the Natural transform. Interestingly, our proposed $\mathbb{J}$-transform can be applied successfully to solve complex problems that are ordinarily beyond the scope of either Sumudu transform or Natural transform. As a proof of concept, we consider some classic examples and highlight the limitations of the previously reported integral transforms and lastly demonstrate the superiority of the proposed $\mathbb{J}$-transform in addressing those limitations.

Keywords Laplace transform, Sumudu transform, Natural transform, Elzaki transform, ordinary and partial differential equations.

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1. Introduction

Integral transform method is one of the most employed technique for solving differential and integral equations [1–6, 9, 10, 12–14, 18, 20, 21]. Besides, the Laplace transform is the most popular in the literature [16]. The Laplace transform maps a function $f(t)$ in $t-domain$ to a function $F(s)$ in $s-domain$ and the variables s and $F(s)$ are considered as dummies in the transform. In the literature, the Laplace transform of the function $f(t)$ is defined as

$$\mathcal{L}[f(t)] = F(s) = \lim_{\alpha\to\infty}\int_0^{\alpha} \exp(-st)f(t)dt, \tag{1.1}$$

provided the limit of the integral exists for some s, where $\mathcal{L}$ is the Laplace transform operator.

In 1993, Watugala introduced the Sumudu transform which is closely connected with the p-multiplied form of the standard Laplace transform (popularly known as the Laplace-Carson transform [11]). The Sumudu transform was successfully applied to solved controlled engineering problems [19]. Sumudu transform maps a

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function $f(t)$ in $t - domain$ to a function $F(u)$ in $u - domain$ but the variables u and $F(u)$ are not dummies and are considered as the replicas of t and the function $f(t)$ respectively. The Sumudu transform of the function $f(t)$ is defined as

$$S[f(t)] = G(u) = \frac{1}{u} \lim_{\alpha\to\infty} \int_0^{\alpha} \exp\left(\frac{-t}{u}\right) f(t)dt,\ u \in (-\tau_1,\ \tau_2), \tag{1.2}$$

provided the limit of the integral exists for some u, where S is the Sumudu transform operator.

In 2008, N-transform (also known as the Natural transform [6,7]) which is similar to Laplace-Carson transform and the Sumudu integral transform was introduced. N-transform gives both Laplace, and Sumudu integral transforms by the changing of variables and it was successfully applied to unsteady fluid flow problem over a plane wall [15]. The N-transform of the function $f(t)$ is defined as

$$\mathbb{N}^{+}[f(t)] = R(s,u) = \frac{1}{u} \lim_{\alpha\to\infty} \int_0^{\alpha} \exp\left(\frac{-st}{u}\right) f(t)dt,\ s > 0,\ u > 0, \tag{1.3}$$

provided the limit of the integral exists for some u, and s, where $\mathbb{N}^{+}$ is the Natural transform operator.

In 2011, another integral transform similar to both Laplace and the Sumudu transform called the Elzaki transform was introduced. The Elzaki integral transform is defined as

$$E\left[f(t)\right] = T(v) = v \lim_{\alpha\to\infty} \int_0^{\alpha} \exp\left(\frac{-t}{v}\right) f(t)dt,\ t > 0, \tag{1.4}$$

provided the limit of the integral exists [12]. Recently, a new integral transform (Laplace-type) for solving steady heat transfer problems was introduced by Yang. The new integral transform is defined as [22, 23]

$$Y[\phi(\tau)] = \phi(w) = \lim_{\alpha\to\infty} \int_0^{\alpha} \exp\left(\frac{-\tau}{w}\right) \phi(\tau)d\tau, \quad \tau \geq 0, \tag{1.5}$$

provided the limit of the integral exists for some w.

However, despite the potential of the reported integral transforms, they are not universal techniques for solving some of the existing problems in applied physical science and engineering, especially those that are multifaceted.

In this paper, we introduce an integral transform called the $\mathbb{J}$-transform by modification of both Sumudu and Natural transform. $\mathbb{J}$-transform can be considered as an alternative to Sumudu or Natural transform and becomes Laplace's transform when the variable $u = 1$. New theorems and properties (or known) of the proposed integral are introduced. To demonstrate its efficiency and high accuracy, the integral is applied to ordinary and partial differential equations. Furthermore, the relationship of $\mathbb{J}$-transform to other integral transforms are illustrated. In the Appendix section, we computed the $\mathbb{J}$-transform of some useful functions in Table 1 and Table 2.

Throughout this paper, the $\mathbb{J}$-transform is denoted by an operator $\mathbb{J}[\cdot]$.

2. Main Results: $\mathbb{J}$-transform

Definition 2.1 ($\mathbb{J}$-transform)**.** The $\mathbb{J}$-transform of the function $v : [0,\infty) \times \mathbb{R}$ of exponential order is defined over the set of functions,

$$A = \left\{ v(t) : \exists N, \eta_1, \eta_2 > 0, \text{ such that } |v(t)| < N \exp\left(\frac{|t|}{\eta_i}\right) \text{ for } t \in (-1)^i \times [0,\infty) \right\},$$

by

$$\mathbb{J}\left[v(t)\right](s,u) = V(s,u) = u \int_0^\infty \exp\left(\frac{-st}{u}\right) v(t) dt = u \lim_{\alpha \to \infty} \int_0^\alpha \exp\left(\frac{-st}{u}\right) v(t) dt \tag{2.1}$$

for $s > 0$ and $u > 0$, provided the limit of the integral exists, where s and u are the $\mathbb{J}$-transform variables.

Notice that $\mathbb{J}$-transform in (2.1) can also be written as

$$\mathbb{J}\left[v(t)\right](s,u) = V(s,u) = u^2 \int_0^\infty \exp\left(-st\right) v(tu) dt. \tag{2.2}$$

For simplicity, throughout this paper we will also use $\mathbb{J}\left[v(t)\right]$ to denote $\mathbb{J}\left[v(t)\right](s,u)$.

In the next theorem, we prove the sufficient condition for the existence of the $\mathbb{J}$-transform.

Theorem 2.1. *If the function $v(t)$ is piecewise continuous in every finite interval $0 \le t \le \alpha$ and of exponential order β for $t > \alpha$. Then its $\mathbb{J}$-transform $V(s,u)$ exists.*

Proof. For any positive number α, it holds

$$u \int_0^\infty \exp\left(-\frac{st}{u}\right) v(t) dt = u \int_0^\alpha \exp\left(-\frac{st}{u}\right) v(t) dt + u \int_\alpha^\infty \exp\left(-\frac{st}{u}\right) v(t) dt. \tag{2.3}$$

Since the function $v(t)$ is piecewise continuous in every finite interval $0 \le t \le \alpha$, then the first integral on the right hand side exists. And further the second integral on the right hand side also exists, since the function $v(t)$ is of exponential order β for $t > \alpha$. To verify this, we consider the following case

$$\begin{aligned} \left| u \int_\alpha^\infty \exp\left(-\frac{st}{u}\right) v(t) dt \right| &\le u \int_0^\infty \exp\left(-\frac{st}{u}\right) |v(t)| \, dt \\ &\le u \int_0^\infty \exp\left(-\frac{st}{u}\right) N \exp\left(\beta t\right) dt = \frac{u^2 N}{s - \beta u}. \end{aligned}$$

□

3. Some properties of the $\mathbb{J}$-transform

Property 3.1 (Linearity)**.** *Let v and w be in set A. It holds*

$$\mathbb{J}\left[\alpha v(t) + \beta w(t)\right] = \alpha \mathbb{J}\left[v(t)\right] + \beta \mathbb{J}\left[w(t)\right], \tag{3.1}$$

where α and β are two constants.

Proof. Linearity property follows directly from Definition 2.1. □

Property 3.2 (First translation or shifting property of $\mathbb{J}$-transform)**.** *Let* $\exp(\alpha t)\, v(t) \in$ *A, where* α *is constant. Then*

$$\mathbb{J}\left[\exp(\alpha t)\, v(t)\right] = \frac{s-\alpha u}{s} V\left(s, \frac{su}{s-\alpha u}\right). \tag{3.2}$$

Proof. By Definition 2.1, we have

$$\mathbb{J}\left[\exp(\alpha t)\, v(t)\right] = u^2 \int_0^\infty \exp\left(-(s-\alpha u)t\right) v(ut)dt. \tag{3.3}$$

Let $\eta s = (s-\alpha u)t$ which implies $t = \frac{s\eta}{s-\alpha u}$ and $dt = \frac{s}{s-\alpha u} d\eta$ in Equ. (3.3), we deduce

$$\begin{aligned}
\mathbb{J}\left[\exp(\alpha t)\, v(t)\right] &= \frac{su^2}{s-\alpha u} \int_0^\infty \exp(-\eta s)\, v\left(\frac{us\eta}{s-\alpha u}\right) d\eta \\
&= \frac{su^2}{s-\alpha u} \int_0^\infty \exp(-st)\, v\left(\frac{ust}{s-\alpha u}\right) dt \\
&= \frac{u^2 s}{s-\alpha u} \left(\frac{s-\alpha u}{us}\right)^2 V\left(s, \frac{su}{s-\alpha u}\right) \\
&= \frac{s-\alpha u}{s} V\left(s, \frac{su}{s-\alpha u}\right).
\end{aligned}$$

This ends the proof. □

Based on variables transformation in property 3.2, we deduce

$$\mathbb{J}\left[\exp(\alpha t)\, v(t)\right] = \begin{cases} \frac{s-\alpha}{s} V\left(s, \frac{s}{s-\alpha}\right), \; u = 1 \; \text{(Laplace transform)} \\ (1-\alpha u) V\left(1, \frac{u}{1-\alpha u}\right), \; s = 1 \; \text{(Elzaki transform).} \end{cases} \tag{3.4}$$

Property 3.3 (Scaling property)**.** *Let* $V = V(s,u)$ *be the* $\mathbb{J}$*-transform of the function* $v = v(t)$*, and* $\alpha > 0$*. Then we have the scaling property*

$$\mathbb{J}\left[v(\alpha t)\right] = \frac{1}{\alpha} V\left(\frac{s}{\alpha}, u\right). \tag{3.5}$$

Proof. By Equ. (2.1), we deduce

$$\mathbb{J}\left[v(\alpha t)\right] = u \int_0^\infty \exp\left(\frac{-st}{u}\right) v(\alpha t) dt. \tag{3.6}$$

Substituting $\eta = \alpha t$ which implies $t = \frac{\eta}{\alpha}$ and $dt = \frac{d\eta}{\alpha}$ in Equ. (3.6) we obtain

$$\begin{aligned}
\mathbb{J}\left[v(\alpha t)\right] &= \frac{u}{\alpha} \int_0^\infty \exp\left(\frac{-s\eta}{u\alpha}\right) v(\eta) d\eta \\
&= \frac{u}{\alpha} \int_0^\infty \exp\left(\frac{-st}{u\alpha}\right) v(t) dt \\
&= \frac{u}{\alpha} \int_0^\infty \exp\left(\frac{-\left(\frac{s}{\alpha}\right) t}{u}\right) v(t) dt
\end{aligned}$$

$$= \frac{1}{\alpha} V\left(\frac{s}{\alpha}, u\right).$$

The proof ends. □

Theorem 3.1. *Assume that* $v^{(i)} \in A, i = 0, 1, \ldots, n$. *Let* $V(s,u)$ *and* $V_n(s,u)$ *be the* $\mathbb{J}$*-transforms of* $v \in A$ *and* $v^{(n)}$*, respectively. Then*

$$V_n(s,u) = \mathbb{J}\left[v^{(n)}(t)\right] = \frac{s^n}{u^n} V(s,u) - \sum_{k=0}^{n-1} \frac{s^{n-(k+1)}}{u^{n-(k+2)}} v^{(k)}(0). \tag{3.7}$$

Proof. Using Equ. (2.1) for $n = 1$ and integration by parts, we deduce

$$\begin{aligned}
\mathbb{J}\left[v'(t)\right] &= \lim_{\alpha\to\infty} \int_0^{\alpha} \exp\left(\frac{-st}{u}\right) v'(t)\, dt \\
&= u \lim_{\alpha\to\infty} \left[\exp\left(-\frac{st}{u}\right) v(t)\right]_0^{\alpha} + \frac{s}{u} \lim_{\alpha\to\infty} \int_0^{\alpha} u \exp\left(\frac{-st}{u}\right) v(t) dt \\
&= -uv(0) + \frac{s}{u} \mathbb{J}\left[v(t)\right]. \qquad (3.8)
\end{aligned}$$

Since Equ. (3.8) above is true for $n = k$. Then using induction hypothesis we deduce

$$\begin{aligned}
\mathbb{J}\left[(v^{(k)}(t))'\right] &= \frac{s}{u} \mathbb{J}\left[v^{(k)}(t)\right] - uv^{(k)}(0) \\
&= \frac{s}{u} \left[\frac{s^k}{u^k} \mathbb{J}\left[v(t)\right] - \sum_{i=0}^{k-1} \frac{s^{k-(i+1)}}{u^{k-(i+2)}} v^{(i)}(0)\right] - uv^{(k)}(0) \\
&= \left(\frac{s}{u}\right)^{k+1} \mathbb{J}\left[v(t)\right] - \sum_{i=0}^{k} \frac{s^{k-i}}{u^{k-(i+1)}} v^{(i)}(0),
\end{aligned}$$

which implies that Equ. (3.7) holds true for $n = k+1$. By induction we have proved the theorem. □

By Theorem 2 and noting the fact that

$$\mathbb{J}\left[\frac{\partial^n v(x,t)}{\partial x^n}\right] = \int_0^{\infty} u \exp\left(\frac{-st}{u}\right) \frac{\partial^n v(x,t)}{\partial x^n}\, dt = \frac{\partial^n}{\partial x^n} \int_0^{\infty} u \exp\left(\frac{-st}{u}\right) v(x,t)\, dt,$$

we have the following propositions.

Proposition 3.1. *Assume* $v = v(x,t)$, $\frac{\partial^n v(x,t)}{\partial t^n}$ *and* $\frac{\partial^n v}{\partial x^n}$ *be in set* A. *Let* $V(x,s,u)$ *and* $V_n(x,s,u)$ *be the* $\mathbb{J}$*-transforms of* $v(x,t)$ *and* $\frac{\partial^n v(x,t)}{\partial t^n}$ *with respect to* t. *Then*

$$\mathbb{J}\left[\frac{\partial^n v(x,t)}{\partial x^n}\right] = \frac{d^n}{dx^n}\left[V(x,s,u)\right], \tag{3.9}$$

$$V_n(x,s,u) = \mathbb{J}\left[\frac{\partial^n v(x,t)}{\partial t^n}\right] = \frac{s^n}{u^n} V(x,s,u) - \sum_{k=0}^{n-1} \frac{s^{n-(k+1)}}{u^{n-(k+2)}} v^{(k)}(x,0). \tag{3.10}$$

By Theorem 2 and Proposition 3.1, we explicitly give the formulas of (3.10) for $n = 1, 2, 3$.

$$\mathbb{J}\left[\frac{\partial v(x,t)}{\partial t}\right] = \frac{s}{u} V(x,s,u) - uv(x,0). \tag{3.11}$$

$$\mathbb{J}\left[\frac{\partial^2 v(x,t)}{\partial t^2}\right] = \frac{s^2}{u^2}V(x,s,u) - sv(x,0) - uv'(x,0). \tag{3.12}$$

$$\mathbb{J}\left[\frac{\partial^3 v(x,t)}{\partial t}\right] = \frac{s^3}{u^3}V(x,s,u) - \frac{s^2}{u}v(x,0) - sv'(x,0) - uv''(x,0). \tag{3.13}$$

Similarly, using Leibniz's rule, we deduce

$$\begin{aligned}\mathbb{J}\left[\frac{\partial v(x,t)}{\partial x}\right] &= \int_0^\infty u\exp\left(\frac{-st}{u}\right)\frac{\partial v(x,t)}{\partial x}\,dt = \frac{\partial}{\partial x}\int_0^\infty u\exp\left(\frac{-st}{u}\right)v(x,t)\,dt\\ &= \frac{\partial}{\partial x}\left[V(x,s,u)\right] \Rightarrow \mathbb{J}\left[\frac{\partial v(x,t)}{\partial x}\right] = \frac{d}{dx}\left[V(x,s,u)\right],\end{aligned}$$

$$\begin{aligned}\mathbb{J}\left[\frac{\partial^2 v(x,t)}{\partial x^2}\right] &= \int_0^\infty u\exp\left(\frac{-st}{u}\right)\frac{\partial^2 v(x,t)}{\partial x^2}\,dt = \frac{\partial^2}{\partial x^2}\int_0^\infty u\exp\left(\frac{-st}{u}\right)v(x,t)dt\\ &= \frac{\partial^2}{\partial x^2}\left[V(x,s,u)\right] \Rightarrow \mathbb{J}\left[\frac{\partial^2 v(x,t)}{\partial x^2}\right] = \frac{d^2}{dx^2}\left[V(x,s,u)\right].\end{aligned}$$

For the general n,

$$\begin{aligned}\mathbb{J}\left[\frac{\partial^n v(x,t)}{\partial x^n}\right] &= \int_0^\infty u\exp\left(\frac{-st}{u}\right)\frac{\partial^n v(x,t)}{\partial x^n}\,dt = \frac{\partial^n}{\partial x^n}\int_0^\infty u\exp\left(\frac{-st}{u}\right)v(x,t)\,dt\\ &= \frac{\partial^n}{\partial x^n}\left[V(x,s,u)\right] \Rightarrow \mathbb{J}\left[\frac{\partial^n v(x,t)}{\partial x^n}\right] = \frac{d^n}{dx^n}\left[V(x,s,u)\right].\end{aligned}$$

Proposition 3.2. *Let n be a non-negative integer. Assume that $v(t)$, $tv^{(n)}(t)$, and $t^2v^{(n)}(t)$ are functions in set A. Let $V(s,u)$ and $V_n(s,u)$ be the $\mathbb{J}$-transforms of $v(t)$ and $v^{(n)}(t)$, respectively. Then*

$$\mathbb{J}\left[tv^{(n)}(t)\right] = \frac{u^2}{s}\frac{d}{du}\left[V_n(s,u)\right] - \frac{u}{s}\left[V_n(s,u)\right], \tag{3.14}$$

$$\mathbb{J}\left[t^2v^{(n)}(t)\right] = \frac{u^4}{s^2}\frac{d^2}{du^2}\left[V_n(s,u)\right]. \tag{3.15}$$

Proof. Applying Equ. (2.1) and Leibniz's rule, we deduce

$$\begin{aligned}\frac{d}{du}V(s,u) &= \frac{d}{du}\int_0^\infty u\exp\left(\frac{-st}{u}\right)v(t)\,dt = \int_0^\infty \frac{\partial}{\partial u}\left[u\exp\left(\frac{-st}{u}\right)v(t)\right]\,dt\\ &= \int_0^\infty \exp\left(\frac{-st}{u}\right)v(t)dt + \frac{s}{u}\int_0^\infty t\exp\left(\frac{-st}{u}\right)v(t)\,dt\\ &= \frac{1}{u}V(s,u) + \frac{s}{u^2}\mathbb{J}\left[tv(t)\right],\end{aligned}$$

which is the formula (3.14) with $n = 0$. For $n = 1$, we have

$$\begin{aligned}\mathbb{J}\left[tv'(t)\right] =& \frac{u^2}{s}\frac{d}{du}\left[\frac{s}{u}V(s,u) - uv(0)\right] - \frac{u}{s}\left[\frac{s}{u}V(s,u) - uv(0)\right]\\ =& \frac{u^2}{s}\frac{d}{du}\left[V_1(s,u)\right] - \frac{u}{s}\left[V_1(s,u)\right].\end{aligned} \tag{3.16}$$

Then, by induction we obtain the formula (3.14). Similarly, the formula (3.15) can be proved in the same fashion. The proof ends. □

In the next theorem, we prove the convolution theorem of the proposed integral transform.

Theorem 3.2. *Let the functions $f(t)$, $g(t) \in A$. If $V(s,u)$ and $W(s,u)$ are the respective $\mathbb{J}$-transforms of the functions $f(t)$ and $g(t)$. Then the convolution theorem of $\mathbb{J}$-transform is given by*

$$\mathbb{J}\left[(f*g)(t)\right] = \frac{1}{u}V(s,u)W(s,u), \tag{3.17}$$

where $f*g$ is the convolution of two functions $f(t)$ and $g(t)$ which is defined by

$$\int_0^t f(\alpha)g(t-\alpha)d\alpha = \int_0^t f(t-\alpha)g(\alpha)d\alpha. \tag{3.18}$$

Proof. Let first recall that Laplace transform of $(f*g)$ is defined as [16]:

$$\mathcal{L}\left[(f*g)(t)\right] = F(s)G(s). \tag{3.19}$$

Then by the Laplace transform-$\mathbb{J}$-transform duality property (3.4), we deduce

$$\mathbb{J}\left[(f*g)(t)\right] = u\mathcal{L}\left[f*g)(t)\right], \tag{3.20}$$

and since, $V(s,u) = uF\left(\frac{s}{u}\right)$, and $W(s,u) = uG\left(\frac{s}{u}\right)$. Then $\mathbb{J}\left[(f*g)(t)\right]$ is given by

$$\begin{aligned}\mathbb{J}\left[(f*g)(t)\right] &= uF\left(\frac{s}{u}\right)\times G\left(\frac{s}{u}\right)\\ &= \frac{1}{u}uF\left(\frac{s}{u}\right)\times uG\left(\frac{s}{u}\right)\\ &= \frac{1}{u}V(s,u)W(s,u).\end{aligned}$$

This ends the proof. □

The duality property of $\mathbb{J}$-transform with the Laplace transform, the Sumudu transform, and the Natural transform are given below.

Property 3.4 (Laplace transform-$\mathbb{J}$-transform duality)**.** *Let $\mathcal{L}[v(t)] = F(s)$ and $\mathbb{J}\left[(v(t)\right] = V(s,u)$ be the respective Laplace and $\mathbb{J}$-transforms of $v(t) \in A$. Then*

$$V(s,u) = uF\left(\frac{s}{u}\right). \tag{3.21}$$

Proof. By Definition 2.1,

$$\mathbb{J}\left[v(t)\right] = u^2\int_0^\infty \exp\left(-st\right)v(ut)dt. \tag{3.22}$$

Setting $\alpha = ut \Rightarrow t = \frac{\alpha}{u}$ and $dt = \frac{d\alpha}{u}$. Then

$$\frac{1}{u}\left\{u^2\int_0^\infty \exp\left(-\frac{s\alpha}{u}\right)v(\alpha)d\alpha\right\} = u\int_0^\infty \exp\left(-\frac{s\alpha}{u}\right)v(\alpha)d\alpha = uF\left(\frac{s}{u}\right). \tag{3.23}$$

The proof ends. □

Property 3.5 ($\mathbb{J}$-transform-Sumudu transform duality)**.** *Let $V(s,u)$ and $G(u)$ be the $\mathbb{J}$ and Sumudu transforms of $v(t) \in A$, respectively. Then*

$$V(s,u) = u^2G\left(\frac{u}{s}\right). \tag{3.24}$$

Proof. By Definition 2.1, we have

$$\mathbb{J}\left[v(t)\right] = V(s,u) = u^2\int_0^\infty \exp\left(-st\right)v(ut)dt. \tag{3.25}$$

For $u > 0$, let $\alpha = ut \Rightarrow t = \frac{\alpha}{u}$ and $dt = \frac{d\alpha}{u}$. Then

$$\begin{aligned}\mathbb{J}\left[v(t)\right] =& u^2\frac{1}{u}\int_0^\infty \exp\left(-\frac{s}{u}\alpha\right)v(\alpha)d\alpha\\ =& u^2G\left(\frac{u}{s}\right).\end{aligned} \tag{3.26}$$

□

Property 3.6 ($\mathbb{J}$-transform-Natural transform duality)**.** *Let $V(s,u)$ and $R(s,u)$ be the $\mathbb{J}$ and natural transforms of the function $v(t) \in A$. Then*

$$V(s,u) = u^2R(s,u). \tag{3.27}$$

Proof. By Definition 2.1, we deduce

$$\mathbb{J}\left[v(t)\right] = u^2\int_0^\infty \exp\left(-st\right)v(ut)dt = u^2\left\{\frac{1}{u}\int_0^\infty \exp\left(-\frac{st}{u}\right)v(t)dt\right\}, \tag{3.28}$$

which implies (3.27). □

Property 3.7. *Let the function $v(t) = \frac{t^n\exp(\alpha t)}{n!}$ $n = 0, 1, 2, \cdots$ be in set A. Then its $\mathbb{J}$-transform is given by*

$$\mathbb{J}\left[\frac{t^n\exp\left(\alpha t\right)}{n!}\right] = \frac{u^{n+2}}{(s-\alpha u)^{n+1}}. \tag{3.29}$$

Proof. By Definition 2.1 and integration by parts, we deduce

$$\begin{aligned}\mathbb{J}\left[t^n\exp\left(\alpha t\right)\right] &= u\int_0^\infty t^n\exp\left(-\frac{(s-\alpha u)}{u}t\right)dt\\ &= \frac{u^2n}{(s-\alpha u)}\int_0^\infty t^{n-1}\exp\left(-\frac{(s-\alpha)}{u}t\right)dt\\ &= \frac{u^3n(n-1)}{(s-\alpha u)^2}\int_0^\infty t^{n-2}\exp\left(-\frac{(s-\alpha)}{u}t\right)dt = \cdots = \frac{n!u^{n+2}}{(s-\alpha u)^{n+1}}.\end{aligned}$$

□

More results are available in Table 1 and Table 2. Before we illustrate the applications of $\mathbb{J}$-transform. It is important to study the inverse property of $\mathbb{J}$-transform. We first recall the following important results.

Theorem 3.3 (Cauchy's Integral Formula)**.** *Suppose C is a simple closed curve and the function $f(z)$ is analytic on a region containing C and its interior. We assume C is oriented counterclockwise. Then for any z_0 inside C [17]*

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz. \tag{3.30}$$

Theorem 3.4 (Cauchy's residue theorem)**.** *If C is a simple closed, positively oriented contour in the complex plane and f is analytic except for some points $z_1, z_2, \cdots, z_n$ inside the contour C, then*

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} Res_f(z_k). \tag{3.31}$$

Cauchy's residue theorem is a consequence of Cauchy's integral formula, and is very useful in computing real integral by applying appropriate contour in the complex plane [17].

Theorem 3.5 (Inverse Laplace transform)**.** *Let $F(s)$ be the Laplace transform of the function $f(t)$, then $\mathcal{L}^{-1}$ is called the inverse Laplace transform, that is [16]*

$$\mathcal{L}^{-1}[F(s)] = f(t), \text{ for } t \geq 0. \tag{3.32}$$

Equivalently, based on Theorems 3.3 and 3.4, the complex inverse Laplace transform is defined as [16]

$$\mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i} \lim_{\beta\to\infty} \int_{\alpha-i\beta}^{\alpha+i\beta} \exp(st) F(s) ds. \tag{3.33}$$

Theorem 3.6 (Inverse Sumudu transform)**.** *Let $G(u)$ be the Laplace transform of the function $f(t)$, then S^{-1} is called the inverse Sumudu transform, that is [19]*

$$S^{-1}[G(u)] = f(t), \text{ for } t \geq 0. \tag{3.34}$$

Equivalently, based on Theorems 3.3 and 3.4, the complex inverse Sumudu transform is defined as [4]

$$S^{-1}[F(s)] = \frac{1}{2\pi i} \lim_{\beta\to\infty} \int_{\alpha-i\beta}^{\alpha+i\beta} \exp(st) G\left(\frac{1}{s}\right) \frac{ds}{s}. \tag{3.35}$$

Theorem 3.7 (Inverse Natural transform)**.** *Let $R(s,u)$ be the Laplace transform of the function $f(t)$, then $\mathbb{N}^{-1}$ is called the inverse Natural transform, that is [6, 7]*

$$\mathbb{N}^{-1}[(R(s,u)] = f(t), \text{ for } t \geq 0. \tag{3.36}$$

Equivalently, based on Theorems 3.3 and 3.4, the complex inverse Natural transform is defined as [7]

$$\mathbb{N}^{-1}[(R(s,u)] = \frac{1}{2\pi i} \lim_{\beta\to\infty} \int_{\alpha-i\beta}^{\alpha+i\beta} \exp\left(\frac{st}{u}\right) R(s,u) ds. \tag{3.37}$$

Theorem 3.8 (Inverse $\mathbb{J}$-transform)**.** *Let $V(s,u)$ be the $\mathbb{J}$-transform of the function $v(t)$, then $\mathbb{J}^{-1}$ is called the inverse $\mathbb{J}$-transform of $V(s,u)$, that is,*

$$\mathbb{J}^{-1}[V(s,u)] = v(t), \text{ for } t \geq 0. \tag{3.38}$$

Equivalently, based on Theorems 3.3 and 3.4, the complex inverse $\mathbb{J}$-transform is defined as

$$\begin{aligned} v(t) &= \lim_{\beta\to\infty} \frac{1}{2\pi i} \int_{\alpha-i\beta}^{\alpha+i\beta} \frac{1}{u^2} \exp\left(\frac{st}{u}\right) V(s,u) ds \\ &= \sum \text{residues of } \frac{1}{u^2} \exp\left(\frac{st}{u}\right) V(s,u) \text{ at the poles of } V(s,u). \end{aligned} \tag{3.39}$$

Proof. *The proof follows from the $\mathbb{J}$-transform-Natural transform duality Property 3.6.* □

4. Applications

In this section, we illustrate some applications of $\mathbb{J}$-transform on ordinary and partial differential equations to demonstrate its efficiency and high accuracy.

Example 4.1. Compute the inverse $\mathbb{J}$-transform of

$$V(s,u)=\frac{u^2s^2+\alpha^2u^4+su^3\alpha}{s(s^2+\alpha^2u^2)}. \tag{4.1}$$

We observe that $V(s,u)$ has a simple poles at $s=0,\ -iu\alpha,\ \textit{and}\ +iu\alpha$ respectively. Then using Equ. (3.39), we have

$$\begin{aligned}
v(t)&=\sum\text{residues of }\frac{1}{u^2}\exp\left(\frac{st}{u}\right)\frac{u^2s^2+\alpha^2u^4+su^3\alpha}{s(s^2+\alpha^2u^2)}.\\
&=\lim_{s\to 0}(s-0)\frac{\exp\left(\frac{st}{u}\right)}{u^2}\left(\frac{u^2s^2+\alpha^2u^4+su^3\alpha}{s\,(s-iu\alpha)\,(s+iu\alpha)}\right)\\
&\quad+\lim_{s\to +iu\alpha}(s-iu\alpha)\frac{\exp\left(\frac{st}{u}\right)}{u^2}\left(\frac{u^2s^2+\alpha^2u^4+su^3\alpha}{s\,(s-iu\alpha)\,(s+iu\alpha)}\right)\\
&\quad+\lim_{s\to -iu\alpha}(s+iu\alpha)\frac{\exp\left(\frac{st}{u}\right)}{u^2}\left(\frac{u^2s^2+\alpha^2u^4+su^3\alpha}{s\,(s-iu\alpha)\,(s+iu\alpha)}\right)\\
&=1+\frac{1}{2i}\exp(i\alpha t)-\frac{1}{2i}\exp(i\alpha t)=1+\sin(\alpha t).
\end{aligned}$$

Example 4.2. Consider the following partial differential equation

$$\frac{\partial v(x,t)}{\partial t}-\frac{\partial^2 v(x,t)}{\partial x^2}+8v(x,t)=\exp(x)+x, \tag{4.2}$$

subject to the boundary and initial conditions

$$v(0,t)=0,\quad v\left(\frac{\pi}{2},t\right)=0,\quad v(x,0)=12\cos(x)-16\cos(2x). \tag{4.3}$$

Applying the $\mathbb{J}$-transform on both sides of Equ. (4.2), we get

$$\frac{s}{u}V(x,s,u)-uv(x,0)-\frac{d^2V(x,s,u)}{dx^2}+8V(x,s,u)=\frac{u^2\exp(x)}{s}+\frac{u^2x}{s}. \tag{4.4}$$

Substituting the given initial condition and simplifying, we deduce

$$\begin{aligned}\tfrac{d^2V(x,s,u)}{dx^2}-V(x,s,u)\tfrac{(s+8u)}{u}&=-\tfrac{u^2x}{s}-\tfrac{u^2\exp(x)}{s}\\&\quad+12\cos(x)-16\cos(2x).\end{aligned} \tag{4.5}$$

The general solution of Equ. (4.5) can be written as

$$V(x,s,u)=V_h(x,s,u)+V_p(x,s,u), \tag{4.6}$$

where $V_h(x,s,u)$ is the solution of the homogeneous part which is given by

$$V_h(x,s,u)=\alpha_1\exp\left(\sqrt{\frac{s+8u}{u}}x\right)+\alpha_2\exp\left(-\sqrt{\frac{s+8u}{u}}x\right), \tag{4.7}$$

and $V_p(x,s,u)$ is the solution of the non-homogeneous part which is given by

$$V_p(x,s,u) = \beta_1 x + \beta_2 e^x + \beta_3 \cos(x) + \beta_4 \cos(2x). \tag{4.8}$$

Applying the boundary conditions on Equ. (4.7), we get

$$\alpha_1 + \alpha_2 = 0 \text{ and } \alpha_1 \exp\left(\sqrt{\frac{s+8u}{u}}\frac{\pi}{2}\right) + \alpha_2 \exp\left(-\sqrt{\frac{s+8u}{u}}\frac{\pi}{2}\right) = 0$$
$$\Rightarrow V_h(x,s,u) = 0, \text{ since } \alpha_1 = \alpha_2 = 0. \tag{4.9}$$

Using the method of undetermined coefficients on the non-homogeneous part, we obtain

$$V_p(x,s,u) = \frac{xu^3}{s(s+8u)} + \frac{e^x u^3}{s(s+7u)} - \frac{12u^2\cos(x)}{s+9u} + \frac{16u^2\cos(2x)}{s+12u}, \tag{4.10}$$

since, $\beta_1 = \frac{u^3}{s(s+8u)}$, $\beta_2 = \frac{u^3}{s(s+7u)}$, $\beta_3 = -\frac{12u^2}{s+9u}$, and $\beta_4 = \frac{16u^2}{s+12u}$.

Then Equ. (4.6) will become

$$V(x,s,u) = \frac{xu^3}{s(s+8u)} + \frac{\exp(x)u^3}{s(s+7u)} - \frac{12u^2\cos(x)}{s+9u} + \frac{16u^2\cos(2x)}{s+12u}. \tag{4.11}$$

Computing the inverse $\mathbb{J}$-transform of Equ. (4.11), we obtain the solution

$$v(x,t) = \frac{x}{8}\left(1-\exp(-8t)\right) + \frac{\exp(x)}{7}\left(1-\exp(-7t)\right)$$
$$-12\cos(x)\exp(-9t) + 16\cos(2x)\exp(-12t). \tag{4.12}$$

Example 4.3. Consider the following wave equation

$$\frac{\partial^2 v(x,t)}{\partial t^2} = \theta^2 \frac{\partial^2 v(x,t)}{\partial x^2} - v(x,t) + 32x + 48\sin(2x), \tag{4.13}$$

subject to the boundary and initial conditions

$$v(0,t) = 0, \quad v(\pi,t) = 0, \quad \frac{\partial v(x,0)}{\partial t} = 0,$$
$$v(x,0) = 32x + 8\sin(4x) - 4\sin(5x), \quad \theta = 2. \tag{4.14}$$

Applying the $\mathbb{J}$-transform on both sides of Equ. (4.13), we get

$$\frac{s^2}{u^2}V(x,s,u) - sv(x,0) - uv'(x,0)$$
$$= \frac{d^2V(x,s,u)}{dx^2} + 48\frac{u^2\sin(2x)}{s} + \frac{32xu^2}{s}. \tag{4.15}$$

Substituting the given initial condition and simplifying, we deduce

$$\frac{d^2V(x,s,u)}{dx^2} - \frac{(s^2+u^2)}{4u^2}V(x,s,u)$$
$$= -\frac{12u^2\sin(2x)}{s} - \frac{8x(s^2+u^2)}{s} - 2s\sin(4x) + s\sin(5x). \tag{4.16}$$

The general solution of Equ. (4.16) can be written as

$$V(x,s,u)=V_h(x,s,u)+V_p(x,s,u), \tag{4.17}$$

where $V_h(x,s,u)$ is the solution of the homogeneous part which is given by

$$V_h(x,s,u)=\lambda_1\exp\left(\frac{\sqrt{s^2+u^2}}{2u}x\right)+\lambda_2\exp\left(-\frac{\sqrt{s^2+u^2}}{2u}x\right), \tag{4.18}$$

and $V_p(x,s,u)$ is the solution of the non-homogeneous part which is given by

$$V_p(x,s,u)=\eta_1 x+\eta_2\sin(2x)+\eta_3\sin(4x)+\eta_4\sin(5x). \tag{4.19}$$

Applying the boundary conditions on Equ. (4.18), yield

$$\begin{aligned}\lambda_1+\lambda_2=0\ \text{ and }\ \lambda_1\exp\left(\frac{\sqrt{s^2+u^2}}{2u}\pi\right)+\lambda_2\exp\left(-\frac{\sqrt{s^2+u^2}}{2u}\pi\right)=0\\ \Rightarrow V_h(x,s,u)=0,\ \text{ since }\ \lambda_1=\lambda_2=0.\end{aligned} \tag{4.20}$$

Using the method of undetermined coefficients on the non-homogeneous part, we have

$$V_p(x,s,u)=\frac{32xu^2}{s}+\frac{48u^4\sin(2x)}{s(s^2+17u^2)}+\frac{4su^2\sin(4x)}{s^2+65u^2}-\frac{4su^2\sin(5x)}{s^2+101u^2}, \tag{4.21}$$

since,

$$\eta_1=\frac{32u^2}{s},\ \eta_2=\frac{48u^4}{s(s^2+17u^2)},\ \eta_3=\frac{4su^2}{s^2+65u^2},\ \eta_4=-\frac{4su^2}{s^2+101u^2}. \tag{4.22}$$

Then Equ. (4.17) will become

$$V(x,s,u)=\frac{32xu^2}{s}+\frac{48u^4\sin(2x)}{s(s^2+17u^2)}+\frac{4su^2\sin(4x)}{s^2+65u^2}-\frac{4su^2\sin(5x)}{s^2+101u^2}. \tag{4.23}$$

Computing the inverse $\mathbb{J}$-transform of Equ. (4.23) yield

$$\begin{aligned}v(x,t)=32x+\tfrac{48}{17}\sin(2x)(1-\cos(\sqrt{17}t)\\ +4\sin(4x)\cos(\sqrt{65}t)-4\sin(5x)\cos(\sqrt{101}t).\end{aligned} \tag{4.24}$$

Example 4.4. Consider the following Bessel's differential equation with polynomial coefficients

$$t^2\frac{d^2v(t)}{dt^2}+\frac{dv(t)}{dt}+6tv(t)=0 \tag{4.25}$$

subject to the initial conditions

$$v(0)=\alpha,\ \ \frac{dv(0)}{dt}=\beta. \tag{4.26}$$

Applying the $\mathbb{J}$-transform on both sides of Equ. (4.25), we get

$$\frac{u^2}{s}\frac{d}{du}\left[\frac{s^2}{u^2}V(s,u)-sv(0)-uv'(0)\right]$$

$$
\begin{aligned}
&-\frac{u}{s}\left[\frac{s^2}{u^2}V(s,u)-sv(0)-uv'(0)\right]\\
&+6\left[\frac{u^2}{s}\frac{d}{du}V(s,u)-\frac{u}{s}V(s,u)\right]\\
=&\ \frac{24u^4}{s^3}+\frac{6u^3}{s^2}+\frac{2u^2}{s}. \qquad (4.27)
\end{aligned}
$$

Substituting the given initial conditions and simplifying, we deduce

$$\frac{dV(s,u)}{du}-V(s,u)\left(\frac{2s^2+6u^2}{u(s^2+6u^2)}\right)=0. \qquad (4.28)$$

Solving Equ. (4.28), we obtain

$$V(s,u)=\frac{\gamma u^2}{\sqrt{s^2+6u^2}}, \qquad (4.29)$$

where γ is a constant. Taking the inverse $\mathbb{J}^{-1}$ to Equ. (4.29) we obtain

$$v(t)=\alpha J_0(\sqrt{6}t), \qquad (4.30)$$

since, $\alpha=\gamma$, using the initial conditions. See Table 2.

Example 4.5. Consider the following ordinary differential equation with variable coefficients

$$t^2\frac{d^2v(t)}{dt^2}+4t\frac{dv(t)}{dt}+2v(t)=12t^2+6t+2 \qquad (4.31)$$

with the initial conditions $v(0)=\frac{dv(0)}{dt}=0$.

Applying the $\mathbb{J}$-transform on both sides of Equ. (4.31), yield

$$
\begin{aligned}
&\frac{u^4}{s^2}\frac{d^2}{du^2}\left[\frac{s^2}{u^2}V(s,u)-sv(0)-uv'(0)\right]\\
&+\frac{4u^2}{s}\frac{d}{du}\left[\frac{s}{u}V(s,u)-uv(0)\right]\\
&-\frac{4u}{s}\left[\frac{s}{u}V(s,u)-uv(0)\right]+2V(s,u)\\
=&\ \frac{24u^4}{s^3}+\frac{6u^3}{s^2}+\frac{2u^2}{s}. \qquad (4.32)
\end{aligned}
$$

Substituting the given initial conditions, we deduce

$$\frac{d^2V(s,u)}{du^2}=\frac{24u^2}{s^3}+\frac{6u}{s^2}+\frac{2}{s}. \qquad (4.33)$$

Integrating Equ. (4.33) with respect to u, we obtain

$$V(s,u)=\frac{2u^4}{s^3}+\frac{u^3}{s^2}+\frac{u^2}{s}+\alpha u+\beta, \qquad (4.34)$$

where α and β are the constants. Then

$$V(s,u)=\frac{2u^4}{s^3}+\frac{u^3}{s^2}+\frac{u^2}{s} \qquad (4.35)$$

since, $\alpha = \beta = 0$ using the initial conditions. Now computing the inverse $\mathbb{J}$-transform of Equ. (4.35), gives us the solution

$$v(t) = t^2 + t + 1. \tag{4.36}$$

Example 4.6. Consider the following ordinary differential equation with variable coefficients

$$t^2 \frac{d^3 v(t)}{dt^3} + 6t \frac{d^2 v(t)}{dt^2} + 6 \frac{dv(t)}{dt} = 120t^5 - 60t^3 - 3\delta(t) \tag{4.37}$$

with the initial conditions $v(0) = \frac{dv(0)}{dt} = \frac{d^2 v(0)}{dt^2} = 0$.

Taking the $\mathbb{J}$-transform on both sides of Equ. (4.37), we obtain

$$\begin{aligned} &\frac{u^4}{s^2} \frac{d^2}{du^2} \left[\frac{s^3}{u^3} V(s,u) - \frac{s^2}{u} v(0) - s v'(0) - u v''(0) \right] \\ &+ \frac{6u^2}{s} \frac{d}{du} \left[\frac{s^2}{u^2} V(s,u) - s v(0) - u v'(0) \right] \\ &- \frac{6u}{s} \left[\frac{s^2}{u^2} V(s,u) - s v(0) - u v'(0) \right] + \frac{6s}{u} V(s,u) \\ = &\frac{14400 u^7}{s^6} - \frac{360 u^5}{s^4} + 3u. \end{aligned} \tag{4.38}$$

Substituting the given initial conditions and simplifying leads to

$$\frac{d^2 V(s,u)}{du^2} = \frac{14400 u^6}{s^7} - \frac{360 u^5}{s^6} + \frac{3}{s}. \tag{4.39}$$

Integrating Equ. (4.39) with respect to u, we deduce

$$V(s,u) = \frac{1800 u^8}{7 s^7} - \frac{60 u^7}{7 s^6} + \frac{3u^2}{2s} + \alpha u + \beta, \tag{4.40}$$

where α and β are the constants, then

$$V(s,u) = \frac{1800 u^8}{7 s^7} - \frac{60 u^7}{7 s^6} + \frac{3u^2}{2s}, \tag{4.41}$$

since, $\alpha = \beta = 0$ using the initial conditions. Computing the inverse $\mathbb{J}$-transform of Equ. (4.41), we get the solution

$$v(t) = \frac{3}{2} \left(\frac{15}{63} t^6 - \frac{1}{21} t^5 + 1 \right). \tag{4.42}$$

Remark 4.1. Both example 5 and example 6 cannot be solve using the Sumudu transform and the natural transform, since in each case we algebraically obtained the original problem. Details are shown below.

For example 5: Applying the Natural transform on Equ. (4.31) we get

$$u^2 \frac{d^2}{du^2} \left(V(s,u) \right) + 4u \frac{d}{du} \left(V(s,u) \right) + 2V(s,u) = \frac{24u^2}{s^3} + \frac{6u}{s^2} + \frac{2}{s}. \tag{4.43}$$

Applying the Sumudu transform on Equ. (4.31) we have:

$$u^4 \frac{d^2}{du^2} \left(G_2(u) \right) + 4u^3 \frac{d}{du} \left(G_2(u) \right) + 4u^2 \frac{d}{du} \left(G_1(u) \right)$$

$$\begin{aligned}&+2u^2G_2(u)+4uG_1(u)+2G(u)\\&=24u^2+6u+\frac{2}{u},\end{aligned}\tag{4.44}$$

where $G_1(u),$ and $G_2(u)$ are the Sumudu first, and second derivatives.

For example 6: Applying the Natural transform on Equ. (4.37) we get

$$us\frac{d^2}{du^2}\left(V(s,u)\right)+4s\frac{d}{du}\left(V(s,u)\right)+\frac{6s}{u}V(s,u)=\frac{14400u^5}{s^6}-\frac{360u^3}{s^4}+\frac{3}{u}.\tag{4.45}$$

Applying the Sumudu transform on Equ. (4.37) yield

$$\begin{aligned}&u^4\frac{d^2}{du^2}\left(G_3(u)\right)+4u^3\frac{d}{du}\left(G_3(u)\right)+6u^2\frac{d}{du}\left(G_2(u)\right)+6uG_2(u)+6G_1(u)\\&=14400u^5-360u^3+\frac{3}{u},\end{aligned}\tag{4.46}$$

where $G_1(u),\ G_2(u),$ and $G_3(u)$ are the Sumudu first,second, and third derivatives.

Conclusion

In this paper, a powerful integral transform called the $\mathbb{J}$-transform for solving differential equations in time domain is introduced by modifications of the popular Sumudu transform and Natural transform. $\mathbb{J}$-transform possesses distinct properties that make its applications to applied physical sciences and engineering problems easier. Through some variables transformation, $\mathbb{J}$-transform becomes Laplace's transform when the variable $u=1$, and becomes Elzaki's transform when the variable $s=1$. Many interesting properties including inverse property, linearity, shifting, change of scaling, and convolution properties of the proposed integral transform are clearly presented. Applications which cannot be solved using both Sumudu and Natural transform are successfully solved using the proposed $\mathbb{J}$-transform. Thus, $\mathbb{J}$-transform can be regarded as a modification of both Sumudu, and the Natural transform. $\mathbb{J}$-transform can easily be extended to study many applications in physical science and engineering. We discussed the relationship of $\mathbb{J}$-transform with some integral transforms in the literature. Our goal in the near future is to study the extended properties and applications of the proposed integral transform.

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Appendix

In this section, the $\mathbb{J}$-transform of some special functions are given.

Table 1. $\mathbb{J}$-transform of some special functions

S/N.	Definitions	Functions	$\mathbb{J}$-transforms
1	Gamma function	$\Gamma(n)$	$\frac{u^2\Gamma(n)}{s}$
2	Beta function	$B(n,m)$	$\frac{u^2 B(n,m)}{s}$
3	Error function	$erf(t)$	$\frac{u^2}{s}e^{\frac{s^2}{4u^2}}Erfc(\frac{s}{2u})$
		$erf(\alpha t)$	$\frac{u^2}{s}e^{\frac{s^2}{4u^2\alpha^2}}Erfc(\frac{s}{2u\alpha})$
4	Complementary error function	$erfcf(t)$	$\frac{u}{s}\left(u-ue^{\frac{s^2}{4u^2}}Erfc(\frac{s}{2u})\right)$
		$erfcf(\alpha t)$	$\frac{u}{s}\left(u-ue^{\frac{s^2}{4u^2\alpha^2}}Erfc(\frac{s}{2u\alpha})\right)$
5	Bessel function	$J_n(t)$	$\frac{u(\frac{s}{u})^{-1-n}\left(1+\sqrt{1+\frac{u^2}{s^2}}\right)^{-n}}{\sqrt{1+\frac{u^2}{s^2}}}$
		$J_n(\alpha t)$	$\frac{u(\frac{s}{u})^{-1-n}\alpha^n\left(1+\sqrt{1+\frac{u^2\alpha^2}{s^2}}\right)^{-n}}{\sqrt{1+\frac{u^2\alpha^2}{s^2}}}$
6	Modified Bessel function	$I_n(t)$	$\frac{u\left(\sqrt{\frac{s^2}{u^2}-1}+\frac{s}{u}\right)^{-n}}{\sqrt{\frac{s^2}{u^2}-1}}$
		$I_n(\alpha t)$	$\frac{u^2\left(\sqrt{\frac{s^2}{u^2}-\alpha^2}+\lvert\frac{s}{u}\rvert\right)^{-n}\lvert\alpha\rvert^n}{s\sqrt{\frac{s^2\alpha^2}{u^2}-1}}$
7	Sine Integral function	$Si(\alpha t)$	$\frac{u^2}{s}\arctan(\frac{\alpha u}{s})$
8	Hyperbolic Sine function	$Shi(t)$	$\frac{u}{\lvert s\rvert}\lvert u\rvert\arctan(\frac{\alpha u}{s})$
		$Shi(\alpha t)$	$\frac{u}{\alpha}\lvert\frac{u\alpha}{s}\rvert\arctan(\frac{\alpha u}{s})$
9	Cosine Integral	$Ci(t)$	$\frac{-u^2}{2s}\log(\frac{u^2}{u^2+s^2})$
	Hyperbolic Cosine Integral	Chi	$\frac{u^2}{2s}\log(\frac{u^2}{s^2-u^2})$
10	Laguerre Polynomial	$L_n(t)$	$ue^{in\pi}\left(1-\frac{s}{u}\right)^{-1-n}$
		$L_n(\alpha t)$	$u(\frac{s}{u})^{-1-n}\left(1-\frac{s}{\alpha u}\right)^n(-\alpha)^n$
11	Sinc	$Sinc(t)$	$u\arctan(\frac{u}{s})$
		$Sinc(\alpha t)$	$\frac{u}{\alpha}\arctan(\frac{u\alpha}{s})$

Table 2. $\mathbb{J}$-transform of some functions

S/N.	Functions	$\mathbb{J}$-transforms
1	1	$\frac{u^2}{s}$
2	t	$\frac{u^3}{s^2}$
3	$\exp(\alpha t)$	$\frac{u^2}{s-\alpha u}$
4	$\frac{\sin(\alpha t)}{\alpha}$	$\frac{u^3}{s^2+\alpha^2 u^2}$
5	$\cos(\alpha t)$	$\frac{u^2 s}{s^2+\alpha^2 u^2}$
6	$\cosh(\alpha t)$	$\frac{u^2 s}{s^2-u^2}$
7	$\frac{t^n}{n!}\ n=0,1,2...$	$\frac{u^{n+2}}{s^{n+1}}$
8	$\frac{t^n}{\Gamma(n+1)}\ n=0,1,2,...$	$\frac{u^{n+2}}{s^{n+1}}$
9	$\cos(t)$	$\frac{u^2 s}{s^2+u^2}$
10	$\sin(t)$	$\frac{u^3}{s^2+u^2}$
11	$\frac{\sinh(\alpha t)}{\alpha}$	$\frac{u^3}{s^2-\alpha^2 u^2}$
12	$\cosh(\alpha t)$	$\frac{u^2 s}{s^2-\alpha^2 u^2}$
13	$\exp(\beta t)\cosh(\alpha t)$	$\frac{u^2(s-\beta u)}{(s-\beta u)^2-\alpha^2 u^2}$
14	$\frac{\exp(\beta t)\sinh(\alpha t)}{\alpha}$	$\frac{u^3}{(s-\beta u)^2-\alpha^2 u^2}$
15	$\frac{t\sin(\alpha t)}{2\alpha}$	$\frac{u^4 s}{(s^2+\alpha^2 u^2)^2}$
16	$t\cos(\alpha t)$	$\frac{u^3(s^2-\alpha^2 u^2)}{(s^2+\alpha^2 u^2)^2}$
17	$\frac{\sin(\alpha t)+\alpha t\cos(\alpha t)}{2\alpha}$	$\frac{u^3 s^2}{(s^2+\alpha^2 u^2)^2}$
18	$\cos(\alpha t)-\frac{\alpha t\sin(\alpha t)}{2}$	$\frac{u^2 s^3}{(s^2+\alpha^2 u^2)^2}$
19	$\frac{\sin(\alpha t)-\alpha t\cos(\alpha t)}{2\alpha^3}$	$\frac{u^5}{(s^2+\alpha^2 u^2)^2}$
20	$\frac{\alpha t\cosh(\alpha t)-\sinh(\alpha t)}{2\alpha^3}$	$\frac{u^5}{(s^2-\alpha^2 u^2)^2}$
21	$\frac{t\sinh(\alpha t)}{2\alpha}$	$\frac{u^4 s}{(s^2-\alpha^2 u^2)^2}$
22	$\frac{\sinh(\alpha t)+\alpha t\cosh(\alpha t)}{2\alpha}$	$\frac{u^3 s^2}{(s^2-\alpha^2 u^2)^2}$
23	$\cosh(\alpha t)+\frac{\alpha t\sinh(\alpha t)}{2}$	$\frac{u^2 s^3}{(s^2-\alpha^2 u^2)^2}$
24	$t\cosh(\alpha t)$	$\frac{u^3(s^2+\alpha^2 u^2)}{(s^2-\alpha^2 u^2)^2}$
25	$\frac{(3-\alpha^2 t^2)\sin(\alpha t)-3\alpha t\cos(\alpha t)}{8\alpha^5}$	$\frac{u^8}{(s^2+\alpha^2 u^2)^3}$
26	$\frac{(3-\alpha^2 t^2)\sin(\alpha t)-3\alpha t\cos(\alpha t)}{8\alpha}$	$\frac{u^3 s^4}{(s^2+\alpha^2 u^2)^3}$
27	$\frac{(8-\alpha^2 t^2)\cos(\alpha t)-7\alpha t\sin(\alpha t)}{8}$	$\frac{u^2 s^5}{(s^2+\alpha^2 u^2)^3}$
28	$\frac{t^2\sin(\alpha t)}{2\alpha}$	$\frac{u^5(3s^2-\alpha^2)}{(s^2+\alpha^2 u^2)^3}$
29	$\frac{t^2\cos(\alpha t)}{2}$	$\frac{u^4(s^3-3\alpha^2 u^2 s)}{(s^2+\alpha^2 u^2)^3}$
30	$\frac{t^3\sin(\alpha t)}{24\alpha}$	$\frac{u^6(s^3-\alpha^2 u^2 s)}{(s^2+\alpha^2 u^2)^4}$
31	$\frac{\exp(\alpha t)-\exp(\beta t)}{\alpha-\beta}\ \alpha\neq\beta$	$\frac{u^3}{(s-\alpha u)(s-\beta u)}$
31	$\frac{\alpha\exp(\alpha t)-\beta\exp(\beta t)}{\alpha-\beta}\ \alpha\neq\beta$	$\frac{u^2 s}{(s-\beta u)(s-\alpha u)}$
33	$I_0(\alpha t)$	$\frac{u^2}{\sqrt{s^2-\alpha^2 u^2}}$
34	$\delta(t-\alpha)$	$u^2 e^{\frac{-\alpha s}{u}}$
35	$J_0(\alpha t)$	$\frac{u^2}{\sqrt{s^2+\alpha^2 u^2}}$

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