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STOCHASTIC PARTITIONED AVERAGED VECTOR FIELD METHODS FOR STOCHASTIC DIFFERENTIAL EQUATIONS WITH A CONSERVED QUANTITY

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Abstract In this paper, stochastic differential equations in the Stratonovich sense with a conserved quantity are considered. A stochastic partitioned averaged vector field method is proposed and analyzed. We prove this numerical method is able to preserve the conserved quantity of the original system. Then the convergence analysis is carried out in detail and we derive the method is convergent with order 1 in the mean-square sense. Finally some numerical examples are reported to verify the effectiveness and flexibility of the proposed method.

Keywords Stochastic differential equations, stochastic partitioned averaged vector field methods, conserved quantity, convergence analysis.

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1. Introduction

Stochastic differential equations (SDEs) are widely used to model phenomena in many fields like dynamics, economics, biology, and so on [19]. Since most SDEs cannot be solved analytically, along with the rapid development of computers, the study of numerical methods for the approximation of SDEs has played a more and more important role in recent years (see [2, 3, 8, 13, 14, 21, 25–27] and references therein).

Numerical integrators that can preserve the intrinsic properties such as geometrical or physical properties of the underlying flow are usually called geometric numerical integration methods, which have drawn a lot of attention recently for their good performance especially in a long-term numerical simulation. Since conserved quantity is intrinsic for some systems, it is natural to construct numerical methods which can preserve the conserved quantity of the original system. Many numerical methods that can preserve a single conserved quantity or multiple conserved quantities for ordinary differential equations (ODEs) have been proposed in the last three decades (see [1, 6, 10, 15, 20, 23, 24] and references therein), and some

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of them have been extended to stochastic cases. [22] proposes a difference scheme for one-dimensional stochastic canonical Hamiltonian system which can preserve the conserved quantity. [12] constructs discrete gradient methods which preserve a conserved quantity based on an equivalent skew-gradient (SG) form of the original SDEs. [11] gives conditions for stochastic Runge-Kutta methods to preserve quadratic invariants. Based on generalized averaged vector field methods, [7] proposes energy-preserving schemes for stochastic Poisson systems, [17] and [5] construct conservative schemes for SDEs with a conserved quantity. [28] designs projection methods preserving single or multiple conserved quantities for SDEs which can achieve high strong convergence order. [16] constructs a class of discrete gradient methods and linear projection methods for SDEs with a conserved quantity and studies their relationship.

In applications, many systems are represented by partitioned differential equations, for example, the Hamiltonian system

$$\begin{cases} \dot{p} = -\frac{\partial H(p,q)}{\partial q}, \\ \dot{q} = \frac{\partial H(p,q)}{\partial p}. \end{cases} \tag{1.1}$$

When a system can be represented in a partitioned form, it is worth trying a partitioned method. Partitioned methods are usually used to approximate the solution trajectory by using different formulas for different parts of a partitioned differential equation, and the importance of partitioned methods is mentioned in [10, 18]. [4] studies partitioned averaged vector field methods for preserving Hamiltonian function of the deterministic Hamiltonian system (1.1), which inspires us to extend the idea to stochastic cases. Considering that the stochastic Hamiltonian system with an invariant Hamiltonian function is very special, in this work we are concerned with the more general case, that is, stochastic partitioned differential equations with a conserved quantity.

The rest of the paper is organized as follows. In Section 2, the stochastic partitioned averaged vector field (SPAVF) method is proposed and proved to preserve the conserved quantity. In Section 3, we analyze the convergence order of the proposed SPAVF method and derive the method is convergent with mean-square order 1. In Section 4, numerical experiments are displayed to show the effectiveness of the proposed method in preserving the conserved quantity and show the convergence order results.

2. SPAVF method

Consider a system that can be represented by the following stochastic partitioned differential equation in the Stratonovich sense

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = S(x,y) \begin{pmatrix} I_x(x,y) \\ I_y(x,y) \end{pmatrix} dt + T(x,y) \begin{pmatrix} I_x(x,y) \\ I_y(x,y) \end{pmatrix} \circ dW(t), \ t \in [0,T], \tag{2.1}$$

where $x = (x^1, \dots, x^{d_1})^T \in \mathbb{R}^{d_1}$, $y = (y^1, \dots, y^{d_2})^T \in \mathbb{R}^{d_2}$, the notations $I_x(x,y) = \partial I(x,y)/\partial x$, $I_y(x,y) = \partial I(x,y)/\partial y$, and $W(t)$ is a one-dimensional Wiener process. We assume that $S(x,y)$ and $T(x,y)$ are two $(d_1 + d_2) \times (d_1 + d_2)$ smooth skew-symmetric matrix-valued functions and $I(x, y)$ is a sufficiently smooth scalar function such that the exact solution of (2.1) uniquely exists for all time. Notice the structure matrices $S(x, y)$ and $T(x, y)$ in (2.1) are not necessary to be constant matrices, so (2.1) is more general than the case that S and T are constant skew-symmetric matrices. It's easy to verify that I is a conserved quantity of (2.1) as

$$\begin{aligned} dI &= I_x^T(x,y)dx + I_y^T(x,y)dy \\ &= (I_x^T(x,y), I_y^T(x,y))S(x,y)\begin{pmatrix} I_x(x,y) \\ I_y(x,y) \end{pmatrix} dt \\ &\quad + (I_x^T(x,y), I_y^T(x,y))T(x,y)\begin{pmatrix} I_x(x,y) \\ I_y(x,y) \end{pmatrix} \circ dW(t) \\ &= 0. \end{aligned}$$

For an equidistant discretization $0 = t_0 < t_1 < \ldots < t_N = T$ with a fixed step size $h > 0$, on the premise of no confusion, we always denote the numerical solution at $t_n = nh$ by $(x_n, y_n) \approx (x(t_n), y(t_n))$ hereinafter. Now we define two stochastic partitioned numerical methods for solving (2.1) as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} + hS(x_n, y_n)\begin{pmatrix} f_n^1 \\ f_n^2 \end{pmatrix} + \Delta W(h)T(\tfrac{x_n+x_{n+1}}{2}, \tfrac{y_n+y_{n+1}}{2})\begin{pmatrix} f_n^1 \\ f_n^2 \end{pmatrix}, \tag{2.2}$$

and

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} + hS(x_n, y_n)\begin{pmatrix} g_n^1 \\ g_n^2 \end{pmatrix} + \Delta W(h)T(\tfrac{x_n+x_{n+1}}{2}, \tfrac{y_n+y_{n+1}}{2})\begin{pmatrix} g_n^1 \\ g_n^2 \end{pmatrix}, \tag{2.3}$$

with

$$\begin{pmatrix} f_n^1 \\ f_n^2 \end{pmatrix} = \begin{pmatrix} \int_0^1 I_x(\xi x_{n+1} + (1-\xi)x_n, y_n)d\xi \\ \int_0^1 I_y(x_{n+1}, \xi y_{n+1} + (1-\xi)y_n)d\xi \end{pmatrix},$$
$$\begin{pmatrix} g_n^1 \\ g_n^2 \end{pmatrix} = \begin{pmatrix} \int_0^1 I_x(\xi x_{n+1} + (1-\xi)x_n, y_{n+1})d\xi \\ \int_0^1 I_y(x_n, \xi y_{n+1} + (1-\xi)y_n)d\xi \end{pmatrix},$$

where $\Delta W(h) = W(t_{n+1}) - W(t_n)$ $(n = 0, 1, \ldots, N-1)$ are independent Gaussian random variables with $N(0, h)$ distribution. Next we will prove the method (2.2) and method (2.3) both preserve the conserved quantity I of (2.1).

Theorem 2.1. *If $S(x, y)$ and $T(x, y)$ are skew-symmetric for all (x, y), $I(x, y) \in C^1(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}, \mathbb{R})$, then the method (2.2) and method (2.3) both preserve the conserved quantity I of (2.1), i.e., $I(x_{n+1}, y_{n+1}) = I(x_n, y_n)$ for all $n \geq 0$.*

Proof. From (2.2) we get

$$\begin{pmatrix} x_{n+1}-x_n \\ y_{n+1}-y_n \end{pmatrix} = hS(x_n,y_n)\begin{pmatrix} f_n^1 \\ f_n^2 \end{pmatrix} + \Delta W(h)T(\frac{x_n+x_{n+1}}{2},\frac{y_n+y_{n+1}}{2})\begin{pmatrix} f_n^1 \\ f_n^2 \end{pmatrix}. \tag{2.4}$$

By taking product with the row vector $\big((f_n^1)^T,\ (f_n^2)^T\big)$ on both sides of (2.4), and using the skew-symmetry of S and T, we obtain that

$$\begin{aligned} 0 &= \textstyle\int_0^1 I_x^T(\xi x_{n+1}+(1-\xi)x_n,y_n)d\xi(x_{n+1}-x_n) \\ &\quad + \textstyle\int_0^1 I_y^T(x_{n+1},\xi y_{n+1}+(1-\xi)y_n)d\xi(y_{n+1}-y_n) \\ &= \textstyle\int_0^1 \frac{d}{d\xi}I(\xi x_{n+1}+(1-\xi)x_n,y_n)d\xi + \int_0^1 \frac{d}{d\xi}I(x_{n+1},\xi y_{n+1}+(1-\xi)y_n)d\xi \\ &= I(x_{n+1},y_n)-I(x_n,y_n)+I(x_{n+1},y_{n+1})-I(x_{n+1},y_n) \\ &= I(x_{n+1},y_{n+1})-I(x_n,y_n), \end{aligned}$$

which shows that the method (2.2) preserves the conserved quantity I. Similarly, we can prove the method (2.3) preserves I too. □

Based on the methods (2.2) and (2.3), we put forward to the SPAVF method for solving (2.1) as following

$$\begin{aligned} \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} &= \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \frac{1}{2}hS(x_n,y_n)\begin{pmatrix} f_n^1+g_n^1 \\ f_n^2+g_n^2 \end{pmatrix} \\ &\quad + \frac{1}{2}\Delta W(h)T(\frac{x_n+x_{n+1}}{2},\frac{y_n+y_{n+1}}{2})\begin{pmatrix} f_n^1+g_n^1 \\ f_n^2+g_n^2 \end{pmatrix}. \end{aligned} \tag{2.5}$$

Obviously, if we consider all variables as one group, the SPAVF method (2.5) reduces to a non-partitioned one, i.e., the stochastic averaged vector field method.

Theorem 2.2. *If $S(x,y)$ and $T(x,y)$ are skew-symmetric for all (x,y), $I(x,y)\in C^1(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2},\mathbb{R})$, then the SPAVF method (2.5) can preserve the conserved quantity I of (2.1), which is*

$$I(x_{n+1},y_{n+1}) = I(x_n,y_n).$$

Proof. The proof is similar to that of Theorem 2.1. From (2.5) we get

$$\begin{aligned} \begin{pmatrix} x_{n+1}-x_n \\ y_{n+1}-y_n \end{pmatrix} &= \frac{1}{2}hS(x_n,y_n)\begin{pmatrix} f_n^1+g_n^1 \\ f_n^2+g_n^2 \end{pmatrix} \\ &\quad + \frac{1}{2}\Delta W(h)T(\frac{x_n+x_{n+1}}{2},\frac{y_n+y_{n+1}}{2})\begin{pmatrix} f_n^1+g_n^1 \\ f_n^2+g_n^2 \end{pmatrix}. \end{aligned} \tag{2.6}$$

Then take product with the row vector $\big((f_n^1+g_n^1)^T,(f_n^2+g_n^2)^T\big)$ on both sides of (2.6). Following the way in the proof of Theorem 2.1, we can definitely derive the conclusion. □

Actually, if $T(x,y)$ vanishes, then the three methods (2.2), (2.3) and (2.5) reduce to the PAVF method, the PAVF method's adjoint method and the PAVF-P method for solving the deterministic Hamiltonian system (1.1) proposed in [4], respectively. In the next section, we will show the SPAVF method (2.5) is convergent while the methods (2.2) and (2.3) are not convergent for solving (2.1) generally.

3. Convergence analysis

Convergence analysis is important for numerical methods. A numerical method that is not convergent is ineffective. In this section, we will analyze the mean-square convergence order of the proposed SPAVF method (2.5) for solving (2.1) by comparing the Taylor expansions of the numerical solution and the exact solution term by term.

Theorem 3.1. *Consider the SPAVF method (2.5) for solving (2.1). Assume the matrix-valued functions* $S(x,y), T(x,y) \in C^2(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}, \mathbb{R}^{(d_1+d_2)\times(d_1+d_2)})$ *have uniformly bounded derivatives up to order* 2, *the scalar function* $I(x,y) \in C^3(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}, \mathbb{R})$ *has uniformly bounded derivatives up to order* 3. *Then the SPAVF method (2.5) has mean-square convergence order* 1 *for solving (2.1).*

Proof. For convenience, we denote $S = S(x,y),\ T = T(x,y)$. Partition the skew-symmetric matrices S and T as

$$S = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}, T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix},$$

where $S_1 = (S_1^{i,j})_{d_1\times d_1}$, $S_2 = (S_2^{i,j})_{d_1\times d_2}$, $S_3 = (S_3^{i,j})_{d_2\times d_1}$, $S_4 = (S_4^{i,j})_{d_2\times d_2}$, $T_1 = (T_1^{i,j})_{d_1\times d_1}$, $T_2 = (T_2^{i,j})_{d_1\times d_2}$, $T_3 = (T_3^{i,j})_{d_2\times d_1}$, $T_4 = (T_4^{i,j})_{d_2\times d_2}$. Then

$$S = \begin{pmatrix} S_1^{1,1} & \cdots & S_1^{1,d_1} & S_2^{1,1} & \cdots & S_2^{1,d_2} \\ \vdots & & \vdots & \vdots & & \vdots \\ S_1^{d_1,1} & \cdots & S_1^{d_1,d_1} & S_2^{d_1,1} & \cdots & S_2^{d_1,d_2} \\ S_3^{1,1} & \cdots & S_3^{1,d_1} & S_4^{1,1} & \cdots & S_4^{1,d_2} \\ \vdots & & \vdots & \vdots & & \vdots \\ S_3^{d_2,1} & \cdots & S_3^{d_2,d_1} & S_4^{d_2,1} & \cdots & S_4^{d_2,d_2} \end{pmatrix}, T = \begin{pmatrix} T_1^{1,1} & \cdots & T_1^{1,d_1} & T_2^{1,1} & \cdots & T_2^{1,d_2} \\ \vdots & & \vdots & \vdots & & \vdots \\ T_1^{d_1,1} & \cdots & T_1^{d_1,d_1} & T_2^{d_1,1} & \cdots & T_2^{d_1,d_2} \\ T_3^{1,1} & \cdots & T_3^{1,d_1} & T_4^{1,1} & \cdots & T_4^{1,d_2} \\ \vdots & & \vdots & \vdots & & \vdots \\ T_3^{d_2,1} & \cdots & T_3^{d_2,d_1} & T_4^{d_2,1} & \cdots & T_4^{d_2,d_2} \end{pmatrix}.$$

Rewrite (2.1) in the following component form as

$$\begin{aligned} dx^k &= \Big(\sum_{i=1}^{d_1} S_1^{k,i} I_{x^i} + \sum_{i=1}^{d_2} S_2^{k,i} I_{y^i}\Big)dt + \Big(\sum_{i=1}^{d_1} T_1^{k,i} I_{x^i} + \sum_{i=1}^{d_2} T_2^{k,i} I_{y^i}\Big) \circ dW(t),\ k=1,\dots,d_1, \\ dy^k &= \Big(\sum_{i=1}^{d_1} S_3^{k,i} I_{x^i} + \sum_{i=1}^{d_2} S_4^{k,i} I_{y^i}\Big)dt + \Big(\sum_{i=1}^{d_1} T_3^{k,i} I_{x^i} + \sum_{i=1}^{d_2} T_4^{k,i} I_{y^i}\Big) \circ dW(t),\ k=1,\dots,d_2, \end{aligned} \tag{3.1}$$

where $S_1^{k,i}$, $S_2^{k,i}$, $S_3^{k,i}$, $S_4^{k,i}$, $T_1^{k,i}$, $T_2^{k,i}$, $T_3^{k,i}$, $T_4^{k,i}$, I_{x^i}, I_{y^i} are short notations for $S_1^{k,i}(x,y)$, $S_2^{k,i}(x,y)$, $S_3^{k,i}(x,y)$, $S_4^{k,i}(x,y)$, $T_1^{k,i}(x,y)$, $T_2^{k,i}(x,y)$, $T_3^{k,i}(x,y)$, $T_4^{k,i}(x,y)$, $I_{x^i}(x,y)$, $I_{y^i}(x,y)$, respectively.

For a fixed step size $h>0$, we use the one-step approximation to prove the theorem, by assuming $(x_n,y_n)=(x(t_n),y(t_n))$ at t_n. First we analyze the component expansion of the numerical solution (x_{n+1},y_{n+1}) at $t_{n+1}=t_n+h$. From (2.5) we have

$$\begin{aligned}
x_{n+1}^k =& x_n^k+\frac{1}{2}h\Big[\sum_{i=1}^{d_1}S_1^{k,i}(x_n,y_n)\Big(\int_0^1 I_{x^i}(\xi x_{n+1}+(1-\xi)x_n,y_n)d\xi \\
&+\int_0^1 I_{x^i}(\xi x_{n+1}+(1-\xi)x_n,y_{n+1})d\xi\Big) \\
&+\sum_{i=1}^{d_2}S_2^{k,i}(x_n,y_n)\Big(\int_0^1 I_{y^i}(x_{n+1},\xi y_{n+1}+(1-\xi)y_n)d\xi \\
&+\int_0^1 I_{y^i}(x_n,\xi y_{n+1}+(1-\xi)y_n)d\xi\Big)\Big] \\
&+\frac{1}{2}\Delta W(h)\Big[\sum_{i=1}^{d_1}T_1^{k,i}(\frac{x_n+x_{n+1}}{2},\frac{y_n+y_{n+1}}{2})\Big(\int_0^1 I_{x^i}(\xi x_{n+1}+(1-\xi)x_n,y_n)d\xi \\
&+\int_0^1 I_{x^i}(\xi x_{n+1}+(1-\xi)x_n,y_{n+1})d\xi\Big) \\
&+\sum_{i=1}^{d_2}T_2^{k,i}(\frac{x_n+x_{n+1}}{2},\frac{y_n+y_{n+1}}{2})\Big(\int_0^1 I_{y^i}(x_{n+1},\xi y_{n+1}+(1-\xi)y_n)d\xi \\
&+\int_0^1 I_{y^i}(x_n,\xi y_{n+1}+(1-\xi)y_n)d\xi\Big)\Big],\ k=1,\ldots,d_1,
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
y_{n+1}^k =& y_n^k+\frac{1}{2}h\Big[\sum_{i=1}^{d_1}S_3^{k,i}(x_n,y_n)\Big(\int_0^1 I_{x^i}(\xi x_{n+1}+(1-\xi)x_n,y_n)d\xi \\
&+\int_0^1 I_{x^i}(\xi x_{n+1}+(1-\xi)x_n,y_{n+1})d\xi\Big) \\
&+\sum_{i=1}^{d_2}S_4^{k,i}(x_n,y_n)\Big(\int_0^1 I_{y^i}(x_{n+1},\xi y_{n+1}+(1-\xi)y_n)d\xi \\
&+\int_0^1 I_{y^i}(x_n,\xi y_{n+1}+(1-\xi)y_n)d\xi\Big)\Big] \\
&+\frac{1}{2}\Delta W(h)\Big[\sum_{i=1}^{d_1}T_3^{k,i}(\frac{x_n+x_{n+1}}{2},\frac{y_n+y_{n+1}}{2})\Big(\int_0^1 I_{x^i}(\xi x_{n+1}+(1-\xi)x_n,y_n)d\xi \\
&+\int_0^1 I_{x^i}(\xi x_{n+1}+(1-\xi)x_n,y_{n+1})d\xi\Big) \\
&+\sum_{i=1}^{d_2}T_4^{k,i}(\frac{x_n+x_{n+1}}{2},\frac{y_n+y_{n+1}}{2})\Big(\int_0^1 I_{y^i}(x_{n+1},\xi y_{n+1}+(1-\xi)y_n)d\xi
\end{aligned}$$

$$+\int_0^1 I_{y^i}(x_n,\xi y_{n+1}+(1-\xi)y_n)d\xi\Big)\Big],\ k=1,\ldots,d_2. \tag{3.3}$$

For dealing with the integral terms in (3.2) and (3.3), we expand $I_{x^i}(\xi x_{n+1}+(1-\xi)x_n,y_n)$, $I_{x^i}(\xi x_{n+1}+(1-\xi)x_n,y_{n+1})$, $I_{y^i}(x_{n+1},\xi y_{n+1}+(1-\xi)y_n)$ and $I_{y^i}(x_n,\xi y_{n+1}+(1-\xi)y_n)$ around $\xi=0$ to yield

$$\begin{aligned}&I_{x^i}(\xi x_{n+1}+(1-\xi)x_n,y_n)\\=&I_{x^i}(x_n,y_n)+\xi\sum_{j=1}^{d_1}I_{x^ix^j}(x_n,y_n)(x_{n+1}^j-x_n^j)\\&+\frac{\xi^2}{2}\sum_{j=1}^{d_1}\sum_{l=1}^{d_1}I_{x^ix^jx^l}(\hat{x}_n,y_n)(x_{n+1}^j-x_n^j)(x_{n+1}^l-x_n^l),\ i=1,\ldots,d_1,\end{aligned} \tag{3.4}$$

$$\begin{aligned}&I_{x^i}(\xi x_{n+1}+(1-\xi)x_n,y_{n+1})\\=&I_{x^i}(x_n,y_{n+1})+\xi\sum_{j=1}^{d_1}I_{x^ix^j}(x_n,y_{n+1})(x_{n+1}^j-x_n^j)\\&+\frac{\xi^2}{2}\sum_{j=1}^{d_1}\sum_{l=1}^{d_1}I_{x^ix^jx^l}(\check{x}_n,y_{n+1})(x_{n+1}^j-x_n^j)(x_{n+1}^l-x_n^l),\ i=1,\ldots,d_1,\end{aligned} \tag{3.5}$$

$$\begin{aligned}&I_{y^i}(x_{n+1},\xi y_{n+1}+(1-\xi)y_n)\\=&I_{y^i}(x_{n+1},y_n)+\xi\sum_{j=1}^{d_2}I_{y^iy^j}(x_{n+1},y_n)(y_{n+1}^j-y_n^j)\\&+\frac{\xi^2}{2}\sum_{j=1}^{d_2}\sum_{l=1}^{d_2}I_{y^iy^jy^l}(x_{n+1},\hat{y}_n)(y_{n+1}^j-y_n^j)(y_{n+1}^l-y_n^l),\ i=1,\ldots,d_2,\end{aligned} \tag{3.6}$$

$$\begin{aligned}&I_{y^i}(x_n,\xi y_{n+1}+(1-\xi)y_n)\\=&I_{y^i}(x_n,y_n)+\xi\sum_{j=1}^{d_2}I_{y^iy^j}(x_n,y_n)(y_{n+1}^j-y_n^j)\\&+\frac{\xi^2}{2}\sum_{j=1}^{d_2}\sum_{l=1}^{d_2}I_{y^iy^jy^l}(x_n,\check{y}_n)(y_{n+1}^j-y_n^j)(y_{n+1}^l-y_n^l),\ i=1,\ldots,d_2,\end{aligned} \tag{3.7}$$

where $\hat{x}_n$ and $\check{x}_n$ depend on x_n and x_{n+1}, $\hat{y}_n$ and $\check{y}_n$ depend on y_n and y_{n+1}, the notation $I_{x^ix^j}$ is the second order partial derivative with respect to x^i and x^j, $I_{x^ix^jx^l}$ is the third order partial derivative with respect to x^i, x^j and x^l. After a straightforward computation by substituting (3.4)-(3.7) into (3.2) and (3.3), then expanding the functions

$$\begin{aligned}&I_{x^i}(x_n,y_{n+1}),\ I_{y^i}(x_{n+1},y_n),\ I_{x^ix^j}(x_n,y_{n+1}),\ I_{y^iy^j}(x_{n+1},y_n),\\&T_1^{k,i}(\frac{x_n+x_{n+1}}{2},\frac{y_n+y_{n+1}}{2}),\ T_2^{k,i}(\frac{x_n+x_{n+1}}{2},\frac{y_n+y_{n+1}}{2}),\end{aligned}$$

$$T_3^{k,i}(\frac{x_n+x_{n+1}}{2},\frac{y_n+y_{n+1}}{2}),\ T_4^{k,i}(\frac{x_n+x_{n+1}}{2},\frac{y_n+y_{n+1}}{2}),$$

around (x_n, y_n), with plugging the expressions of $x^k_{n+1}-x^k_n$ and $y^k_{n+1}-y^k_n$, we obtain

$$\begin{aligned}
x^k_{n+1} =& x^k_n + h\Big[\sum_{i=1}^{d_1} S_1^{k,i} I_{x^i} + \sum_{i=1}^{d_2} S_2^{k,i} I_{y^i}\Big] + \Delta W(h)\Big[\sum_{i=1}^{d_1} T_1^{k,i} I_{x^i} + \sum_{i=1}^{d_2} T_2^{k,i} I_{y^i}\Big]\\
&+\frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_1} T_1^{k,i}\sum_{j=1}^{d_1} I_{x^ix^j}\Big[\sum_{l=1}^{d_1} T_1^{j,l} I_{x^l} + \sum_{l=1}^{d_2} T_2^{j,l} I_{y^l}\Big]\\
&+\frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_1} T_1^{k,i}\sum_{j=1}^{d_2} I_{x^iy^j}\Big[\sum_{l=1}^{d_1} T_3^{j,l} I_{x^l} + \sum_{l=1}^{d_2} T_4^{j,l} I_{y^l}\Big]\\
&+\frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_1}\sum_{j=1}^{d_1} T_{1x^j}^{k,i} I_{x^i}\Big[\sum_{l=1}^{d_1} T_1^{j,l} I_{x^l} + \sum_{l=1}^{d_2} T_2^{j,l} I_{y^l}\Big]\\
&+\frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_1}\sum_{j=1}^{d_2} T_{1y^j}^{k,i} I_{x^i}\Big[\sum_{l=1}^{d_1} T_3^{j,l} I_{x^l} + \sum_{l=1}^{d_2} T_4^{j,l} I_{y^l}\Big]\\
&+\frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_2} T_2^{k,i}\sum_{j=1}^{d_1} I_{y^ix^j}\Big[\sum_{l=1}^{d_1} T_1^{j,l} I_{x^l} + \sum_{l=1}^{d_2} T_2^{j,l} I_{y^l}\Big]\\
&+\frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_2} T_2^{k,i}\sum_{j=1}^{d_2} I_{y^iy^j}\Big[\sum_{l=1}^{d_1} T_3^{j,l} I_{x^l} + \sum_{l=1}^{d_2} T_4^{j,l} I_{y^l}\Big]\\
&+\frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_2}\sum_{j=1}^{d_1} T_{2x^j}^{k,i} I_{y^i}\Big[\sum_{l=1}^{d_1} T_1^{j,l} I_{x^l} + \sum_{l=1}^{d_2} T_2^{j,l} I_{y^l}\Big]\\
&+\frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_2}\sum_{j=1}^{d_2} T_{2y^j}^{k,i} I_{y^i}\Big[\sum_{l=1}^{d_1} T_3^{j,l} I_{x^l} + \sum_{l=1}^{d_2} T_4^{j,l} I_{y^l}\Big] + R_1,\ k=1,\ldots,d_1,
\end{aligned}\tag{3.8}$$

with

$$\begin{aligned}
R_1 =& \frac{1}{2}h\Delta W(h)\sum_{i=1}^{d_1} S_1^{k,i}\sum_{j=1}^{d_1} I_{x^ix^j}\Big[\sum_{l=1}^{d_1} T_1^{j,l} I_{x^l} + \sum_{l=1}^{d_2} T_2^{j,l} I_{y^l}\Big]\\
&+\frac{1}{2}h\Delta W(h)\sum_{i=1}^{d_2} S_2^{k,i}\sum_{j=1}^{d_2} I_{y^iy^j}\Big[\sum_{l=1}^{d_1} T_3^{j,l} I_{x^l} + \sum_{l=1}^{d_2} T_4^{j,l} I_{y^l}\Big] + \cdots,
\end{aligned}\tag{3.9}$$

where the remainder R_1 consists of some terms of mean-square order greater than 1, say, terms with $h\Delta W(h)$, $\Delta W^3(h)$, h^2, etc. In view of the the smoothness and boundedness hypotheses on the functions S, T, I and their derivatives, as well as the properties of Wiener process that $|E(h\Delta W(h))| = 0$, $(E(h\Delta W(h))^2)^{1/2} = O(h^{3/2})$, $|E(\Delta W^3(h))| = 0$, $(E(\Delta W^3(h))^2)^{1/2} = O(h^{3/2})$, using the fundamental inequality yields

$$|ER_1| = O(h^2),\ (ER_1^2)^{1/2} = O(h^{3/2}).$$

Notice we omit the variables (x_n, y_n) of all the functions in (3.8) for the sake of simplicity, for example, $S_1^{k,i} = S_1^{k,i}(x_n, y_n)$, $I_{x^i} = I_{x^i}(x_n, y_n)$, etc. Similarly, we can deduce that

$$
\begin{aligned}
y_{n+1}^k =& y_n^k + h\Big[\sum_{i=1}^{d_1} S_3^{k,i} I_{x^i} + \sum_{i=1}^{d_2} S_4^{k,i} I_{y^i}\Big] + \Delta W(h)\Big[\sum_{i=1}^{d_1} T_3^{k,i} I_{x^i} + \sum_{i=1}^{d_2} T_4^{k,i} I_{y^i}\Big]\\
&+ \frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_1} T_3^{k,i}\sum_{j=1}^{d_1} I_{x^i x^j}\Big[\sum_{l=1}^{d_1} T_1^{j,l} I_{x^l} + \sum_{l=1}^{d_2} T_2^{j,l} I_{y^l}\Big]\\
&+ \frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_1} T_3^{k,i}\sum_{j=1}^{d_2} I_{x^i y^j}\Big[\sum_{l=1}^{d_1} T_3^{j,l} I_{x^l} + \sum_{l=1}^{d_2} T_4^{j,l} I_{y^l}\Big]\\
&+ \frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_1}\sum_{j=1}^{d_1} T_{3x^j}^{k,i} I_{x^i}\Big[\sum_{l=1}^{d_1} T_1^{j,l} I_{x^l} + \sum_{l=1}^{d_2} T_2^{j,l} I_{y^l}\Big]\\
&+ \frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_1}\sum_{j=1}^{d_2} T_{3y^j}^{k,i} I_{x^i}\Big[\sum_{l=1}^{d_1} T_3^{j,l} I_{x^l} + \sum_{l=1}^{d_2} T_4^{j,l} I_{y^l}\Big]\\
&+ \frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_2} T_4^{k,i}\sum_{j=1}^{d_1} I_{y^i x^j}\Big[\sum_{l=1}^{d_1} T_1^{j,l} I_{x^l} + \sum_{l=1}^{d_2} T_2^{j,l} I_{y^l}\Big]\\
&+ \frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_2} T_4^{k,i}\sum_{j=1}^{d_2} I_{y^i y^j}\Big[\sum_{l=1}^{d_1} T_3^{j,l} I_{x^l} + \sum_{l=1}^{d_2} T_4^{j,l} I_{y^l}\Big]\\
&+ \frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_2}\sum_{j=1}^{d_1} T_{4x^j}^{k,i} I_{y^i}\Big[\sum_{l=1}^{d_1} T_1^{j,l} I_{x^l} + \sum_{l=1}^{d_2} T_2^{j,l} I_{y^l}\Big]\\
&+ \frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_2}\sum_{j=1}^{d_2} T_{4y^j}^{k,i} I_{y^i}\Big[\sum_{l=1}^{d_1} T_3^{j,l} I_{x^l} + \sum_{l=1}^{d_2} T_4^{j,l} I_{y^l}\Big] + R_2,\ k = 1, \dots, d_2,
\end{aligned}
\tag{3.10}
$$

with

$$
\begin{aligned}
R_2 =& \frac{1}{2}h\Delta W(h)\sum_{i=1}^{d_1} S_3^{k,i}\sum_{j=1}^{d_1} I_{x^i x^j}\Big[\sum_{l=1}^{d_1} T_1^{j,l} I_{x^l} + \sum_{l=1}^{d_2} T_2^{j,l} I_{y^l}\Big]\\
&+ \frac{1}{2}h\Delta W(h)\sum_{i=1}^{d_2} S_4^{k,i}\sum_{j=1}^{d_2} I_{y^i y^j}\Big[\sum_{l=1}^{d_1} T_3^{j,l} I_{x^l} + \sum_{l=1}^{d_2} T_4^{j,l} I_{y^l}\Big] + \cdots,
\end{aligned}
$$

where $|ER_2| = O(h^2)$, $(ER_2^2)^{1/2} = O(h^{3/2})$. Also the variables (x_n, y_n) of all the functions in (3.10) are omitted.

Next we will show the component expression of the exact solution $(x(t_n + h), y(t_n + h))$ at $t_{n+1} = t_n + h$. Integrating on both sides of the first equation in (3.1), we get the component of the exact solution $x(t_n + h)$ is

$$
x^k(t_n + h) = x_n^k + \int_{t_n}^{t_n+h}\sum_{i=1}^{d_1} S_1^{k,i} I_{x^i}\, dt + \int_{t_n}^{t_n+h}\sum_{i=1}^{d_2} S_2^{k,i} I_{y^i}\, dt
$$

$$+\int_{t_n}^{t_n+h}\sum_{i=1}^{d_1}T_1^{k,i}I_{x^i}\circ dW(t)+\int_{t_n}^{t_n+h}\sum_{i=1}^{d_2}T_2^{k,i}I_{y^i}\circ dW(t),\ \ k=1,\ldots,d_1.$$

By using the Stratonovich-Taylor expansion, we get

$$\begin{aligned}
&x^k(t_n+h)\\
=&x_n^k+h\sum_{i=1}^{d_1}S_1^{k,i}(x_n,y_n)I_{x^i}(x_n,y_n)\\
&+\int_{t_n}^{t_n+h}\int_{t_n}^{t_n+t}\sum_{i=1}^{d_1}\sum_{j=1}^{d_1}(S_{1x^j}^{k,i}I_{x^i}+S_1^{k,i}I_{x^ix^j})(\sum_{l=1}^{d_1}S_1^{j,l}I_{x^l}+\sum_{l=1}^{d_2}S_2^{j,l}I_{y^l})dsdt\\
&+\int_{t_n}^{t_n+h}\int_{t_n}^{t_n+t}\sum_{i=1}^{d_1}\sum_{j=1}^{d_1}(S_{1x^j}^{k,i}I_{x^i}+S_1^{k,i}I_{x^ix^j})(\sum_{l=1}^{d_1}T_1^{j,l}I_{x^l}+\sum_{l=1}^{d_2}T_2^{j,l}I_{y^l})\circ dW(s)dt\\
&+\int_{t_n}^{t_n+h}\int_{t_n}^{t_n+t}\sum_{i=1}^{d_1}\sum_{j=1}^{d_2}(S_{1y^j}^{k,i}I_{x^i}+S_1^{k,i}I_{x^iy^j})(\sum_{l=1}^{d_1}S_3^{j,l}I_{x^l}+\sum_{l=1}^{d_2}S_4^{j,l}I_{y^l})dsdt\\
&+\int_{t_n}^{t_n+h}\int_{t_n}^{t_n+t}\sum_{i=1}^{d_1}\sum_{j=1}^{d_2}(S_{1y^j}^{k,i}I_{x^i}+S_1^{k,i}I_{x^iy^j})(\sum_{l=1}^{d_1}T_3^{j,l}I_{x^l}+\sum_{l=1}^{d_2}T_4^{j,l}I_{y^l})\circ dW(s)dt\\
&+h\sum_{i=1}^{d_2}S_2^{k,i}(x_n,y_n)I_{y^i}(x_n,y_n)\\
&+\int_{t_n}^{t_n+h}\int_{t_n}^{t_n+t}\sum_{i=1}^{d_2}\sum_{j=1}^{d_1}(S_{2x^j}^{k,i}I_{y^i}+S_2^{k,i}I_{y^ix^j})(\sum_{l=1}^{d_1}S_1^{j,l}I_{x^l}+\sum_{l=1}^{d_2}S_2^{j,l}I_{y^l})dsdt\\
&+\int_{t_n}^{t_n+h}\int_{t_n}^{t_n+t}\sum_{i=1}^{d_2}\sum_{j=1}^{d_1}(S_{2x^j}^{k,i}I_{y^i}+S_2^{k,i}I_{y^ix^j})(\sum_{l=1}^{d_1}T_1^{j,l}I_{x^l}+\sum_{l=1}^{d_2}T_2^{j,l}I_{y^l})\circ dW(s)dt\\
&+\int_{t_n}^{t_n+h}\int_{t_n}^{t_n+t}\sum_{i=1}^{d_2}\sum_{j=1}^{d_2}(S_{2y^j}^{k,i}I_{y^i}+S_2^{k,i}I_{y^iy^j})(\sum_{l=1}^{d_1}S_3^{j,l}I_{x^l}+\sum_{l=1}^{d_2}S_4^{j,l}I_{y^l})dsdt\\
&+\int_{t_n}^{t_n+h}\int_{t_n}^{t_n+t}\sum_{i=1}^{d_2}\sum_{j=1}^{d_2}(S_{2y^j}^{k,i}I_{y^i}+S_2^{k,i}I_{y^iy^j})(\sum_{l=1}^{d_1}T_3^{j,l}I_{x^l}+\sum_{l=1}^{d_2}T_4^{j,l}I_{y^l})\circ dW(s)dt\\
&+\Delta W(h)\sum_{i=1}^{d_1}T_1^{k,i}(x_n,y_n)I_{x^i}(x_n,y_n)\\
&+\int_{t_n}^{t_n+h}\int_{t_n}^{t_n+t}\sum_{i=1}^{d_1}\sum_{j=1}^{d_1}(T_{1x^j}^{k,i}I_{x^i}+T_1^{k,i}I_{x^ix^j})(\sum_{l=1}^{d_1}S_1^{j,l}I_{x^l}+\sum_{l=1}^{d_2}S_2^{j,l}I_{y^l})ds\circ dW(t)\\
&+\int_{t_n}^{t_n+h}\int_{t_n}^{t_n+t}\sum_{i=1}^{d_1}\sum_{j=1}^{d_1}(T_{1x^j}^{k,i}I_{x^i}+T_1^{k,i}I_{x^ix^j})(\sum_{l=1}^{d_1}T_1^{j,l}I_{x^l}+\sum_{l=1}^{d_2}T_2^{j,l}I_{y^l})\circ dW(s)\circ dW(t)\\
&+\int_{t_n}^{t_n+h}\int_{t_n}^{t_n+t}\sum_{i=1}^{d_1}\sum_{j=1}^{d_2}(T_{1y^j}^{k,i}I_{x^i}+T_1^{k,i}I_{x^iy^j})(\sum_{l=1}^{d_1}S_3^{j,l}I_{x^l}+\sum_{l=1}^{d_2}S_4^{j,l}I_{y^l})ds\circ dW(t)
\end{aligned}$$

$$
\begin{aligned}
&+\int_{t_n}^{t_n+h}\int_{t_n}^{t_n+t}\sum_{i=1}^{d_1}\sum_{j=1}^{d_2}(T_{1y^j}^{k,i}I_{x^i}+T_1^{k,i}I_{x^iy^j})(\sum_{l=1}^{d_1}T_3^{j,l}I_{x^l}+\sum_{l=1}^{d_2}T_4^{j,l}I_{y^l})\circ dW(s)\circ dW(t)\\
&+\Delta W(h)\sum_{i=1}^{d_2}T_2^{k,i}(x_n,y_n)I_{y^i}(x_n,y_n)\\
&+\int_{t_n}^{t_n+h}\int_{t_n}^{t_n+t}\sum_{i=1}^{d_2}\sum_{j=1}^{d_1}(T_{2x^j}^{k,i}I_{y^i}+T_2^{k,i}I_{y^ix^j})(\sum_{l=1}^{d_1}S_1^{j,l}I_{x^l}+\sum_{l=1}^{d_2}S_2^{j,l}I_{y^l})ds\circ dW(t)\\
&+\int_{t_n}^{t_n+h}\int_{t_n}^{t_n+t}\sum_{i=1}^{d_2}\sum_{j=1}^{d_1}(T_{2x^j}^{k,i}I_{y^i}+T_2^{k,i}I_{y^ix^j})(\sum_{l=1}^{d_1}T_1^{j,l}I_{x^l}+\sum_{l=1}^{d_2}T_2^{j,l}I_{y^l})\circ dW(s)\circ dW(t)\\
&+\int_{t_n}^{t_n+h}\int_{t_n}^{t_n+t}\sum_{i=1}^{d_2}\sum_{j=1}^{d_2}(T_{2y^j}^{k,i}I_{y^i}+T_2^{k,i}I_{y^iy^j})(\sum_{l=1}^{d_1}S_3^{j,l}I_{x^l}+\sum_{l=1}^{d_2}S_4^{j,l}I_{y^l})ds\circ dW(t)\\
&+\int_{t_n}^{t_n+h}\int_{t_n}^{t_n+t}\sum_{i=1}^{d_2}\sum_{j=1}^{d_2}(T_{2y^j}^{k,i}I_{y^i}+T_2^{k,i}I_{y^iy^j})(\sum_{l=1}^{d_1}T_3^{j,l}I_{x^l}+\sum_{l=1}^{d_2}T_4^{j,l}I_{y^l})\circ dW(s)\circ dW(t).
\end{aligned}
$$

We repeat the Stratonovich-Taylor expanding procedure and take the terms of higher order (such as terms containing $\int_{t_n}^{t_n+h}\int_{t_n}^{t_n+t}\cdot\, ds\circ dW(t)$, $\int_{t_n}^{t_n+h}\int_{t_n}^{t_n+t}\cdot \circ dW(s)dt$, $\int_{t_n}^{t_n+h}\int_{t_n}^{t_n+t}\cdot\, dsdt$) as remainder terms, then we have

$$
\begin{aligned}
&x^k(t_n+h)\\
=&x_n^k+h\Big[\sum_{i=1}^{d_1}S_1^{k,i}(x_n,y_n)I_{x^i}(x_n,y_n)+\sum_{i=1}^{d_2}S_2^{k,i}(x_n,y_n)I_{y^i}(x_n,y_n)\Big]\\
&+\Delta W(h)\Big[\sum_{i=1}^{d_1}T_1^{k,i}(x_n,y_n)I_{x^i}(x_n,y_n)+\sum_{i=1}^{d_2}T_2^{k,i}(x_n,y_n)I_{y^i}(x_n,y_n)\Big]\\
&+\frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_1}\sum_{j=1}^{d_1}T_{1x^j}^{k,i}(x_n,y_n)I_{x^i}(x_n,y_n)\Big[\sum_{l=1}^{d_1}T_1^{j,l}(x_n,y_n)I_{x^l}(x_n,y_n)\\
&+\sum_{l=1}^{d_2}T_2^{j,l}(x_n,y_n)I_{y^l}(x_n,y_n)\Big]\\
&+\frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_1}\sum_{j=1}^{d_1}T_1^{k,i}(x_n,y_n)I_{x^ix^j}(x_n,y_n)\Big[\sum_{l=1}^{d_1}T_1^{j,l}(x_n,y_n)I_{x^l}(x_n,y_n)\\
&+\sum_{l=1}^{d_2}T_2^{j,l}(x_n,y_n)I_{y^l}(x_n,y_n)\Big]\\
&+\frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_1}\sum_{j=1}^{d_2}T_{1y^j}^{k,i}(x_n,y_n)I_{x^i}(x_n,y_n)\Big[\sum_{l=1}^{d_1}T_3^{j,l}(x_n,y_n)I_{x^l}(x_n,y_n)\\
&+\sum_{l=1}^{d_2}T_4^{j,l}(x_n,y_n)I_{y^l}(x_n,y_n)\Big]
\end{aligned}
$$

$$+\frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_1}\sum_{j=1}^{d_2}T_1^{k,i}(x_n,y_n)I_{x^iy^j}(x_n,y_n)\Big[\sum_{l=1}^{d_1}T_3^{j,l}(x_n,y_n)I_{x^l}(x_n,y_n)$$
$$+\sum_{l=1}^{d_2}T_4^{j,l}(x_n,y_n)I_{y^l}(x_n,y_n)\Big]$$
$$+\frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_2}\sum_{j=1}^{d_1}T_{2x^j}^{k,i}(x_n,y_n)I_{y^i}(x_n,y_n)\Big[\sum_{l=1}^{d_1}T_1^{j,l}(x_n,y_n)I_{x^l}(x_n,y_n)$$
$$+\sum_{l=1}^{d_2}T_2^{j,l}(x_n,y_n)I_{y^l}(x_n,y_n)\Big]$$
$$+\frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_2}\sum_{j=1}^{d_1}T_2^{k,i}(x_n,y_n)I_{y^ix^j}(x_n,y_n)\Big[\sum_{l=1}^{d_1}T_1^{j,l}(x_n,y_n)I_{x^l}(x_n,y_n)$$
$$+\sum_{l=1}^{d_2}T_2^{j,l}(x_n,y_n)I_{y^l}(x_n,y_n)\Big]$$
$$+\frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_2}\sum_{j=1}^{d_2}T_{2y^j}^{k,i}(x_n,y_n)I_{y^i}(x_n,y_n)\Big[\sum_{l=1}^{d_1}T_3^{j,l}(x_n,y_n)I_{x^l}(x_n,y_n)$$
$$+\sum_{l=1}^{d_2}T_4^{j,l}(x_n,y_n)I_{y^l}(x_n,y_n)\Big]$$
$$+\frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_2}\sum_{j=1}^{d_2}T_2^{k,i}(x_n,y_n)I_{y^iy^j}(x_n,y_n)\Big[\sum_{l=1}^{d_1}T_3^{j,l}(x_n,y_n)I_{x^l}(x_n,y_n)$$
$$+\sum_{l=1}^{d_2}T_4^{j,l}(x_n,y_n)I_{y^l}(x_n,y_n)\Big]+\tilde{R}_1,\ \ k=1,\ldots,d_1, \tag{3.11}$$

with

$$\tilde{R}_1=\int_{t_n}^{t_n+h}\int_{t_n}^{t_n+t}\sum_{i=1}^{d_1}\sum_{j=1}^{d_1}(S_{1x^j}^{k,i}I_{x^i}+S_1^{k,i}I_{x^ix^j})(\sum_{l=1}^{d_1}T_1^{j,l}I_{x^l}+\sum_{l=1}^{d_2}T_2^{j,l}I_{y^l})\circ dW(s)dt$$
$$+\int_{t_n}^{t_n+h}\int_{t_n}^{t_n+t}\sum_{i=1}^{d_1}\sum_{j=1}^{d_2}(S_{1y^j}^{k,i}I_{x^i}+S_1^{k,i}I_{x^iy^j})(\sum_{l=1}^{d_1}T_3^{j,l}I_{x^l}+\sum_{l=1}^{d_2}T_4^{j,l}I_{y^l})\circ dW(s)dt$$
$$+\int_{t_n}^{t_n+h}\int_{t_n}^{t_n+t}\sum_{i=1}^{d_2}\sum_{j=1}^{d_1}(S_{2x^j}^{k,i}I_{y^i}+S_2^{k,i}I_{y^ix^j})(\sum_{l=1}^{d_1}T_1^{j,l}I_{x^l}+\sum_{l=1}^{d_2}T_2^{j,l}I_{y^l})\circ dW(s)dt$$
$$+\cdots,$$

where the remainder $\tilde{R}_1$ consists of some terms of mean-square order greater than 1 such as terms with $\int_{t_n}^{t_n+h}\int_{t_n}^{t_n+t}\cdot\circ dW(s)dt$, $\int_{t_n}^{t_n+h}\int_{t_n}^{t_n+t}\int_{t_n}^{t_n+\tau}\cdot\circ dW(s)\circ dW(\tau)\circ dW(t)$, $\int_{t_n}^{t_n+h}\int_{t_n}^{t_n+t}\cdot\, dsdt$, etc. In view of the smoothness and boundedness hypotheses on the functions S, T, I and their derivatives, as well as the properties of multiple Stratonovich stochastic integrals that

$$
\begin{aligned}
&|E(\int_{t_n}^{t_n+h}\int_{t_n}^{t_n+t}\cdot\ \circ dW(s)dt)|=0,\\
&(E(\int_{t_n}^{t_n+h}\int_{t_n}^{t_n+t}\cdot\ \circ dW(s)dt)^2)^{1/2}=O(h^{3/2}),\\
&|E(\int_{t_n}^{t_n+h}\int_{t_n}^{t_n+t}\int_{t_n}^{t_n+\tau}\cdot\ \circ dW(s)\circ dW(\tau)\circ dW(t))|=0,\\
&(E(\int_{t_n}^{t_n+h}\int_{t_n}^{t_n+t}\int_{t_n}^{t_n+\tau}\cdot\ \circ dW(s)\circ dW(\tau)\circ dW(t))^2)^{1/2}=O(h^{3/2}),
\end{aligned}
$$

application of the fundamental inequality yields

$$
|E\tilde{R}_1|=O(h^2),\ (E\tilde{R}_1^2)^{1/2}=O(h^{3/2}).
$$

Similarly, we obtain

$$
\begin{aligned}
&y^k(t_n+h)\\
=&y_n^k+h\Big[\sum_{i=1}^{d_1}S_3^{k,i}(x_n,y_n)I_{x^i}(x_n,y_n)+\sum_{i=1}^{d_2}S_4^{k,i}(x_n,y_n)I_{y^i}(x_n,y_n)\Big]\\
&+\Delta W(h)\Big[\sum_{i=1}^{d_1}T_3^{k,i}(x_n,y_n)I_{x^i}(x_n,y_n)+\sum_{i=1}^{d_2}T_4^{k,i}(x_n,y_n)I_{y^i}(x_n,y_n)\Big]\\
&+\frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_1}\sum_{j=1}^{d_1}T_{3x^j}^{k,i}(x_n,y_n)I_{x^i}(x_n,y_n)\Big[\sum_{l=1}^{d_1}T_1^{j,l}(x_n,y_n)I_{x^l}(x_n,y_n)\\
&+\sum_{l=1}^{d_2}T_2^{j,l}(x_n,y_n)I_{y^l}(x_n,y_n)\Big]\\
&+\frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_1}\sum_{j=1}^{d_1}T_3^{k,i}(x_n,y_n)I_{x^ix^j}(x_n,y_n)\Big[\sum_{l=1}^{d_1}T_1^{j,l}(x_n,y_n)I_{x^l}(x_n,y_n)\\
&+\sum_{l=1}^{d_2}T_2^{j,l}(x_n,y_n)I_{y^l}(x_n,y_n)\Big]\\
&+\frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_1}\sum_{j=1}^{d_2}T_{3y^j}^{k,i}(x_n,y_n)I_{x^i}(x_n,y_n)\Big[\sum_{l=1}^{d_1}T_3^{j,l}(x_n,y_n)I_{x^l}(x_n,y_n)\\
&+\sum_{l=1}^{d_2}T_4^{j,l}(x_n,y_n)I_{y^l}(x_n,y_n)\Big]\\
&+\frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_1}\sum_{j=1}^{d_2}T_3^{k,i}(x_n,y_n)I_{x^iy^j}(x_n,y_n)\Big[\sum_{l=1}^{d_1}T_3^{j,l}(x_n,y_n)I_{x^l}(x_n,y_n)\\
&+\sum_{l=1}^{d_2}T_4^{j,l}(x_n,y_n)I_{y^l}(x_n,y_n)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_2}\sum_{j=1}^{d_1}T_{4x^j}^{k,i}(x_n,y_n)I_{y^i}(x_n,y_n)\Big[\sum_{l=1}^{d_1}T_1^{j,l}(x_n,y_n)I_{x^l}(x_n,y_n)\\
&+\sum_{l=1}^{d_2}T_2^{j,l}(x_n,y_n)I_{y^l}(x_n,y_n)\Big]\\
&+\frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_2}\sum_{j=1}^{d_1}T_4^{k,i}(x_n,y_n)I_{y^ix^j}(x_n,y_n)\Big[\sum_{l=1}^{d_1}T_1^{j,l}(x_n,y_n)I_{x^l}(x_n,y_n)\\
&+\sum_{l=1}^{d_2}T_2^{j,l}(x_n,y_n)I_{y^l}(x_n,y_n)\Big]\\
&+\frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_2}\sum_{j=1}^{d_2}T_{4y^j}^{k,i}(x_n,y_n)I_{y^i}(x_n,y_n)\Big[\sum_{l=1}^{d_1}T_3^{j,l}(x_n,y_n)I_{x^l}(x_n,y_n)\\
&+\sum_{l=1}^{d_2}T_4^{j,l}(x_n,y_n)I_{y^l}(x_n,y_n)\Big]\\
&+\frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_2}\sum_{j=1}^{d_2}T_4^{k,i}(x_n,y_n)I_{y^iy^j}(x_n,y_n)\Big[\sum_{l=1}^{d_1}T_3^{j,l}(x_n,y_n)I_{x^l}(x_n,y_n)\\
&+\sum_{l=1}^{d_2}T_4^{j,l}(x_n,y_n)I_{y^l}(x_n,y_n)\Big]+\tilde{R}_2,\ \ k=1,\ldots,d_2,
\end{aligned}\tag{3.12}
$$

with

$$
\begin{aligned}
\tilde{R}_2=&\int_{t_n}^{t_n+h}\int_{t_n}^{t_n+t}\sum_{i=1}^{d_1}\sum_{j=1}^{d_1}(S_{3x^j}^{k,i}I_{x^i}+S_3^{k,i}I_{x^ix^j})(\sum_{l=1}^{d_1}T_1^{j,l}I_{x^l}+\sum_{l=1}^{d_2}T_2^{j,l}I_{y^l})\circ dW(s)dt\\
&+\int_{t_n}^{t_n+h}\int_{t_n}^{t_n+t}\sum_{i=1}^{d_1}\sum_{j=1}^{d_2}(S_{3y^j}^{k,i}I_{x^i}+S_3^{k,i}I_{x^iy^j})(\sum_{l=1}^{d_1}T_3^{j,l}I_{x^l}+\sum_{l=1}^{d_2}T_4^{j,l}I_{y^l})\circ dW(s)dt\\
&+\int_{t_n}^{t_n+h}\int_{t_n}^{t_n+t}\sum_{i=1}^{d_2}\sum_{j=1}^{d_1}(S_{4x^j}^{k,i}I_{y^i}+S_4^{k,i}I_{y^ix^j})(\sum_{l=1}^{d_1}T_1^{j,l}I_{x^l}+\sum_{l=1}^{d_2}T_2^{j,l}I_{y^l})\circ dW(s)dt\\
&+\cdots,
\end{aligned}
$$

where $|E\tilde{R}_2|=O(h^2)$, $(E\tilde{R}_2^2)^{1/2}=O(h^{3/2})$.

Comparing (3.8) with (3.11), (3.10) with (3.12), respectively, we derive that

$$
\begin{aligned}
&|E(x^k(t_n+h)-x_{n+1}^k)|=O(h^2),\ \ k=1,\ldots,d_1,\\
&(E(x^k(t_n+h)-x_{n+1}^k)^2)^{\frac{1}{2}}=O(h^{\frac{3}{2}}),\ \ k=1,\ldots,d_1,\\
&|E(y^k(t_n+h)-y_{n+1}^k)|=O(h^2),\ \ k=1,\ldots,d_2,\\
&(E(y^k(t_n+h)-y_{n+1}^k)^2)^{\frac{1}{2}}=O(h^{\frac{3}{2}}),\ \ k=1,\ldots,d_2.
\end{aligned}
$$

Therefore, the SPAVF method (2.5) has mean-square convergence order 1 according to [21]. □

Remark 3.1. If we replace $S(x_n,y_n)$ by $S(x_{n+1},y_{n+1})$ or $S\big((x_n+x_{n+1})/2,(y_n+$

$y_{n+1})/2\big)$ in the method (2.5), the method is still convergent with mean-square order 1.

Remark 3.2. Although the method (2.2) can preserve the conserved quantity I, following the way in proving Theorem 3.1, we get that

$$\begin{aligned} x^k(t_n+h)-x_{n+1}^k =&\frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_1}\sum_{j=1}^{d_2}T_1^{k,i}I_{x^iy^j}\left[\sum_{l=1}^{d_1}T_3^{j,l}I_{x^l}+\sum_{l=1}^{d_2}T_4^{j,l}I_{y^l}\right]\\ &-\frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_2}\sum_{j=1}^{d_1}T_2^{k,i}I_{y^ix^j}\left[\sum_{l=1}^{d_1}T_1^{j,l}I_{x^l}+\sum_{l=1}^{d_2}T_2^{j,l}I_{y^l}\right]+\cdots, \end{aligned}$$

$$\begin{aligned} y^k(t_n+h)-y_{n+1}^k =&\frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_1}\sum_{j=1}^{d_2}T_3^{k,i}I_{x^iy^j}\left[\sum_{l=1}^{d_1}T_3^{j,l}I_{x^l}+\sum_{l=1}^{d_2}T_4^{j,l}I_{y^l}\right]\\ &-\frac{1}{2}\Delta W^2(h)\sum_{i=1}^{d_2}\sum_{j=1}^{d_1}T_4^{k,i}I_{y^ix^j}\left[\sum_{l=1}^{d_1}T_1^{j,l}I_{x^l}+\sum_{l=1}^{d_2}T_2^{j,l}I_{y^l}\right]+\cdots, \end{aligned}$$

so the method (2.2) is not convergent for solving (2.1) generally. Similarly, the method (2.3) is not convergent generally either.

For a more general case of stochastic partitioned system which contains s partitions as following

$$\begin{aligned} \begin{pmatrix} dx^1\\ dx^2\\ \vdots\\ dx^s \end{pmatrix} =&S(x^1,x^2,\ldots,x^s)\begin{pmatrix} I_{x^1}(x^1,x^2,\ldots,x^s)\\ I_{x^2}(x^1,x^2,\ldots,x^s)\\ \vdots\\ I_{x^s}(x^1,x^2,\ldots,x^s) \end{pmatrix}dt\\ &+T(x^1,x^2,\ldots,x^s)\begin{pmatrix} I_{x^1}(x^1,x^2,\ldots,x^s)\\ I_{x^2}(x^1,x^2,\ldots,x^s)\\ \vdots\\ I_{x^s}(x^1,x^2,\ldots,x^s) \end{pmatrix}\circ dW(t), \end{aligned} \tag{3.13}$$

where I is a conserved quantity, S and T are skew-symmetry matrices, $x^i\in\mathbb{R}^{d_i},\ i=1,\ldots,s$. Similar with (2.2) and (2.3), we can define two numerical methods as

$$\begin{pmatrix} x_{n+1}^1\\ x_{n+1}^2\\ \vdots\\ x_{n+1}^s \end{pmatrix}=\begin{pmatrix} x_n^1\\ x_n^2\\ \vdots\\ x_n^s \end{pmatrix}+hS_n\begin{pmatrix} \int_0^1 I_{x^1}(\xi x_{n+1}^1+(1-\xi)x_n^1,x_n^2,\ldots,x_n^s)d\xi\\ \int_0^1 I_{x^2}(x_{n+1}^1,\xi x_{n+1}^2+(1-\xi)x_n^2,\ldots,x_n^s)d\xi\\ \vdots\\ \int_0^1 I_{x^s}(x_{n+1}^1,x_{n+1}^2,\ldots,\xi x_{n+1}^s+(1-\xi)x_n^s)d\xi \end{pmatrix}$$

$$+\Delta W(h)T_{\frac{n+(n+1)}{2}}\begin{pmatrix}\int_0^1 I_{x^1}(\xi x_{n+1}^1+(1-\xi)x_n^1,x_n^2,\dots,x_n^s)d\xi\\ \int_0^1 I_{x^2}(x_{n+1}^1,\xi x_{n+1}^2+(1-\xi)x_n^2,\dots,x_n^s)d\xi\\ \vdots\\ \int_0^1 I_{x^s}(x_{n+1}^1,x_{n+1}^2,\dots,\xi x_{n+1}^s+(1-\xi)x_n^s)d\xi\end{pmatrix}, \tag{3.14}$$

and

$$\begin{pmatrix}x_{n+1}^1\\x_{n+1}^2\\\vdots\\x_{n+1}^s\end{pmatrix}=\begin{pmatrix}x_n^1\\x_n^2\\\vdots\\x_n^s\end{pmatrix}+hS_n\begin{pmatrix}\int_0^1 I_{x^1}(\xi x_{n+1}^1+(1-\xi)x_n^1,x_{n+1}^2,\dots,x_{n+1}^s)d\xi\\ \int_0^1 I_{x^2}(x_n^1,\xi x_{n+1}^2+(1-\xi)x_n^2,\dots,x_{n+1}^s)d\xi\\ \vdots\\ \int_0^1 I_{x^s}(x_n^1,x_n^2,\dots,\xi x_{n+1}^s+(1-\xi)x_n^s)d\xi\end{pmatrix}$$
$$+\Delta W(h)T_{\frac{n+(n+1)}{2}}\begin{pmatrix}\int_0^1 I_{x^1}(\xi x_{n+1}^1+(1-\xi)x_n^1,x_{n+1}^2,\dots,x_{n+1}^s)d\xi\\ \int_0^1 I_{x^2}(x_n^1,\xi x_{n+1}^2+(1-\xi)x_n^2,\dots,x_{n+1}^s)d\xi\\ \vdots\\ \int_0^1 I_{x^s}(x_n^1,x_n^2,\dots,\xi x_{n+1}^s+(1-\xi)x_n^s)d\xi\end{pmatrix}, \tag{3.15}$$

where S_n and $T_{\frac{n+(n+1)}{2}}$ are $S(x_n^1,\dots,x_n^s)$ and $T\big((x_n^1+x_{n+1}^1)/2,\dots,(x_n^s+x_{n+1}^s)/2\big)$ for short, respectively. Then we define the corresponding SPAVF method by

$$\begin{pmatrix}x_{n+1}^1\\x_{n+1}^2\\\vdots\\x_{n+1}^s\end{pmatrix}=\begin{pmatrix}x_n^1\\x_n^2\\\vdots\\x_n^s\end{pmatrix}+\frac{1}{2}hS_n\begin{pmatrix}f_n^1+g_n^1\\f_n^2+g_n^2\\\vdots\\f_n^s+g_n^s\end{pmatrix}+\frac{1}{2}\Delta W(h)T_{\frac{n+(n+1)}{2}}\begin{pmatrix}f_n^1+g_n^1\\f_n^2+g_n^2\\\vdots\\f_n^s+g_n^s\end{pmatrix}, \tag{3.16}$$

with

$$\begin{pmatrix}f_n^1\\f_n^2\\\vdots\\f_n^s\end{pmatrix}=\begin{pmatrix}\int_0^1 I_{x^1}(\xi x_{n+1}^1+(1-\xi)x_n^1,x_n^2,\dots,x_n^s)d\xi\\ \int_0^1 I_{x^2}(x_{n+1}^1,\xi x_{n+1}^2+(1-\xi)x_n^2,\dots,x_n^s)d\xi\\ \vdots\\ \int_0^1 I_{x^s}(x_{n+1}^1,x_{n+1}^2,\dots,\xi x_{n+1}^s+(1-\xi)x_n^s)d\xi\end{pmatrix},$$
$$\begin{pmatrix}g_n^1\\g_n^2\\\vdots\\g_n^s\end{pmatrix}=\begin{pmatrix}\int_0^1 I_{x^1}(\xi x_{n+1}^1+(1-\xi)x_n^1,x_{n+1}^2,\dots,x_{n+1}^s)d\xi\\ \int_0^1 I_{x^2}(x_n^1,\xi x_{n+1}^2+(1-\xi)x_n^2,\dots,x_{n+1}^s)d\xi\\ \vdots\\ \int_0^1 I_{x^s}(x_n^1,x_n^2,\dots,\xi x_{n+1}^s+(1-\xi)x_n^s)d\xi\end{pmatrix}.$$

It is easy to prove the method (3.16) preserves the conserved quantity I of (3.13) along the way proving Theorem 2.2. Furthermore, similar with Theorem 3.1, we can derive that the SPAVF method (3.16) is of order 1 in the mean-square sense for solving (3.13) under certain conditions.

4. Numerical examples

In this section, we will employ the proposed SPAVF methods to solve several common stochastic systems with a conserved quantity. Three examples are given below to demonstrate the effectiveness of the SPAVF methods in preserving the conserved quantity and the convergence order.

Example 4.1.

Consider the Kubo stochastic oscillator

$$\begin{cases} dp(t) = -aq(t)dt - bq(t) \circ dW(t),\ t \in [0,T], \\ dq(t) = ap(t)dt + bp(t) \circ dW(t),\ t \in [0,T], \end{cases} \tag{4.1}$$

which can be rewritten in the form of (2.1) as

$$\begin{pmatrix} dp(t) \\ dq(t) \end{pmatrix} = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \begin{pmatrix} p(t) \\ q(t) \end{pmatrix} dt + \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix} \begin{pmatrix} p(t) \\ q(t) \end{pmatrix} \circ dW(t),\ t \in [0,T], \tag{4.2}$$

where $I(p,q) = (p^2+q^2)/2$ is the conserved quantity. We use this example to demonstrate the convergence order of the proposed SPAVF method because compared to those equations whose exact solutions cannot be expressed explicitly, (4.1) has the following explicit exact solution

$$\begin{aligned} p(t) &= p_0 \cos(at + bW(t)) - q_0 \sin(at + bW(t)), \\ q(t) &= p_0 \sin(at + bW(t)) + q_0 \cos(at + bW(t)), \end{aligned}$$

where $p_0 = p(0),\ q_0 = q(0)$ are initial values, so that the convergence order results we derive are more convincing.

We employ the SPAVF method (2.5) to solve (4.2). Choose the initial values $p_0 = 0.5,\ q_0 = 0$ and the coefficients $a = 1,\ b = 0.5$. Figure 1 demonstrates the convergence rate of the SPAVF method (2.5) for solving (4.2), where we use 1000 independent sample paths, and for each path, the SPAVF method (2.5) is implemented with five different step sizes: $h = 2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}$. We calculate the mean-square errors at the terminal $T = 1$ by

$$\sqrt{\sum_{i=1}^{1000} (|p(1,\omega_i) - p_N(\omega_i)|^2 + |q(1,\omega_i) - q_N(\omega_i)|^2)/1000},$$

and show the results in a log-log plot in Figure 1. By comparing with the reference line with slope 1, we see the proposed SPAVF method (2.5) is of mean-square order 1. To demonstrate the long-term behavior of the proposed method, we set

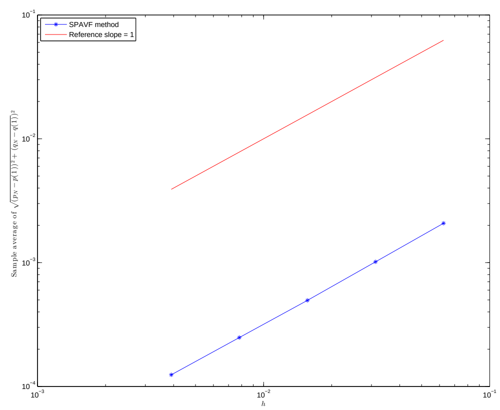

Figure 1. The convergence rate of the SPAVF method (2.5) for solving (4.2).

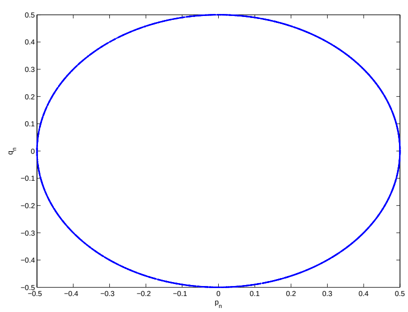

Figure 2. Phase space plot of the numerical solution computed by the SPAVF method (2.5) solving (4.2) with $h = 0.1$ on $[0, 500]$.

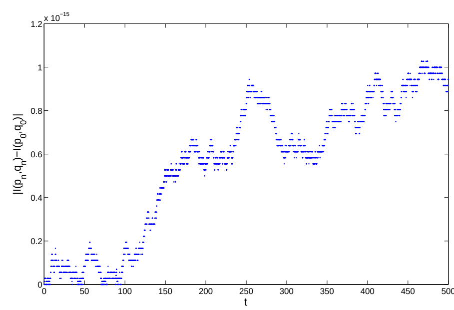

Figure 3. Errors $|I(p_n, q_n) - I(p_0, q_0)|$ computed by the SPAVF method (2.5) solving (4.2) with $h = 0.1$ on $[0, 500]$.

a long computational interval $[0, 500]$. Figure 2 reports the numerical solutions of a sample phase trajectory of (4.2) simulated by the SPAVF method (2.5) on the interval $[0, 500]$ with step size $h = 0.1$, from which we find the numerical solutions lie on the circle determined by the conserved quantity. Figure 3 exhibits the errors $|I(p_n, q_n) - I(p_0, q_0)|$ of the proposed SPAVF method (2.5) on the interval $[0, 500]$ with step size $h = 0.1$, which shows the SPAVF method (2.5) can preserve the conserved quantity $I(p, q)$ exactly.

Example 4.2.

This model [9,12] describes the dynamical behavior of the fluid system by using the dimensionless equations in the Stratonovich sense

$$\begin{cases} dr = vdt, \ t \geq 0, \\ dv = (F(r) - v)dt + (2\alpha\varepsilon)^{\frac{1}{2}} \circ dW(t), \ t \geq 0, \\ d\varepsilon = v^2 dt - v(2\alpha\varepsilon)^{\frac{1}{2}} \circ dW(t), \ t \geq 0, \end{cases} \tag{4.3}$$

where r denotes the position, v denotes the velocity, ε denotes the energy of the fluid system, $F(r) = -\partial V(r)/\partial r$ denotes the conservative force, and α denotes the dimensionless heat capacity of the fluid. The system (4.3) has a conserved quantity $E(r, v, \varepsilon) = V(r) + v^2/2 + \varepsilon$. Rewrite (4.3) in the form of (3.13) as

$$\begin{pmatrix} dr \\ dv \\ d\varepsilon \end{pmatrix} = S(r, v, \varepsilon) \begin{pmatrix} -F(r) \\ v \\ 1 \end{pmatrix} dt + T(r, v, \varepsilon) \begin{pmatrix} -F(r) \\ v \\ 1 \end{pmatrix} \circ dW(t), \ t \geq 0, \tag{4.4}$$

where

$$S(r, v, \varepsilon) = \begin{pmatrix} 0 & v^2 & v - v^3 \\ -v^2 & 0 & F(r) - v - v^2 F(r) \\ v^3 - v & v^2 F(r) - F(r) + v & 0 \end{pmatrix},$$

$$T(r, v, \varepsilon) = (2\alpha\varepsilon)^{\frac{1}{2}} \begin{pmatrix} 0 & -v & v^2 \\ v & 0 & 1 + vF(r) \\ -v^2 & -1 - vF(r) & 0 \end{pmatrix}.$$

Notice (4.3) is nonlinear, the conserved quantity E is not quadratic, and the skew-symmetric matrices S and T are not constant, so that this example is very different from Example 4.1. Based on the three-partition form (4.4), we apply the SPAVF method (3.16) to solving (4.4). The experiments are performed using the bistable potential $V(r) = \beta(r^4 - 2r^2)$ with the coefficients $\alpha = 1/4, \ \beta = 1$. The initial values are chosen as $r(0) = r_0 = 0, \ v(0) = v_0 = 0, \ \varepsilon(0) = \varepsilon_0 = 1$. Figure 4 reports the phase portrait by using the SPAVF method (3.16) to simulate a sample path on the long interval $[0, 500]$ with step size $h = 0.1$, which shows the numerical solutions lie on the manifold. Figure 5 reports the errors $|E(r_n, v_n, \varepsilon_n) - E(r_0, v_0, \varepsilon_0)|$ computed by the SPAVF method (3.16) on the interval $[0, 500]$ with step size $h = 0.1$, which indicates the SPAVF method (3.16) can preserve the conserved quantity E exactly.

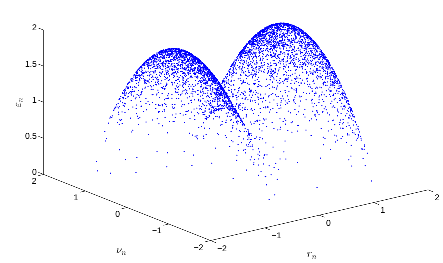

Figure 4. Phase space plot of the numerical solution computed by the SPAVF method (3.16) solving (4.4) with $h = 0.1$ on $[0, 500]$.

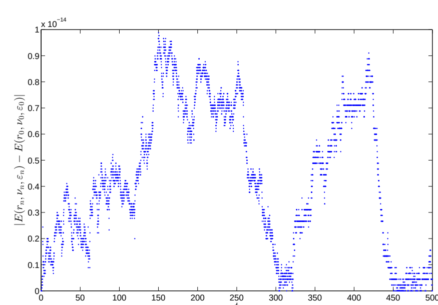

Figure 5. Errors $|E(r_n, v_n, \varepsilon_n) - E(r_0, v_0, \varepsilon_0)|$ computed by the SPAVF method (3.16) solving (4.4) with $h = 0.1$ on $[0, 500]$.

Example 4.3.

Consider the Hénon-Heiles system with perturbation as

$$dx = J\nabla H(x)dt + \sigma J\nabla H(x)\circ dW(t),\ t\geq 0, \tag{4.5}$$

where

$$x=(q_1,q_2,p_1,p_2)^T,\ J=\begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix},$$

I_2 is a 2×2 identity matrix, $H(x)=(q_1^2+q_2^2+p_1^2+p_2^2)/2+q_1^2q_2-q_2^3/3$ is the conserved quantity. This model describes stellar motion with perturbation inside the gravitational potential of a galaxy. Notice the equation (4.5) is also nonlinear and the conserved quantity H is not quadratic. We rewrite (4.5) in the following form with four partitions

$$\begin{pmatrix} dq_1 \\ dq_2 \\ dp_1 \\ dp_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} \nabla H_{q_1} \\ \nabla H_{q_2} \\ \nabla H_{p_1} \\ \nabla H_{p_2} \end{pmatrix} dt + \begin{pmatrix} 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & \sigma \\ -\sigma & 0 & 0 & 0 \\ 0 & -\sigma & 0 & 0 \end{pmatrix}\begin{pmatrix} \nabla H_{q_1} \\ \nabla H_{q_2} \\ \nabla H_{p_1} \\ \nabla H_{p_2} \end{pmatrix}\circ dW(t),\ t\geq 0. \tag{4.6}$$

Employ the SPAVF method (3.16) to solve (4.6). Mention that here we rewrite (4.5) in a four-partition form but not a two-partition form (i.e., take (q_1,q_2) as a partition and (p_1,p_2) as a partition) because our choice leads to a easier iterative scheme, which shows the flexibility of the proposed SPAVF method through dividing variables into different groups. Choose the coefficient $\sigma=0.5$, the initial values $q_1(0)=q_{1,0}=0.1$, $q_2(0)=q_{2,0}=0.5$, $p_2(0)=p_{2,0}=0$, while $p_1(0)=p_{1,0}$ is determined by $H_0=(q_{1,0}^2+q_{2,0}^2+p_{1,0}^2+p_{2,0}^2)/2+q_{1,0}^2q_{2,0}-q_{2,0}^3/3$ and $H_0=1/6$. Figure 6 reports the phase portrait by using the SPAVF method (3.16) to simulate a sample path on the interval $[0,500]$ with step size $h=0.1$, which shows the numerical solutions lie on the triangle manifold. Figure 7 reports the errors $|H(q_{1,n},q_{2,n},p_{1,n},p_{2,n})-H(q_{1,0},q_{2,0},p_{1,0},p_{2,0})|$ computed by the SPAVF method (3.16) on the interval $[0,500]$ with step size $h=0.1$, where we can see the proposed SPAVF method (3.16) has good performance in preserving the conserved quantity.

We mention that it seems the examples above don't satisfy the hypotheses on the boundedness in Theorem 3.1. In fact, the hypotheses on the boundedness can be relaxed on the invariant manifold, see [7] for more details. Local boundedness is sufficient to get the mean-square order 1 thanks to the conservative property of the proposed SPAVF method.

Remark 4.1. Since the SPAVF methods (2.5) and (3.16) exactly preserve the conserved quantity, one can find, for problems as presented in this section, for any given initial value (x_0,y_0), a convex subset of the phase space containing almost surely the numerical trajectories starting from (x_0,y_0) on which the functions S, T and their derivatives up to order 2 as well as the function I and its derivatives up to order 3 are bounded. Hence, the conclusion of Theorem 3.1 extends to these cases straightforwardly.

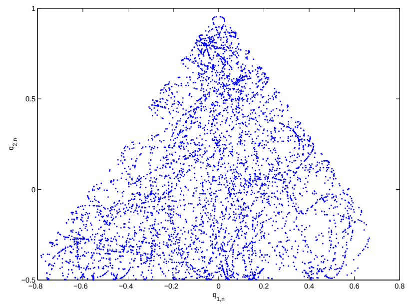

Figure 6. Phase space plot of the numerical solution computed by the SPAVF method (3.16) solving (4.6) with $h = 0.1$ on $[0, 500]$.

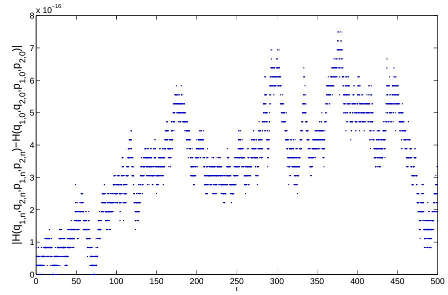

Figure 7. Errors $|H(q_{1,n}, q_{2,n}, p_{1,n}, p_{2,n}) - H(q_{1,0}, q_{2,0}, p_{1,0}, p_{2,0})|$ computed by the S-PAVF method (3.16) solving (4.6) with $h = 0.1$ on $[0, 500]$.

5. Conclusions

This work is an extension of the deterministic partitioned averaged vector field methods [4] into the stochastic counterpart. An SPAVF method for more general SDEs with a conserved quantity is proposed in this paper. We prove the SPAVF method can preserve the conserved quantity, then elaborately analyze the convergence order and derive the SPAVF method is convergent with mean-square order 1. In addition, as a partitioned method, the SPAVF method is flexible in the choices of variables grouping strategy, which could lead to more efficient schemes. Three examples of linear/nonlinear equations with a quadratic/non-quadratic conserved quantity are presented. Numerical experiments show the ability of the SPAVF method in preserving the conserved quantity, and verify the convergence order result as well as the flexibility of the proposed method.

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References

[1] L. Brugnano, F. Iavernaro and D. Trigiante, *Energy and quadratic invariants-preserving integrators based upon Gauss collocation formulae*, SIAM J. Numer. Anal., 2012, 50(6), 2897–2916.

[2] K. Burrage and P. M. Burrage, *High strong order explicit Runge-Kutta methods for stochastic ordinary differential equations*, Appl. Numer. Math., 1996, 22(1-3), 81–101.

[3] K. Burrage, P. M. Burrage and T. Tian, *Numerical methods for strong solutions of stochastic differential equations: an overview*, Proc. R. Soc. Lond. Ser. A, 2004, 460, 373–402.

[4] W. Cai, H. Li and Y. Wang, *Partitioned averaged vector field methods*, J. Comput. Phy., 2018, 370, 25–42.

[5] C. Chen, D. Cohen and J. Hong, *Conservative methods for stochastic differen-*

tial equations with a conserved quantity, Int. J. Numer. Anal. Mod., 2016, 13, 435–456.

[6] Y. Chen, Y. Sun and Y. Tang, *Energy-preserving numerical methods for Landau-Lifshitz equation*, J. Phys. A: Math. Theor., 2011, 44(29), 295207.

[7] D. Cohen, G. Dujardin, *Energy-preserving integrators for stochastic Poisson systems*, Commun. Math. Sci., 2014, 12(8), 1523–1539.

[8] X. Ding, Q. Ma and L. Zhang, *Convergence and stability of the split-step θ-method for stochastic differential equations*, Comput. Math. Appl., 2010, 60(5), 1310–1321.

[9] G. D. Fabritiis, M. Serrano, P. Español and P.V. Conency, *Efficient numerical integrators for stochastic models*, Phys. A, 2006, 361(2), 429–440.

[10] E. Hairer, C. Lubich and G. Wanner, *Geometric Numerical Integration*, Springer-Verlag, Berlin, 2006.

[11] J. Hong, D. Xu and P. Wang, *Preservation of quadratic invariants of stochastic differential equations via Runge-Kutta methods*, Appl. Numer. Math., 2015, 87, 38–52.

[12] J. Hong, S. Zhai and J. Zhang, *Discrete gradient approach to stochastic differential equations with a conserved quantity*, SIAM J. Numer. Anal., 2011, 49(5), 2017–2038.

[13] C. Huang, *Exponential mean square stability of numerical methods for systems of stochastic differential equations*, J. Comput. Appl. Math., 2012, 236(16), 4016–4026.

[14] P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*, Springer, Berlin, 1992.

[15] H. Li, Y. Wang and M. Qin, *A sixth order averaged vector field method*, J. Comput. Math., 2016, 34(5), 479–498.

[16] X. Li, C. Zhang, Q. Ma and X. Ding, *Discrete gradient methods and linear projection methods for preserving a conserved quantity of stochastic differential equations*, Int. J. Comput. Math., 2018, 95(12), 2511–2524.

[17] Q. Ma, *Structure-preserving numerical methods for several classes of stochastic differential equations*, PhD thesis, Harbin Institute of Technology, 2013.

[18] Q. Ma and X. Ding, *Stochastic symplectic partitioned Runge-Kutta methods for stochastic Hamiltonian systems with multiplicative noise*, Appl. Math. Comput., 2015, 252, 520–534.

[19] X. Mao, *Stochastic Differential Equations and Their Applications*, Horwood Publishing, Chichester, 1997.

[20] R. I. McLachlan, G. R. W. Quispel and N. Robidoux, *Geometric integration using discrete gradients*, Philos. Trans. Ser. A-Math. Phys. Eng. Sci., 1999, 357(1754), 1021–1045.

[21] G. N. Milstein, *Numerical Integration of Stochastic Differential Equations*, Kluwer Academic Publishers, Dordrecht, 1995.

[22] T. Misawa, *Energy conservative stochastic difference scheme for stochastic Hamilton dynamical systems*, Jpn. J. Ind. Appl. Math., 2000, 17(1), 119–128.

[23] G. R. W. Quispel and D. I. McLaren, *A new class of energy-preserving numerical integration methods*, J. Phys. A: Math. Theor., 2008, 41(4), 045206.

[24] G. R. W. Quispel and G. S. Turner, *Discrete gradient methods for solving ODEs numerically while preserving a first integral*, J. Phys. A: Math. Gen., 1996, 29(13), L341–L349.

[25] X. Wang and S. Gan, *The improved split-step backward Euler method for stochastic differential delay equations*, Int. J. Comput. Math., 2011, 88(11), 2359–2378.

[26] X. Wang, S. Gan and D. Wang, *A family of fully implicit Milstein methods for stiff stochastic differential equations with multiplicative noise*, BIT Numer. Math., 2012, 52(3), 741–772.

[27] A. Xiao and X. Tang, *High strong order stochastic Runge-Kutta methods for Stratonovich stochastic differential equations with scalar noise*, Numer. Algor., 2016, 72(2), 1–38.

[28] W. Zhou, L. Zhang, J. Hong and S. Song, *Projection methods for stochastic differential equations with conserved quantities*, BIT Numer. Math., 2016, 56(4), 1497–1518.