Journal of Applied Analysis and Computation
Volume 9, Number 3, June 2019, 1032–1052
Website:http://jaac.ijournal.cn
DOI:10.11948/2156-907X.20180248

DELAY-DEPENDENT STABILITY OF RUNGE-KUTTA METHODS FOR NEUTRAL SYSTEMS WITH DISTRIBUTED DELAYS*

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Abstract The aim of this paper is to analyze the asymptotic stability of Runge-Kutta (RK) methods for neutral systems with distributed delays. With an adaptation of the argument principle, some sufficient criteria for weak delay-dependent stability of numerical solutions are proposed. Several numerical examples are performed to confirm the effectiveness of our theoretical results.

Keywords Neutral systems with distributed delays, delay-dependent stability, Pouzet-type Runge-Kutta methods, argument principle.

MSC(2010) 65L05, 65L07, 65L20.

1. Introduction

Neutral delay-integro-differential equations (NDIDEs) have widely arisen in many scientific and technological fields, such as economics, biology, medicine, physics, engineering, control theory and so on (see, e.g., [8,15,17]). In the last several decades, the stability analysis of numerical methods for delay integro-differential equations (DIDEs) and neutral delay differential equations (NDDEs) has been investigated intensively by lots of researchers and a large number of results have been acquired (see, e.g., [1,2,11,16,18,22]). However, only a few results have been obtained as for the case of NDIDEs. Moreover, most of the results having no relationship with delays are referred to as delay-independent stability, others containing information on delays are called delay-dependent. As declared in [2,9,14,21,25], delay-dependent stability of numerical methods for delay differential systems is difficult to handle. One of the reasons is that the region of delay-independent stability is only a subset of the delay-dependent stability, and the latter gives a more complete description of the asymptotic behavior of numerical methods. Furthermore, it is proved in [2,20] that no A-stable natural RK methods for a system of delay differential equations (DDEs) is D-stable. That is, almost all the standard RK methods are not available in the sense of the D-stability. It indicates that the definition of D-stability is too rigid.

To avoid the shortage of D-stability of the RK methods for DDEs, more recently, Hu and Mitsui [13] introduced a new definition of delay-dependent stability for

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*The authors were supported by National Natural Science Foundation of China (11471217).

numerical methods called weak delay-dependent stability. In particular, the weak delay-dependent stability only requires the difference system yielded by a numerical method is asymptotically stable with a certain integer m giving in the step-size $h = \tau/m$. Based on this novel definition, some new sufficient conditions of delay-dependent stability of RK and linear mutistep (LM) methods for DDEs with neutral type are derived, respectively.

Stimulated by the work of [13] , in this paper, we deal with the weak delay-dependent stability of Pouzet-type Runge-Kutta (PRK) methods for the neutral systems with distributed delays described by

$$\begin{cases} \dot{x}(t) = Lx(t) + Mx(t-\tau) + N\dot{x}(t-\tau) \\ \qquad\qquad + \displaystyle\int_{-\tau}^{0} K(s)x(t+s)ds + \int_{-\tau}^{0} R(s)\dot{x}(t+s)ds, \qquad t > 0, \\ x(t) = \varphi(t), \qquad -\tau \leq t \leq 0 \end{cases} \tag{1.1}$$

with the condition

$$\|N\| + \int_{-\tau}^{0} \|R(s)\| ds \leq \alpha < 1, \tag{1.2}$$

where $x(t) \in R^d$, parameter matrices $L, M, N, K(s), R(s) \in R^{d\times d}$ and the constant delay $\tau > 0$. Here φ is a given C^1-function, the elements $k_{ij}(s)$ of the matrix $K(s)$ and the elements $r_{ij}(s)$ of the matrix $R(s)$ are continuous on $[-\tau, 0]$, respectively. Some sufficient criteria of weak delay-dependent stability for PRK methods will be obtained.

Notice that $\dot{x}(t)$ appears in the integral term on the right hand side of (1.1), which contains information of the solution on the interval $[t-\tau, t]$. It is worth pointing that Hu [12] derived asymptotic stability criteria, which are necessary and sufficient for analytical solutions of the neutral systems. Brunner [5] investigated the attainable order of local superconvergence of continuous Volterra-Runge-Kutta methods for the initial value problem of a class of neutral Volterra delay integro-differential equations. Then, Enright and Hu [7] studied convergence of explicit and implicit continuous-Runge-Kutta methods. Zhang and Qin [24] further introduced a type of mixed RK by combining the underlying RK methods and the compound quadrature rules for a class of nonlinear functional-integro-differential equations and derived a global stability criterion. Nevertheless, few investigation has been devoted to the delay-dependent stability of the corresponding numerical discretization for the neutral system (1.1).

The organization of the present paper is as follows. Several definitions and lemmas are reviewed in Section 2, which are crucial in the derivation of the stability results given in Section 3. In Section 3, some new sufficient criteria of weak delay-dependent stability for PRK methods are formulated. Numerical examples in Section 4 are provided to validate the effectiveness of the theoretical results, and some conclusions are presented.

2. Preliminaries

In this section, we recall several definitions and lemmas, which are available to prove the main results of the paper.

Throughout this paper, I_d stands for identity matrix with d-dimensions, $\|F\|$ is 2-norm of the matrix F. $\lambda_i(F)$ denotes the i^{th} eigenvalue of F . Res and Ims mean the real part and the imaginary part of a complex s. Furthermore, the open left half-plane $\{s : \text{Re}s < 0\}$ is denoted by C^- and the right half-plane $\{s : \text{Re}s \geq 0\}$ by C^+. The symbols $\otimes$ and $\circ$ stand for the Kronecker product and the Hadamard product, respectively.

The following well-known argument principle plays a major part in this paper.

Lemma 2.1 ([3]). *Suppose that*

(i) a function $P(s)$ is meromorphic in the domain interior to a positively oriented simple closed contour l;

(ii) $P(s)$ is analytic and nonzero on l;

(iii) counting multiplicities, Z is the number of zeros and Y is the number of poles of $P(s)$ inside l.

Then

$$\frac{1}{2\pi}\Delta_l \arg P(s) = Z - Y,$$

where $\Delta_l \arg P(s)$ means the change of the argument of $P(s)$ along l.

Definition 2.1 ([10]). For two matrices A and B with same dimension, the Hadamard product is a matrix of the same dimension as the operands, with elements given by

$$(A \circ B)_{i,j} = (A)_{i,j}(B)_{i,j}.$$

Definition 2.2 ([10]). If A is an $m \times n$ matrix and B is a $p \times q$ matrix, then the Kronecker procuct $A \otimes B$ is the $mp \times nq$ matrix:

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{pmatrix}.$$

It is known that the asymptotic stability of the system (1.1) is decided directly by the location of the roots of its characteristic equation

$$P(\lambda) = \det[\lambda I - L - Me^{-\tau\lambda} - \int_{-\tau}^{0} K(s)e^{s\lambda}ds - \lambda N e^{-\tau\lambda} - \lambda \int_{-\tau}^{0} R(s)e^{s\lambda}ds] = 0. \tag{2.1}$$

The system (1.1) satisfying (1.2) is asymptotically stable if the characteristic equation (2.1) has no roots in the right half-plane C^+ [15].

On the contrary, if there exist characteristic roots satisfying $\text{Re}\lambda \geq 0$, then they are located in a bounded semi-circular region. It is stated in the following Lemma, which is a special case of the conclusion in the literature [12].

Lemma 2.2 ([12]). *Suppose that condition (1.2) holds. Let λ be an unstable characteristic root of equation (2.1), then*

$$|\lambda| \leq \gamma := \frac{\beta}{1-\alpha},$$

where $\beta = \|L\| + \|M\| + \tau\|\widetilde{K}\|$. *Furthermore,* $\widetilde{K}$ *is a nonnegative constant matrix assumed to satisfy* $|K(s)| \le \widetilde{K}$, *i.e.,* $k_{ij}(s)$ *and* $\tilde{k}_{ij}$, *which are the entries of matrix* $K(s), \widetilde{K}$, *such that* $|k_{ij}(s)| \le \tilde{k}_{ij}$ *for all* $s \in [-\tau, 0], i, j = 1, 2, \ldots, d$.

Remark 2.1. By Lemma 2.2, we obtain that there exists a bounded semi-circular region in the right half complex plane C^+ which includes all the unstable characteristic roots of (2.1). We denote this semi-circular region as

$$D = \{\lambda : |\lambda| \le \gamma \quad \text{and} \quad \mathrm{Re}\lambda \ge 0\}$$

and the boundary of D as Γ.

Remark 2.2. In fact, in the practical computation in Section 4, we can choose $\alpha = \|N\| + \tau\|\widetilde{R}\|$, where the matrix $\widetilde{R}$ has the similar meaning with $\widetilde{K}$ in Lemma 2.2.

We proceed to utilize the asymptotic stability criteria established by Hu [12] for further work.

Lemma 2.3 ([12]). *The system (1.1) satisfying (1.2) is asymptotically stable if and only if*

$$P(s) \neq 0, \qquad s \in \Gamma$$

and

$$\triangle_\Gamma \arg P(s) = 0,$$

where $\triangle_\Gamma \arg P(s)$ *denotes the change of the argument of* $P(s)$ *along the closed semi-circumference* Γ, *and* Γ *is defined in Remark 2.1.*

3. Delay-dependent stability of Runge-Kutta methods

In this section, we are concerned with the delay-dependent stability of PRK methods (cf. Brunner, van der Houwen [4]). By means of the argument principle, sufficient conditions of the delay-dependent stability for the system (1.1) satisfying (1.2) are obtained.

Considering the initial value problem of ordinary differential equations (ODEs)

$$\dot{y}(t) = f(t, y(t)), \quad f: \quad I \times R^n \to R^n \quad (I = [t_0, T]), \tag{3.1}$$

an s-stage Runge-Kutta method for ODEs (3.1) is defined (e.g., [19]) by

$$y_{n+1} = y_n + h\sum_{i=1}^{s} b_i k_i \tag{3.2a}$$

with

$$k_i = f(t_n + c_i h, y_n + h\sum_{j=1}^{s} a_{ij} k_j) \quad (i = 1, 2, \ldots, s). \tag{3.2b}$$

We shall always assume that the following condition (the row-sum condition) holds:

$$c_i = \sum_{j=1}^{s} a_{ij} (i = 1, 2, \ldots, s),$$

where $a_{ij}, b_i, c_i (1 \le i, j \le s)$ are referred to as the parameters of an s-stage RK method (3.2) and c_i are supposed to satisfy $0 \le c_i \le 1$. Let $t_n = nh, n \in Z$, denote step points with a stepsize $h = \tau/m$ $(m \in N)$.

Now we extend the PRK methods to the system (1.1), which is first introduced by Zhang [23] to discuss numerical stability of a class of nonlinear functional integro-differential equations. For simplicity, we try to find a solution of (1.1) on the interval $0 \le t \le \tau N$ for some integer N. And we acquire the following numerical scheme: for $1 \le i \le s$,

$$X_{n,i} - NX_{n-m,i} - Z_{n,i} = x_n - Nx_{n-m} - z_n + h\sum_{j=1}^{s} a_{ij}(LX_{n,j} + MX_{n-m,j} + G_{n,j}), \tag{3.3}$$

$$\begin{aligned} Z_{n,i} = {} & h\sum_{j=1}^{s} a_{ij} R(t_n + c_j h - (t_n + c_i h))X_{n,j} \\ & + h\sum_{k=1}^{m}\sum_{j=1}^{s} b_j R(t_{n-k} + c_j h - (t_n + c_i h))X_{n-k,j} \\ & - h\sum_{j=1}^{s} a_{ij} R(t_{n-m} + c_j h - (t_n + c_i h))X_{n-m,j}, \end{aligned} \tag{3.4}$$

$$\begin{aligned} G_{n,i} = {} & h\sum_{j=1}^{s} a_{ij} K(t_n + c_j h - (t_n + c_i h))X_{n,j} \\ & + h\sum_{k=1}^{m}\sum_{j=1}^{s} b_j K(t_{n-k} + c_j h - (t_n + c_i h))X_{n-k,j} \\ & - h\sum_{j=1}^{s} a_{ij} K(t_{n-m} + c_j h - (t_n + c_i h))X_{n-m,j}, \end{aligned} \tag{3.5}$$

$$x_{n+1} - Nx_{n+1-m} - z_{n+1} = x_n - Nx_{n-m} - z_n + h\sum_{i=1}^{s} b_i(LX_{n,i} + MX_{n-m,i} + G_{n,i}), \tag{3.6}$$

and

$$z_{n+1} = h\sum_{k=1}^{m}\sum_{j=1}^{s} b_j R(t_{n+1-k} + c_j h - t_{n+1})X_{n+1-k,j}, n = 0, 1, \dots, mN - 1, \tag{3.7}$$

where $x_n, z_n, X_{n,i}, Z_{n,i}$ are approximations to $x(t_n), z(t_n), x(t_n + c_i h), z(t_n + c_i h)$, respectively, and $z(t)$ denotes the memory term

$$z(t) := \int_{t-\tau}^{t} R(s-t)x(s)ds.$$

Among the numerical scheme (3.3)–(3.7), the integral approximations $z_n, Z_{n,i}$ and $G_{n,i}$ are generated by Pouzet quadrature rules. We describe them in detail: each $Z_{n,i}$ gives an approximate value of the integral

$$\int_{t_n+c_i h-\tau}^{t_n+c_i h} R(s - (t_n + c_i h))x(s)ds = \int_{t_n}^{t_n+c_i h} R(s - (t_n + c_i h))x(s)ds$$

$$+\int_{t_{n-m}}^{t_n} R(s-(t_n+c_ih))x(s)ds - \int_{t_{n-m}}^{t_{n-m}+c_ih} R(s-(t_n+c_ih))x(s)ds.$$

Similarly, $G_{n,i}$ gives an approximate value of the integral

$$\int_{t_n+c_ih-\tau}^{t_n+c_ih} K(s-(t_n+c_ih))x(s)ds = \int_{t_n}^{t_n+c_ih} K(s-(t_n+c_ih))x(s)ds$$

$$+\int_{t_{n-m}}^{t_n} K(s-(t_n+c_ih))x(s)ds - \int_{t_{n-m}}^{t_{n-m}+c_ih} K(s-(t_n+c_ih))x(s)ds.$$

In order to simplify the notation and presentation, let

$$R_{j-i-k} := R(t_{n-k}+c_jh-(t_n+c_ih)),$$

$$K_{j-i-k} := K(t_{n-k}+c_jh-(t_n+c_ih)),$$

and

$$r_{j-k} := R(t_{n+1-k}+c_jh-t_{n+1}).$$

Therefore $Z_{n,i}$ and $G_{n,i}$ are presented with a simple form:

$$Z_{n,i} = h\sum_{j=1}^{s} a_{ij}R_{j-i}X_{n,j} + h\sum_{k=1}^{m}\sum_{j=1}^{s} b_jR_{j-i-k}X_{n-k,j} - h\sum_{j=1}^{s} a_{ij}R_{j-i-m}X_{n-m,j}, \tag{3.8}$$

$$G_{n,i} = h\sum_{j=1}^{s} a_{ij}K_{j-i}X_{n,j} + h\sum_{k=1}^{m}\sum_{j=1}^{s} b_jK_{j-i-k}X_{n-k,j} - h\sum_{j=1}^{s} a_{ij}K_{j-i-m}X_{n-m,j}. \tag{3.9}$$

Moreover, an initial condition is supposed to satisfy

$$X_{n,i} = \varphi(t_n+c_ih), \qquad -m \le n \le -1, \qquad x_0 = \varphi(t_0).$$

Now, we present the definition of weak delay-dependent stability of numerical methods for the system (1.1), which was first introduced by Hu and Mitsui [13].

Definition 3.1 ([13]). Suppose that the system (1.1) is asymptotically stable for given matrices $L, M, N, K(s), R(s)$ and a fixed delay τ. A numerical method is called weakly delay-dependently stable for the system (1.1) if there exists a positive integer m such that the the numerical solution x_n with step-size $h = \tau/m$ satisfies

$$\lim_{n\to\infty} x_n = 0$$

for any initial function.

Next we investigate the stability of the numerical solution of PRK methods in the context of the weak delay-dependent stability.

Lemma 3.1. *The characteristic polynomial of the PRK method* (3.3)–(3.7) *is given by*

$$P_{PRK}(\lambda) = \det\Phi(\lambda), \tag{3.10}$$

where $\Phi(\lambda)$ *represents the matrix:*

$$
\begin{aligned}
&\left\{\begin{pmatrix}
I_{sd}-h(A\otimes L) & -I_{sd} & -h(A\otimes I_d) & 0 & 0\\
-h(A\otimes \overline{I}_d)\circ\overline{R}_0 & I_{sd} & 0 & 0 & 0\\
-h(A\otimes \overline{I}_d)\circ\overline{K}_0 & 0 & I_{sd} & 0 & 0\\
-h(\mathbf{b}^T\otimes L) & 0 & -h(\mathbf{b}^T\otimes I_d) & I_d & -I_d\\
(\mathbf{b}^T\otimes \overline{I}_d)\circ\tilde{R}(1) & 0 & 0 & 0 & I_d
\end{pmatrix}\lambda^{m+1}\right.\\
&+\begin{pmatrix}
0 & 0 & 0 & -\mathbf{e}\otimes I_d & \mathbf{e}\otimes I_d\\
-h(\mathbf{b}^T\otimes\mathbf{e}\otimes \overline{I}_d)\circ\overline{R}_1 & 0 & 0 & 0 & 0\\
-h(\mathbf{b}^T\otimes\mathbf{e}\otimes \overline{I}_d)\circ\overline{K}_1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -I_d & I_d\\
(\mathbf{b}^T\otimes \overline{I}_d)\circ\tilde{R}(2) & 0 & 0 & 0 & 0
\end{pmatrix}\lambda^{m}\\
&+\begin{pmatrix}
0 & 0 & 0 & 0 & 0\\
-h(\mathbf{b}^T\otimes\mathbf{e}\otimes \overline{I}_d)\circ\overline{R}_2 & 0 & 0 & 0 & 0\\
-h(\mathbf{b}^T\otimes\mathbf{e}\otimes \overline{I}_d)\circ\overline{K}_2 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0\\
(\mathbf{b}^T\otimes \overline{I}_d)\circ\tilde{R}(3) & 0 & 0 & 0 & 0
\end{pmatrix}\lambda^{m-1}+\dots\\
&+\begin{pmatrix}
0 & 0 & 0 & 0 & 0\\
-h(\mathbf{b}^T\otimes\mathbf{e}\otimes \overline{I}_d)\circ\overline{R}_{m-1} & 0 & 0 & 0 & 0\\
-h(\mathbf{b}^T\otimes\mathbf{e}\otimes \overline{I}_d)\circ\overline{K}_{m-1} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0\\
(\mathbf{b}^T\otimes \overline{I}_d)\circ\tilde{R}(m) & 0 & 0 & 0 & 0
\end{pmatrix}\lambda^{2}\\
&+\begin{pmatrix}
-h(A\otimes M)-I_s\otimes N & 0 & 0 & 0 & 0\\
h[A\otimes \overline{I}_d)\circ\overline{R}_m-(\mathbf{b}^T\otimes\mathbf{e}\otimes \overline{I}_d)\circ\overline{R}_m] & 0 & 0 & 0 & 0\\
h[A\otimes \overline{I}_d)\circ\overline{K}_m-(\mathbf{b}^T\otimes\mathbf{e}\otimes \overline{I}_d)\circ\overline{K}_m] & 0 & 0 & 0 & 0\\
-h(\mathbf{b}^T\otimes M) & 0 & 0 & -N & 0\\
0 & 0 & 0 & 0 & 0
\end{pmatrix}\lambda\\
&+\left.\begin{pmatrix}
0 & 0 & 0 & \mathbf{e}\otimes N & 0\\
0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & N & 0\\
0 & 0 & 0 & 0 & 0
\end{pmatrix}\right\},
\end{aligned}
$$

where the s-dimensional vectors $\mathbf{e} = (1, 1, \ldots, 1)^T, \mathbf{b} = (b_1, b_2, \ldots, b_s)^T$, *s-dimensional matrix* A *is denoted by* $A := (a_{ij})$, $\overline{I}_d$ $\overline{K}_i$, $\overline{R}_i$ *and* $\tilde{R}$ *respectively stand for d-dimensional, (ds)-dimensional ,(ds)-dimensional and* $d \times ds$ *matrices for* $i = 1, 2, \ldots, m$,

$$\overline{K}_i = \begin{pmatrix} K_{1-1-i} & K_{2-1-i} & \ldots & K_{s-1-i} \\ K_{1-2-i} & K_{2-2-i} & \ldots & K_{s-2-i} \\ \vdots & \vdots & \ddots & \vdots \\ K_{1-s-i} & K_{2-s-i} & \ldots & K_{s-s-i} \end{pmatrix},$$

$$\overline{R}_i = \begin{pmatrix} R_{1-1-i} & R_{2-1-i} & \ldots & R_{s-1-i} \\ R_{1-2-i} & R_{2-2-i} & \ldots & R_{s-2-i} \\ \vdots & \vdots & \ddots & \vdots \\ R_{1-s-i} & R_{2-s-i} & \ldots & R_{s-s-i} \end{pmatrix},$$

$$\overline{I}_d = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{pmatrix},$$

$$\tilde{R}(i) = (r_{1-i}, r_{2-i}, \ldots, r_{s-i}).$$

Proof. By means of the Kronecker product and the Hadamard product, the difference system (3.3)–(3.7) can be equivalently rewritten as:

$$\begin{pmatrix} I_{sd} - h(A \otimes L) & -I_{sd} & -h(A \otimes I_d) & 0 & 0 \\ -h(A \otimes \overline{I}_d) \circ \overline{R}_0 & I_{sd} & 0 & 0 & 0 \\ -h(A \otimes \overline{I}_d) \circ \overline{K}_0 & 0 & I_{sd} & 0 & 0 \\ -h(\mathbf{b}^T \otimes L) & 0 & -h(\mathbf{b}^T \otimes I_d) & I_d & -I_d \\ (\mathbf{b}^T \otimes \overline{I}_d) \circ \tilde{R}(1) & 0 & 0 & 0 & I_d \end{pmatrix} \begin{pmatrix} X_n \\ Z_n \\ G_n \\ x_{n+1} \\ z_{n+1} \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & 0 & 0 & -\mathbf{e} \otimes I_d & \mathbf{e} \otimes I_d \\ -h(\mathbf{b}^T \otimes \mathbf{e} \otimes \overline{I}_d) \circ \overline{R}_1 & 0 & 0 & 0 & 0 \\ -h(\mathbf{b}^T \otimes \mathbf{e} \otimes \overline{I}_d) \circ \overline{K}_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_d & I_d \\ (\mathbf{b}^T \otimes \overline{I}_d) \circ \tilde{R}(2) & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_{n-1} \\ Z_{n-1} \\ G_{n-1} \\ x_n \\ z_n \end{pmatrix}$$

$$
+\begin{pmatrix}
0 & 0\,0\,0\,0 \\
-h(\mathbf{b}^T\otimes\mathbf{e}\otimes\overline{I}_d)\circ\overline{R}_2 & 0\,0\,0\,0 \\
-h(\mathbf{b}^T\otimes\mathbf{e}\otimes\overline{I}_d)\circ\overline{K}_2 & 0\,0\,0\,0 \\
0 & 0\,0\,0\,0 \\
(\mathbf{b}^T\otimes\overline{I}_d)\circ\tilde{R}(3) & 0\,0\,0\,0
\end{pmatrix}
\begin{pmatrix} X_{n-2} \\ Z_{n-2} \\ G_{n-2} \\ x_{n-1} \\ z_{n-1} \end{pmatrix}
$$

$$
+\dots+\begin{pmatrix}
0 & 0\,0\,0\,0 \\
-h(\mathbf{b}^T\otimes\mathbf{e}\otimes\overline{I}_d)\circ\overline{R}_{m-1} & 0\,0\,0\,0 \\
-h(\mathbf{b}^T\otimes\mathbf{e}\otimes\overline{I}_d)\circ\overline{K}_{m-1} & 0\,0\,0\,0 \\
0 & 0\,0\,0\,0 \\
(\mathbf{b}^T\otimes\overline{I}_d)\circ\tilde{R}(m) & 0\,0\,0\,0
\end{pmatrix}
\begin{pmatrix} X_{n-m+1} \\ Z_{n-m+1} \\ G_{n-m+1} \\ x_{n-m+2} \\ z_{n-m+2} \end{pmatrix}
$$

$$
+\begin{pmatrix}
-h(A\otimes M)-I_s\otimes N & 0 & 0 & 0 & 0 \\
h[A\otimes\overline{I}_d)\circ\overline{R}_m-(\mathbf{b}^T\otimes\mathbf{e}\otimes\overline{I}_d)\circ\overline{R}_m] & 0 & 0 & 0 & 0 \\
h[A\otimes\overline{I}_d)\circ\overline{K}_m-(\mathbf{b}^T\otimes\mathbf{e}\otimes\overline{I}_d)\circ\overline{K}_m] & 0 & 0 & 0 & 0 \\
-h(\mathbf{b}^T\otimes M) & 0 & 0 & -N & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix} X_{n-m} \\ Z_{n-m} \\ G_{n-m} \\ x_{n-m+1} \\ z_{n-m+1} \end{pmatrix}
$$

$$
+\begin{pmatrix}
0 & 0 & 0 & \mathbf{e}\otimes N & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & N & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix} X_{n-m-1} \\ Z_{n-m-1} \\ G_{n-m-1} \\ x_{n-m} \\ z_{n-m} \end{pmatrix} = 0, \quad (3.11)
$$

where the (ds)-dimensional vectors X_n, Z_n, G_n stand for

$$X_n = [X_{n,1}^T, X_{n,2}^T, \dots, X_{n,s}^T]^T,$$

$$Z_n = [Z_{n,1}^T, Z_{n,2}^T, \dots, Z_{n,s}^T]^T,$$

$$G_n = [G_{n,1}^T, G_{n,2}^T, \dots, G_{n,s}^T]^T.$$

Therefore, the dimension of the vector $\begin{bmatrix} X_n \\ Z_n \\ G_n \\ x_{n+1} \\ z_{n+1} \end{bmatrix}$ becomes $d(3s+2)$.

Using the Z-transform to (3.11) and introducing as $Z\left\{\begin{bmatrix} X_{n-m-1} \\ Z_{n-m-1} \\ G_{n-m-1} \\ x_{n-m} \\ z_{n-m} \end{bmatrix}\right\} = V(\lambda)$,

we acquire

$$
\left\{ \begin{pmatrix} I_{sd} - h(A \otimes L) & -I_{sd} & -h(A \otimes I_d) & 0 & 0 \\ -h(A \otimes \overline{I}_d) \circ \overline{R}_0 & I_{sd} & 0 & 0 & 0 \\ -h(A \otimes \overline{I}_d) \circ \overline{K}_0 & 0 & I_{sd} & 0 & 0 \\ -h(\mathbf{b}^T \otimes L) & 0 & -h(\mathbf{b}^T \otimes I_d) & I_d & -I_d \\ (\mathbf{b}^T \otimes \overline{I}_d) \circ \tilde{R}(1) & 0 & 0 & 0 & I_d \end{pmatrix} \lambda^{m+1} \right.
$$

$$
+ \begin{pmatrix} 0 & 0 & 0 & -\mathbf{e} \otimes I_d & \mathbf{e} \otimes I_d \\ -h(\mathbf{b}^T \otimes \mathbf{e} \otimes \overline{I}_d) \circ \overline{R}_1 & 0 & 0 & 0 & 0 \\ -h(\mathbf{b}^T \otimes \mathbf{e} \otimes \overline{I}_d) \circ \overline{K}_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_d & I_d \\ (\mathbf{b}^T \otimes \overline{I}_d) \circ \tilde{R}(2) & 0 & 0 & 0 & 0 \end{pmatrix} \lambda^{m}
$$

$$
+ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -h(\mathbf{b}^T \otimes \mathbf{e} \otimes \overline{I}_d) \circ \overline{R}_2 & 0 & 0 & 0 & 0 \\ -h(\mathbf{b}^T \otimes \mathbf{e} \otimes \overline{I}_d) \circ \overline{K}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ (\mathbf{b}^T \otimes \overline{I}_d) \circ \tilde{R}(3) & 0 & 0 & 0 & 0 \end{pmatrix} \lambda^{m-1} + \dots
$$

$$
+ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -h(\mathbf{b}^T \otimes \mathbf{e} \otimes \overline{I}_d) \circ \overline{R}_{m-1} & 0 & 0 & 0 & 0 \\ -h(\mathbf{b}^T \otimes \mathbf{e} \otimes \overline{I}_d) \circ \overline{K}_{m-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ (\mathbf{b}^T \otimes \overline{I}_d) \circ \tilde{R}(m) & 0 & 0 & 0 & 0 \end{pmatrix} \lambda^{2}
$$

$$
+ \begin{pmatrix} -h(A \otimes M) - I_s \otimes N & 0 & 0 & 0 & 0 \\ h[A \otimes \overline{I}_d) \circ \overline{R}_m - (\mathbf{b}^T \otimes \mathbf{e} \otimes \overline{I}_d) \circ \overline{R}_m] & 0 & 0 & 0 & 0 \\ h[A \otimes \overline{I}_d) \circ \overline{K}_m - (\mathbf{b}^T \otimes \mathbf{e} \otimes \overline{I}_d) \circ \overline{K}_m] & 0 & 0 & 0 & 0 \\ -h(\mathbf{b}^T \otimes M) & 0 & 0 & -N & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \lambda
$$

$$+\begin{pmatrix} 0 & 0 & 0 & \mathbf{e}\otimes N & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & N & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\Bigg\} V(\lambda) = 0.$$

Whence, we get the characteristic polynomial (3.10) of the difference system (3.3)–(3.7). This completes the proof. □

For an explicit PRK method, *i.e.* $a_{ij} = 0$ for $i \le j$, we have the following result.

Theorem 3.1. *For an s-stage explicit PRK method with the step-size $h = \tau/m$, where m is a positive integer. If*
(H1) the system (1.1) satisfying (1.2) is asymptotically stable for known matrices $L, M, N, K(s), R(s)$ and a fixed delay τ (that is, the conditions of Lemma 2.3 hold);
(H2) the characteristic polynomial $P_{PRK}(\lambda)$ never vanishes on the unit circle $\mu = \{\lambda : |\lambda| = 1\}$ and its change of argument admits

$$\frac{1}{2\pi}\Delta_\mu \arg P_{PRK}(\lambda) = d(m+1)(3s+2). \tag{3.12}$$

Then the PRK method for the system (1.1) satisfying (1.2) is weakly delay-dependently stable.

Proof. It is well known that the difference system (3.11) is asymptotically stable if and only if all the characteristic roots of $P_{PRK}(\lambda) = 0$ satisfy $|\lambda| < 1$. Note that the coefficient matrix of the term λ^{m+1} in $\Phi(\lambda)$ is

$$\begin{pmatrix} I_{sd} - h(A\otimes L) & -I_{sd} & -h(A\otimes I_d) & 0 & 0 \\ -h(A\otimes \overline{I}_d)\circ \overline{R}_0 & I_{sd} & 0 & 0 & 0 \\ -h(A\otimes \overline{I}_d)\circ \overline{K}_0 & 0 & I_{sd} & 0 & 0 \\ -h(\mathbf{b}^T\otimes L) & 0 & -h(\mathbf{b}^T\otimes I_d) & I_d & -I_d \\ (\mathbf{b}^T\otimes \overline{I}_d)\circ \tilde{R}(1) & 0 & 0 & 0 & I_d \end{pmatrix}.$$

In order to investigate the singularity of the coefficient matrix of the term λ^{m+1} in $\Phi(\lambda)$, we perform the elementary row transformation. Then, the coefficient matrix of the term λ^{m+1} is converted to a lower triangular matrix:

$$\begin{pmatrix} I_{sd} & I_{sd} & h(A\otimes I_d) & 0 & 0 \\ 0 & I_{sd} & 0 & 0 & 0 \\ 0 & 0 & I_{sd} & 0 & 0 \\ 0 & 0 & 0 & I_d & I_d \\ 0 & 0 & 0 & 0 & I_d \end{pmatrix}\begin{pmatrix} I_{sd} - h(A\otimes L) & -I_{sd} & -h(A\otimes I_d) & 0 & 0 \\ -h(A\otimes \overline{I}_d)\circ \overline{R}_0 & I_{sd} & 0 & 0 & 0 \\ -h(A\otimes \overline{I}_d)\circ \overline{K}_0 & 0 & I_{sd} & 0 & 0 \\ -h(\mathbf{b}^T\otimes L) & 0 & -h(\mathbf{b}^T\otimes I_d) & I_d & -I_d \\ (\mathbf{b}^T\otimes \overline{I}_d)\circ \tilde{R}(1) & 0 & 0 & 0 & I_d \end{pmatrix}$$

$$= \begin{pmatrix} Q & 0 & 0 & 0 & 0 \\ -h(A\otimes \overline{I}_d)\circ \overline{R}_0 & I_{sd} & 0 & 0 & 0 \\ -h(A\otimes \overline{I}_d)\circ \overline{K}_0 & 0 & I_{sd} & 0 & 0 \\ -h(\mathbf{b}^T\otimes L)+(\mathbf{b}^T\otimes \overline{I}_d)\circ \tilde{R}(1) & 0 & -h(\mathbf{b}^T\otimes I_d) & I_d & 0 \\ (\mathbf{b}^T\otimes \overline{I}_d)\circ \tilde{R}(1) & 0 & 0 & 0 & I_d \end{pmatrix}, \tag{3.13}$$

where the matrix Q is expressed as $I_{sd}-h(A\otimes L)-h(A\otimes \overline{I}_d)\circ \overline{R}_0-h^2(A\otimes I_d)[(A\otimes \overline{I}_d)\circ \overline{K}_0]$.

Since the PRK method is explicit, *i.e.*, A is a lower triangular matrix where all diagonal elements are zero, then $A\otimes L$, $(A\otimes \overline{I}_d)\circ \overline{R}_0$, $A\otimes I_d, (A\otimes \overline{I}_d)\circ \overline{K}_0$ are all lower triangular matrices in which all diagonal elements are zero. It follows that $h(A\otimes L)+h(A\otimes \overline{I}_d)\circ \overline{R}_0+h^2(A\otimes I_d)[(A\otimes \overline{I}_d)\circ \overline{K}_0]$ is also a lower triangular matrix whose diagonal elements are zero, which implies that all the eigenvalues of matrix $h(A\otimes L)+h(A\otimes \overline{I}_d)\circ \overline{R}_0+h^2(A\otimes I_d)[(A\otimes \overline{I}_d)\circ \overline{K}_0]$ vanish. As a result, the matrix Q is nonsingular. Thus the lower triangular matrix

$$\begin{pmatrix} Q & 0 & 0 & 0 & 0 \\ -h(A\otimes \overline{I}_d)\circ \overline{R}_0 & I_{sd} & 0 & 0 & 0 \\ -h(A\otimes \overline{I}_d)\circ \overline{K}_0 & 0 & I_{sd} & 0 & 0 \\ -h(\mathbf{b}^T\otimes L)+(\mathbf{b}^T\otimes \overline{I}_d)\circ \tilde{R}(1) & 0 & -h(\mathbf{b}^T\otimes I_d) & I_d & 0 \\ (\mathbf{b}^T\otimes \overline{I}_d)\circ \tilde{R}(1) & 0 & 0 & 0 & I_d \end{pmatrix}$$

is nonsingular.

It is obvious that the coefficient matrix of the term λ^{m+1} in $\Phi(\lambda)$ has the same singularity with the right side of equality (3.13). Therefore the coefficient matrix of the term λ^{m+1} in $\Phi(\lambda)$ is also nonsingular. Counting multiplicities, we know that $P_{PRK}(\lambda)=\det\Phi(\lambda)=0$ has $d(m+1)(3s+2)$ roots in total. By condition (H2) and the argument principle again, we get that all roots of $P_{PRK}(\lambda)=0$ are located in the open unit circular region $\mu=\{\lambda:|\lambda|=1\}$. Hence, the PRK method (3.3)–(3.7) is weakly delay-dependently stable. This completes the proof. □

As for an s-stage implicit PRK for the system (1.1) satisfying (1.2), it is not difficult to derive an analogous result.

Theorem 3.2. *For an implicit PRK method of s-stage with the step-size $h=\tau/m$, where m is a positive integer. Assume that*
(H3) the conditions in Theorem 3.1 hold;
(H4) the eigenvalue λ_i of matrix $h(A\otimes L)+h(A\otimes \overline{I}_d)\circ \overline{R}_0+h^2(A\otimes I_d)[(A\otimes \overline{I}_d)\circ \overline{K}_0]$ never equal to unity for all $i(1\le i\le sd)$.
Then the PRK method for the system (1.1) satisfying (1.2) is weakly delay-dependently stable.

Proof. Along the line of the proof of Theorem 3.1, the coefficient matrix of the term λ^{m+1} in $\Phi(\lambda)$ can be converted to a lower triangular matrix with the elementary row transformation as (3.13). It follows from the condition (H4) that the matrix Q is nonsingular. Therefore, the coefficient matrix of the term λ^m in $\Phi(\lambda)$ is also nonsingular due to the property of the row elementary transformation. Hence, the degree of the polynomial $P_{PRK}(\lambda) = \det \Phi(\lambda)$ becomes $d(m+1)(3s+2)$. Then similar to the proof of the Theorem 3.1, the PRK method for the system (1.1) satisfying condition (1.2) is weakly delay-dependently stable. This completes the proof. $\square$

Remark 3.1. When a semi-implicit PRK method, in which $a_{ij} = 0$, as $i < j$ and $a_{ij} \neq 0$ for $i = j$, Theorem 3.2 is applied to the system (1.1) satisfying (1.2). Since matrix A is lower triangular, then $h(A\otimes L)+h(A\otimes \overline{I}_d)\circ \overline{R}_0+h^2(A\otimes I_d)[(A\otimes \overline{I}_d)\circ \overline{K}_0]$ is a block lower triangular matrix whose diagonal elements are $ha_{ii}L+ha_{ii}\overline{I}_d R(0)+h^2 a_{ii}^2 \overline{I}_d K(0)$, hence for asymptotically stable system (1.1) satisfying (1.2) a semi-implicit PRK method is weakly delay-dependently stable, if there is no eigenvalues of matrices $ha_{ii}L + ha_{ii}\overline{I}_d R(0) + h^2 a_{ii}^2 \overline{I}_d K(0)$ equal to unity for $i = 1, 2, \ldots, s$.

Remark 3.2. When N and $R(s)$ are null matrices, the system (1.1) becomes a delay differential system. Cong et al. [6] have derived sufficient conditions for the delay-dependent stability of this case . Hence Theorem 3.1 and Theorem 3.2 are extensions of the results in [6].

This section ends with an algorithm to examine the conditions of Theorem 3.1 and Theorem 3.2.

Algorithm 3.3. The stability of numerical solutions:

Step 1. Choosing a sufficiently large positive integer n, we distribute n node points λ_l $(l = 0, 1, \ldots, n-1)$ on the unit circle μ of λ-plane satisfying $\arg\lambda_l = (2\pi)l/n$. For each λ_l, we estimate the characteristic polynomial of the difference system by calculating

$$P_{PRK}(\lambda_l) = \det \Phi(\lambda_l),\ l = 0, 1, \ldots, n-1.$$

Also we decompose $P(\lambda_l)$ into its real and imaginary parts.

Step 2. For each λ_l $(l = 0, 1, \ldots, n-1)$, we examine whether $P_{PRK}(\lambda_l) = 0$ by evaluating its magnitude with the preassigned small positive tolerance ε_1. If $|P_{PRK}(\lambda_l)| \leq \varepsilon_1$ fulfills, then $\lambda_l \in \mu$ is a root of $P_{PRK}(\lambda) = 0$. As a result, the PRK scheme for the system (1.1) is not asymptotically stable and stop the algorithm. Otherwise we go to the next step.

Step 3. Check whether $\frac{1}{2\pi}\Delta_\mu \arg P(\lambda_l) = d(m+1)(3s+2)$ by examining $|\frac{1}{2\pi}\Delta_\mu \arg P(\lambda_l) - d(m+1)(3s+2)| \leq \varepsilon_2$ for each λ_l $(l = 0, 1, \ldots, n-1)$ with the preassigned tolerance ε_2. If it holds, then the PRK method for the system (1.1) is weakly delay-dependently stable. Or else, it is not weakly delay-dependently stable.

4. Numerical examples

This section gives two examples to confirm the effectiveness of the stability criteria acquired in Section 3. All the experiments were implemented in MATLAB. We adopt the underlying PRK scheme based on the following classical fourth-order RK method for ODEs (3.1):

$$k_1 = f(t_n, y_n),$$

$$
\begin{aligned}
k_2 &= f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1), \\
k_3 &= f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2), \\
k_4 &= f(t_{n+1}, y_n + hk_3), \\
y_{n+1} &= y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4).
\end{aligned}
$$

Example 4.1. We take a two-dimensional neutral system with distributed delays, where the parameter matrices are given by

$$
L = \begin{bmatrix} -2 & 3 \\ -4 & -2.8 \end{bmatrix}, \qquad M = \begin{bmatrix} -0.03 & -3 \\ 2 & -0.01 \end{bmatrix}, \qquad N = \begin{bmatrix} 0.05 & 0 \\ 0 & -0.01 \end{bmatrix},
$$

$$
K(s) = \begin{bmatrix} \sin 2s & 0 \\ -1 & -\cos s \end{bmatrix}, \qquad R(s) = \begin{bmatrix} 0.1\sin s & 0 \\ 0 & 0.1\cos s \end{bmatrix},
$$

where $s \in [-\tau, 0]$.

The case of $\tau = 1$. According to Lemma 2.2 and Remark 2.2, if we choose

$$
\widetilde{R} = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix},
$$

then

$$
\|N\| + \int_{-\tau}^{0} \|R(s)\| ds \leq \|N\| + \int_{-\tau}^{0} \|\widetilde{R}\| ds = 0.15 = \alpha < 1,
$$

i.e., condition (1.2) holds. Furthermore, we can select

$$
\widetilde{K} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.
$$

By direct computation, we have

$$
\|L\| = 4.8841, \quad \|M\| = 3.0002, \quad \|N\| = 0.0500, \quad \|\widetilde{K}\| = 1.6180.
$$

And $\beta = \|L\| + \|M\| + \tau\|\widetilde{K}\| = 9.5023$. By Lemma 2.2, the radius of the unstable region is given by

$$
\gamma := \frac{\beta}{1-\alpha} = 11.1792.
$$

Our computation gives that $\triangle_\Gamma \arg P(\lambda) = 0$, which indicates that the system is asymptotically stable with the given parameter matrices by Lemma 2.3.

Now we apply Algorithm 3.3 to investigate the weak delay-dependent stability of the numerical solutions generated by the PRK method.

First, let the initial vector function be

$$
x(t) = \begin{bmatrix} \sin(t/2) + t \\ \cos t \end{bmatrix}, \qquad t \in [-\tau, 0].
$$

Taking $m = 2$, we acquire that

$$\frac{1}{2\pi}\Delta_\mu \arg P_{PRK}(\lambda) = 81.9989 \neq 84 = d(m+1)(3s+2).$$

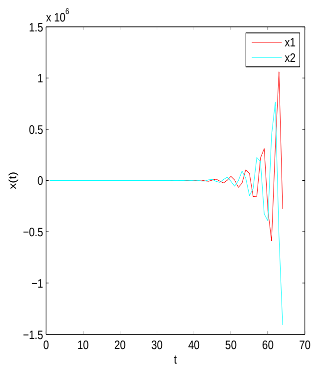
Thus the conditions of Theorem 3.1 does not hold. Whence, the PRK method is not stable and the numerical solution is divergent, which is simulated in Fig. 1(a). Nevertheless, if we choose $m = 5$ and $m = 10$, according to Algorithm 3.3 again, we get that

$$\frac{1}{2\pi}\Delta_\mu \arg P_{PRK}(\lambda) = 167.9977 \approx 168 = d(m+1)(3s+2)$$

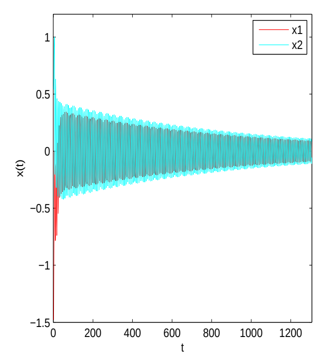
and

$$\frac{1}{2\pi}\Delta_\mu \arg P_{PRK}(\lambda) = 307.9958 \approx 308 = d(m+1)(3s+2)$$

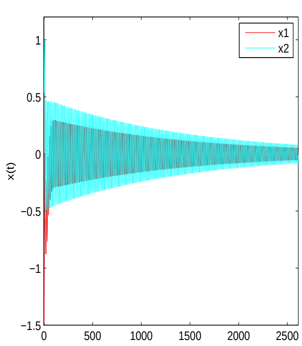
respectively. So Theorem 3.1 asserts that the numerical solutions are stable, which are depicted in Fig. 1(b) and Fig. 1(c). The same case is that for $m = 100$ and the corresponding behavior of numerical solution is presented in Fig. 1(d).



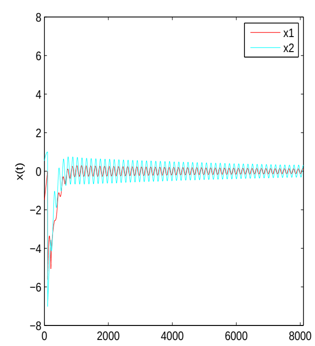
(a) Numerical solution with $m = 2$



(b) Numerical solution with $m = 5$



(c) Numerical solution with $m = 10$



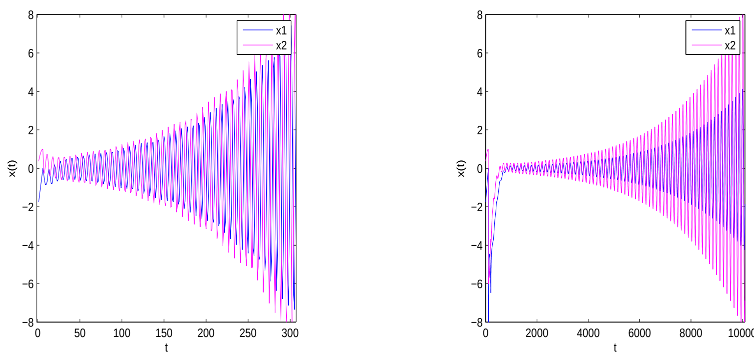
(d) Numerical solution with $m = 100$

Figure 1. Numerical solution when $\tau = 1$ in Example 4.1

The case of $\tau = 1.2$, we obtain

$$\|N\| + \int_{-\tau}^{0} \|R(s)\|ds \leq \|N\| + \int_{-\tau}^{0} \|\widetilde{R}\|ds = 0.17 = \alpha < 1,$$

i.e., condition (1.2) fulfills. And $\beta = \|L\| + \|M\| + \tau\|\widetilde{K}\| = 9.8259$. Lemma 2.3 is employed to check the stability of the system. It reveals that the system is not asymptotically stable as our evaluation $\Delta_\Gamma \arg P(\lambda) = 2$ along the curve Γ. Thus the conditions of Theorem 3.1 are not satisfied. As a consequence, the weak delay-dependent stability is not guaranteed. Moreover, our numerical experiments for $m = 5$ and $m = 100$ indicates that the numerical solutions are divergent, which are presented in Fig. 2. Furthermore, several other numerical experiments for $m > 5$ have been implemented, and the same results are found, *i.e.*, the numerical solutions of which are still divergent.



(a) Numerical solution with $m = 5$ (b) Numerical solution with $m = 100$

Figure 2. Numerical solution are unstable when $\tau = 1.2$ in Example 4.1

Example 4.2. Consider the following four-dimensional neutral system with distributed delays, where the parameter matrices are

$$L = \begin{bmatrix} 0 & 0.8 & -0.9 & 0 \\ -3.35 & -2.6 & 2 & -2 \\ -3.6 & 0 & -1.68 & 0 \\ -2.66 & 0 & 0 & -5.89 \end{bmatrix}, \quad M = \begin{bmatrix} -0.9 & 2 & 1.8 & -1 \\ 3 & 2.9 & -1.65 & 0 \\ 1 & 2 & -0.88 & 1 \\ 2 & 2.85 & 1 & -3 \end{bmatrix},$$

$$N = \begin{bmatrix} 0.03 & -0.02 & 0.06 & -0.09 \\ -0.032 & 0.009 & -0.014 & -0.0025 \\ -0.3 & 0.009 & 0.018 & 0.1 \\ -0.1 & 0.22 & 0.06 & 0.1 \end{bmatrix},$$

$$K(s) = \begin{bmatrix} 2\cos s & -3 & -1.5 & 1 \\ 0 & 0.8 & 4 & 2 \\ 0 & 0 & 0.6 & 0.5\sin s \\ 0 & 0 & 0 & 2.3 \end{bmatrix}, \qquad R(s) = \begin{bmatrix} 0.001 & -0.2\sin s & 0 & 0 \\ -0.04 & 0.025 & 0 & 0.1 \\ 0 & 0.12 & 0.13 & 0 \\ 0.004 & 0 & 0 & 0.06 \end{bmatrix}.$$

Here $s \in [-\tau, 0]$.

Now, we perform a similar process as Example 4.1. The case of $\tau = 0.23$. If we take

$$\widetilde{R} = \begin{pmatrix} 0.001 & 0.2 & 0 & 0 \\ 0.04 & 0.025 & 0 & 0.1 \\ 0 & 0.12 & 0.13 & 0 \\ 0.004 & 0 & 0 & 0.06 \end{pmatrix},$$

then

$$\|N\| + \int_{-\tau}^{0} \|R(s)\| ds \le \|N\| + \int_{-\tau}^{0} \|\widetilde{R}\| ds = 0.4215 = \alpha < 1,$$

that is, condition (1.2) is satisfied. We choose

$$\widetilde{K} = \begin{pmatrix} 2 & 3 & 1.5 & 1 \\ 0 & 0.8 & 4 & 2 \\ 0 & 0 & 0.6 & 0.5 \\ 0 & 0 & 0 & 2.3 \end{pmatrix}$$

to satisfy the conditions of Lemma 2.2 and Remark 2.2. By simple calculation, we get

$$\|L\| = 7.7340, \quad \|M\| = 6.1897, \quad \|N\| = 0.3648, \quad \|\widetilde{K}\| = 5.5271,$$

and $\beta = \|L\| + \|M\| + \tau\|\widetilde{K}\| = 15.2502$. So by Lemma 2.2, the radius of the unstable region is given by

$$\gamma := \frac{\beta}{1-\alpha} = 26.4760.$$

Following our computation, we get that $\triangle_\Gamma \arg P(\lambda) = 0$. According to Lemma 2.3, the system with the known parameter matrices is asymptotically stable.

Now we use Algorithm 3.3 again to examine whether the PRK method for the system is delay-dependently stable or not.

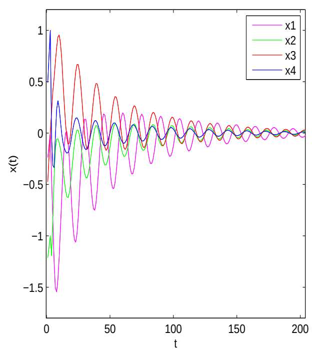
We first set the initial condition

$$x(t) = \begin{bmatrix} \sin t \\ \exp(t) - 2 \\ 2\sin t \\ 2t + \cos t \end{bmatrix}, \qquad t \in [-\tau, 0].$$

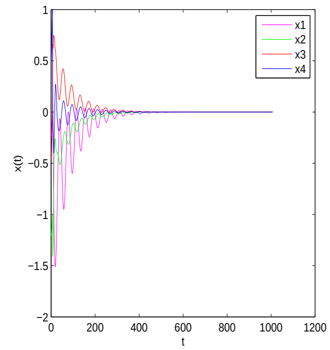
Next, we take $m = 2$ and get that

$$\frac{1}{2\pi} \Delta_\mu \arg P_{PRK}(\lambda) = 167.9977 \approx 168 = d(m+1)(3s+2).$$

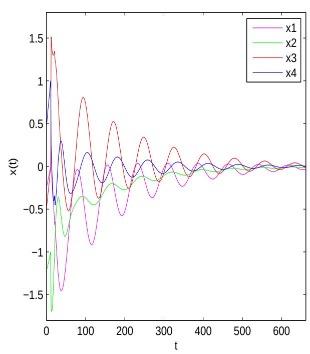
Thus, using Theorem 3.1, the numerical solution is stable as depicted in Fig. 3(a). When we take $m = 5, m = 10$ and $m = 80$, the conditions $\frac{1}{2\pi}\Delta_\mu \arg P_{PRK}(\lambda) = d(m+1)(3s+2)$ also hold, then the system with given parameters matrices is still weakly delay-dependently stable and the numerical solutions are shown in Fig. 3(b), Fig. 3(c) and Fig. 3(d).



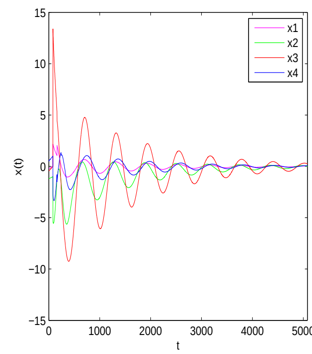
(a) Numerical solution with $m = 2$



(b) Numerical solution with $m = 5$



(c) Numerical solution with $m = 10$



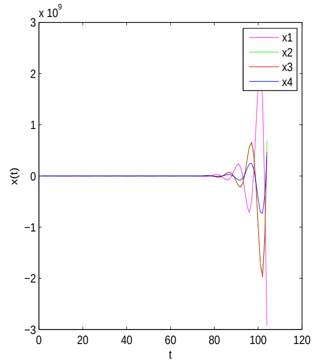
(d) Numerical solution with $m = 80$

Figure 3. Numerical solutions are asymptotically stable when $\tau = 0.23$ in Example 4.2

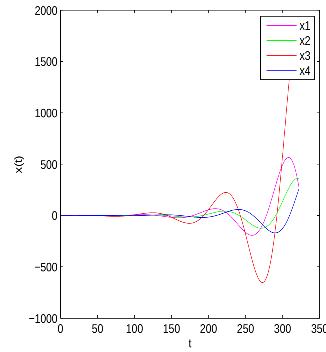
The case of $\tau = 0.5$, we have that

$$\|N\| + \int_{-\tau}^{0} \|R(s)\| ds \leq \|N\| + \int_{-\tau}^{0} \|\widetilde{R}\| ds = 0.4880 = \alpha < 1,$$

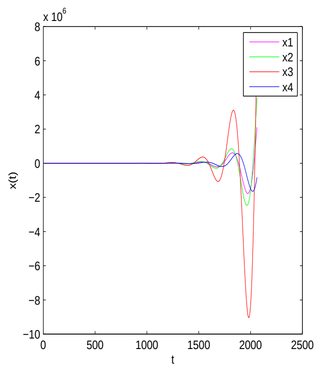
that is, condition (1.2) holds. In this case, $\beta = \|L\| + \|M\| + \tau\|\widetilde{K}\| = 16.6872$, and $\Delta_\Gamma \arg P(\lambda) = 2$. So the system with the known parameter matrices is not asymptotically stable by Lemma 2.3. Then the assumptions of Theorem 3.1 do not hold. Therefore, it is not sure whether the numerical solution is stable or not. Actually, if we choose $m = 2, 20, 60, 100$, the numerical solutions are unstable, which are depicted in Fig. 4. And many other numerical experiments for $m > 100$ have been implemented, whose numerical solutions are still divergent.



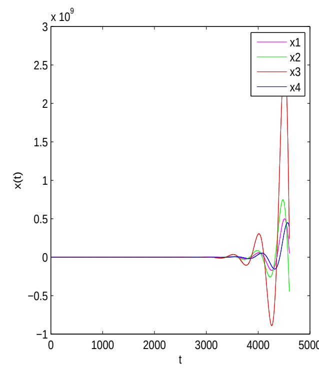
(a) Numerical solution with $m = 2$



(b) Numerical solution with $m = 20$



(c) Numerical solution with $m = 60$



(d) Numerical solution with $m = 100$

Figure 4. Numerical solutions are not stable when $\tau = 0.5$ in Example 4.2

5. Conclusion

The present work analyzes the delay-dependent stability of the PRK methods for the neutral systems with distributed delays. We obtain that the PRK methods can preserve weak delay-dependent stability under some conditions. Whereas the stability criteria presented here are sufficient but not necessary. Numerical experiments show that the theoretical results are effective.

LM methods are widely used to solve the DDEs. Thus, the weak delay-dependent stability of LM methods for the system (1.1) attracts our attention and the relevant work will be accomplished later.

Acknowledgements. The authors are grateful to the anonymous referees for their useful suggestions which improve the contents of this article.

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