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EXISTENCE AND MULTIPLICITY OF SOLUTIONS TO A CLASS OF NONCOOPERATIVE ELLIPTIC SYSTEMS WITH SUPERLINEAR NONLINEAR TERMS*

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Abstract We study a class of noncooperative elliptic systems. By applying a new superlinear condition, it is shown that there exists a nontrivial weak solution. Moreover, infinitely many solutions are obtained.

Keywords Noncooperative elliptic systems, new superlinear condition.

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1. Introduction and main result

Consider the following noncooperative elliptic system

$$\begin{cases} -\triangle u = |v|^{q-2}v & \text{in } \Omega, \\ -\triangle v = f(u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $q > 1$, $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, and $f \in C(\mathbb{R})$.

Problem (1.1) arises naturally a steady states in reaction diffusion process. When $f(u) = |u|^{r-2}u$ (where $r > 1$), problem (1.1) is also referred as Lane-Emden system because it is a natural extension of Lane-Emden equation

$$-\triangle u = |u|^{\gamma-2}u \text{ (where } \gamma \text{ is a constant) in } \Omega.$$

In the case $N = 3$, this equation arises in astrophysics and is used to model the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics. In the "model case", that is, $q > 2$ and $f(x, s) = |s|^{r-2}s$ with $r > 2$, there are many results about non-existence or existence of solutions to problem (1.1). It is known (see [6, 7, 10]) that problem (1.1) admits a nontrivial solution provided

$$1 > \frac{1}{q} + \frac{1}{r} > 1 - \frac{2}{N}. \tag{1.2}$$

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If $N = 2$, this condition holds for any $q > 2$ and $r > 2$. If $N \geq 3$, on the "critical hyperbola", i.e., the curve of $(q,\ r) \in \mathbb{R}^2$ satisfying $\frac{1}{q} + \frac{1}{r} = 1 - \frac{2}{N}$, non-existence of solutions has been proved in [15] and [20] by the Pohozaev type argument because of the lack of compactness. In addition, existence of solution to problem (1.1) has been established in [9] in the case $0 < (q-1)(r-1) < 1$.

For general f, Kristály in [12] has proved that there are infinitely many solutions to problem (1.1) if f has a suitable oscillatory behavior. In the border-line case $q = \frac{N}{N-2}$, a critical growth of exponential type for f has been studied in [17]. Existence and multiplicity of solutions to the subquadratic problem have been established in [1–3]. As for the superquadratic problem, by assuming the Ambrosetti-Rabinowitz type condition

(AR) there exist constants $\theta > \frac{q}{q-1}$ and $s_1' \geq 0$ such that

$$0 < \theta F(s) \leq s f(s), \quad \forall\ |s| \geq s_1',$$

where $F(s) := \int_0^s f(t)dt$,

de Figueiredo and Ruf in [8] have obtained a nontrivial solution in a fractional Sobolev space via the variational method. Salvatore in [18] has used the algebraic approach based on the Pohozaev's fibering method to obtain a nontrivial solution provided that f further satisfies

(**R**) there exist two functions $\lambda_\pm(u) \in C^1(S)$ with $\lambda_-(u) < 0 < \lambda_+(u)$ such that

$$|\lambda_\pm(u)|^{\frac{q}{q-1}} - \int_\Omega f(\lambda_\pm(u)u)\lambda_\pm(u)u dx = 0,$$

where S is the unit sphere in the Banach space $W^{2,\ \frac{q}{q-1}}(\Omega) \cap W_0^{1,\ \frac{q}{q-1}}(\Omega)$.

Moreover, if f is odd in $s \in \mathbb{R}$, infinitely many pairs of weak solutions have been obtained. Subsequently, dropping the assumption (**R**), Salvatore in [19] has proved that problem (1.1) has infinitely many pairs of weak solutions via the variational method. In addition, when $1 < q \leq 2$ if $N = 2, 3$, or $1 < q < \frac{N}{N-2}$ if $N \geq 4$, Chen and Zou in [5] have generalized Salvatore's results in [18,19] by replacing (AR) with

(GAR) there exist a constant $s_2' \geq 0$ and a function $\theta \in C(\mathbb{R}\backslash(-s_2', s_2'), \mathbb{R}^+)$ such that

$$0 < \left(\frac{q}{q-1} + \theta(s)\right) F(s) \leq s f(s), \quad \forall\ |s| \geq s_2',$$

where θ satisfies $\lim\limits_{|s|\to+\infty} |s|\theta(s) = \lim\limits_{|s|\to+\infty} \int_{s_0}^{|s|} \frac{\theta(t)}{t} dt = +\infty.$

Obviously, (GAR) is essentially weaker than (AR). From $1 < q < 2$ if $N = 2, 3$ or $1 < q < \frac{N}{N-2}$ if $N \geq 4$, one can deduces $\frac{q}{q-1} > \frac{N}{2}$. When $N \geq 4$, from $\frac{N}{N-2} \leq q \leq 2$ it follows that $2 \leq \frac{q}{q-1} \leq \frac{N}{2}$. In this paper, setting $p := \frac{q}{q-1}$, our main results are the following theorems.

Theorem 1.1. *Assume that* $p > \frac{N}{2}$ *and* f *satisfies*

(f_1) $\lim\limits_{s\to 0} \frac{F(s)}{|s|^p} = 0,$

(f$_2$) $\lim\limits_{|s|\to+\infty} \dfrac{F(s)}{|s|^p} = +\infty$,

(f$_3$) *there exist constants* $s_1 > 0$, $\alpha > 0$ *and* $\beta > 0$ *such that*

$$\frac{F(s)}{|s|^p} \leq \alpha\left(sf(s) - pF(s)\right) + \beta$$

for $|s| \geq s_1$.

Then problem (1.1) *has a nontrivial weak solution. If* f *satisfies the additional condition*

(f$_*$) $f(s)$ *is odd in* $s \in \mathbb{R}$,

then problem (1.1) *has infinitely many pairs of weak solutions.*

Remark 1.1. Theorem 1.1 generalizes Theorem 1.1 in [18] and Theorem 1.1 in [19], and complements Theorems 1.1 in [5]. In fact,

(1) (AR) implies (f$_2$) and (f$_3$). Indeed, from (AR), one can deduce that

$$F(s) \geq a_1|s|^\theta - a_2, \quad \forall\, s \in \mathbb{R},$$

where a_1 and a_2 are positive constants, which implies that (f$_2$) holds. In addition,

$$sf(s) - 2F(s) \geq (\theta-2)F(s) \geq (\theta-2)(s_1')^p \cdot \frac{F(s)}{|s|^p}, \quad \forall\, |s| \geq s_1'.$$

That is, (f$_3$) holds.

(2) There are functions which satisfy the assumptions in Theorem 1.1 but don't satisfy (GAR). For example, let

$$F(s) = |s|^\theta + (\theta-p)|s|^{\theta-\tau} \cdot \sin^2\left(\frac{|s|^\tau}{\tau}\right)$$

with $\theta > p+1$ and $\tau \in (1, \min\{p, \theta-p\})$. It is easy to check that (f$_1$), (f$_2$) and (f$_*$) are satisfied. Additional, by a simple calculation, we have

$$f(s) = \theta s|s|^{\theta-2} + (\theta-p)(\theta-\tau)s|s|^{\theta-\tau-2} \cdot \sin^2\left(\frac{|s|^\tau}{\tau}\right) + (\theta-p)s|s|^{\theta-2} \cdot \sin\left(\frac{2|s|^\tau}{\tau}\right),$$

and

$$sf(s) - pF(s) = (\theta-p)|s|^\theta\left[1 + \sin\left(\frac{2|s|^\tau}{\tau}\right)\right] + (\theta-p)(\theta-p-\tau)|s|^{\theta-\tau} \cdot \sin^2\left(\frac{|s|^\tau}{\tau}\right).$$

Then

$$\frac{sf(s) - pF(s)}{F(s)} = (\theta-p) \cdot \frac{\left[1 + \sin\left(\frac{2|s|^\tau}{\tau}\right)\right] \cdot |s|^\tau + (\theta-p-\tau)\sin^2\left(\frac{|s|^\tau}{\tau}\right)}{|s|^\tau + (\theta-p)\sin^2\left(\frac{|s|^\tau}{\tau}\right)}.$$

Setting $s_k = \left[\tau(k\pi + \frac{3\pi}{4})\right]^{\frac{1}{\tau}}$, $k \in \mathbb{N}^+$, it is easy to verify that

$$s_k \to +\infty, \quad s_k \cdot \frac{s_k f(s_k) - pF(s_k)}{F(s_k)} \to 0$$

as $k \to \infty$. Hence, there is no function $\theta(s)$ such that $\lim_{|s|\to+\infty} |s|\theta(s) = +\infty$ and $(p + \theta(s))F(s) \le sf(s)$. However, for $|s|$ large, we have

$$\frac{F(s)}{|s|^p} \le a_3|s|^{\theta-p} \le a_3|s|^{\theta-\tau} \le a_4(sf(s) - pF(s)),$$

where a_3 and a_4 are positive constants, so (f_3) is satisfied.

Theorem 1.2. *Assume that $p \le \frac{N}{2}$ and f satisfies (f_1), (f_2), and*

(f_4) *there exist constants $s_2 > 0$, $\alpha > 0$, $\beta > 0$ and $\sigma > \frac{N}{2p}$ such that*

$$\left(\frac{F(s)}{|s|^p}\right)^\sigma \le \alpha\left(sf(s) - pF(s)\right) + \beta$$

for $|s| \ge s_2$,

(f_5) *there exist positive constants a_5, a_6 and r such that (1.2) holds and*

$$|f(s)| \le a_5|s|^{r-1} + a_6$$

for $s \in \mathbb{R}$.

Then problem (1.1) has a nontrivial weak solution. If f satisfies the additional condition (f_*), then problem (1.1) has infinitely many pairs of weak solutions.

Remark 1.2. Theorem 1.2 generalizes Theorem 1.4 in [18] and Theorem 1.2 in [19]. In fact,

(1) Under the condition (f_5), (AR) implies (f_2) and (f_4). As is shown in item (1) in Remark 1.1, (f_2) holds. In addition, from (1.2), one has

$$p < r < \begin{cases} \frac{Np}{N-2p}, \, p < \frac{N}{2}, \\ +\infty, \quad p = \frac{N}{2}. \end{cases}$$

Therefore,

$$\frac{r}{r-p} > \begin{cases} \frac{N}{2p}, \, p < \frac{N}{2}, \\ 1, \quad p = \frac{N}{2}. \end{cases}$$

Taking arbitrarily $\sigma \in \left(\frac{N}{2p}, \frac{r}{r-p}\right)$ when $p < \frac{N}{2}$ or $\sigma \in \left(1, \frac{r}{r-p}\right)$ when $p = \frac{N}{2}$, we have $\sigma > 1$ and $\frac{p\sigma}{\sigma-1} > r$. Then from (f_5) it follows that

$$\lim_{|s|\to\infty} \frac{F(s)}{|s|^{\frac{p\sigma}{\sigma-1}}} = 0.$$

Hence, there exists a constant $s_3' > s_1'$ such that

$$0 < \frac{F(s)}{|s|^{\frac{p\sigma}{\sigma-1}}} \leq (\theta - p)^{\frac{1}{\sigma-1}}$$

for $|s| \geq s_3'$, from this and (AR) we obtain

$$\left(\frac{F(s)}{|s|^p}\right)^{\sigma} \leq (\theta - p)F(s) \leq sf(s) - pF(s)$$

for $|s| \geq s_3'$.

(2) There exists functions which satisfy the assumptions in Theorem 1.2 but don't satisfy (AR). Indeed, for the function F listed in Remark 1.1 with $\tau \in (0, p)$, and $\theta \in \left(p, \frac{(N-2\tau)p}{N-2p}\right)$ when $p < \frac{N}{2}$ or $\theta \in (p, +\infty)$ when $p = \frac{N}{2}$, it is not difficult to check that (f_1), (f_2), (f_5) and (f_*) are satisfied. Besides this, for s_k given in Remark 1.1, one has

$$\frac{s_k f(s_k) - pF(s_k)}{F(s_k)} \to 0$$

as $k \to \infty$, hence (AR) is not satisfied. However, for $|s|$ large, there exists positive constant a_7 such that

$$\left(\frac{F(s)}{|s|^p}\right)^{\frac{\theta-\tau}{\theta-p}} \leq a_7 \left(|s|^{\theta-p}\right)^{\frac{\theta-\tau}{\theta-p}} = a_7|s|^{\theta-\tau} \leq sf(s) - pF(s).$$

Setting $\sigma := \frac{\theta-\tau}{\theta-p}$, then $\sigma > \frac{N}{2p}$ when $p = \frac{N}{2}$. When $p < \frac{N}{2}$, from $\theta \in \left(p, \frac{(N-2\tau)p}{N-2p}\right)$ it follows that $\sigma > \frac{N}{2p}$. To sum up, we can conclude that (f_4) is satisfied.

2. Preliminaries

System (1.1) is equivalent to the following $p-$biharmonic equation with Navier boundary conditions

$$\begin{cases} \triangle(|\triangle u|^{p-2}\triangle u) = f(u) & \text{in } \Omega, \\ u = \triangle u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

which arises in the study of traveling waves in suspension bridges (see [13, 14]) and the study of the static deflection of an elastic plate in a fluid. Let $E := W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ endowed with the norm

$$\|u\| = \left(\int_{\Omega} |\triangle u|^p dx\right)^{\frac{1}{p}}$$

which is equivalent to the usual intersection norm

$$\|u\|_E = \max\left\{\|u\|_{W^{2,p}(\Omega)}, \|u\|_{W_0^{1,p}(\Omega)}\right\}.$$

Thus, these following embedding mappings

$$E \hookrightarrow \begin{cases} L^{\nu}(\Omega),\ \nu < \frac{Np}{N-2p}, \text{ when } p < \frac{N}{2} \\ L^{\nu}(\Omega),\ \nu < +\infty, \quad \text{when } p = \frac{N}{2} \\ C_B(\Omega), \qquad\qquad\quad \text{when } p > \frac{N}{2} \end{cases}$$

are compact, where $L^{\nu}(\Omega)$ is the Lebesgue space with the norm $|u|_{\nu} = \left(\int_{\Omega} |u|^{\nu} dx\right)^{\frac{1}{\nu}}$, $C_B(\Omega)$ is the space of the continuous bounded functions on Ω with the norm $\|u\|_{C_B(\Omega)} = \sup\limits_{x\in\Omega} |u(x)|$. We denote by S_{ν} (respectively S_{∞}) the imbedding constant of E in $L^{\nu}(\Omega)$ (respectively in $C_B(\Omega)$), that is,

$$S_{\nu}|u|_{\nu} \leq \|u\| \text{ (respectively } S_{\infty}|u|_{C_B(\Omega)} \leq \|u\|),\ \forall\ u \in E.$$

Additionally, the functional Φ defined as

$$\Phi(u) = \frac{1}{p}\int_{\Omega} |\triangle u|^p dx - \int_{\Omega} F(u)dx, \quad u \in E$$

belongs to $C^1(E,\ \mathbb{R})$,

$$\langle \Phi'(u),\ v\rangle = \int_{\Omega} |\triangle u|^{p-2}\triangle u \triangle v dx - \int_{\Omega} f(u)v dx, \quad \forall u,\ v \in E,$$

and the weak solutions of problem (2.1) are exactly the critical points of Φ in E.

3. Proof of Theorems

In the following statement, we use $C_i (i \in \mathbb{N}^+)$ to represent suitable positive constants.

Lemma 3.1. *When $p > \frac{N}{2}$, assume that f satisfies* (f_1), *then there exist positive constants ρ and a such that $\Phi_{\partial B_{\rho}} \geq a$, where and in what follows $B_{\varrho} := \{u \in E : \|u\| \leq \varrho\}$ and $\partial B_{\varrho} := \{u \in E : \|u\| = \varrho\}$ for $\varrho > 0$.*

Proof. We denote by $|\Omega|$ the Lebesgue measure of Ω, then from (f_1), for $\varepsilon \in \left(0, \frac{S_{\infty}^p}{2p|\Omega|}\right)$, there exists a constant $\delta > 0$ such that

$$|F(s)| \leq \varepsilon |s|^p$$

for $|s| < \delta$. Thus, arbitrarily taking $\rho \in (0, S_{\infty}\delta)$, we have

$$|u(x)| \leq \frac{\|u\|}{S_{\infty}} \leq \frac{\rho}{S_{\infty}} < \delta \text{ in } \Omega, \quad \forall\ u \in B_{\rho},$$

this leads to

$$\Phi(u) \geq \frac{1}{p}\|u\|^p - \varepsilon \int_{\Omega} |u|^p dx \geq \left(\frac{1}{p} - \frac{\varepsilon|\Omega|}{S_{\infty}^p}\right)\|u\|^p \geq \frac{1}{2p}\|u\|^p, \quad \forall\ u \in B_{\rho}.$$

Setting $a := \frac{1}{2p} \cdot \rho^p > 0$, then we have $\Phi_{\partial B_{\rho}} \geq a$. □

Lemma 3.2. *When $p \leq \frac{N}{2}$, assume that f satisfies* (f_1) *and* (f_5)*, then there exist positive constants ρ and a such that $\Phi_{\partial B_\rho} \geq a$.*

Proof. By (f_1) and (f_5), for $\varepsilon \in \left(0, \frac{S_p^p}{2p|\Omega|}\right)$, there exists a constant $C(\varepsilon) > 0$ such that

$$|F(s)| \leq \varepsilon |s|^p + C(\varepsilon)|s|^r$$

for $s \in \mathbb{R}$. Then we have

$$\begin{aligned}\Phi(u) &\geq \frac{1}{p}\|u\|^p - \varepsilon \int_\Omega |u|^p dx - C(\varepsilon)\int_\Omega |u|^r dx \\ &\geq \left(\frac{1}{p} - \frac{\varepsilon|\Omega|}{S_p^p}\right)\|u\|^p - C(\varepsilon)\int_\Omega |u|^r dx \\ &\geq \left(\frac{1}{2p} - \frac{C(\varepsilon)}{S_r^r}\|u\|^{r-p}\right)\|u\|^p.\end{aligned}$$

Setting $\rho = \left(\frac{S_r^r}{4pC(\varepsilon)}\right)^{\frac{1}{r-p}}$ and $a = \frac{1}{4p} \cdot \rho^p > 0$, then we have $\Phi_{\partial B_\rho} \geq a$. □

Lemma 3.3. *Assume that f satisfies* (f_2)*, then there exists $u_0 \in E$ with $\|u_0\| > \rho$ such that $\Phi(u_0) < 0$.*

Proof. From (f_2) it follows that for every $K > 0$, there exists a positive constant C_K such that

$$F(s) \geq \frac{K}{p} \cdot |s|^p - C_K \tag{3.1}$$

for $s \in \mathbb{R}$. Hence, for arbitrary $\phi \in E$ with $\|\phi\| = 1$, fixing $K > \frac{1}{|\phi|_p^p}$, we have

$$\Phi(t\phi) \leq \frac{|t|^p}{p}\left(\|\phi\|^p - K\int_\Omega |\phi|^p dx\right) + C_K \cdot |\Omega| = \frac{|t|^p}{p}\left(1 - K\int_\Omega |\phi|^p dx\right) + C_K \cdot |\Omega|.$$

Setting $t_0 = \left(\frac{2pC_K \cdot |\Omega|}{K|\phi|_p^p - 1}\right)^{\frac{1}{p}}$ and $u_0 := t_0\phi$, then one has $\Phi(u_0) \leq -C_K|\Omega| < 0$. □

Lemma 3.4. *When $p > \frac{N}{2}$, assume that* (f_2) *and* (f_3) *hold, then Φ satisfies the* (C) *condition, that is, for every $c \in \mathbb{R}$ and any sequence $\{u_n\}$ such that*

$$\|\Phi'(u_n)\|(1 + \|u_n\|) \to 0 \quad \text{and} \quad \Phi(u_n) \to c \quad \text{as } n \to \infty \tag{3.2}$$

has a convergent subsequence.

Proof. Firstly, we show that $\{u_n\}$ is bounded in E. We argue by contradiction. If $\{u_n\}$ is unbounded, then $\|u_n\| \to +\infty$ as $n \to \infty$ after passing to a subsequence. Setting $w_n = \frac{u_n}{\|u_n\|}$, then $\|w_n\| = 1$. Up to a subsequence, we may assume that

$$w_n \rightharpoonup w \text{ weakly in } E \quad \text{and} \quad w_n \to w \text{ strongly in } C_B(\Omega). \tag{3.3}$$

By (f_2), there exists a positive constant $M_1 > s_1$ such that

$$\frac{F(s)}{|s|^p} \geq 1 \tag{3.4}$$

for $|s| \geq M_1$. Thus, there is a positive constant L_{M_1} such that

$$|F(s)| \leq L_{M_1} \tag{3.5}$$

for $|s| \leq M_1$. From (3.4) and (3.5), we have

$$F(s) \geq |s|^p - L_{M_1} - M_1^p \geq -L_{M_1} - M_1^p \tag{3.6}$$

for $s \in \mathbb{R}$.

Let $\Omega^{'} := \{x \in \Omega : w(x) \neq 0\}$, then for $x \in \Omega^{'}$, we have $u_n(x) = w_n(x)\|u_n\| \to \infty$ as $n \to \infty$, which implies that

$$\lim_{n\to\infty} \frac{F(u_n(x))}{|u_n(x)|^p} = +\infty. \tag{3.7}$$

If $|\Omega^{'}| > 0$, (3.2) together with (3.6) leads to

$$\begin{aligned}\frac{1}{p} - \frac{c+o(1)}{\|u_n\|^p} &= \int_\Omega \frac{F(u_n)}{\|u_n\|^p} dx \\ &= \int_{\Omega'} \frac{F(u_n)}{|u_n|^p} |w_n|^p dx + \int_{\Omega\setminus\Omega'} \frac{F(u_n)}{\|u_n\|^p} dx \\ &\geq \int_{\Omega'} \frac{F(u_n)}{|u_n|^p} |w_n|^p dx - \frac{(L_{M_1} + M_1^p)\cdot|\Omega|}{\|u_n\|^p}.\end{aligned}$$

Then from (3.3) and (3.7), applying Fatou's lemma gets

$$\frac{1}{p} \geq \liminf_{n\to\infty} \left(\int_{\Omega'} \frac{F(u_n)}{|u_n|^p} |w_n|^p - \frac{(L_{M_1} + M_1^p)\cdot|\Omega|}{\|u_n\|^p} \right) \geq +\infty,$$

a contradiction. Hence $|\Omega^{'}| = 0$, that is, $w = 0$.

Setting

$$\kappa := \max_{|s|\leq M_1} |sf(s) - pF(s)|, \quad \Omega_n := \{x \in \Omega : |u_n(x)| \geq M_1\},$$

we derive from (3.2), (3.5) and (f_3) that

$$\begin{aligned}\frac{1}{p} - \frac{c+o(1)}{\|u_n\|^p} &= \int_\Omega \frac{F(u_n)}{|u_n|^p} |w_n|^p dx \\ &= \int_{\Omega\setminus\Omega_n} \frac{F(u_n)}{\|u_n\|^p} dx + \int_{\Omega_n} \frac{F(u_n)}{|u_n|^p} |w_n|^p dx \\ &\leq \frac{L_{M_1}|\Omega|}{\|u_n\|^p} + \int_\Omega \left[\alpha\left(u_n f(u_n) - pF(u_n)\right) + \beta + \alpha\kappa\right] |w_n|^p dx \\ &\leq \frac{L_{M_1}|\Omega|}{\|u_n\|^p} + \left[\left(\alpha\kappa + \beta\right)|\Omega| + \alpha\left(p\Phi(u_n) - \Phi'(u_n)u_n\right)\right] \cdot \|w_n\|^p_{C_B(\Omega)}.\end{aligned}$$

By (3.3), letting $n \to \infty$ in the above inequality gives

$$\frac{1}{p} \leq 0,$$

a contradiction. Hence, $\|u_n\|$ is bounded in E.

Let $I(u) := \frac{1}{p}\int_\Omega |\triangle u|^p dx$ and $J(u) := \int_\Omega F(u)dx$, then $I, J \in C^1(E, \mathbb{R})$ with

$$\langle I'(u),\ h\rangle = \int_\Omega |\triangle u|^{p-2}(\triangle u)(\triangle h)dx, \quad \langle J'(u),\ h\rangle = \int_\Omega f(u)hdx, \quad \forall\ u, h \in E.$$

Moreover, $J' : E \to E^*$ is compact, and

$$\|I'(u_n) - I'(u_m)\|_* \le \|\Phi'(u_n) - \Phi'(u_m)\|_* + \|J'(u_n) - J'(u_m)\|_*, \tag{3.8}$$

where E^* is the dual space of E and $\|\cdot\|_*$ denotes the norm in E^*. In addition, we note that (see [11]), for $\xi,\ \eta \in \mathbb{R}^N$,

$$\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta,\ \xi - \eta\right) \ge \begin{cases} C_1|\xi - \eta|^p, & p \ge 2, \\ C_2\left(1 + |\xi| + |\eta|\right)^{p-2}|\xi - \eta|^2, & 1 < p < 2, \end{cases}$$

where $(\cdot, \cdot)$ denotes the Euclidean inner product in $\mathbb{R}^N$. When $p \ge 2$, we have

$$\langle I'(u_n) - I'(u_m), u_n - u_m\rangle \ge C_1\|u_n - u_m\|^p,$$

which implies that

$$\|I'(u_n) - I'(u_m)\|_* \ge C_1\|u_n - u_m\|^{p-1}. \tag{3.9}$$

When $1 < p < 2$, applying the Hölder inequality, we have

$$\begin{aligned}\|u_n - u_m\|^p \le C_3 \int_\Omega & \left\{\left[\left(|\triangle u_n|^{p-2}\triangle u_n - |\triangle u_m|^{p-2}\triangle u_m\right)(\triangle u_n - \triangle u_m)\right]^{\frac{p}{2}}\right\} \\ & \times (1 + |\triangle u_n| + |\triangle u_m|)^{\frac{p(2-p)}{2}} dx \\ \le{} & C_4\left[\int_\Omega (|\triangle u_n|^{p-2}\triangle u_n - |\triangle u_m|^{p-2}\triangle u_m)(\triangle u_n - \triangle u_m)dx\right]^{\frac{p}{2}} \\ & \times \left[\int_\Omega (1 + |\triangle u_n| + |\triangle u_m|)^p dx\right]^{\frac{2-p}{2}} \\ \le{} & C_5\|I'(u_n) - I'(u_m)\|_*^{\frac{p}{2}}\|u_n - u_m\|^{\frac{p}{2}}(1 + \|u_n\| + \|u_m\|)^{\frac{p(2-p)}{2}},\end{aligned}$$

which implies

$$\begin{aligned}\|u_n - u_m\| &\le C_6\|I'(u_n) - I'(u_m)\|_*(1 + \|u_n\| + \|u_m\|)^{2-p} \\ &\le C_7\|I'(u_n) - I'(u_m)\|_*.\end{aligned} \tag{3.10}$$

From $\{u_n\}$ is bounded and J' is compact, one deduces $J'(u_n)$ has a convergent subsequence. Then from (3.2), (3.8), (3.9) or (3.10) it follows that $\{u_n\}$ has a convergent subsequence. □

Lemma 3.5. *Assume that* (f_2), (f_4) *and* (f_5) *hold, then* Φ *satisfies the* (C) *condition.*

Proof. For any sequence $\{u_n\}$ satisfying (3.2), setting $w_n = \frac{u_n}{\|u_n\|}$, then $\|w_n\| = 1$. Up to a subsequence, we may assume that

$$w_n \rightharpoonup w \text{ weakly in } E,\ w_n \to w \text{ strongly in } L^\nu(\Omega),\ w_n(x) \to w(x) \text{ a.e. } x \in \Omega, \tag{3.11}$$

where $\nu < \frac{Np}{N-2p}$ when $p < \frac{N}{2}$ or $\nu < +\infty$ when $p = \frac{N}{2}$. Through a discussion the same as that in proof of Lemma 3.4, we deduce from (f_2) that $w = 0$.

Setting

$$\kappa' := \max_{|s|\leq M_1} |sf(s) - pF(s)|, \quad \Omega'_n := \{x \in \Omega : |u_n(x)| \geq M_1\},$$

we derive from (3.2), (3.4), (3.5) and (f_4) that

$$\begin{aligned}
\frac{1}{p} - \frac{c + o(1)}{\|u_n\|^p} &= \int_\Omega \frac{F(u_n)}{|u_n|^p}|w_n|^p dx \\
&= \int_{\Omega\setminus\Omega'_n} \frac{F(u_n)}{\|u_n\|^p} dx + \int_{\Omega'_n} \frac{F(u_n)}{|u_n|^p}|w_n|^p dx \\
&\leq \int_{\Omega\setminus\Omega'_n} \frac{F(u_n)}{\|u_n\|^p} dx + \left[\int_{\Omega'_n} \left(\frac{F(u_n)}{|u_n|^p}\right)^\sigma dx\right]^{\frac{1}{\sigma}} \left[\int_{\Omega'_n} |w_n|^{\frac{p\sigma}{\sigma-1}} dx\right]^{\frac{\sigma-1}{\sigma}} \\
&\leq \frac{L_{M_1}|\Omega|}{\|u_n\|^p} + \left\{\int_\Omega [\alpha\left(u_n f(u_n) - pF(u_n)\right) + \beta + \alpha\kappa'] dx\right\}^{\frac{1}{\sigma}} \cdot |u_n|^p_{\frac{p\sigma}{\sigma-1}} \\
&\leq \frac{L_{M_1}|\Omega|}{\|u_n\|^p} + [(\alpha\kappa' + \beta)|\Omega| + \alpha\left(p\Phi(u_n) - \Phi'(u_n)u_n\right)]^{\frac{1}{\sigma}} \cdot |u_n|^p_{\frac{p\sigma}{\sigma-1}}.
\end{aligned}$$

Noting $\frac{p\sigma}{\sigma-1} < \frac{Np}{N-2p}$ when $p < \frac{N}{2}$. Then by (3.11), letting $n \to \infty$ in the above inequality gives

$$\frac{1}{p} \leq 0,$$

a contradiction. Hence, $\|u_n\|$ is bounded in E. The reminders is just the same as that in proof of Lemma 3.4. □

In addition, similar to the argument of Theorem 9.12 in [16], one can prove the following Z_2 version of the Mountain Pass Theorem under the (C) condition.

Lemma 3.6. *Let E be an infinite dimensional Banach space and $\Phi \in C^1(E, \mathbb{R})$ be even, satisfy the* (C) *condition, and $\Phi(0) = 0$. If $E = V \oplus X$, where V finite dimensional, and Φ satisfies*
(Φ_1) *there are constants $\rho,\ b > 0$ such that $\Phi_{\partial B_\rho \cap X} \geq b$,*
(Φ_2) *for each finite dimensional subspace $\widetilde{E} \subset E$, there is an $R = R(\widetilde{E})$ such that $\Phi \leq 0$ on $\widetilde{E} \setminus B_{R(\widetilde{E})}$,*
then I possesses an unbounded sequence of critical values.

Proof of Theorem 1.1. By Lemmas 3.1, 3.3 and 3.4, Φ possesses a mountain pass geometry and satisfies the (C) condition. Then there is a nontrivial solution for problem (2.1) as well as problem (1.1) by Theorem 2.6 in [4]. Moreover, Lemma 3.1 obviously implies (Φ_1). For each finite dimensional subspace $\widetilde{E} \subset E$, the set $\partial B_1 \cap \widetilde{E}$ is a compact subset of $\widetilde{E}$. Hence the continuous functional $\int_\Omega |u|^p dx : \partial B_1 \cap \widetilde{E} \to \mathbb{R}$ attains its minimum. So $\mu := \min_{u \in \partial B_1 \cap \widetilde{E}} \int_\Omega |u|^p dx > 0$. Fixing $K > \frac{1}{\mu}$, one gets $R(\widetilde{E}) := \left(\frac{2pC_K \cdot |\Omega|}{K\mu - 1}\right)^{\frac{1}{p}} > 0$. Thus, for $u \in \partial B_1 \cap \widetilde{E}$ and $t \geq R(\widetilde{E})$, it follows from (3.1) that

$$\Phi(tu) \leq \frac{t^p}{p}\left(1 - K\int_\Omega |u|^p dx\right) + C_K \cdot |\Omega| \leq \frac{(R(\widetilde{E}))^p}{p}(1 - K\mu) + C_K \cdot |\Omega| \leq 0,$$

that is, $\Phi \leq 0$ on $\widetilde{E} \setminus B_{R(\widetilde{E})}$. Therefore, we can obtain infinitely many solutions for problem (2.1) as well as problem (1.1) by applying Lemma 3.6. □

Proof of Theorem 1.2. By Lemmas 3.2, 3.3 and 3.5, Φ possesses a mountain pass geometry and satisfies the (C) condition. The reminders is just the same as that in proof of Theorem 1.1. □

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