

STABILITY ANALYSIS OF HIGHLY NONLINEAR HYBRID MULTIPLE-DELAY STOCHASTIC DIFFERENTIAL EQUATIONS*

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Abstract Stability criteria for stochastic differential delay equations (SD-DEs) have been studied intensively for the past few decades. However, most of these criteria can only be applied to delay equations where their coefficients are either linear or nonlinear but bounded by linear functions. Recently, the stability of highly nonlinear hybrid stochastic differential equations with a single delay is investigated in [Fei, Hu, Mao and Shen, *Automatica*, 2017], whose work, in this paper, is extended to highly nonlinear hybrid stochastic differential equations with variable multiple delays. In other words, this paper establishes the stability criteria of highly nonlinear hybrid variable multiple-delay stochastic differential equations. We also discuss an example to illustrate our results.

Keywords Variable multiple-delay stochastic differential equation, nonlinear growth condition, asymptotic stability, Markovian switching, Lyapunov functional.

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1. Introduction

In many practical systems, such as science, industry, economics and finance etc., we will encounter the systems with time delay. Differential delay equations (DDEs) have been employed to model such time-delay systems. Since the time-delay often causes the instability of systems, stability of DDEs has been explored intensively for more than 50 years. Generally, the stability criteria are classified into the delay-independent and delay-dependent stability criteria. When the size of delays of the systems is incorporated into the delay-dependent stability criteria, the delay-dependent systems are generally less conservative than the delay-independent ones

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which work for any size of delays. There exists a very rich literature in this topics (see, e.g., [3, 11–13, 16, 17, 21, 36]).

Since 1980's, stochastic differential delay equations were investigated in order to model practical systems which are subject to external noises (see, e.g., [27]). Since then, the study of the stability on SDDEs has been one of the most important topics (see, e.g., [5, 10, 15, 19, 20, 24]).

In 1990's, hybrid SDDEs (called also SDDEs with Markovian switching) were developed to model real-world systems since they may experience abrupt changes in their parameters and structures in addition to uncertainties and time lags. One of the important issues in the research of hybrid SDDEs is the analysis of stability of control systems. Moreover the delay-dependent stability criteria have been erected by many authors (see, e.g., [2, 4, 22, 23, 25, 26, 28, 33–35]). To our best knowledge, the existing delay-dependent stability criteria are mainly created for the hybrid SDDEs where their coefficients are either linear or nonlinear but bounded by linear functions. Based on highly nonlinear hybrid SDDEs (see, e.g., [6–9, 14, 15, 30–32]), [7] has recently established the delay-dependent stability criterion where they solve the stability of a single delay system. However, many real systems has multiple time-delay states (see, e.g., [1, 18, 29]). Therefore we further develop the stability criteria of highly nonlinear hybrid SDDEs with variable multiple delays.

Specifically, we first discuss the following SDDE with two delays $\delta_1(t), \delta_2(t)$ with $\delta_1(t) \leq \tau$ (see Example 4.1)

$$dx(t) = \begin{cases} (-10x^3(t) - x(t - \delta_1(t)))dt + \frac{1}{4}x^2(t - \delta_2(t))dB(t), & \text{if } r(t) = 1, \\ (-4x^3(t) + \frac{1}{2}x(t - \delta_1(t)))dt + \frac{1}{2}x^2(t - \delta_2(t))dB(t), & \text{if } r(t) = 2, \end{cases} \quad (1.1)$$

on $t \geq 0$ with initial data

$$\{x(u) = 2 + \sin(u) : -\tau \leq u \leq 0\} \in C([-\tau, 0]; \mathbb{R}), \quad r(0) = i_0 \in \mathbb{S}. \quad (1.2)$$

Here $B(t)$ is a scalar Brownian motion, $r(t)$ is a Markovian chain with space $\mathbb{S} = \{1, 2\}$ and its generator Γ given by

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 8 & -8 \end{pmatrix}. \quad (1.3)$$

The above system (1.1) will switch from one mode to the other according to the probability law of the Markovian chain. If $\delta_1(t) \leq \tau = 0.01$, the computer simulation shows it is asymptotically stable (see Figure 4.1). If the time-delay is large, say $\delta_1(t) \leq \tau = 2$, the computer simulation shows that the hybrid multiple-delays stochastic differential equation(SDE) (1.1) is unstable (see Figure 4.2). In other words, whether the hybrid multiple-delay SDE is stable or not depends on how small or large the time-delay is. On the other hand, both drift and diffusion coefficients of the hybrid SDE with multiple delays affect the stability of systems due to highly nonlinear. However, there is no delay dependent criterion which can be applied to the SDE with multiple delays to derive a sufficient bound on the time-delay τ such that the SDDE is stable, although the stability criteria of the highly nonlinear hybrid SDE with single delay have been created in [7]. This paper first established delay dependent criteria for highly nonlinear hybrid SDEs with variable multiple delays.

In comparison with [7], the key contributions in this paper are highlighted below:

- This paper takes the variable multiple delays into account to develop a new theory on the robust stability and boundedness for highly nonlinear hybrid SDDEs.
- The new theory established in this paper is applicable to hybrid SDDEs with different delays in drift and diffusion coefficient of SDDEs with multiple delays (see (2.1)). Especially, we found that the sizes of delays in drift coefficient only affect the stability of the system, but the sizes of delays in the diffusion coefficient do NOT. This result has a significant importance.
- A significant amount of new mathematics has been developed to deal with the difficulties due to different delays in drift and diffusion coefficient of SDDEs with multiple delays and those without the linear growth condition. For example, a more complicated Lyapunov function will be designed in order to deal with the effects of the different delays. A lot of effort has also been put into showing the bounds of the sizes of delays.

To develop our new theory, we will introduce some necessary notation in Section 2. We will discuss in Section 3 the delay-dependent asymptotic stability of SDEs with variable multiple delays, and give main results on robust boundedness and stability. We will present an example in Section 4 to illustrate our theory. We will finally conclude our paper in Section 5.

2. Notation and Assumptions

Throughout this paper, unless otherwise specified, we use the following notation. If A is a vector or matrix, its transpose is denoted by A^\top . If $x \in \mathbb{R}^d$, then $|x|$ is its Euclidean norm. For a matrix A , we let $|A| = \sqrt{\text{trace}(A^\top A)}$ be its trace norm and $\|A\| = \max\{|Ax| : |x| = 1\}$ be the operator norm. Let $\mathbb{R}_+ = [0, \infty)$. For $\tau > 0$, denote by $C([- \tau, 0]; \mathbb{R}^d)$ the family of continuous functions η from $[- \tau, 0] \rightarrow \mathbb{R}^d$ with the norm $\|\eta\| = \sup_{-\tau \leq u \leq 0} |\eta(u)|$. If A is a subset of Ω , denote by I_A its indicator function. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions. Let $B(t) = (B_1(t), \dots, B_m(t))^\top$ be an m -dimensional Brownian motion defined on the probability space. Let $r(t)$, $t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $\mathbb{S} = \{1, 2, \dots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$ given by

$$\mathbb{P}\{r(t + \Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$. Let $\tau_j, \bar{\delta}_j \in [0, 1]$, $j = 1, \dots, n$, be constants with $\tau =: \max_{j=1}^{n_1} \tau_j$. The delays $\delta_j(\cdot)$ are differential functions from $\mathbb{R}_+ \rightarrow [0, \tau]$, such that $\dot{\delta}_j(t) := d\delta_j(t)/dt \leq \bar{\delta}_j$ for all $t \leq 0$, and $\tau_j \geq \delta_j(t)$. For Borel measurable functions

$$f : \mathbb{R}^{d(n_1+1)} \times \mathbb{S} \times \mathbb{R}_+ \rightarrow \mathbb{R}^d \quad \text{and} \quad g : \mathbb{R}^{d(n-n_1+1)} \times \mathbb{S} \times \mathbb{R}_+ \rightarrow \mathbb{R}^{d \times m},$$

we consider a d -dimensional hybrid SDE with n -delays

$$dx(t) = f(x(t), x(t - \delta_1(t)), \dots, x(t - \delta_{n_1}(t)), r(t), t)dt$$

$$+ g(x(t), x(t - \delta_{n_1+1}(t)), \dots, x(t - \delta_n(t)), r(t), t) dB(t) \quad (2.1)$$

on $t \geq 0$ with initial data

$$\{x(t) : -\tau \leq t \leq 0\} = \eta \in C([- \tau, 0]; \mathbb{R}^d), \quad r(0) = i_0 \in \mathbb{S}. \quad (2.2)$$

The classical conditions for the existence and uniqueness of the global solution are the local Lipschitz condition and the linear growth condition (see, e.g., [24]). In this paper, we need only the local Lipschitz condition. However, we will consider highly nonlinear hybrid SDEs with multiple delays which, in general, do not satisfy the linear growth condition in this paper. Therefore, we impose the polynomial growth condition, instead of the linear growth condition. Let us state these conditions as an assumption for our aim.

Assumption 2.1. Assume that for any $h > 0$, there exists a positive constant K_h such that

$$\begin{aligned} & |f(x, y_1, \dots, y_{n_1}, i, t) - f(\bar{x}, \bar{y}_1, \dots, \bar{y}_{n_1}, i, t)| \\ & \quad + |g(x, y_{n_1+1}, \dots, y_n, i, t) - g(\bar{x}, \bar{y}_{n_1+1}, \dots, \bar{y}_n, i, t)| \\ & \leq K_h (|x - \bar{x}| + \sum_{j=1}^n |y_j - \bar{y}_j|) \end{aligned}$$

for all $x, y_1, \dots, y_n, \bar{x}, \bar{y}_1, \dots, \bar{y}_n \in \mathbb{R}^d$ with $|x| \vee |y_1| \vee \dots \vee |y_n| \vee |\bar{x}| \vee |\bar{y}_1| \vee \dots \vee |\bar{y}_n| \leq h$ and all $(i, t) \in \mathbb{S} \times \mathbb{R}_+$. Assume moreover that there exist three constants $K > 0$, $q_1 \geq 1$ and $q_2 \geq 1$ such that

$$\begin{aligned} |f(x, y_1, \dots, y_{n_1}, i, t)| & \leq K(1 + |x|^{q_1} + \sum_{j=1}^{n_1} |y_j|^{q_1}), \\ |g(x, y_{n_1+1}, \dots, y_n, i, t)| & \leq K(1 + |x|^{q_2} + \sum_{j=n_1+1}^n |y_j|^{q_2}) \end{aligned} \quad (2.3)$$

for all $x, y_1, \dots, y_n \in \mathbb{R}^d$, $(i, t) \in \mathbb{S} \times \mathbb{R}_+$.

If $q_1 = q_2 = 1$, then condition (2.3) is the familiar linear growth condition. However, we emphasise once again that we are here interested in highly nonlinear multiple-delay SDEs which have either $q_1 > 1$ or $q_2 > 1$. We will refer condition (2.3) as the polynomial growth condition. It is known that Assumption 2.1 only guarantees that the SDDE (2.1) with the initial data (2.2) has a unique maximal solution, which may explode to infinity at a finite time. To avoid such a possible explosion, we need to impose an additional condition in terms of Lyapunov functions. For this purpose, we need more notation.

Let $C^{2,1}(\mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+; \mathbb{R}_+)$ denote the family of non-negative functions $U(x, i, t)$ defined on $(x, i, t) \in \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+$ which are continuously twice differentiable in x and once in t . For such a function $U(x, i, t)$, let $U_t = \frac{\partial U}{\partial t}$, $U_x = \left(\frac{\partial U}{\partial x_1}, \dots, \frac{\partial U}{\partial x_d} \right)$, and $U_{xx} = \left(\frac{\partial^2 U}{\partial x_k \partial x_l} \right)_{d \times d}$. Let $C(\mathbb{R}^d \times [-\tau, \infty); \mathbb{R}_+)$ denote the family of all continuous functions from $\mathbb{R}^d \times [-\tau, \infty)$ to \mathbb{R}_+ . We can now state another assumption.

Assumption 2.2. Assume that there exists a pair of functions $\bar{U} \in C^{2,1}(\mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+; \mathbb{R}_+)$ and $G \in C(\mathbb{R}^d \times [-\tau, \infty); \mathbb{R}_+)$, as well as positive numbers $c_1, c_2, c_{3,j}$ and $q \geq 2(q_1 \vee q_2)$, such that

$$\sum_{j=1}^n \frac{c_{3,j}}{1 - \delta_j} < c_2, \quad |x|^q \leq \bar{U}(x, i, t) \leq G(x, t)$$

for $\forall(x, i, t) \in \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+$,
and

$$\begin{aligned} \mathbb{L}\bar{U}(x, y_1, \dots, y_n, i, t) &:= \bar{U}_t(x, i, t) + \bar{U}_x(x, i, t)f(x, y_1, \dots, y_n, i, t) \\ &+ \frac{1}{2} \text{trace}[g^T(x, y_{n_1+1}, \dots, y_n, i, t)\bar{U}_{xx}(x, i, t)g(x, y_{n_1+1}, \dots, y_n, i, t)] \\ &+ \sum_{j=1}^N \gamma_{ij}\bar{U}(x, j, t) \\ &\leq c_1 - c_2G(x, t) + \sum_{j=1}^n c_{3,j}G(y_j, t - \delta_j(t)) \end{aligned}$$

for all $x, y_1, \dots, y_n \in \mathbb{R}^d, (i, t) \in \mathbb{S} \times \mathbb{R}_+$.

Similar to the discussion in [14], we have the following claim.

Lemma 2.1. Under Assumptions 2.1 and 2.2, the variable multiple-delay SDE (2.1) with the initial data (2.2) has the unique global solution $x(t)$ on $t \geq -\tau$ and the solution has the property that

$$\sup_{-\tau \leq t < \infty} \mathbb{E}|x(t)|^q < \infty.$$

3. Delay-Dependent Asymptotic Stability of SDEs with Variable Multiple Delays

In Lemma 2.1, we used the method of Lyapunov functions to study the existence and uniqueness of the solution of the highly nonlinear hybrid SDE (2.1). In this section, we will use the method of Lyapunov functionals to investigate the delay-dependent asymptotic stability. We define two segments $\bar{x}_t := \{x(t + s) : -2\tau \leq s \leq 0\}$ and $\bar{r}_t := \{r(t + s) : -2\tau \leq s \leq 0\}$ for $t \geq 0$. For \bar{x}_t and \bar{r}_t to be well defined for $0 \leq t < 2\tau$, we set $x(s) = \eta(-\tau)$ for $s \in [-2\tau, -\tau)$ and $r(s) = r_0$ for $s \in [-2\tau, 0)$. We construct the Lyapunov functional as follows

$$\begin{aligned} V(\bar{x}_t, \bar{r}_t, t) &= U(x(t), r(t), t) \\ &+ \sum_{j=1}^{n_1} \theta_j \int_{-\tau_j}^0 \int_{t+s}^t \left[\tau_j |f(x(v), x(v - \delta_1(v)), \dots, x(v - \delta_{n_1}(v)), r(v), v)|^2 \right. \\ &\left. + |g(x(v), x(v - \delta_{n_1+1}(v)), \dots, x(v - \delta_n(v)), r(v), v)|^2 \right] dv ds \end{aligned}$$

for $t \geq 0$, where $U \in C^{2,1}(\mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+; \mathbb{R}_+)$ such that

$$\lim_{|x| \rightarrow \infty} \left[\inf_{(t,i) \in \mathbb{R}_+ \times \mathbb{S}} U(x, t, i) \right] = \infty,$$

and $\theta_j, j = 1, \dots, n$ are positive numbers to be determined later while we set

$$\begin{aligned} f(x, y_1, \dots, y_{n_1}, i, s) &= f(x, y_1, \dots, y_{n_1}, i, 0), \\ g(x, y_{n_1+1}, \dots, y_n, i, s) &= g(x, y_{n_1+1}, \dots, y_n, i, 0) \end{aligned}$$

for all $x, y_1, \dots, y_n \in \mathbb{R}^d, (i, s) \in \mathbb{S} \times [-2\tau, 0)$. Applying the generalized Itô formula (see, e.g., [26, Theorem 1.45 on page 48]) to $U(x(t), r(t), t)$, we get

$$\begin{aligned} dU(x(t), r(t), t) &= \left(U_t(x(t), r(t), t) + U_x(x(t), r(t), t) f(x(t), x(t - \delta_1(t)), \dots, x(t - \delta_{n_1}(t)), r(t), t) \right. \\ &\quad + \frac{1}{2} \text{trace}[g^T(x(t), x(t - \delta_{n_1+1}(t)), \dots, x(t - \delta_n(t)), r(t), t) \\ &\quad \times U_{xx}(x(t), r(t), t) g(x(t), x(t - \delta_{n_1+1}(t)), \dots, x(t - \delta_n(t)), r(t), t)] \\ &\quad \left. + \sum_{j=1}^N \gamma_{r(t), j} U(x(t), j, t) \right) dt + dM(t), \end{aligned}$$

for $t \geq 0$, where $M(t)$ (see, e.g., [26, Theorem 1.45 on page 48]) is a continuous local martingale with $M(0) = 0$. Rearranging terms gives

$$\begin{aligned} dU(x(t), r(t), t) &= \left(U_x(x(t), r(t), t) [f(x(t), x(t - \delta_1(t)), \dots, x(t - \delta_{n_1}(t)), r(t), t) \right. \\ &\quad - f(x(t), x(t), \dots, x(t), r(t), t)] \\ &\quad \left. + \mathcal{L}U(x(t), x(t - \delta_{n_1+1}(t)), \dots, x(t - \delta_n(t)), r(t), t) \right) + dM(t), \end{aligned}$$

where the function $\mathcal{L}U : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is defined by

$$\begin{aligned} \mathcal{L}U(x, y_{n_1+1}, \dots, y_n, i, t) &= U_t(x, i, t) + U_x(x, i, t) f(x, x, \dots, x, i, t) \\ &\quad + \frac{1}{2} \text{trace}[g^T(x, y_{n_1+1}, \dots, y_n, i, t) U_{xx}(x, i, t) g(x, y_{n_1+1}, \dots, y_n, i, t)] + \sum_{j=1}^N \gamma_{ij} U(x, j, t). \end{aligned} \tag{3.1}$$

Moreover, the fundamental theory of calculus shows, for $j = 1, \dots, n$,

$$\begin{aligned} &d \left(\int_{-\tau_j}^0 \int_{t+s}^t \left[\tau_j |f(x(v), x(v - \delta_1(v)), \dots, x(v - \delta_{n_1}(v)), r(v), v)|^2 \right. \right. \\ &\quad \left. \left. + |g(x(v), x(v - \delta_{n_1+1}(v)), \dots, x(v - \delta_n(v)), r(v), v)|^2 \right] dv ds \right) \\ &= \left(\tau_j \left[\tau_j |f(x(t), x(t - \delta_1(v)), \dots, x(t - \delta_{n_1}(v)), r(t), t)|^2 \right. \right. \\ &\quad \left. \left. + |g(x(t), x(t - \delta_{n_1+1}(v)), \dots, x(t - \delta_n(v)), r(t), t)|^2 \right] \right. \\ &\quad \left. - \int_{t-\tau_j}^t \left[\tau_j |f(x(v), x(v - \delta_1(v)), \dots, x(v - \delta_{n_1}(v)), r(v), v)|^2 \right. \right. \\ &\quad \left. \left. + |g(x(v), x(v - \delta_{n_1+1}(v)), \dots, x(v - \delta_n(v)), r(v), v)|^2 \right] dv \right) dt. \end{aligned}$$

Lemma 3.1. *With the notation above, $V(\bar{x}_t, \bar{r}_t, t)$ is an Itô process on $t \geq 0$ with its Itô differential*

$$dV(\bar{x}_t, \bar{r}_t, t) = LV(\bar{x}_t, \bar{r}_t, t)dt + dM(t),$$

where $M(t)$ is a continuous local martingale with $M(0) = 0$ and

$$\begin{aligned} LV(\bar{x}_t, \bar{r}_t, t) = & U_x(x(t), r(t), t)[f(x(t), x(t - \delta_1(t)), \dots, x(t - \delta_{n_1}(t)), r(t), t) \\ & - f(x(t), x(t), \dots, x(t), r(t), t)] \\ & + \mathcal{L}U(x(t), x(t - \delta_1(t)), \dots, x(t - \delta_n(t)), r(t), t) \\ & + \sum_{j=1}^{n_1} \theta_j \tau_j \left[\tau_j |f(x(t), x(t - \delta_1(t)), \dots, x(t - \delta_{n_1}(t)), r(t), t)|^2 \right. \\ & \left. + |g(x(t), x(t - \delta_{n_1+1}(t)), \dots, x(t - \delta_n(t)), r(t), t)|^2 \right] \\ & - \sum_{j=1}^{n_1} \theta_j \int_{t-\tau_j}^t \left[\tau_j |f(x(v), x(v - \delta_1(v)), \dots, x(v - \delta_{n_1}(v)), r(v), v)|^2 \right. \\ & \left. + |g(x(v), x(v - \delta_{n_1+1}(v)), \dots, x(v - \delta_n(v)), r(v), v)|^2 \right] dv. \end{aligned}$$

To study the delay-dependent asymptotic stability of the SDDE (2.1), we need to impose several new assumptions.

Assumption 3.1. Assume that there are functions $U \in C^{2,1}(\mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+; \mathbb{R}_+)$, $U_1 \in C(\mathbb{R}^d \times [-\tau, \infty); \mathbb{R}_+)$, and positive numbers α, α_j ($j = 1, \dots, n$) and β_k ($k = 1, 2, 3$) such that

$$\sum_{j=1}^n \frac{\alpha_j}{1 - \delta_j} < \alpha \tag{3.2}$$

and

$$\begin{aligned} & \mathcal{L}U(x, y_1, \dots, y_n, i, t) + \beta_1 |U_x(x, i, t)|^2 \\ & + \beta_2 |f(x, y_1, \dots, y_{n_1}, i, t)|^2 + \beta_3 |g(x, y_{n_1+1}, \dots, y_n, i, t)|^2 \\ & \leq -\alpha U_1(x, t) + \sum_{j=1}^n \alpha_j U_1(y_j, t - \delta_j(t)), \end{aligned} \tag{3.3}$$

for all $x, y_1, \dots, y_n \in \mathbb{R}^d, (i, t) \in \mathbb{S} \times \mathbb{R}_+$.

Assumption 3.2. Assume that there exists positive numbers $w_j, j = 1, \dots, n_1$ such that

$$|f(x, x, \dots, x, i, t) - f(x, y_1, \dots, y_{n_1}, i, t)| \leq \sum_{j=1}^{n_1} w_j |x - y_j|$$

for all $x, y_1 \dots, y_{n_1} \in \mathbb{R}^d, (i, t) \in \mathbb{S} \times [-2\tau, \infty)$.

Theorem 3.3. Let Assumptions 2.1, 2.2, 3.1 and 3.2 hold. Assume also that

$$n_1 \sum_{j=1}^{n_1} w_j^2 \tau_j^2 \leq 2\beta_1 \beta_2 \quad \text{and} \quad n_1 \sum_{j=1}^{n_1} w_j^2 \tau_j \leq 2\beta_1 \beta_3. \tag{3.4}$$

Then for any given initial data (2.2), the solution of the SDDE (2.1) has the properties that

$$\int_0^\infty \mathbb{E}U_1(x(t), t) dt < \infty \tag{3.5}$$

and

$$\sup_{0 \leq t < \infty} \mathbb{E}U(x(t), r(t), t) < \infty.$$

Proof. Fix the initial data $\eta \in C([-\tau, 0]; \mathbb{R}^d)$ and $r_0 \in \mathbb{S}$ arbitrarily. Let $k_0 > 0$ be a sufficiently large integer such that $\|\eta\| := \sup_{-\tau \leq s \leq 0} |\eta(s)| < k_0$. For each integer $k > k_0$, define the stopping time

$$\sigma_k = \inf\{t \geq 0 : |x(t)| \geq k\},$$

where throughout this paper we set $\inf \emptyset = \infty$ (as usual \emptyset denotes the empty set). It is easy to see that σ_k is increasing as $k \rightarrow \infty$ and $\lim_{k \rightarrow \infty} \sigma_k = \infty$ a.s. By the generalized Itô formula we obtain from Lemma 3.1 that

$$\mathbb{E}V(\bar{x}_{t \wedge \sigma_k}, \bar{r}_{t \wedge \sigma_k}, t \wedge \sigma_k) = V(\bar{x}_0, \bar{r}_0, 0) + \mathbb{E} \int_0^{t \wedge \sigma_k} LV(\bar{x}_s, \bar{r}_s, s) ds \tag{3.6}$$

for any $t \geq 0$ and $k \geq k_0$. Let $\theta_j = n_1 w_j^2 / (2\beta_1)$. By Assumption 3.2, it is easy to see that

$$\begin{aligned} & U_x(x(t), r(t), t)[f(x(t), x(t - \delta_1(t)), \dots, x(t - \delta_{n_1}(t)), r(t), t) \\ & \quad - f(x(t), x(t), \dots, x(t), r(t), t)] \\ & \leq \beta_1 |U_x(x(t), r(t), t)|^2 + n_1 \sum_{j=1}^{n_1} \frac{w_j^2}{4\beta_1} |x(t) - x(t - \delta_j(t))|^2. \end{aligned} \tag{3.7}$$

By condition (3.4), we also have

$$\sum_{j=1}^{n_1} \theta_j \tau_j^2 \leq \beta_2 \quad \text{and} \quad \sum_{j=1}^{n_1} \theta_j \tau_j \leq \beta_3.$$

It then follows from Lemma 3.1 that

$$\begin{aligned} LV(\bar{x}_s, \bar{r}_s, s) & \leq \mathcal{L}U(x(s), x(s - \delta_1(s)), \dots, x(s - \delta_n(s)), r(s), s) + \beta_1 |U_x(x(s), r(s), s)|^2 \\ & \quad + \beta_2 |f(x(s), x(s - \delta_1(s)), \dots, x(s - \delta_{n_1}(s)), r(s), s)|^2 \\ & \quad + \beta_3 |g(x(s), x(s - \delta_{n_1+1}(s)), \dots, x(s - \delta_n(s)), r(s), s)|^2 \\ & \quad + n_1 \sum_{j=1}^{n_1} \frac{w_j^2}{4\beta_1} |x(s) - x(s - \delta_j(s))|^2 \\ & \quad - n_1 \sum_{j=1}^{n_1} \frac{w_j^2}{2\beta_1} \int_{s - \tau_j}^s \left[\tau_j |f(x(v), x(v - \delta_1(v)), \dots, x(v - \delta_{n_1}(v)), r(v), v)|^2 \right. \\ & \quad \left. + |g(x(v), x(v - \delta_{n_1+1}(v)), \dots, x(v - \delta_n(v)), r(v), v)|^2 \right] dv. \end{aligned}$$

By Assumption 3.1, we then have

$$\begin{aligned} LV(\bar{x}_s, \bar{r}_s, s) & \leq -\alpha U_1(x(s), s) + \sum_{j=1}^n \alpha_j U_1(x(s - \delta_j(s)), s - \delta_j(s)) \\ & \quad + n_1 \sum_{j=1}^{n_1} \frac{w_j^2}{4\beta_1} |x(s) - x(s - \delta_j(s))|^2 \end{aligned}$$

$$\begin{aligned}
 & - n_1 \sum_{j=1}^{n_1} \frac{w_j^2}{2\beta_1} \int_{s-\tau_j}^s \left[\tau_j |f(x(v), x(v-\delta_1(v)), \dots, x(v-\delta_{n_1}(v)), r(v), v)|^2 \right. \\
 & \left. + |g(x(v), x(v-\delta_{n_1+1}(v)), \dots, x(v-\delta_n(v)), r(v), v)|^2 \right] dv.
 \end{aligned}$$

Substituting this into (3.6) implies

$$\mathbb{E}V(\bar{x}_{t \wedge \sigma_k}, \bar{r}_{t \wedge \sigma_k}, t \wedge \sigma_k) \leq V(\bar{x}_0, \bar{r}_0, 0) + H_1 + \sum_{j=1}^{n_1} (H_2^j - H_3^j), \tag{3.8}$$

where

$$\begin{aligned}
 H_1 &= \mathbb{E} \int_0^{t \wedge \sigma_k} \left[-\alpha U_1(x(s), s) + \sum_{l=1}^n \alpha_l U_1(x(s - \delta_l(s)), s - \delta_l(s)) \right] ds, \\
 H_2^j &= \frac{n_1 w_j^2}{4\beta_1} \mathbb{E} \int_0^{t \wedge \sigma_k} |x(s) - x(s - \delta_j(s))|^2 ds, \\
 H_3^j &= \frac{n_1 w_j^2}{2\beta_1} \mathbb{E} \int_0^{t \wedge \sigma_k} \int_{s-\tau_j}^s \left[\tau_j |f(x(v), x(v - \delta_1(v)), \dots, x(v - \delta_{n_1}(v)), r(v), v)|^2 \right. \\
 & \left. + |g(x(v), x(v - \delta_{n_1+1}(v)), \dots, x(v - \delta_n(v)), r(v), v)|^2 \right] dv ds.
 \end{aligned}$$

Noting that, for $l = 1, \dots, n$,

$$\begin{aligned}
 \int_0^{t \wedge \sigma_k} U_1(x(s - \delta_l(s)), s - \delta_l(s)) ds &\leq \int_{-\delta_l(0)}^{t \wedge \sigma_k - \delta_l(t \wedge \sigma_k)} \frac{U_1(x(v), v)}{1 - \bar{\delta}_l} dv \\
 &\leq \int_{-\tau_l}^{t \wedge \sigma_k} \frac{U_1(x(v), v)}{1 - \bar{\delta}_l} dv,
 \end{aligned}$$

we have

$$H_1 \leq \sum_{l=1}^n \frac{\alpha_l}{1 - \bar{\delta}_l} \int_{-\tau_l}^0 U_1(\eta(v), v) dv - \bar{\alpha} \mathbb{E} \int_0^{t \wedge \sigma_k} U_1(x(s), s) ds,$$

where $\bar{\alpha} = \alpha - \sum_{l=1}^n \alpha_l / (1 - \bar{\delta}_l) > 0$ by Assumption 3.1. Substituting this into (3.8) yields

$$\bar{\alpha} \mathbb{E} \int_0^{t \wedge \sigma_k} U_1(x(s), s) ds \leq C_1 + \sum_{j=1}^{n_1} (H_2^j - H_3^j), \tag{3.9}$$

where C_1 is a constant defined by

$$C_1 = V(\bar{x}_0, \bar{r}_0, 0) + \sum_{l=1}^n \frac{\alpha_l}{1 - \bar{\delta}_l} \int_{-\tau_l}^0 U_1(\eta(s), s) ds.$$

Applying the classical Fatou lemma and let $k \rightarrow \infty$ in (3.9) to obtain

$$\bar{\alpha} \mathbb{E} \int_0^t U_1(x(s), s) ds \leq C_1 + \sum_{j=1}^{n_1} (\bar{H}_2^j - \bar{H}_3^j), \tag{3.10}$$

where

$$\begin{aligned}\bar{H}_2^j &= \frac{n_1 w_j^2}{4\beta_1} \mathbb{E} \int_0^t |x(s) - x(s - \delta_j(s))|^2 ds, \\ \bar{H}_3^j &= \frac{n_1 w_j^2}{2\beta_1} \mathbb{E} \int_0^t \int_{s-\tau_j}^s \left[\tau_j |f(x(v), x(v - \delta_1(v)), \dots, x(v - \delta_{n_1}(v)), r(v), v)|^2 \right. \\ &\quad \left. + |g(x(v), x(v - \delta_{n_1+1}(v)), \dots, x(v - \delta_n(v)), r(v), v)|^2 \right] dv ds.\end{aligned}\quad (3.11)$$

By the well-known Fubini theorem, we have

$$\bar{H}_2^j = \frac{n_1 w_j^2}{4\beta_1} \int_0^t \mathbb{E} |x(s) - x(s - \delta_j(s))|^2 ds.$$

For $t \in [0, \tau_j]$, we have

$$\begin{aligned}\bar{H}_2^j &\leq \frac{n_1 w_j^2}{2\beta_1} \int_0^{\tau_j} (\mathbb{E} |x(s)|^2 + \mathbb{E} |x(s - \delta_j(s))|^2) ds \\ &\leq \frac{n_1 w_j^2 \tau_j}{\beta_1} \left(\sup_{-\tau_j \leq v \leq \tau_j} \mathbb{E} |x(v)|^2 \right) \\ &\leq \frac{n_1 w_j^2 \tau_j}{\beta_1} \left(\sup_{-\tau \leq v \leq \tau} \mathbb{E} |x(v)|^2 \right).\end{aligned}$$

For $t > \tau_j$, we have

$$\bar{H}_2^j \leq \frac{n_1 w_j^2 \tau_j}{\beta_1} \left(\sup_{-\tau \leq v \leq \tau} \mathbb{E} |x(v)|^2 \right) + \frac{n_1 w_j^2}{4\beta_1} \int_{\tau_j}^t \mathbb{E} |x(s) - x(s - \delta_j(s))|^2 ds.\quad (3.12)$$

Noting that

$$\begin{aligned}& |x(s) - x(s - \delta_j(s))| \\ & \leq \left| \int_{s-\tau_j}^s f(x(v), x(v - \delta_1(v)), \dots, x(v - \delta_{n_1}(v)), r(v), v) dv \right. \\ & \quad \left. + \int_{s-\tau_j}^s g(x(v), x(v - \delta_{n_1+1}(v)), \dots, x(v - \delta_n(v)), r(v), v) dB(v) \right|,\end{aligned}$$

we have

$$\begin{aligned}& \mathbb{E} |x(s) - x(s - \delta_j(s))|^2 \\ & \leq 2\mathbb{E} \int_{s-\tau_j}^s \left[\tau_j |f(x(v), x(v - \delta_1(v)), \dots, x(v - \delta_{n_1}(v)), r(v), v)|^2 \right. \\ & \quad \left. + |g(x(v), x(v - \delta_{n_1+1}(v)), \dots, x(v - \delta_n(v)), r(v), v)|^2 \right] dv.\end{aligned}$$

Notice also that

$$\begin{aligned}& \int_{\tau_j}^t \mathbb{E} |x(s) - x(s - \delta_j(s))|^2 \\ & \leq 2\mathbb{E} \int_{\tau_j}^t \int_{s-\tau_j}^s \left[\tau_j |f(x(v), x(v - \delta_1(v)), \dots, x(v - \delta_{n_1}(v)), r(v), v)|^2 \right.\end{aligned}$$

$$+ |g(x(v), x(v - \delta_{n_1+1}(v)), \dots, x(v - \delta_n(v)), r(v), v)|^2] dv ds.$$

Thus from (3.11) and (3.12) we get

$$\bar{H}_2^j \leq \frac{n_1 w_j^2 \tau_j}{\beta_1} \left(\sup_{-\tau \leq v \leq \tau} \mathbb{E}|x(v)|^2 \right) + \bar{H}_3^j. \tag{3.13}$$

Substituting (3.13) into (3.10) yields

$$\bar{\alpha} \mathbb{E} \int_0^t U_1(x(s), s) ds \leq C_1 + 2\beta_3 \sup_{-\tau \leq v \leq \tau} \mathbb{E}|x(v)|^2 := C_2.$$

Letting $t \rightarrow \infty$ gives

$$\mathbb{E} \int_0^\infty U_1(x(s), s) ds \leq \frac{C_2}{\bar{\alpha}}. \tag{3.14}$$

Now we see from (3.8) that

$$\mathbb{E}U\left(x(t \wedge \sigma_k), r(t \wedge \sigma_k), t \wedge \sigma_k\right) \leq C_1 + \sum_{j=1}^{n_1} (H_2^j - H_3^j). \tag{3.15}$$

Letting $k \rightarrow \infty$ we get

$$\mathbb{E}U(x(t), r(t), t) \leq C_2 < \infty,$$

which shows

$$\sup_{0 \leq t < \infty} \mathbb{E}U(x(t), r(t), t) < \infty. \tag{3.16}$$

Thus the proof is complete. □

Corollary 3.1. *Let the conditions of Theorem 3.3 hold. If there moreover exists a pair of positive constants c and p such that*

$$c|x|^p \leq U_1(x, t), \quad \forall (x, t) \in \mathbb{R}^d \times \mathbb{R}_+,$$

then for any given initial data (2.2), the solution of the multiple-delay SDE (2.1) satisfies

$$\int_0^\infty \mathbb{E}|x(t)|^p dt < \infty. \tag{3.17}$$

That is, the multiple-delay SDE (2.1) is H_∞ -stable in L^p .

This corollary follows from Theorem 3.3 obviously. However, it does not follow from (3.17) that $\lim_{t \rightarrow \infty} \mathbb{E}|x(t)|^p = 0$.

Theorem 3.4. *Let the conditions of Corollary 3.1 hold. If, moreover,*

$$p \geq 2 \quad \text{and} \quad (p + q_1 - 1) \vee (p + 2q_2 - 2) \leq q,$$

then the solution of the multiple-delay SDE (2.1) satisfies

$$\lim_{t \rightarrow \infty} \mathbb{E}|x(t)|^p = 0$$

for any initial data (2.2). That is, the variable multiple-delay SDE (2.1) is asymptotically stable in L^p .

Proof. Fix the initial data (2.2) arbitrarily. For any $0 \leq t_1 < t_2 < \infty$, by the Itô formula, we get

$$\begin{aligned} & \mathbb{E}|x(t_2)|^p - \mathbb{E}|x(t_1)|^p \\ = & \mathbb{E} \int_{t_1}^{t_2} \left(p|x(t)|^{p-2} x(t)^\top f(x(t), x(t - \delta_1(t)), \dots, x(t - \delta_{n_1}(t)), r(t), t) \right. \\ & + \frac{p}{2} |x(t)|^{p-2} |g(x(t), x(t - \delta_{n_1+1}(t)), \dots, x(t - \delta_n(t)), r(t), t)|^2 \\ & \left. + \frac{p(p-2)}{2} |x(t)|^{p-4} |(x(t)^\top g(x(t), x(t - \delta_{n_1+1}(t)), \dots, x(t - \delta_n(t)), r(t), t))|^2 \right) dt, \end{aligned}$$

which implies

$$\begin{aligned} & |\mathbb{E}|x(t_2)|^p - \mathbb{E}|x(t_1)|^p| \\ \leq & \mathbb{E} \int_{t_1}^{t_2} \left(p|x(t)|^{p-1} |f(x(t), x(t - \delta_1(t)), \dots, x(t - \delta_{n_1}(t)), r(t), t)| \right. \\ & \left. + \frac{p(p-1)}{2} |x(t)|^{p-2} |g(x(t), x(t - \delta_{n_1+1}(t)), \dots, x(t - \delta_n(t)), r(t), t)|^2 \right) dt \\ \leq & \mathbb{E} \int_{t_1}^{t_2} \left(pK|x(t)|^{p-1} [1 + |x(t)|^{q_1} + \sum_{j=1}^{n_1} |x(t - \delta_j(t))|^{q_1}] \right. \\ & \left. + \frac{(n - n_1 + 2)p(p-1)K^2}{2} |x(t)|^{p-2} [1 + |x(t)|^{2q_2} + \sum_{j=n_1+1}^n |x(t - \delta_j(t))|^{2q_2}] \right) dt. \end{aligned}$$

By inequalities,

$$\begin{aligned} |x(t)|^{p-1} |x(t - \delta_j(t))|^{q_1} & \leq |x(t)|^{p+q_1-1} + |x(t - \delta_j(t))|^{p+q_1-1}, \\ |x(t)|^{p-1} & \leq 1 + |x(t)|^q, \end{aligned}$$

we can obtain

$$|\mathbb{E}|x(t_2)|^p - \mathbb{E}|x(t_1)|^p| \leq C_3(t_2 - t_1),$$

where

$$\begin{aligned} C_3 = & pK(1 + 2(n_1 + 1)) \sup_{-\tau \leq t < \infty} \mathbb{E}|x(t)|^q \\ & + \frac{1}{2}(n - n_1 + 2)p(p-1)K^2(1 + 2(n - n_1 + 1)) \sup_{-\tau \leq t < \infty} \mathbb{E}|x(t)|^q < \infty. \end{aligned}$$

Thus we have $\mathbb{E}|x(t)|^p$ is uniformly continuous in t on \mathbb{R}_+ . By (3.17), there is a sequence $\{t_l\}_{l=1}^\infty$ in \mathbb{R} such that $\mathbb{E}|x(t_l)|^p \rightarrow 0$, which easily show the claim. Thus the proof is complete. \square

4. An Example for Muptiple-delay SDEs

Let us now discuss an example to illustrate our theory.

Example 4.1. Let us consider the SDDE with two delays (1.1), we consider two case: $\delta_1(t) \leq \tau = 0.01$ and $\delta_1(t) \leq \tau = 2$ for all $t \geq 0$. Let $\bar{\delta}_1 = \bar{\delta}_2 = 0.1$ and $\delta_2(t) = 2$ (in fact, the stability of system is independent on the size of $\delta_2(t)$). In case $\tau = 0.01$, let the initial data $x(u) = 2 + \sin(u)$ for $u \in [-0.01, 0]$, $r(0) = 2$, the sample paths of the Markovian chain and the solution of the multiple delay SDE are shown in Figure 1, which indicates that the multiple delay SDE is asymptotically stable. In the case $\tau = 2$, let the initial data $x(u) = 2 + \sin(u)$ for $u \in [-2, 0]$, $r(0) = 2$, the sample paths of the Markovian chain and the solution of the multiple-delay SDE are plotted in Figure 2, which indicates that the multiple-delay SDE is asymptotically unstable. From the example we can see SDDE (1.1) is stable or not depends on how long or short the time-delay is.

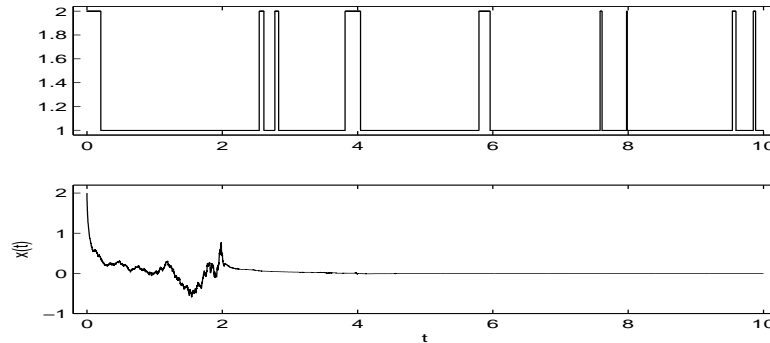


Figure 1. The computer simulation of the sample paths of the Markovian chain and the SDDE (1.1) with $\tau = 0.01$ using the Euler–Maruyama method with step size 10^{-3} .

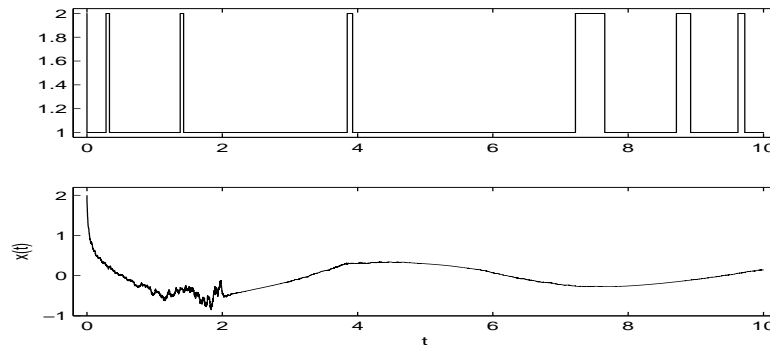


Figure 2. The computer simulation of the sample paths of the Markovian chain and the SDDE (1.1) with $\tau = 2$ using the Euler–Maruyama method with step size 10^{-3} .

We can see coefficients defined by (1.1) satisfy Assumption 2.1 with $q_1 = 3$ and $q_2 = 2$. Define $\bar{U}(x, i, t) = |x|^6$ for $(x, i, t) \in \mathbb{R} \times \mathbb{S} \times \mathbb{R}_+$. It is easy to show that

$$\mathbb{L}\bar{U}(x, y_1, y_2, i, t) = 6x^5 f(x, y_1, i, t) + 15x^4 |g(x, y_2, i, t)|^2$$

for $(x, y_1, y_2, i, t) \in \mathbb{R}^3 \times \mathbb{S} \times \mathbb{R}_+$. We have

$$\mathbb{L}\bar{U}(x, y_1, y_2, 1, t) = 6x^5(-y_1 - 10x^3) + \frac{15}{16}x^4\left(\frac{1}{4}y_2^2\right)^2$$

$$\leq 5x^6 + y_1^6 + \frac{15}{128}y_2^6 - (60 - \frac{15}{128})x^8$$

and

$$\begin{aligned} \mathbb{L}\bar{U}(x, y_1, y_2, 2, t) &= 6x^5\left(\frac{1}{2}y_1 - 4x^3\right) + \frac{15}{4}x^4(y_2^2)^2 \\ &\leq 2.5x^6 + 0.5y_1^6 - 22.125x^8 + 1.875y_2^8. \end{aligned}$$

Thus, we can obtain

$$\begin{aligned} \mathbb{L}\bar{U}(x, y_1, y_2, i, t) &\leq 5x^6 + y_1^6 - 22.125x^8 + 1.875y_2^8 \\ &\leq c_1 - 10(1 + x^8) + (1 + y_1^8) + 2(1 + y_2^8), \end{aligned}$$

where

$$c_1 = \sup_{x \in \mathbb{R}} \{8 + 5x^6 - 12.125x^8\} < \infty$$

and $G(x, t) = 1 + x^8$, $c_2 = 10$, $c_{3,1} = 1$, $c_{3,2} = 2$. Therefore, Assumption 2.2 is satisfied. From Lemma 2.1, solution of the SDDE (1.1) has the that

$$\sup_{-\tau \leq t < \infty} \mathbb{E}|x(t)|^6 < \infty.$$

To verify Assumption 3.1, we define

$$U(x, i, t) = \begin{cases} x^2 + x^4, & \text{if } i = 1, \\ 2x^2 + 3x^4, & \text{if } i = 2, \end{cases} \quad (4.1)$$

which shows

$$U_x(x, i, t) = \begin{cases} 2x + 4x^3, & \text{if } i = 1, \\ 4x + 12x^3, & \text{if } i = 2, \end{cases}$$

for $(x, i, t) \in \mathbb{R} \times \mathbb{S} \times \mathbb{R}_+$. By the equation (3.1), we have

$$\begin{aligned} \mathcal{L}U(x, y_2, 1, t) &= (2x + 4x^3)(-x - 10x^3) + \frac{1}{32}(y_2^2)^2(2 + 12x^2) - (x^2 + x^4) + (2x^2 + 3x^4) \\ &\leq -x^2 - 22x^4 - 39.875x^6 + \frac{1}{16}y_2^4 + 0.25y_2^6 \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}U(x, y_2, 2, t) &= (4x + 12x^3)\left(\frac{1}{2}x - 4x^3\right) + \frac{1}{8}(y_2^2)^2(4 + 36x^2) + 8(x^2 + x^4) - 8(2x^2 + 3x^4) \\ &\leq -6x^2 - 26x^4 - 46.5x^6 + \frac{1}{2}y_2^4 + 3y_2^6. \end{aligned}$$

Moreover

$$|U_x(x, i, t)|^2 = \begin{cases} 4x^2 + 16x^4 + 16x^6, & \text{if } i = 1, \\ 16x^2 + 96x^4 + 144x^6, & \text{if } i = 2. \end{cases} \quad (4.2)$$

$$|f(x, y_1, i, t)|^2 = \begin{cases} |-y_1 - 10x^3|^2 \leq 2y_1^2 + 200x^6, & \text{if } i = 1, \\ |\frac{1}{2}y_1 - 4x^3|^2 \leq \frac{1}{2}y_1^2 + 32x^6, & \text{if } i = 2. \end{cases} \tag{4.3}$$

$$|g(x, y_2, 1, t)|^2 = \begin{cases} \frac{1}{16}|y_2^2|^2, & \text{if } i = 1, \\ \frac{1}{4}|y_2^2|^2, & \text{if } i = 2. \end{cases} \tag{4.4}$$

Setting $\beta_1 = 0.05, \beta_2 = 0.1, \beta_3 = 4$, using (4.2)-(4.4), we obtain that

$$\begin{aligned} & \mathcal{L}U(x, y_1, y_2, i, t) + \beta_1|U_x(x, i, t)|^2 + \beta_2|f(x, y_1, y_2, i, t)|^2 + \beta_3|g(x, y_1, y_2, i, t)|^2 \\ & \leq \begin{cases} -0.8x^2 - 21.2x^4 - 19.075x^6 + 0.2y_1^2 + \frac{5}{16}y_2^4 + \frac{1}{4}y_2^6, & \text{if } i = 1, \\ -5.2x^2 - 21.1x^4 - 36.1x^6 + 0.05y_1^2 + 1.5y_2^4 + 3y_2^6, & \text{if } i = 2. \end{cases} \end{aligned}$$

This implies

$$\begin{aligned} & \mathcal{L}U(x, y_2, i, t) + \beta_1|U_x(x, i, t)|^2 + \beta_2|f(x, y_1, i, t)|^2 + \beta_3|g(x, y_2, i, t)|^2 \\ & \leq -0.8x^2 - 21.1x^4 - 19.075x^6 + 0.2y_1^2 + 1.5y^4 + 3y^6 \\ & \leq -6(0.1x^2 + 3x^4 + 3x^6) + 2(0.1y_1^2 + 3y_1^4 + 3y_1^6) + 0.1y_2^2 + 3y_2^4 + 3y_2^6. \end{aligned}$$

Letting $U_1(x, t) = 0.1x^2 + 3x^4 + 3x^6, \alpha = 6, \alpha_1 = 2, \alpha_2 = 1,$, we get condition (3.2). Noting that $n_1 = 1, n = 2$ and $w_1 = 1$, then condition (3.4) becomes

$$\tau \leq 0.1.$$

By Theorem 3.3, we can therefore conclude that the solution of the SDDE (1.1)

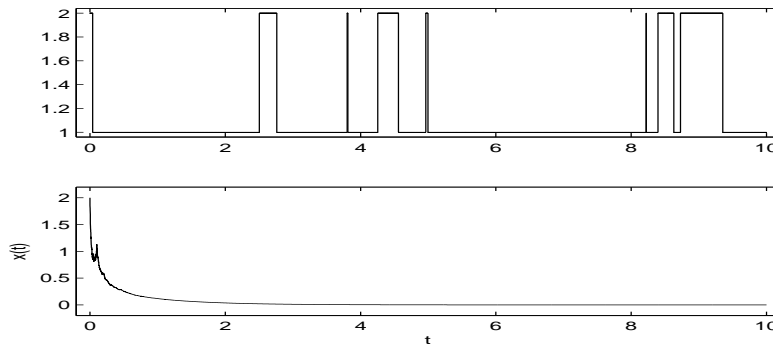


Figure 3. The computer simulation of the sample paths of the Markovian chain and the SDDE (1.1) with $\tau = 0.1$ using the Euler-Maruyama method with step size 10^{-3} .

has the properties that

$$\begin{aligned} & \int_0^\infty (x^2(t) + x^4(t) + x^6(t))dt < \infty \text{ a.s. and} \\ & \int_0^\infty \mathbb{E}(x^2(t) + x^4(t) + x^6(t))dt < \infty. \end{aligned}$$

Moreover, as $|x(t)|^p \leq x^2(t) + x^4(t) + x^6(t)$ for any $p \in [2, 6]$, we have

$$\int_0^\infty \mathbb{E}|x(t)|^p dt < \infty.$$

Recalling $q_1 = 3$, $q_2 = 2$ and $q = 6$, we see that for $p = 4$, all conditions of Theorem 3.4 are satisfied and hence we have

$$\lim_{t \rightarrow \infty} \mathbb{E}|x(t)|^4 = 0.$$

We perform a computer simulation with the time-delay $\tau = 0.1$ for all $t \geq 0$ and the initial data $x(u) = 2 + \sin(u)$ for $u \in [-0.1, 0]$ and $r(0) = 2$. The sample paths of the Markovian chain and the solution of the SDDE (1.1) are plotted in Figure 3. The simulation supports our theoretical results.

5. Conclusion

In real world applications, the stability and boundedness of stochastic differential delay equations are interesting topics. In this paper, we established the criteria of stability and boundedness of the solutions to SDDEs with variable multiple delays. To this end, we investigated the highly nonlinear hybrid multiple-delay SDEs. In fact, the stability of SDDEs have been studied for many years, most of the results in this topic require that the coefficients of equations are linear or nonlinear but bounded by linear functions. Recently, without the linear growth condition, Fei et al. [7] was the first to establish the delay-dependent stability criteria for highly nonlinear SDDEs by the method of Lyapunov function with a single time delay. In this paper, we obtained the results of hybrid highly nonlinear SDE with variable multiple delays. An illustrative example was given for our theory.

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