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GROUND STATE AND NODAL SOLUTIONS FOR A CLASS OF BIHARMONIC EQUATIONS WITH SINGULAR POTENTIALS*

Hongliang Liu[1], Qizhen Xiao[1,†], Hongxia Shi[2], Haibo Chen[3] and Zhisu Liu[1]

Abstract In this paper, we are concerned with a class of fourth order elliptic equations of Kirchhoff type with singular potentials in $\mathbb{R}^N$. The existence of ground state and nodal solutions are obtained by using variational methods and properties of Hessian matric. Furthermore, the "energy doubling" property of nodal solutions is also explored.

Keywords Biharmonic equation, ground state and nodal solution, variational methods, Hessian matric, energy doubling.

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1. Introduction and main results

In this paper, we are interested in the existence of ground state and nodal solutions for the following fourth order elliptic equations

$$\begin{cases} \Delta^2 u - \left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\Delta u + V(x)u = |u|^{p-2}u, \quad \text{in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $5 \leq N < 8$, $4 < p < 2_* = 2N/(N-4)$ with $N > 4$ is the Sobolev exponent, the paraments $a > 0$, $b \geq 0$ and the potential $V(x)$ satisfies the following condition:

(V) $V(x)$ is a continuous function and satisfies

$$V(r) + (\overline{\lambda} - \alpha)\frac{1}{r^4} \geq 0, \quad \lim_{r\to 0} r^4 V(r) = \lim_{r\to\infty} r^4 V(r) = +\infty,$$

where $\alpha > 0$ is a constant, $r = |x|$ and $\overline{\lambda} = [N^2(N-4)^2]/16$.

†the corresponding author. Email address:math_lhliang@163.com(H. Liu)
[1]School of Mathematics and Physics, University of South China, Hengyang, 421001, China
[2]School of Mathematics and Computational Science, Hunan First Normal University, Changsha, 410205, China
[3]School of Mathematics and Statistics, Central South University, Changsha, 410083, China
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The problem (1.1) or the more general one

$$\begin{cases} \Delta^2 u - \left(a + b\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\Delta u + V(x)u = f(x,u), \quad \text{in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N) \end{cases} \tag{1.2}$$

has a strong physical meaning. Indeed, replacing $\mathbb{R}^N$ by a bounded domain $\Omega \subset \mathbb{R}^N$ and let $V(x) = 0$, problem (1.2) becomes the following fourth order elliptic equation of Kirchhoff type Dirichlet problem

$$\begin{cases} \Delta^2 u - \left(a + b\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\Delta u = f(x,u), & \text{in } \Omega, \\ u = 0, \quad \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

which is related to the stationary analog of the evolution equation of Kirchhoff type

$$u_{tt} + \Delta^2 u - \left(a + b\int_{\Omega} |\nabla u|^2 dx\right)\Delta u = f(x,t), \tag{1.4}$$

where Δ^2 is the biharmonic operator and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous. Dimensions one and two are relevant from the point of view of physics and engineering becausse in those situations model (1.4) is considered as a good approximation describing nonlinear vibrations of beams or plates [1, 3].

Recently, there are many works on the existence of nontrivial solutions to these types of problems by using variational methods. For example, F. Wang et al. [16] obtained that the problem (1.3) has a nontrivial solution by using mountain pass techniques and the truncation method. For the problem (1.3) with $b = 0$, the authors in [11, 19] studied the existence of multiple nontrivial solutions by applying the mountain pass theorem and employing the Morse theory, respectively. In addition, the existence of infinitely many sign-changing solutions of the problem (1.3) with $b = 0$ was obtained in [20] via the sign-changing critical point theorem. On the other hand, F. Wang et al. [17] studied the positive solutions to the problem (1.2) by using variational methods and truncations methods. For the problem (1.2) with $b = 0$, Ye and Tang [18] considered the existence and multiplicity of solutions by applying the mountain pass theorem when the potential is positive. Their results unify and sharply improve the results of Liu, Chen and Wu [6]. Besides, the infinitely many solutions were obtained in [22] via the symmetric mountain pass theorem for the problem (1.2) with $b = 0$ and a sign-changing potential. For other interesting results on fourth order elliptic equations and Kirchhoff type equations, see, for example [4, 8, 12, 14, 22, 23] and references therein.

However, as far as we know, there seems no results on the problem (1.1) when $V(x)$ is a singular potential. Motivated by the above facts, the aim of this paper is to consider the existence of ground state and nodal solutions for problem (1.1). Moreover, the "energy doubling" property of nodal solution is explored in the present paper.

Before stating our main results, we first give some preliminaries.

As usual, we denote by $L^p(\mathbb{R}^N), 1 \le p < +\infty$ a Lebesgue space with the usual norm

$$\|u\|_p = \left(\int_{\mathbb{R}^N} |u|^p dx\right)^{\frac{1}{p}}$$

and denote by $D_0^{1,2}(\mathbb{R}^N)$ the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_D := \|u\|_{D_0^{1,2}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{\frac{1}{2}}.$$

Recall that Hardy-Rellich's inequality implies that for all $u \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |\Delta u|^2 dx \geq \overline{\lambda} \int_{\mathbb{R}^N} \frac{u^2}{r^4} dx, \tag{1.5}$$

where $\overline{\lambda}$ (defined in (V)) is the optimal constant and $\frac{1}{r^4}$ cannot be improved.

Let X be a weight Sobolev space which is defined as the subspace of the radially symmetric function in the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the inner product and norm

$$\langle u, v\rangle = \int_{\mathbb{R}^N} \left[\Delta u \Delta v + a\nabla u \nabla v + V(r)uv\right] dx, \quad \|u\| = \langle u, u\rangle^{\frac{1}{2}},$$

where $\int_{\mathbb{R}^N} V(r)u^2 dx < \infty$. Then X is a Hilbert space. Moreover, we have the following compactness result.

Lemma 1.1 (Theorem 1.1, [21])**.** *Under assumption (V), the embedding $X \hookrightarrow L^p(\mathbb{R}^N)$ is compact for $2 < p < 2_*$.*

Now we define a functional I on X by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} [|\Delta u|^2 + a|\nabla u|^2 + V(r)u^2]dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^2 - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p dx, \tag{1.6}$$

for all $u \in X$. Under assumption (V), it is easy to prove that the functional I is of class C^1. Consequently, the solutions of (1.1) are the critical points of I. Moreover, we have

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^N} [\Delta u \Delta v + a\nabla u \nabla v + V(r)uv]dx \tag{1.7}$$

$$+b\int_{\mathbb{R}^N} |\nabla u|^2 dx \int_{\mathbb{R}^N} \nabla u \nabla v dx - \int_{\mathbb{R}^N} |u|^{p-2}uv dx. \tag{1.8}$$

Notation. Throughout this paper, we denote $u^+ = \max\{u(x), 0\}$ and $u^- = \min\{u(x), 0\}$, then $u = u^+ + u^-$. For any $\rho > 0$ and for any $z \in \mathbb{R}^N$, $B_\rho(z)$ denotes the ball of radius ρ centered at z. C and C_i denote various positive constants, which may vary from line to line. $\mathcal{N}$ denotes the Nehari manifold

$$\mathcal{N} = \{u \in X \setminus \{0\} : \langle I'(u), u\rangle = 0\}. \tag{1.9}$$

Definition 1.1. If $u \in X$ is a ground state solution of problem (1.1) we mean that u is such a solution of (1.1) which has the least energy among all nontrivial solutions of (1.1) in X. If $u \in X$ is a solution of problem (1.1) with $u^\pm \neq 0$, then we call that u is a nodal solution of (1.1). Furthermore, if u is a nodal solution of problem (1.1) with $I(u) = \inf\{I(v) : v \text{ is the nodal solution of } (1.1)\}$, then we call that u is the least energy nodal solution of (1.1).

Now, we state our first main result as follows.

Theorem 1.1. *Let $5 \le N < 8$ and suppose that condition (V) is satisfied. Then problem (1.1) possesses a ground state solution $\bar{u}$ with $I(\bar{u}) = c = \inf_{u\in\mathcal{N}} I(u)$.*

To find nodal solutions for the problem (1.1), we define the set

$$\overline{\mathcal{N}}_{\pm} := \left\{u \in X : u^{\pm} \neq 0, \langle I'(u), u^{+}\rangle = 0, \langle I'(u), u^{-}\rangle = 0\right\}. \tag{1.10}$$

Then, the second main result of this paper is the following Theorem.

Theorem 1.2. *Let $5 \le N < 8$ and suppose that condition (V) is satisfied. Then problem (1.1) has a least energy nodal solution u with exactly two nodal domains, that is, there exist $t_*^+, t_*^- > 0$ such that $t_*^+u^+ + t_*^-u^- \in \overline{\mathcal{N}}_{\pm}$ and $\bar{c} = \inf_{u\in\overline{\mathcal{N}}_{\pm}} I(u) > 0$.*

The other purpose of the present paper is to study the "energy doubling" property of the nodal solutions of problem (1.1), that is, the energy of each nodal solution of (1.1) is larger than two times the energy of the ground state solution. Now, we give the third main result of this paper as follows.

Theorem 1.3. *Assume that all conditions of Theorem 1.2 hold. Then for all $v \in \overline{\mathcal{N}}_{\pm}$, there exists $\varepsilon > 0$ such that $I(v) \ge 2c + \varepsilon$. In particular, for each nodal solution u of problem (1.1) in X, we have $I(u) \ge 2c + \varepsilon$.*

Remark 1.1. The condition (V) is due to [21] in which the authors showed that the function $V(x)$ is a singular potential and $V(r) = \frac{|\log r|}{r^4}$ satisfies condition (V).

Here, we give the sketch of how to prove the main results. Followed the ideas of [13], we can verify that the Nehari manifold $\mathcal{N}$ is a natural constraint which ensures us to get the ground state solution. Moreover, inspired by [15], in which the authors investigated the "energy doubling" property of the nodal solutions of the Schrödinger equation $-\Delta u + u = |u|^{p-2}u$, in $\mathbb{R}^3$, we get some important properties on the set $\overline{\mathcal{N}}_{\pm}$. These properties allow us not only to obtain the nodal solution of the problem (1.1) but also to explore the "energy doubling" property of the nodal solutions. To the best of our knowledge, Theorems 1.1-1.3 seem to be the novel results on the problem (1.1) no matter in the entire space or in bounded domains.

The remainder of this paper is organized as follows. In sections $2-4$, we give the proofs of the main results.

2. Proof of Theorem 1.1.

We begin this section by introducing the following variate version of the saddle point theorem.

Lemma 2.1 (Lemma 2.1, [10])**.** *Let X be a Banach space and M_0 be a closed subspace of the metric space M and $\Gamma_0 \subset C(M_0, X)$. Define*

$$\Gamma = \{\gamma \in C(M, X) : \gamma|_{M_0} \in \Gamma_0\}.$$

If $J \in C^1(X, \mathbb{R})$ satisfies

$$\infty > b := \inf_{\gamma\in\Gamma}\sup_{t\in M} J(\gamma(t)) > a := \sup_{\gamma_0\in\Gamma_0}\sup_{t\in M_0} J(\gamma_0(t)),$$

then there exists a sequence $\{u_n\} \subset X$ satisfying

$$J(u_n) \to b, \quad \|J'(u_n)\|(1 + \|u_n\|) \to 0.$$

Lemma 2.2. *Let $5 \le N < 8$ and assume that the condition (V) holds. Then for any $u \in X \setminus \{0\}$, there exists $t(u) > 0$ such that $t(u)u \in \mathcal{N}$.*

Proof. Let $u \in X \setminus \{0\}$ be fixed and define the function $h(t) = I(tu)$ on $[0, \infty)$ as

$$h(t) = I(tu) = \frac{t^2}{2}\|u\|^2 + \frac{bt^4}{4}\|u\|_D^4 - \frac{t^p}{p}\|u\|_p^p.$$

Obviously, we have

$$h'(t) = 0 \Leftrightarrow tu \in \mathcal{N} \Leftrightarrow \|u\|^2 + bt^2\|u\|_D^4 = t^{p-2}\|u\|_p^p.$$

Noting that $5 \le N < 8$, it is easy to verify that $h(0) = 0$, $h(t) > 0$ for $t > 0$ small and $h(t) < 0$ for $t > 0$ large. Therefore, $\max_{t\in[0,\infty)} h(t)$ is achieved at a $t_0 = t(u)$ so that $h'(t_0) = 0$ and $t(u)u \in \mathcal{N}$. The proof is completed. □

Lemma 2.3. *Let $5 \le N < 8$ and assume that the condition (V) holds. Then for any $u \in \mathcal{N}$, there holds $I(u) \ge I(tu)$ for all $t \in [0, \infty)$.*

Proof. For $u \in \mathcal{N}$, it follows from (1.6) that

$$\begin{aligned} I(u) - I(tu) &= \left(\frac{1}{2} - \frac{t^2}{2}\right)\|u\|^2 + b\left(\frac{1}{4} - \frac{t^4}{4}\right)\|u\|_D^4 + \left(\frac{t^p}{p} - \frac{1}{p}\right)\|u\|_p^p \\ &= \left(\frac{1-t^2}{2} + \frac{t^{2_*}-1}{2_*}\right)\|u\|^2 + b\left(\frac{1-t^4}{4} + \frac{t^p-1}{p}\right)\|u\|_D^4. \end{aligned}$$

It is easy to verify that

$$\vartheta(t) := \frac{1-t^\kappa}{\kappa} + \frac{t^p-1}{p} \ge 0, \quad \forall t \in [0,\infty), \kappa = \{2,4\}.$$

Thus, we complete the proof. □

Now, we define

$$c_1 := \inf_{u\in\mathcal{N}} I(u), \quad c_2 := \inf_{u\in X\setminus\{0\}} \max_{t\ge 0} I(tu) \quad \text{and} \quad c := \inf_{\gamma\in\Gamma} \sup_{0\le t\le 1} I(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], X) | \gamma(0) = 0, I(\gamma(1)) < 0\}.$$

Lemma 2.4. *Let $5 \le N < 8$ and assume that the condition (V) holds. Then $c_1 = c_2 = c > 0$ and there exists a sequence $\{u_n\} \subset X$ satisfying*

$$I(u_n) \to c, \quad \|I'(u_n)\|(1 + \|u_n\|) \to 0. \tag{2.1}$$

Proof. By the definition of $\mathcal{N}$, we separate X into the following two components:

$$X^+ := \{u \in X | \langle I'(u), u\rangle > 0\} \cup \{0\} \quad \text{and} \quad X^- := \{u \in X | \langle I'(u), u\rangle < 0\}.$$

Noting that $5 \le N < 8$, it follows from (1.6) and (1.7) that

$$I(u) \ge \frac{1}{4}\langle I'(u), u\rangle, \quad u \in X.$$

This means that $I(u) \ge 0$ for all $u \in X^+$. Moreover, it follows from (1.6) and Sobolev embedding inequality that X^+ contains a small ball around the origin.

Thus, any $\gamma \in \Gamma$ has to cross $\mathcal{N}$ due to the fact that $\gamma(0) \in X^+$ and $\gamma(1) \in X^-$. Therefore, we obtain that $c_1 \leq c$.

On the other hand, from the definition of c_2, one can choose a sequence $\{u_n\} \subset X\backslash\{0\}$ such that

$$c_2 \leq \max_{t\geq 0} I(tu_n) < c_2 + \frac{1}{n}, \quad \forall n \in \mathbb{N}. \tag{2.2}$$

By Lemma 2.2 and Lemma 2.3, for $u \in X\backslash\{0\}$ and t large enough, we have $I(tu) < 0$. Then there exist $t_n = t(u_n) > 0$ and $s_n > t_n$ such that

$$I(t_n u_n) = \max_{t\geq 0} I(tu_n) \quad \text{and} \quad I(s_n u_n) < 0, \quad \forall n \in \mathbb{N}. \tag{2.3}$$

Let $\gamma_n(t) = ts_n u_n$, $t \in [0,1]$, then $\gamma_n \in \Gamma$. So, from (2.2) and (2.3), we have

$$\sup_{t\in[0,1]} I(\gamma_n(t)) = \max_{t\geq 0} I(tu_n) < c_2 + \frac{1}{n}, \quad \forall n \in \mathbb{N},$$

which implies that $c \leq c_2$.

Moreover, by Lemma 2.2 and Lemma 2.3 one can easily get that $c_1 = c_2$. Hence, combining the above arguments, we have that $c_1 = c_2 = c$.

In order to show the second part of this lemma, we apply Lemma 2.1 with $M = [0,1]$, $M_0 = \{0,1\}$ and

$$\Gamma_0 = \{\gamma_0 : M_0 \to X | \gamma_0(0) = 0, I(\gamma_0(1)) < 0\}.$$

From Lemma 2.2 and Lemma 2.3, it is easy to prove that there exists $r > 0$ such that

$$\max_{\|u\|\geq r} I(u) = 0, \qquad \inf_{\|u\|=r} I(u) > 0.$$

Thus, we have

$$c \geq \inf_{\|u\|=r} I(u) > 0 = \sup_{\gamma_0\in\Gamma_0} \sup_{t\in M_0} I(\gamma_0(t)).$$

So, by Lemma 2.1, there exists a sequence $\{u_n\} \subset X$ satisfy (2.1). We complete the proof. $\square$

Lemma 2.5. *Let $5 \leq N < 8$ and assume that the condition (V) holds. Then any sequence $\{u_n\}$ satisfying (2.1) is bounded in X. Moreover, $I(u)$ satisfies the $(PS)-$condition in X.*

Proof. Let $\{u_n\} \in X$ satisfying (2.1). Arguing by contradiction, we suppose that $\|u_n\| \to \infty$ as $n \to \infty$. Noting $5 \leq N < 8$, an immediate consequence of (1.6), (1.7) and (2.1) is that

$$\begin{aligned} c + o(1)\|u_n\| &= I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle \\ &= \frac{1}{4}\|u_n\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right)\|u_n\|_p^p \\ &\geq \frac{1}{4}\|u_n\|^2 \to \infty, \quad \text{as } n \to \infty. \end{aligned}$$

This is impossible. So, any sequence $\{u_n\}$ satisfying (2.1) is bounded in X. Therefore, there exist a subsequence $\{u_n\}$ (still denotes by $\{u_n\}$) and u_0 in X such that $u_n \rightharpoonup u_0$ in X. Then by Lemma 1.1, we have

$$u_n \to u_0, \quad \text{in } L^p(\mathbb{R}^N). \tag{2.4}$$

Now, we prove that $u_n \to u_0$ in X. Let $v_n = u_n - u_0$, then for n sufficiently large, it follows from (1.7) and (2.4) that

$$\begin{aligned} o(1) &= \langle I'(v_n), v_n \rangle \\ &= \|v_n\|^2 + b\|v_n\|_D^4 - \|v_n\|_p^p \\ &\geq \|v_n\|^2 - \|v_n\|_p^p \\ &= \|v_n\|^2 - o(1), \end{aligned}$$

which means that $v_n \to 0$ as $n \to \infty$. We conclude the lemma. □

Now we give the proof of Theorem 1.1. For any $u \in \mathcal{N}$, it follows from (1.7) and Lemma 1.1 that

$$\begin{aligned} 0 &= \langle I'(u), u \rangle \\ &= \|u\|^2 + b\|u\|_D^4 - \|u\|_p^p \\ &\geq \|u\|^2 - \|u\|_p^p \\ &\geq \|u\|^2 - C\|u\|^p. \end{aligned} \tag{2.5}$$

Recall that $u \neq 0$ whenever $u \in \mathcal{N}$, then (2.5) implies that

$$\|u\| \geq \left(\frac{1}{C}\right)^{\frac{1}{p-2}} > 0, \quad \text{for all } u \in \mathcal{N}.$$

Hence, any limit point of a sequence in the Nehari manifold $\mathcal{N}$ is not equal to zero.

Let $\{u_n\} \subset \mathcal{N}$ be such that $I(u_n) \to c_1$ as $n \to \infty$, where c_1 is defined in Lemma 2.4. Following almost the same procedures as the proofs of Lemma 2.5, we can show that $\{u_n\}$ is bounded in X and it has a convergent subsequence, strongly converging to $u_0 \in \mathcal{N}$. Thus, $I(u_0) = c_1$.

Set

$$\Psi(u) = \langle I'(u), u \rangle,$$

then for any $u \in \mathcal{N}$, we have

$$\langle \Psi'(u), u \rangle = (2-p)\|u\|^2 + b(4-p)\|u\|_D^4 < 0,$$

which means that $\mathcal{N}$ is a natural constraint. Therefore, $I'(u_0) = 0$. That is, u_0 is a ground state solution of problem (1.1). The proof is completed. □

3. Proof of Theorem 1.2.

For each $u \in \overline{\mathcal{N}}_\pm$ (defined in (1.10)), we denote

$$g^+(u) := I(u^+) + \frac{b}{4}\|u^+\|_D^2\|u^-\|_D^2,$$

$$g^-(u) := I(u^-) + \frac{b}{4}\|u^+\|_D^2\|u^-\|_D^2,$$

$$G^+(u) := \langle I'(u), u^+ \rangle = \langle I'(u^+), u^+ \rangle + b\|u^+\|_D^2\|u^-\|_D^2 = 0, \tag{3.1}$$

$$G^-(u) := \langle I'(u), u^- \rangle = \langle I'(u^-), u^- \rangle + b\|u^+\|_D^2\|u^-\|_D^2 = 0. \tag{3.2}$$

Then we have

$$g^+(u) = g^+(u) - \frac{1}{4}G^+(u) = \frac{1}{4}\|u^+\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right)\|u^+\|_p^p, \tag{3.3}$$

$$g^-(u) = g^-(u) - \frac{1}{4}G^-(u) = \frac{1}{4}\|u^-\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right)\|u^-\|_p^p. \tag{3.4}$$

$$I(u) = g^+(u) + g^-(u) = I(u) - \frac{1}{4}\langle I'(u), u\rangle. \tag{3.5}$$

Furthermore, borrowing the idea from [15], we have some properties on $\overline{\mathcal{N}}_\pm$ which are given by the following lemmas.

Lemma 3.1. *Let $5 \le N < 8$ and assume that condition (V) holds. Then for each $u \in X$ with $u^\pm \neq 0$, there exists a unique $(t_u, s_u) \in \mathbb{R} \times \mathbb{R}$ with $t_u, s_u > 0$ such that $t_u u^+ + s_u u^- \in \overline{\mathcal{N}}_\pm$ with*

$$I(t_u u^+ + s_u u^-) = \max\{I(tu^+ + su^-) : t, s \ge 0\},$$

and $H_{\beta^u}(t_u, s_u)$ is a negative definite matrix, where $H_{\beta^u}(t, s)$ is the Hessian matrix of $\beta^u(t, s) := I(tu^+ + su^-)$.

Proof. For $u \in X$ with $u^\pm \neq 0$, it can be deduced from the definition of $\beta^u(t, s)$ that

$$\begin{aligned}
\beta^u(t, s) &= I(tu^+ + su^-) \\
&= I(tu^+) + I(su^-) + \frac{b}{2}t^2 s^2 \|u^+\|_D^2 \|u^-\|_D^2 \\
&= \frac{1}{2}t^2\|u^+\|^2 + \frac{b}{4}t^4\|u^+\|_D^4 - \frac{1}{p}|t|^p\|u^+\|_p^p \\
&\quad + \frac{1}{2}s^2\|u^-\|^2 + \frac{b}{4}s^4\|u^-\|_D^4 - \frac{1}{p}|s|^p\|u^-\|_p^p + \frac{b}{2}t^2 s^2\|u^+\|_D^2\|u^-\|_D^2,
\end{aligned}$$

which implies that $\beta^u(t, s) > 0$ for $t, s > 0$ small and $\beta^u(t, s) \to -\infty$ as $|(t, s)| \to \infty$ since $5 \le N < 8$. Noting that $\beta^u(t, s) = \beta^u(|t|, |s|)$, then there exist $t_u, s_u \ge 0$ such that

$$\beta^u(t_u, s_u) = I(t_u u^+ + s_u u^-) = \max\{I(tu^+ + su^-) : t, s \ge 0\},$$

and then we have $G^\pm(t_u u^+ + s_u u^-) = 0$, which means that $t_u u^+ + s_u u^- \in \overline{\mathcal{N}}_\pm$.

Now, we show that $t_u > 0$ and $s_u > 0$. Without loss of generality, we may assume that $s_u = 0$ since $5 \le N < 8$. Then if $s > 0$ is small enough, we have

$$\begin{aligned}
\beta^u(t_u, 0) &\ge \beta^u(t_u, s) \\
&= \beta^u(t_u, 0) + \frac{s^2}{2}\|u^-\|^2 + b\frac{s^4}{4}\|u^-\|_D^4 - \frac{s^p}{p}\|u^-\|_p^p + \frac{b}{2}t^2 s^2\|u^+\|_D^2\|u^-\|_D^2 \\
&\ge \beta^u(t_u, 0) + \frac{s^2}{2}\|u^-\|^2 + b\frac{s^4}{4}\|u^-\|_D^4 - \frac{s^p}{p}\|u^-\|_p^p \\
&> \beta^u(t_u, 0).
\end{aligned}$$

Obviously, this is a contradiction. Therefore, $s_u > 0$. Similarly, we get $t_u > 0$.

Since $t_u u^+ + s_u u^- \in \overline{\mathcal{N}}_{\pm}$, a direct calculation shows that

$$H_{\beta^u}(t_u, s_u) = \begin{bmatrix} A & B \\ B & C \end{bmatrix},$$

where

$$A = 2bt_u^2 \|u^+\|_D^4 - (p-2)t_u^{p-2}\|u^+\|_p^p,$$
$$B = 2bt_u s_u \|u^+\|_D^2 \|u^-\|_D^2,$$
$$C = 2bs_u^2 \|u^-\|_D^4 - (p-2)s_u^{p-2}\|u^-\|_p^p.$$

It follows from $G^+(t_u u^+ + s_u u^-) = 0$ that

$$bt_u^2 \|u^+\|_D^4 - t_u^{p-2}\|u^+\|_p^p = -bs_u^2 \|u^+\|_D^2 \|u^-\|_D^2 - \|u^+\|^2 < 0. \tag{3.6}$$

Then

$$2bt_u^2 \|u^+\|_D^4 < 2t_u^{p-2}\|u^+\|_p^p < (p-2)t_u^{p-2}\|u^+\|_p^p, \tag{3.7}$$

since $5 \leq N < 8$. Similarly, we have

$$bs_u^2 \|u^-\|_D^4 - t_u^{p-2}\|u^-\|_p^p < 0 \tag{3.8}$$

and

$$2bs_u^2 \|u^-\|_D^4 < 2s_u^{p-2}\|u^-\|_p^p < (p-2)s_u^{p-2}\|u^-\|_p^p. \tag{3.9}$$

Thus, (3.7) and (3.9) mean that $A < 0$, $C < 0$. From (3.6) and (3.8), one can easily check that $\det H_{\beta^u}(t_u, s_u) > 0$, that is, $H_{\beta^u}(t_u, s_u)$ is a negative definite matrix.

Next, we shall verify the uniqueness of (t_u, s_u). Suppose, reasoning by contradiction, that there exists another $(\bar{t}_u, \bar{s}_u)$ with $\bar{t}_u > 0$ and $\bar{s}_u > 0$ such that $\bar{t}_u u^+ + \bar{s}_u u^- \in \overline{\mathcal{N}}_{\pm}$. Then we can prove that the Hessian matrix $H_{\beta^u}(\bar{t}_u, \bar{s}_u)$ is negative definite by the almost same procedures above. Therefore, by the properties of Hessian matrix, $(\bar{t}_u, \bar{s}_u)$ is a local maximum point of $\beta^u(t, s)$. Noting that (t_u, s_u) is a global maximum point, then we have $\beta^u(t_u, s_u) \geq \beta^u(\bar{t}_u, \bar{s}_u) > 0$.

Let

$$v^+ = \bar{t}_u u^+, \quad v^- = \bar{s}_u u^-, \quad \widetilde{t_u} = \frac{t_u}{\bar{t}_u}, \quad \widetilde{s_u} = \frac{s_u}{\bar{s}_u}. \tag{3.10}$$

Then $v = v^+ + v^- = \bar{t}_u u^+ + \bar{s}_u u^- \in \overline{\mathcal{N}}_{\pm}$ and $\widetilde{t}_u v^+ + \widetilde{s}_u v^- = t_u u^+ + s_u u^- \in \overline{\mathcal{N}}_{\pm}$. Moreover,

$$\beta^v(\widetilde{t}_u, \widetilde{s}_u) = \beta^u(t_u, s_u) \geq \beta^u(\bar{t}_u, \bar{s}_u) = \beta^v(1, 1). \tag{3.11}$$

Without loss of generality, we may assume that $\widetilde{t}_u \geq \widetilde{s}_u > 0$, then $G^+(v) = 0$ and $G^+(\widetilde{t}_u v^+ + \widetilde{s}_u v^-) = 0$. To be precise,

$$\|v^+\|_p^p = \|v^+\|^2 + b\|v^+\|_D^4 + b\|v^+\|_D^2 \|v^-\|_D^2 \tag{3.12}$$

and

$$\|v^+\|^2 + \widetilde{t}_u^2 b\|v^+\|_D^4 - \widetilde{t}_u^{p-2}\|v^+\|_p^p + b\widetilde{s}_u^2 \|v^+\|_D^2 \|v^-\|_D^2 = 0. \tag{3.13}$$

Noting $\widetilde{t}_u \geq \widetilde{s}_u > 0$ and $\|v\|_D^2 = \|v^+\|_D^2 + \|v^-\|_D^2$, one can easily get from (3.12) and (3.13) that

$$\left(1 - \widetilde{t}_u^{p-2}\right)\|v^+\|^2 + \left(\widetilde{t}_u^2 - \widetilde{t}_u^{p-2}\right) b\|v^+\|_D^2 \|v\|_D^2 \geq 0,$$

which means that $\widetilde{t}_u \leq 1$ since $5 \leq N < 8$ Therefore, $0 < \widetilde{s}_u \leq \widetilde{t}_u \leq 1$.

On the other hand, using (3.3), (3.4) and (3.5), we obtain that

$$\begin{aligned}
\beta^v(1,1) &= I(v) = g^+(v) + g^-(v) \\
&= g^+(\widetilde{t}_u v^+ + \widetilde{s}_u v^-) + g^-(\widetilde{t}_u v^+ + \widetilde{s}_u v^-) \\
&\quad + \frac{1-(\widetilde{t}_u)^2}{4}\|v^+\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right)(1-(\widetilde{t}_u)^p)\|v^+\|_p^p \\
&\quad + \frac{1-(\widetilde{s}_u)^2}{4}\|v^-\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right)(1-(\widetilde{s}_u)^p)\|v^-\|_p^p \\
&= \beta^v(\widetilde{t}_u, \widetilde{s}_u) + \frac{1-(\widetilde{t}_u)^2}{4}\|v^+\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right)(1-(\widetilde{t}_u)^p)\|v^+\|_p^p \\
&\quad + \frac{1-(\widetilde{s}_u)^2}{4}\|v^-\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right)(1-(\widetilde{s}_u)^p)\|v^-\|_p^p,
\end{aligned}$$

which together with (3.11) and $0 < \bar{s}_u \leq \bar{t}_u \leq 1$ implies that $\bar{s}_u = \bar{t}_u = 1$. Then, applying (3.10) yields that $t_u = \bar{t}_u$ and $s_u = \bar{s}_u$. i.e., (t_u, s_u) is unique. The proof is completed. □

Lemma 3.2. *For all $u \in \overline{\mathcal{N}}_\pm$, there exists $C > 0$ such that $\|u^\pm\|_p^p \geq C > 0$. Furthermore, $\bar{c} = \inf_{u \in \overline{\mathcal{N}}_\pm} I(u) > 0$.*

Proof. Suppose, reasoning contradiction, that there exists $\{u_n\} \subset \overline{\mathcal{N}}_\pm$ such that $\|u_n^+\|_p^p \to 0$ or $\|u_n^-\|_p^p \to 0$ as $n \to \infty$. Without loss of generality, we may assume that $\|u_n^+\|_p^p \to 0$ as $n \to \infty$. It follows from $G^+(u_n) = 0$ that $\|u_n\| \to 0$ as $n \to \infty$. On the other hand, by Lemma 1.1 and $G^+(u_n) = 0$ again, we have

$$\|u_n^+\|^2 + b\|u_n^+\|_D^4 + b\|u_n^+\|_D^2\|u_n^-\|_D^2 = \|u_n^+\|_p^p \leq C\|u_n^+\|^p.$$

That is

$$\|u_n^+\|^2 \leq \|u_n^+\|_p^p \leq C\|u_n^+\|^p,$$

which implies that there exists $C > 0$ such that $\|u_n^+\| \geq C$ since $5 \leq N < 8$. This is a contradiction with $\|u_n\| \to 0$ as $n \to \infty$. Hence, there exists $C > 0$ such that $\|u^\pm\|_p^p > C$ for all $u \in \overline{\mathcal{N}}_\pm$ and $\bar{c} > 0$ follows from (3.3), (3.4) and (3.5). We complete the proof. □

Lemma 3.3. *If there exists $u \in \overline{\mathcal{N}}_\pm$ such that $I(u) = \bar{c}$, where $\bar{c}$ is defined in Lemma 3.2, then u is a critical point of problem (1.1) with exactly two nodal domains.*

Proof. The proof of this lemma is almost same to the one of Lemma 2.5 in [7] see also in [2,5,9]. So, we omit it here. □

Remark 3.1. By a similar computation, $\mathcal{N}$ defined by (1.9) has the similar properties of $\overline{\mathcal{N}}_\pm$.

Now we give the proof of Theorem 1.2. Let $\{u_n\} \subset \overline{\mathcal{N}}_\pm$ be a sequence such that $I(u_n) \to \bar{c} > 0$ as $n \to \infty$. Going if necessary to a subsequence, we may assume that $I(u_n) \leq 2\bar{c}$ for all n. Thus, we have

$$2\bar{c} \geq I(u_n) = I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle = \frac{1}{4}\|u_n\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right)\|u_n\|_p^p,$$

which implies that

$$\|u_n\|^2 \leq 8\overline{c}, \quad \|u_n\|_p^p \leq \frac{8p}{p-4}\overline{c}.$$

By Lemma 3.2, there exists $C>0$ such that $\|u_n^\pm\| \geq C$ and $\|u_n^\pm\|_p^p \geq C$. Moreover, in view of Lemma 2.5 and Lemma 1.1, there exists $u \in X$ such that $u_n \rightharpoonup u$ and $u_n^\pm \rightharpoonup u^\pm$ in X as $n \to \infty$ and $u_n^\pm \to u^\pm$ in $L^p(\mathbb{R}^N)$ as $n \to \infty$. Then $u^\pm \neq 0$. It can be deduced from Lemma 3.1 that there exist $t_*^\pm > 0$ such that $t_*^+ u^+ + t_*^- u^- \in \overline{\mathcal{N}}_\pm$. Without loss of generality, we may assume that $t_*^+ \geq t_*^- > 0$. Noting that $\{u_n\} \subset \overline{\mathcal{N}}_\pm$, then $G^+(u_n) = 0$. Then by the weak lower semi-continuity of the norm, we have

$$\|u^+\|^2 + b\|u\|_D^2 \|u^+\|_D^2 \leq \|u^+\|_p^p. \tag{3.14}$$

On the other hand, it follows from $t_*^+ u^+ + t_*^- u^- \in \overline{\mathcal{N}}_\pm$ that

$$(t_*^+)^2\|u^+\|^2 + b(t_*^+)^4\|u^+\|_D^4 + b(t_*^+)^2(t_*^-)^2\|u^+\|_D^2\|u^-\|_D^2 = (t_*^+)^p\|u^+\|_p^p.$$

Furthermore, we have

$$(t_*^+)^{-2}\|u^+\|^2 + b\|u\|_D^2\|u^+\|_D^2 \geq (t_*^+)^{p-4}\|u^+\|_p^p \tag{3.15}$$

since $t_*^+ \geq t_*^- > 0$. Combining (3.14) with (3.15) yields that

$$\left(1 - \frac{1}{(t_*^+)^2}\right)\|u^+\|^2 \leq (1 - (t_*^+)^{p-4})\|u^+\|_p^p,$$

which implies that $t_*^+ \leq 1$. Therefore, $0 < t_*^- \leq t_*^+ \leq 1$. Utilizing (3.3), (3.4) and (3.5), we derive that

$$\begin{aligned}
\overline{c} &\leq I(t_*^+ u^+ + t_*^- u^-) \\
&= g^+(t_*^+ u^+ + t_*^- u^-) + g^-(t_*^+ u^+ + t_*^- u^-) \\
&\leq g^+(u) + g^-(u) \\
&\leq \lim_{n\to\infty} I(u_n) = \overline{c},
\end{aligned}$$

which shows that $I(t_*^+ u^+ + t_*^- u^-) = \overline{c}$. Therefore, by Lemma 3.3, we see that $t_*^+ u^+ + t_*^- u^- \in \overline{\mathcal{N}}_\pm$ is a critical point of I. The proof is completed. $\square$

4. Proof of Theorem 1.3

In view of Lemma 2.5, we know that any $(PS)-$sequence in X is bounded. To prove Theorem 1.3, arguing by contradiction, suppose that there exists $\{u_n\} \subset \overline{\mathcal{N}}_\pm$ satisfying (2.1) such that $I(u_n) \leq 2c + \frac{1}{n}$ for all n. By Lemma 1.1, up to a subsequence, we may assume that $u_n \rightharpoonup u$, $u_n^\pm \rightharpoonup u^\pm$ in X and $u_n^\pm \to u^\pm$ in $L^p(\mathbb{R}^N)$ for some $0 \neq u^\pm \in X$ as $n \to \infty$. Then by Lemma 3.2 and Lemma 1.1, there exists $C > 0$ such that

$$\lim_{n\to\infty} \|u_n^\pm\|_p^p = \|u^\pm\|_p^p \geq C, \quad \liminf_{n\to\infty} \|u_n^\pm\|^2 \geq C. \tag{4.1}$$

Since $0 \neq u^\pm \in X$ as $n \to \infty$, there is $C_1 > 0$ such that

$$\liminf_{n\to\infty} \|u_n^\pm\|_D^2 \geq \|u^\pm\|_D^2 \geq C_1.$$

Hence, for n large enough, (3.1) and (3.2) imply that

$$0 = \langle I'(u_n), u_n^{\pm}\rangle = \langle I'(u_n^{\pm}), u_n^{\pm}\rangle + b\|u_n^+\|_D^2\|u_n^-\|_D^2 \geq \langle I'(u_n^{\pm}), u_n^{\pm}\rangle + \frac{b}{2}C_1^2.$$

Using Lemma 3.1 and Remark 3.1, it should be clear that there exists $\{t_n^{\pm}\}$ such that $t_n^{\pm}u_n^{\pm} \in \mathcal{N}$ with $t_n^{\pm} > 0$. Then, we have

$$\begin{aligned} -\frac{b}{2}C_1^2 &\geq \langle I'(u_n^+), u_n^+\rangle \\ &= \langle I'(u_n^+), u_n^+\rangle - \frac{1}{(t_n^+)^4}\langle I'(t_n^+u_n^+), t_n^+u_n^+\rangle \\ &= \left(1 - \frac{1}{(t_n^+)^2}\right)\|u_n^+\|^2 + \left((t_n^+)^{p-4} - 1\right)\|u_n^+\|_p^p, \end{aligned}$$

which means that $t_n^+ \leq t_*^+ < 1$ holds for some $t_*^+ > 0$ and n large enough. Similarly, we can get that $t_n^- \leq t_*^- < 1$ and n large enough. Thus, by (3.3), (3.4), (3.5) and (4.1), for n large, we have

$$\begin{aligned} 2c + \frac{1}{n} &\geq I(u_n) = g^+(u_n) + g^-(u_n) \\ &= I(t_n^+u_n^+) + I(t_n^-u_n^-) + \frac{1-(t_n^+)^2}{4}\|u_n^+\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right)(1-(t_n^+)^p)\|u_n^+\|_p^p \\ &\quad + \frac{1-(t_n^-)^2}{4}\|u_n^-\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right)(1-(t_n^-)^p)\|u_n^-\|_p^p \\ &\geq 2c + \frac{1-(t_*^+)^2}{4}\|u_n^+\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right)(1-(t_*^+)^p)\|u_n^+\|_p^p + \frac{1-(t_*^-)^2}{4}\|u_n^-\|^2 \\ &\quad + \left(\frac{1}{4} - \frac{1}{p}\right)(1-(t_*^-)^p)\|u_n^-\|_p^p \\ &\geq 2c + C, \end{aligned}$$

which is a contradiction. Therefore, $I(u) \geq 2c + \varepsilon$. We complete the proof.

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