Journal of Applied Analysis and Computation
Volume 9, Number 3, June 2019, 1022–1031

Website:http://jaac.ijournal.cn
DOI:10.11948/2156-907X.20180244

ALL SOLUTIONS OF THE YANG-BAXTER-LIKE MATRIX EQUATION WHEN $A^3 = A$ *

Mansour Saeed Ibrahim Adam[1,2], Jiu Ding[3],
Qianglian Huang[1,†] and Lanping Zhu[1]

Abstract Let A be a square matrix satisfying $A^3 = A$. We find all solutions of the Yang-Baxter matrix equation $AXA = XAX$, based on our previous result on all the solutions of the same equation for a matrix A such that $A^2 = I$.

Keywords Yang-Baxter-like matrix equation, diagonalizable matrix, eigenvalue.

MSC(2010) 15A18, 15A24.

1. Introduction

Let A be an $n \times n$ complex matrix. The quadratic matrix equation

$$AXA = XAX \tag{1.1}$$

is called the *Yang-Baxter-like matrix equation*. An equation of similar pattern first appeared in [11] in 1967 and then in [1] in 1972 independently, and it has been named the Yang-Baxter equation that has found many applications in the fields of quantum theory, statistical mechanics, braid group and knot theory [12]. The Yang-Baxter matrix equation is the counterpart of the classic Yang-Baxter equation in matrix theory, and it has been investigated in recent years, see, for example, [3–5, 10, 13] and the references therein. So far, most obtained solutions are commuting ones, that is, the solutions X satisfying the commutability condition $AX = XA$. Finding all non-commuting solutions and thus all the solutions of the Yang-Baxter matrix equation is a challenging task, and up to now there are only isolated results toward this goal for special classes of matrices.

In a previous paper [6], we were able to find all the solutions of (1.1) when the square of A equals the identity matrix, using various spectral analysis and perturbation results. Our purpose here is to solve the Yang-Baxter-like matrix equation (1.1) for a given matrix A satisfying $A^3 = A$, and our purpose is to find

†the corresponding author. Email address: huangql@yzu.edu.cn (Q. Huang)
1School of Mathematical Sciences, Yangzhou University, China
2Department of Mathematics, Faculty of Education, University of Khartoum P. O. Box 321, Sudan
3Department of Mathematics, University of Southern Mississippi, USA
*The authors were supported by National Natural Science Foundation of China (11771378, 11871064) and the Yangzhou University Foundation for Young Academic Leaders (2016zqn03).

all solutions. Clearly, if $A^2 = I$, then $A^3 = A$, so our previous results should play a role in the more general situation. Indeed, as can be seen from the next section, the process of finding all solutions when A satisfies $A^3 = A$ contains the problem of solving (1.1) with $A^2 = I$.

This paper is organized as follows. We give some preliminary results in Section 2, and we also construct all the commuting solutions of (1.1). Then we solve the matrix equation for different cases of the minimal polynomial. We conclude in Section 5.

2. Preliminary Results

For the purpose of finding all solutions of (1.1) with $A^3 = A$ using solutions of the same equation with $A^2 = I$, we need the following lemma.

Lemma 2.1. *Let $A = \mathrm{diag}\{K, J\}$ be an $n \times n$ matrix such that K is $k \times k$. Then the solutions of (1.1) are*

$$X = \begin{bmatrix} V & C \\ B & W \end{bmatrix}, \tag{2.1}$$

where the sub-matrices V, C, B, W satisfy

$$\begin{cases} KVK = VKV + CJB, \\ KCJ = VKC + CJW, \\ JBK = BKV + WJB, \\ JWJ = BKC + WJW. \end{cases} \tag{2.2}$$

In particular, if $J = 0$, then (2.2) is reduced to

$$\begin{cases} KVK = VKV, \\ VKC = 0, \\ BKV = 0, \\ BKC = 0. \end{cases} \tag{2.3}$$

Proof The lemma follows from

$$AXA = \begin{bmatrix} K & \\ & J \end{bmatrix} \begin{bmatrix} V & C \\ B & W \end{bmatrix} \begin{bmatrix} K & \\ & J \end{bmatrix} = \begin{bmatrix} KVK & KCJ \\ JBK & JWJ \end{bmatrix}$$

and

$$XAX = \begin{bmatrix} V & C \\ B & W \end{bmatrix} \begin{bmatrix} K & \\ & J \end{bmatrix} \begin{bmatrix} V & C \\ B & W \end{bmatrix} = \begin{bmatrix} VKV + CJB & VKC + CJW \\ BKV + WJB & BKC + WJW \end{bmatrix}. \quad \square$$

Throughout the paper we assume that A is an $n \times n$ complex matrix with $n \geq 3$ such that $A^3 = A$. Our purpose is to find all the solutions to the corresponding matrix equation (1.1). Since $A^3 - A = 0$, the polynomial $p(\lambda) = \lambda^3 - \lambda$ is an *annihilator* of A, that is, $p(A) = 0$. Thus from the theory of matrix polynomials [7], the *minimal polynomial* $g(\lambda)$ of A, which is the unique annihilator of A with minimal degree and leading coefficient 1, is a factor of $p(\lambda)$.

It is well-known that the zeros of the minimal polynomial of any square matrix give all the eigenvalues of the matrix and an eigenvalue λ_0 is *semisimple*, that is the algebraic multiplicity of λ_0 equals its geometric multiplicity, if and only if the multiplicity of λ_0, as a zero of the minimal polynomial, is 1. Since $p(\lambda) = \lambda(\lambda-1)(\lambda+1)$, the eigenvalues of A constitute a subset of $\{0, 1, -1\}$. Furthermore, each eigenvalue of A is semisimple, so A is diagonalizable. In other words, there is a nonsingular matrix U such that $AU = UD$, where D is a diagonal matrix.

Now, there are several cases of the given matrix A for us to consider. We omit the trivial cases that the minimal polynomial of A is $g(\lambda) = \lambda$ or $g(\lambda) = \lambda - 1$. In the former $A = 0$ and all matrices of the same order are solutions, and in the latter $A = I$ and the solutions are exactly all idempotents. It is also trivial to see that if $g(\lambda) = \lambda + 1$, then $A = -I$, and so all the solutions of (1.1) are the matrices B such that $B^2 = -B$, which will be called skew-idempotents. The remaining nontrivial cases are listed as follows:

Case I. The minimal polynomial of A is $g(\lambda) = \lambda(\lambda - 1)$.
Case II. The minimal polynomial of A is $g(\lambda) = \lambda(\lambda + 1)$.
Case III. The minimal polynomial of A is $g(\lambda) = (\lambda - 1)(\lambda + 1)$.
Case IV. The minimal polynomial of A is $g(\lambda) = \lambda(\lambda - 1)(\lambda + 1)$.

We give an analysis of each case above in the next section.

3. Solutions of the Matrix Equation

Our task in this section is to find all the solutions of the Yang-Baxter-like matrix equation for each of the four cases listed at the end of the above section. Since A is diagonalizable, we assume that in all cases, there is a nonsingular matrix U such that $AU = UD$, where D is a diagonal matrix. Then, as can be shown easily, solving the original Yang-Baxter-like matrix equation $AXA = XAX$ is equivalent to solving a simpler one

$$DYD = YDY \tag{3.1}$$

in the sense that, any solution Y to the latter gives rise to a solution $X = UYU^{-1}$ to the former, and any solution X of (1.1) can be written as $X = UYU^{-1}$ for some solution Y of (3.1). More over, X is a commuting solution if and only if Y is a commuting solution. Because of the equivalence, the similarity matrix U consisting of linearly independent eigenvectors of A, is assumed to be known, and we only solve (3.1) to get solutions of the original equation.

In the following we denote the rank of A by m. Then the Jordan form of A is

$$D = \operatorname{diag}\{K, 0\}. \tag{3.2}$$

We then partition Y as

$$Y = \begin{bmatrix} V & C \\ B & W \end{bmatrix} \tag{3.3}$$

according to the block structure of D.

3.1. Case I: $A^2 = A$

Suppose $A^2 = A$. Then the Jordan form of the idempotent A is

$$D = \mathrm{diag}\{I_m, 0\},$$

where I_m is the $m \times m$ identity matrix. The equation (1.1) in this case has been studied previously by [2,8], but for the completeness of the presentation, we summarize the main results of [2,8] in a different but equivalent form in the following theorem.

Theorem 3.1. *Let A be an $n \times n$ idempotent with rank m. Then all the solutions of (1.1) are given by*

$$X = U \begin{bmatrix} V & C \\ B & W \end{bmatrix} U^{-1}. \tag{3.4}$$

Here W is an arbitrary $(n-m)\times(n-m)$ matrix and the other sub-matrices V, C, B in (3.4) are constructed as follows: For any $m\times m$ nonsingular matrix S partitioned as

$$S = [S_1 \; S_2] \tag{3.5}$$

and its inverse partitioned as

$$S^{-1} = \begin{bmatrix} \tilde{S}_1 \\ \tilde{S}_2 \end{bmatrix}, \tag{3.6}$$

where S_1 is $m\times s$ and $\tilde{S}_1$ is $s\times m$, the $m\times m$ matrix $V = S_1\tilde{S}_1$, the $m\times(n-m)$ matrix $C = S_2E$, and the $(n-m)\times m$ matrix $B = G\tilde{S}_2$ with arbitrary $(m-s)\times(n-m)$ matrix E and $(n-m)\times(m-s)$ matrix G satisfying $GE = 0$.

In addition, X is a commuting solution if and only if $E = 0$ and $G = 0$.

3.2. Case II: $A^2 = -A$

Under the assumption that $A^2 = -A$, the Jordan form of the skew-idempotent A is $D = \mathrm{diag}\{-I_m, 0\}$. By applying Lemma 2.1 to D, we see that the equation (3.1) with Y given by (3.3) is equivalent to

$$V^2 = -V, \;\; VC = 0, \;\; BV = 0, \;\; BC = 0. \tag{3.7}$$

Using the same approach as in [8], we have the following conclusion.

Theorem 3.2. *Let A be an $n \times n$ matrix such that $A^2 = -A$ with rank m. Then all the solutions of (1.1) are given by (3.4), where the $(n-m)\times(n-m)$ matrix W is arbitrary, and given any $m \times m$ nonsingular matrix S partitioned as (3.5) and its inverse partitioned as (3.6),*

$$V = -S_1\tilde{S}_1, \quad C = S_2E \quad \text{and} \quad B = G\tilde{S}_2$$

with arbitrary $(m-s)\times(n-m)$ *matrix* E *and* $(n-m)\times(m-s)$ *matrix* G *satisfying* $GE = 0$.

In addition, X *is a commuting solution if and only if* $E = 0$ *and* $G = 0$.

3.3. Case III: $A^2 = I$

In this case A is nonsingular with two eigenvalues 1 and -1. Let k be the multiplicity of eigenvalue 1, and let $D = \text{diag}\{I_k, -I_{n-k}\}$ be the Jordan form of A. To solve the equation (3.1), we partition Y as

$$Y = \begin{bmatrix} S & F \\ E & T \end{bmatrix} \tag{3.8}$$

accordingly. Then the system (2.2) in Lemma 2.1 becomes

$$\begin{cases} S = S^2 - FE, \\ F = FT - SF, \\ E = TE - ES, \\ T = EF - T^2. \end{cases} \tag{3.9}$$

All commuting solutions and non-commuting solutions of the above system have been found in [9] and [6] respectively. We summarize the main results as the following theorem.

Theorem 3.3. *Let* A *be an* $n \times n$ *matrix such that* $A^2 = I$*, and let* k *be the multiplicity of eigenvalue* 1*. Then*

1. All the commuting solutions of (1.1) are given by

$$X = U \begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix} U^{-1},$$

where S *and* T *satisfy*

$$S^2 = S \quad \text{and} \quad T^2 = -T.$$

2. All the non-commuting solutions of (1.1) are given by

$$X = U \begin{bmatrix} S & F \\ E & T \end{bmatrix} U^{-1},$$

where S *is any* $k\times k$ *diagonalizable matrix and* T *is any* $(n-k)\times(n-k)$ *diagonalizable matrix such that*

(i) the nonzero matrices F *and* E *have the same rank* r *such that*

$$FEF = \frac{3}{4}F \ \ \text{and} \ \ EFE = \frac{3}{4}E;$$

(ii) S *and* T *have eigenvalue* $-1/2$ *and* $1/2$ *of multiplicity* r*, respectively;*

(iii) the nonzero columns of F and nonzero rows of E are eigenvectors and left eigenvectors of S respectively associated with eigenvalue $-1/2$, and the nonzero columns of E and nonzero rows of F are eigenvectors and left eigenvectors of T respectively associated with eigenvalue $1/2$;

(iv) the other eigenvalues of S and T belong to $\{0,1\}$ and $\{0,-1\}$, respectively.

3.4. Case IV: $A^3 = A$

Now we consider the remaining case that the minimal polynomial of A is $g(\lambda) = \lambda^3 - \lambda$. Assume that the rank of A is m and the multiplicity of eigenvalue 1 is k. Then we can write the Jordan form of A as $D = \mathrm{diag}\{I_k, -I_{m-k}, 0\}$. By the general result of [4], all the commuting solutions of $DYD = YDY$ are $Y = \mathrm{diag}\{S, T, W\}$, where $S = S^2$, $T = -T^2$, and W is arbitrary. Hence we have the following result.

Theorem 3.4. *Let A be an $n\times n$ matrix such that $A^3 = A$ with minimal polynomial $\lambda^3 - \lambda$. Suppose the rank of A is m and the multiplicity of eigenvalue 1 is k. Then all the commuting solutions of (1.1) are given by*

$$X = U\mathrm{diag}\{S, T, W\}U^{-1},$$

where S is any $k \times k$ idempotent, T is any $(m-k) \times (m-k)$ skew idempotent, and W is any $(n-m) \times (n-m)$ matrix.

To find all the non-commuting solutions, we write D as (3.2) with

$$K = \mathrm{diag}\{I_k, -I_{m-k}\}$$

and partition Y in (3.1) as in (3.3), where V has the same size as K. Then from the equivalent system (2.3), we see immediately that W is arbitrary for all of its solutions, so in the remainder of the paper it can be any $(n-m) \times (n-m)$ matrix. Before we investigate the general structure of the solutions of (2.3), we consider several special cases first for which the following result is easy to prove.

Proposition 3.1. *Let A be an $n \times n$ matrix such that $A^3 = A$ with rank m and minimal polynomial $\lambda^3 - \lambda$. Then:*

(i) If $V = 0$, then all solutions $(0, C, B, W)$ of (2.3) are such that $BKC = 0$. In particular, if in addition $C = 0$ or $B = 0$, then all solutions are $(0, 0, B, W)$ or $(0, C, 0, W)$, respectively.

(ii) If $C = 0$, then all solution $(V, 0, B, W)$ of (2.3) are such that V is a solution of the Yang-Baxter-like matrix equation $KVK = VKV$ and all rows of B belong to the left null space of KV. If in addition $B = 0$, then all the solutions are commuting.

(iii) If $B = 0$, then all solution $(V, C, 0, W)$ of (2.3) are such that V is a solution of the Yang-Baxter matrix-like equation $KVK = VKV$ and all columns of C belong to the null space of VK. If in addition $C = 0$, then all the solutions are commuting.

From now on we focus on solving (2.3) for all non-commuting solutions of (3.1). Since the first equation of (2.3) is just the Yang-Baxter-like matrix equation for the nonsingular matrix $K = \mathrm{diag}\{I_k, -I_{m-k}\}$ that satisfies the condition $K^2 = I_m$, its general solution has been constructed in Theorem 3.3. Then for each such obtained solution V, we solve the remaining three equations of (2.3) to get B and C. This verifies our earlier claim that all solutions X of the original matrix equation (1.1) can be obtained from all the solutions Y of the matrix equation (3.1) for a matrix $D = \mathrm{diag}\{K, 0\}$ with $K^2 = I$, thanks to Lemma 2.1.

By Theorem 3.3, all solutions V of the first equation $KVK = VKV$ of (2.3) are

$$V = \begin{bmatrix} S & F \\ E & T \end{bmatrix}, \tag{3.10}$$

where S is any $k \times k$ diagonalizable matrix and T is any $(m-k) \times (m-k)$ diagonalizable matrix such that (i) the matrices F and E have the same rank $r \geq 0$, and satisfy $FEF = \frac{3}{4}F$ and $EFE = \frac{3}{4}E$; (ii) S and T have eigenvalue $-1/2$ and $1/2$ of multiplicity r, respectively; (iii) the nonzero columns of F and nonzero rows of E are eigenvectors and left eigenvectors of S respectively associated with eigenvalue $-1/2$, and the nonzero columns of E and nonzero rows of F are eigenvectors and left eigenvectors of T respectively associated with eigenvalue $1/2$; (iv) the other eigenvalues of S and T belong to $\{0, 1\}$ and $\{0, -1\}$, respectively; (v) a solution V is commuting if and only if $E = 0$ and $F = 0$, and in this case S is an idempotent and T is a skew idempotent.

Let V be any solution of the first equation in (2.3) described as above, so that its four submatrices (S, F, E, T) from the partition (3.10) are all known. We solve the other three equations for C and B. Since $K = \text{diag}\{I_k, -I_{m-k}\}$, we partition C and B as

$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \text{ and } B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}. \tag{3.11}$$

Then the last three equations of (2.3) are

$$\begin{bmatrix} S & -F \\ E & -T \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} S & F \\ -E & -T \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \tag{3.12}$$

and

$$\begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} C_1 \\ -C_2 \end{bmatrix} = 0. \tag{3.13}$$

In summary, we have proved the following theorem.

Theorem 3.5. *Let A be an $n \times n$ matrix such that $A^3 = A$ with minimal polynomial $\lambda^3 - \lambda$. Suppose the rank of A is m and the multiplicity of eigenvalue 1 is k. Then all the solutions of (1.1) are given by*

$$X = U \begin{bmatrix} S & F & C_1 \\ E & T & C_2 \\ B_1 & B_2 & W \end{bmatrix} U^{-1},$$

where the $k \times k$ matrix S, the $k \times (m-k)$ matrix F, the $(m-k) \times k$ matrix E, and the $(m-k) \times (m-k)$ matrix T solve (3.9) and are given by Theorem 3.3 in which n is replaced with m, any nonzero column vector $c = [c_1^T \ c_2^T]^T$ of the $m \times (n-m)$

matrix $[C_1^T\ C_2^T]^T$ *and any nonzero row vector* $b = [b_1\ b_2]$ *of the* $(n-m)\times m$ *matrix* $[B_1\ B_2]$ *are an eigenvector and a left eigenvector of the matrices*

$$\begin{bmatrix} S & -F \\ E & -T \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} S & F \\ -E & -T \end{bmatrix}$$

respectively such that $b_1c_1 = b_2c_2$*, and the* $(n-m)\times(n-m)$ *matrix* W *is arbitrary.*

We present a 4×4 example to illustrate the above theorem. Let

$$A = \begin{bmatrix} 0 & -2 & 3 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -2 & 4 & -1 \\ -1 & -4 & 7 & -2 \end{bmatrix}.$$

Then $A^3 = A$ and the Jordan form of A is $D = \text{diag}\{1, 1, -1, 0\}$ with $AU = UD$, where

$$U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 3 & 2 & 1 & 1 \\ 4 & 3 & 2 & 1 \end{bmatrix} \quad \text{and} \quad U^{-1} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix}.$$

By Theorem 3.4, all commuting solutions of (1.1) are $X = U\text{diag}\{S, t, w\}$, where S is any 2×2 idempotent, t equals 0 or -1, and w is any number. From Theorem 3.5, all non-commuting solutions of (1.1) can be written as

$$X = U \begin{bmatrix} S & f & c_1 \\ e & \frac{1}{2} & c_2 \\ b_1 & b_2 & w \end{bmatrix} U^{-1},$$

where w is an arbitrary number, S is any 2×2 diagonalizable matrix with a simple eigenvalue $-1/2$ and the other simple eigenvalue either 0 or 1 with a nonzero column vector f and a nonzero row vector e right and left eigenvectors of S associated with eigenvalue $-1/2$ such that $ef = 3/4$, and $b = [b_1\ b_2]$ and $c = [c_1^T\ c_2]^T$ are left and right eigenvectors of the matrices

$$\begin{bmatrix} S & -f \\ e & -\frac{1}{2} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} S & f \\ -e & -\frac{1}{2} \end{bmatrix}$$

respectively such that $b_1c_1 = b_2c_2$.

We can find the explicit expressions of

$$S = \begin{bmatrix} \xi & \nu \\ \mu & \eta \end{bmatrix}$$

for the two cases that the eigenvalues of S are $\{-1/2, 0\}$ and $\{-1/2, 1\}$, respectively.

In the first case that the eigenvalues of S are $-1/2$ and 0, by the Vieta formula,

$$\xi + \eta = -\frac{1}{2}, \quad \xi\eta - \mu\nu = 0.$$

Solving the above system gives

$$S = \begin{bmatrix} -\frac{1}{4}\left(1 \pm \sqrt{1 - 16\mu\nu}\right) & \nu \\ \mu & -\frac{1}{4}\left(1 \mp \sqrt{1 - 16\mu\nu}\right) \end{bmatrix}$$

with arbitrary complex numbers μ and ν.

Vieta's formula for the second case that eigenvalues of S are $-1/2$ and 1 implies that

$$\xi + \eta = \frac{1}{2}, \quad \xi\eta - \mu\nu = -\frac{1}{2},$$

from which

$$S = \begin{bmatrix} \frac{1}{4}\left(1 \pm \sqrt{9 - 16\mu\nu}\right) & \nu \\ \mu & \frac{1}{4}\left(1 \mp \sqrt{9 - 16\mu\nu}\right) \end{bmatrix}, \quad \forall\, \mu, \nu.$$

4. Conclusions

We have found all solutions of the Yang-Baxter-like matrix equation (1.1) for a matrix A satisfying $A^3 = A$, which has extended the previous results of [2, 6, 8, 9]. Our approach is direct and simple by means of the digitalization of A and a spectral perturbation result. The same idea and technique in this paper can be applied to find all solutions of (1.1) when A satisfies the condition $A^3 = -A$ or when $A^k = A$ for some $k \in \mathbb{N}$.

Finding the solution set of (1.1) for a general matrix A is a hard task, and it is hoped that special techniques can be employed to find all non-commuting solutions of (1.1) for some other classes of the given matrix, for example the class of all diagonalizable matrices. We hope to solve the general case in the future.

Acknowledgments The authors would like to express their deep gratitude to the referees for their very detailed and constructive comments and suggestions.

References

[1] R. Baxter, *Partition function of the eight-vertex lattice model*, Ann. Phys., 1972, 70(1), 193–228.

[2] A. Cibotarica, J. Ding, J. Kolibal, and N. Rhee, *Solutions of the Yang-Baxter matrix equation for an idempotent*, Numer. Algbra Control Optim., 2013, 3(2), 347–352.

[3] J. Ding and N. Rhee, *Spectral solutions of the Yang-Baxter matrix equation*, J. Math. Anal. Appl., 2013, 402(2), 567–573.

[4] Q. Dong and J. Ding, *Complete commuting solutions of the Yang-Baxter-like matrix equation for diagonalizable matrices*, Computers Math. Appl., 2016, 72(1), 194–201.

[5] Q. Dong, J. Ding and Q. Huang, *Commuting solutions of a quadratic matrix equation for nilpotent matrices*, Algebra Colloquium, 2018, 25(1), 31–44.

[6] Q. Huang, M. Saeed Ibrahim Adam, J. Ding and L. Zhu, *All non-commuting solutions of the Yang-Baxter matrix equation for a class of diagonalizable matrices*, Operators and Matrices., 2019, 13(1), 187–195.

[7] C. Meyer, *Matrix Analysis and Applied Linear Algebra*, SIAM, Cambridge University Press, 2000.

[8] M. Saeed Ibrahim Adam, J. Ding and Q. Huang, *Expresssion of solutions of the Yang-Baxter-like matrix equation for an idempotent*, Appl. Math. Lett., 2017, 63, 71–76.

[9] M. Saeed Ibrahim Adam, J. Ding, Q. Huang and L. Zhu, *Solving a class of quadratic matrix equations*, Appl. Math. Lett., 2018, 82, 58–63.

[10] D. Shen, M. Wei and Z. Jia, *On commuting solutions of the Yang-Baxter-like matrix equation*, J. Math. Anal. Appl., 2018, 462(1), 665–696.

[11] C. Yang, *Some exact results for the many-body problem in one dimension with repulsive delta-function interaction*, Phys. Rev. Lett., 1967, 19(23), 1312–1315.

[12] C. Yang and M. Ge, *Braid Group, Knot Theory, and Statistical Mechanics*, World Scientific, Singapore, 1989.

[13] D. Zhou, G. Chen and J. Ding, *Solving the Yang-Baxter-like matrix equation for rank two matrices*, J. Comput. Appl. Math., 2017, 313, 142–151.