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BOGDANOV-TAKENS BIFURCATION IN A DELAYED MICHAELIS-MENTEN TYPE RATIO-DEPENDENT PREDATOR-PREY SYSTEM WITH PREY HARVESTING*

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Abstract In this paper, we study a delayed Michaelis-Menten Type ratio-dependent predator-prey model with prey harvesting. By considering the characteristic equation associated with the nonhyperbolic equilibrium, the critical value of the parameters for the Bogdanov-Takens bifurcation is obtained. The conditions for the characteristic equation having negative real parts are discussed. Using the normal form theory of Bogdanov-Takens bifurcation for retarded functional differential equations, the corresponding normal form restricted to the associated two-dimensional center manifold is calculated and the versal unfolding is considered. The parameter conditions for saddle-node bifurcation, Hopf bifurcation and homoclinic bifurcation are obtained. Numerical simulations are given to support the analytical results.

Keywords Delayed ratio-dependent predator-prey model, Michaelis-Menten type, prey harvesting, Bogdanov-Takens bifurcation.

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1. Introduction

In recent years, ratio-dependent predator-prey systems have been regarded by some researchers as being more appropriate for predator-prey interactions. When predators have to search seriously for food(or compete for food), the functional response depends on the densities of both prey and predator. Roughly stated, the per capita predator growth rate should be a function of the ratio of prey to predator abundance([4, 10, 11, 18, 19, 22, 23, 33]). Based on the Michaelis-Menten or Holling type II function, Arditi and Ginzburg [1] introduced a Michaelis-Menten type ratio-dependent system of the form

$$\begin{cases} \dot{x} = rx(1-\dfrac{x}{K}) - \dfrac{cxy}{x+my}, \\ \dot{y} = y(-d+\dfrac{fx}{x+my}), \end{cases} \tag{1.1}$$

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where x, y stand for densities of prey and predator respectively. The growth rate of the prey is logistic with carrying capacity K and intrinsic growth rate r, the predator consumes the prey according to functional response $\frac{cx}{x+my}$ with the capturing rate c and the half saturation constant m, d is the mortality rate of predator. r, K, d, f, c, m are all positive constants. Since then, many researchers studied the system and obtained rich results([2, 20, 29]). Xiao and Ruan [29] gave global qualitative analysis for the model depending on all parameters and conditions of existence and non-existence of limit cycles were given. In Li and Kuang [20], the existence of a heteroclinic bifurcation for the Michaelis-Menten type ratio-dependent predator-prey system was rigorously established, limit cycles related to the heteroclinic bifurcation were also discussed.

In [3], Beretta and Kuang proposed a ratio-dependent model with a single discrete positive delay τ

$$\begin{cases} \dot{x}(t) = rx(1-\dfrac{x}{K}) - \dfrac{cxy}{x+my}, \\ \dot{y}(t) = y(-d + \dfrac{fx(t-\tau)}{x(t-\tau)+my(t-\tau)}), \end{cases} \tag{1.2}$$

where $x_0(\theta) = \phi_1(\theta) \geq 0,\ y_0(\theta) = \phi_2(\theta) \geq 0,\ \theta \in [-\tau, 0],\ x_0(0) > 0,\ y_0(0) > 0,$ $\phi = (\phi_1, \phi_2) \in C([-\tau, 0], R^2_+),\ R^2_+ = \{(x, y) : x \geq 0, y \geq 0\}$. They made use of a rather novel and non-trivial way of constructing proper Lyapunov functions to obtain some new and significant global stability or convergence results for (1.2). In [30], Xiao and Li discussed the qualitative behaviour of system (1.2) at the equilibrium in the interior of the first quadrant showed that the interior equilibrium cannot be absolutely stable and there exist non-trivial periodic solutions for the model.

Bogdanov-Takens bifurcation was established by Bogdanov [5] and then introduced in many books and articles(see [13] and refereces therein). In 1995, Faria and Magalhã developed normal forms of Bogdanov-Takens singularity for retarded functional differential equations [8]. Since then Bogdanov-Takens bifurcations of the delayed differential equations have been studied by many researchers (see [7, 14–17, 21, 24–26, 28, 31, 32] and refereces therein).

In this paper , we consider Eq.(1.2) with constant prey harvesting as following form

$$\begin{cases} \dot{x}(t) = rx(1-\dfrac{x}{K}) - \dfrac{cxy}{x+my} - h, \\ \dot{y}(t) = y(-d + \dfrac{fx(t-\tau)}{x(t-\tau)+my(t-\tau)}), \end{cases} \tag{1.3}$$

where $h > 0$ represents the rate of harvesting or removal of the prey and the initial conditions are the same as for (1.2). We find that there is Bogdanov-Takens singularity. We consider the negative real parts of the characteristic equation of the linearized part of system (1.3) so that we can discuss the system on a two dimensional center manifold. By unfolding the model at the Bogdanov-Takens singular point, the parameter conditions for saddle-node bifurcation, Hopf bifurcation and homoclinic bifurcation are obtained. Our results show that the introducing of the prey harvesting plays an important role in the system, even though the prey harvesting parameter h is small, the dynamical behaviors of the model are quite different from which of $h = 0$.

This paper is organized as follows. In Section 2, we discuss the existence of Bogdanov-Takens bifurcation of system (1.3). In Section 3, the normal form of system (1.3) at the Bogdanov-Takens singular point is computed. The dynamical classification near the Bogdanov-Takens point is discussed according to the obtained normal forms. In Section 4, some numerical simulations are given to support the analytical results. A brief conclusion is given in Section 5.

2. The existence of Bogdanov-Takens bifurcation

In this section, by analyzing the characteristic equation of the linearized system of Eq.(1.3) at the positive equilibrium, we investigate the existence of Bogdanov-Takens bifurcation. To guarantee that system (1.3) always has at least one positive equilibrium, we assume that coefficients of system (1.3) satisfy the following conditions:

$$(H1) \quad 0 < \beta < \frac{r}{c - mr}, \quad h \le h_* = \frac{K}{4r}(r - \frac{c\beta}{1 + m\beta})^2,$$

where $\beta = \frac{f-d}{md}$.

The positive equilibria are $E_1(x_1, y_1), E_2(x_2, y_2)$, where

$$\begin{aligned} x_1 &= \frac{K}{2r}\left(r - \frac{c\beta}{1+m\beta} + \sqrt{(r - \frac{c\beta}{1+m\beta})^2 - \frac{4rh}{K}}\right), \\ x_2 &= \frac{K}{2r}\left(r - \frac{c\beta}{1+m\beta} - \sqrt{(r - \frac{c\beta}{1+m\beta})^2 - \frac{4rh}{K}}\right), \\ y_i &= \beta x_i, \ i = 1, 2. \end{aligned}$$

When $h = h_*$, then $E_1 = E_2$.

Let $E_* = E_*(x_*, y_*)$ be an arbitrary positive equilibrium. In order to discuss the properties of system (1.3) in the neighborhood of the equilibrium $E_* = E_*(x_*, y_*)$, let $\bar{x} = x - x_*, \bar{y} = y - y_*$, then E_* is translated to (0, 0), and system (1.3) becomes (still denoting $\bar{x}, \bar{y}$ as x, y)

$$\begin{cases} \dot{x}(t) = (\frac{h}{x_*} - \frac{rx_*}{K} + c\beta\delta)x - c\delta y + g_1x^2 + g_2xy + g_3y^2 + h.o.t, \\ \dot{y}(t) = mf\beta^2\delta x(t-\tau) - mf\beta\delta y(t-\tau) + g_4x^2(t-\tau) + g_5x(t-\tau)y(t-\tau) \\ \qquad + g_6y^2(t-\tau) + g_7x(t-\tau)y + g_8y(t-\tau)y + h.o.t, \end{cases} \tag{2.1}$$

where $h.o.t$ denotes the higher order terms which are equal or higher than third order,

$$\begin{aligned} &g_1 = \frac{2cm\beta^2\delta^{3/2}}{x_*} - \frac{2r}{K}, \ g_2 = -\frac{2cm\beta(1+m\beta)\delta^2}{x_*}, \ g_3 = \frac{2cm\delta^{3/2}}{x_*}, \ g_4 = -\frac{2cm\beta^2\delta^{3/2}}{x_*}, \\ &g_5 = \frac{mf\beta\delta^2(1-m^2\beta^2)}{x_*}, \quad g_6 = \frac{2mf\beta\delta^{3/2}}{x_*}, \quad g_7 = \frac{mf\beta\delta}{x_*}, \quad g_8 = -\frac{mf\delta}{x_*}, \\ &\delta = \frac{1}{(1+m\beta)^2}. \end{aligned}$$

The characteristic equation of the linearized part of system (2.1) is

$$F(\lambda,\tau)=\lambda^2+(mf\beta\delta e^{-\lambda\tau}+\frac{rx_*}{K}-\frac{h}{x_*}-c\beta\delta)\lambda+mf\beta\delta(\frac{rx_*}{K}-\frac{h}{x_*})e^{-\lambda\tau}. \quad (2.2)$$

We note that if $h=\frac{rx_*^2}{K}$, it is $h=h_*$. Let $c_*=mf$, then, we have following lemma.

Lemma 2.1. *Suppose that* $(H1)$ *is satisfied and* $h=h_*$, *then*

(i) *if* $c\neq c_*,\ \lambda=0$ *is a single root of Eq.(2.2);*

(ii) *if* $c=c_*,\ \tau\neq\frac{1}{mf\beta\delta},\ \lambda=0$ *is a double root of Eq.(2.2).*

Proof. From Eq. (2.2), the presence of zero roots follows from the fact that $F(0,\tau)=0$. Taking the partial derivative with respective to λ yields

$$\frac{\partial F(0,\tau)}{\partial\lambda}=mf\beta\delta-c\beta\delta.$$

Clearly, $\frac{\partial F(0,\tau)}{\partial\lambda}\neq 0$ if and only if $\tau>0,\ h=h_*,\ c\neq c_*$, which implies the conclusion of (i). We also have

$$\frac{\partial F^2(0,\tau)}{\partial\lambda^2}=2-2\tau mf\beta\delta.$$

It is easy to know if $\tau\neq\frac{1}{mf\beta\delta},\ h=h_*,\ c=c_*$, then $\frac{\partial F^2(0,\tau)}{\partial\lambda^2}\neq 0$, (ii) holds. This completes the proof. □

Lemma 2.2. *Suppose that* $(H1)$ *is satisfied, then*

(i) When $h=h_*, c<c_*,\ \tau\in[0,\min\{\frac{1}{mf\beta\delta},\tau_0\})$, *all roots of Eq.(2.2), except for the zero roots, have negative real parts, where* $\tau_0=\frac{1}{\omega_0}\arccos(\frac{c}{mf}),\ \omega_0=\sqrt{m^2f^2-c^2}\beta\delta$.

(ii) When $h<h_*, c=c_*$, *there is* τ_1 *such that when* $\tau\in[0,\min\{\frac{1}{mf\beta\delta},\tau_1\})$, *all roots of Eq.(2.2), have negative real parts, where* $\tau_1=\frac{1}{\omega_1}\arccos[\frac{\omega_1^2x_*^2}{(h-h_*)^2+\omega_1^2x_*^2}]$,

$$\omega_1=\frac{2c_*(h-h_*)x_*^2\delta\beta-(h-h_*)^2+\sqrt{((h-h_*)^2-2c_*\delta\beta(h-h_*)x_*^2)^2+4c_*^2\delta^2\beta^2(h-h_*)^2x_*^2}}{2x_*^2}.$$

(iii) When $h<h_*, c<c_*$, *there is* τ_2 *such that when* $\tau\in[0,\min\{\frac{1}{mf\beta\delta},\tau_2\})$, *all roots of Eq.(2.2), have negative real parts, where* $\tau_2=\frac{1}{\omega_2}\arccos[\frac{c\omega_2^2(h-h_*)^2}{c_*((h-h_*)^2+\omega_2^2x_*^2)}]$,

$$\omega_2=\frac{c_*^2\delta^2\beta^2x_*^2-(h-h_*-c\delta\beta)^2+\sqrt{(c_*^2\delta^2\beta^2x_*^2-(h-h_*-c\delta\beta)^2)^2+4c_*^2\delta^2\beta^2(h-h_*)^2x_*^2}}{2x_*^2}.$$

Proof. (i) When $h=h_*, c<c_*$, from (2.2), $F(\lambda,\tau)=0$ has root $\lambda_1=0$ and $\lambda+fm\beta\delta e^{-\lambda\tau}-c\beta\delta=0$.

When $\tau=0$, then

$$\lambda_2=\beta\delta(c-c_*)<0. \quad (2.3)$$

When $\tau\neq 0$. Suppose that $i\omega$ is a root of Eq.(2.2) in the imaginary axis. Substituting it to Eq.(2.2) and separating the real and imaginary parts, we have

$$\begin{cases}\omega=mf\beta\delta\sin\omega\tau,\\ c=mf\cos\omega\tau,\end{cases} \quad (2.4)$$

then, $\omega_0 = \sqrt{m^2 f^2 - c^2}\beta\delta > 0$, if $c < c_*$. We obtain that $\tau_0 = \frac{1}{\omega_0}\arccos(\frac{c}{fm})$, then we know that when $\tau \in [0, min\{\frac{1}{mf\beta\delta}, \tau_0\})$, all roots of Eq.(2.2) have negative real parts except zero roots, which implies the conclusion of (i).

(ii) If $h < h_*, c = c_*$, when $\tau = 0$, $\lambda = \frac{-(h_*-h)\pm\sqrt{(h_*-h)^2-4mf\beta\delta x_*(h_*-h)}}{2x_*}$ which have negative real parts. Suppose that $i\omega$ is a root of Eq.(2.2) in the imaginary axis. Substituting it to Eq.(2.2) and separating the real and imaginary parts, we have

$$\begin{cases} \omega c_*\beta\delta x_* \sin\omega\tau + c_*\beta\delta(h-h_*)\cos\omega\tau = \omega^2 x_*, \\ c_*\beta\delta(h-h_*)\sin\omega\tau - \omega c_*\beta\delta x_*\cos\omega\tau = \omega(h-h_*-c_*\beta\delta), \end{cases} \tag{2.5}$$

then, $\omega_1 = \frac{2c_*(h-h_*)x_*^2\delta\beta-(h-h_*)^2+\sqrt{(2c_*\delta\beta(h-h_*)x_*^2-(h-h_*)^2)^2+4c_*^2\delta^2\beta^2(h-h_*)^2x_*^2}}{2x_*^2} > 0$, if $h < h_*, c = c_*$. We obtain that $\tau_1 = \frac{1}{\omega_1}\arccos[\frac{\omega_1^2 x_*^2}{(h-h_*)^2+\omega_1^2 x_*^2}]$, then we know that when $\tau \in [0, \min\{\frac{1}{mf\beta\delta}, \tau_1\})$, all roots of Eq.(2.2) have negative real parts, which implies the conclusion of (ii).

(iii) If $h < h_*, c < c_*$, we can easy see that when $\tau = 0$, λ has negative real parts. Suppose that $i\omega$ is a root of Eq.(2.2) in the imaginary axis. Substituting it to Eq.(2.2) and separating the real and imaginary parts, we have

$$\begin{cases} \omega mf\beta\delta x_* \sin\omega\tau + mf\beta\delta(h-h_*)\cos\omega\tau = \omega^2 x_*, \\ mf\beta\delta(h-h_*)\sin\omega\tau - \omega mf\beta\delta x_*\cos\omega\tau = \omega(h-h_*-c\beta\delta), \end{cases} \tag{2.6}$$

then, $\omega_2 = \frac{c_*^2\delta^2\beta^2x_*^2-(h-h_*-c\delta\beta)^2+\sqrt{(c_*^2\delta^2\beta^2x_*^2-(h-h_*-c\delta\beta)^2)^2+4c_*^2\delta^2\beta^2(h-h_*)^2x_*^2}}{2x_*^2} > 0$. We obtain that $\tau_2 = \frac{1}{\omega_2}\arccos[\frac{c\omega_2^2(h-h_*)^2}{c_*((h-h_*)^2+\omega_2^2x_*^2)}]$, then we know that when $\tau \in [0, \min\{\frac{1}{mf\beta\delta}, \tau_2\})$, all roots of Eq.(2.2) have negative real parts, which implies the conclusion of (iii).

This completes the proof. □

Theorem 2.1. *Suppose that* $(H1)$ *holds and* $h = h_*$, *there exists B-T singularity near* E_* *if the condition* (ii) *of Lemma 2.1 is satisfied.*

Because of the results of Lemma 2.2, we can discuss Bogdanov-Takens bifurcation on the center manifold of two dimension.

3. Normal form for the Bogdanov-Takens bifurcation

In this section, we focus on the dynamics near nonhyperbolic equilibrium $E_*(x_*, y_*)$ when the parameters h and c varies in a small neighbourhood of the bifurcating point (h_*, c_*) . We employ the method in Faria and Magalhães [8] to obtain the normal forms on the center manifold, which determine the dynamics near the Bogdanov-Takens point.

Bogdanov-Takens bifurcation is a codimension-two bifurcation, requiring two independent parameters for its singularity analysis. Thus, we introduce two perturbation parameters μ_1 and μ_2 by considering $h = h_* + \mu_1$, $c = c_* + \mu_2$, such that system (1.3) undergos a Bogdanov-Takens bifurcation at $\mu = (\mu_1, \mu_2) = (0, 0)$, where $\mu_1 \le 0$, $\mu_2 \le 0$. When $\tau \in [0, \min\{\frac{1}{mf\beta\delta}, \tau_0, \tau_1, \tau_2\})$, we have that all roots of Eq.(2.2) have negative real parts except zero roots from above section. Rescale the time by $t \to t\tau$ to normalize the delay and system (1.3) is transformed into

$$\begin{cases} \dot{x} = \tau[(c_* + \mu_2)(\beta\delta x - \delta y) + \dfrac{1}{x_*}\mu_1 x + F_2^1 + h.o.t.], \\ \dot{y} = \tau[mf\beta^2\delta x(t-1) - mf\beta\delta y(t-1) + F_2^2 + h.o.t.], \end{cases} \tag{3.1}$$

where

$$\begin{aligned} F_2^1 &= g_1 x^2 + g_2 xy + g_3 y^2, \\ F_2^2 &= g_4 x^2(t-1) + g_5 x(t-1)y(t-1) + g_6 y^2(t-1) \\ &\quad + g_7 x(t-1)y + g_8 y(t-1)y. \end{aligned}$$

Let $\eta(\theta) = A\delta(\theta) + B\delta(\theta+1)$, where

$$A = \begin{pmatrix} \tau c_*\beta\delta & -\tau c_*\delta \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ \tau mf\beta^2\delta & -\tau mf\beta\delta \end{pmatrix}, \tag{3.2}$$

and define

$$L_0\varphi = \int_{-1}^{0} d\eta(\theta)\varphi(\theta), \forall \varphi \in C.$$

Define the infinitesimal generator

$$A_0\varphi = \begin{cases} \dot{\varphi}, & -1 \le \theta < 0, \\ \int_{-1}^{0} d\eta(\theta)\varphi(\theta), & \theta = 0. \end{cases}$$

Rewrite system (3.1) as

$$\dot{u}_t = L(\mu)u_t + F(u_t, \mu) + h.o.t = (L_0 + L_1(\mu))u_t + F(u_t, \mu) + h.o.t, \tag{3.3}$$

where

$$L_0 u_t = \tau \begin{pmatrix} c_*\beta\delta x(0) - c_*\delta y(0) \\ mf\beta^2\delta x(-1) - mf\beta\delta y(-1) \end{pmatrix},$$

and

$$\begin{aligned} L_1(\mu)u_t &= \mu_2 \begin{pmatrix} \tau\beta\delta x(0) - \tau\delta y(0) \\ 0 \end{pmatrix} + \mu_1 \begin{pmatrix} \frac{\tau}{x_*} x(0) \\ 0 \end{pmatrix}, \\ \tfrac{1}{2!}F(u_t, \mu) &= \tfrac{1}{2!}\tau \begin{pmatrix} g_1 x^2(0) + g_2 x(0)y(0) + g_3 y^2(0) \\ g_4 x^2(-1) + g_5 x(-1)y(-1) + g_6 y^2(-1) + g_7 x(-1)y(0) + g_8 y(-1)y(0) \end{pmatrix}. \end{aligned} \tag{3.4}$$

The bilinear form on $C \times C^*$ is

$$\langle \psi, \varphi \rangle = \psi(0)\varphi(0) - \int_{-1}^{0}\int_{0}^{\theta} \psi(\xi - \theta) d\eta(\theta)\varphi(\theta),$$

where $\varphi(\theta) = (\varphi_1(\theta), \varphi_2(\theta)) \in C$, $\psi(s) = (\psi_1(s), \psi_2(s))^T \in C^*$. Next we will find $\Phi(\theta)$ and $\Psi(s)$ based on the techniques developed by [32].

Lemma 3.1. *The bases of P and its dual space P^* have the following representation*

$$P = span\Phi, \Phi(\theta) = (\varphi_1(\theta), \varphi_2(\theta)), \qquad -1 \le \theta \le 0,$$
$$P^* = span\Psi, \Psi(s) = col(\Psi_1(s), \Psi_2(s)), \quad 0 \le s \le 1,$$

where $\varphi_1(\theta) = \varphi_1^0 \in R^n \backslash \{0\}, \varphi_2(\theta) = \varphi_2^0 + \varphi_1^0\theta, \varphi_1^0 \in R^n$ *and* $\Psi_2(s) = \Psi_2^0 \in R^{2*}\backslash\{0\}, \Psi_1(s) = \Psi_1^0 - s\Psi_2^0, \Psi_1^0 \in R^{2*}$*,which satisfy*

$$(1)(A+B)\varphi_1^0 = 0, \quad (2)(A+B)\varphi_2^0 = (B+I)\varphi_1^0,$$
$$(3)\psi_2^0(A+B) = 0, \quad (4)\psi_1^0(A+B) = \psi_2^0(B+I),$$
$$(5)\psi_2^0\varphi_2^0 - \tfrac{1}{2}\psi_2^0 B\varphi_1^0 + \psi_2^0 B\varphi_2^0 = 1,$$
$$(6)\psi_1^0\varphi_2^0 - \tfrac{1}{2}\psi_1^0 B\varphi_2^0 + \psi_1^0 B\varphi_2^0 + \tfrac{1}{6}\psi_2^0 B\varphi_1^0 - \tfrac{1}{2}\psi_2^0 B\varphi_2^0 = 0.$$

So it is not difficult to verify that

$$\Phi(\theta) = \begin{pmatrix} 1 & \frac{1}{\tau c_*\delta\beta} + \theta \\ \beta & \beta\theta \end{pmatrix}, \ \Psi(s) = \begin{pmatrix} m_1 + s\beta n & m_2 - sn \\ -\beta n & n \end{pmatrix}, \tag{3.5}$$

where

$$m_1 = \frac{\tau c_*\delta\beta(\tau c_*\delta\beta - 2)}{2(\tau c_*\delta\beta - 1)^2}, \quad m_2 = \frac{1 + (\tau c_*\delta\beta - 1)^2}{2\beta(\tau c_*\delta\beta - 1)^2}, \quad n = \frac{\tau c_*\delta}{\tau c_*\delta\beta - 1}.$$

Let $(x,y)^T = \Phi z + v, \ z = (z_1, z_2)^T, v = (v_1, v_2)^T$, then we have

$$\begin{aligned} &x(0) = z_1 + \frac{z_2}{\tau c_*\delta\beta} + v_1(0), \quad y(0) = \beta z_1 + v_2(0), \\ &x(-1) = z_1 + (\frac{1}{\tau c_*\delta\beta} - 1)z_2 + v_1(-1), \quad y(-1) = \beta(z_1 - z_2) + v_2(-1). \end{aligned} \tag{3.6}$$

System (3.1) can be decomposed as

$$\begin{cases} \dot{z} = Jz + \Psi(0)F(\Phi z + v, \mu), \\ \dot{v} = A_{Q^1}v + (I - \pi)X_0 F(\Phi z + v, \mu), \quad z \in R^2, \ v \in Q^1, \end{cases} \tag{3.7}$$

for $v \in Q^1 = Q \cap C \subset Ker\pi$, where A_{Q^1} is the restriction of A_0 as an operator from Q^1 to the Banach space $Ker\pi$. In view of Taylor expansion, we write (3.7) as the following system

$$\begin{cases} \dot{z} = Jz + \sum\limits_{j\ge 2} \frac{1}{j!} f_j^1(z, v, \mu), \\ \dot{v} = A_{Q^1}v + \sum\limits_{j\ge 2} \frac{1}{j!} f_j^2(z, v, \mu). \end{cases} \tag{3.8}$$

From (3.3), (3.4) we have

$$\frac{1}{2!} f_2^1(z, 0, \mu) = \Psi(0)[L_1(\mu)\Phi z + \frac{1}{2!}F(\Phi z, \mu)] = \Psi(0) \begin{pmatrix} \tilde{F}_2^1(z, 0, \mu_1, \mu_2) \\ \tilde{F}_2^2(z, 0, \mu_1, \mu_2) \end{pmatrix}$$

$$= \begin{pmatrix} m_1 & m_2 \\ -\beta n & n \end{pmatrix} \begin{pmatrix} \tilde{F}_2^1 \\ \tilde{F}_2^2 \end{pmatrix} = \begin{pmatrix} m_1\tilde{F}_2^1 + m_2\tilde{F}_2^2 \\ -\beta n\tilde{F}_2^1 + n\tilde{F}_2^2 \end{pmatrix},$$

$$\frac{1}{2!}f_2^2(z,0,\mu) = (I-\pi)X_0[L_1(\mu)\Phi z + \frac{1}{2!}F(\Phi z,\mu)] = (I-\pi)X_0 \begin{pmatrix} \tilde{F}_2^1(z,0,\mu_1,\mu_2) \\ \tilde{F}_2^2(z,0,\mu_1,\mu_2) \end{pmatrix}, \tag{3.9}$$

where

$$\begin{aligned} \tilde{F}_2^1 =& \frac{\tau}{x_*}\mu_1 z_1 + (\frac{1}{c_*}\mu_2 + \frac{1}{x_* c_* \delta\beta}\mu_1)z_2 + \frac{1}{2}\tau g_1(z_1 + \frac{z_2}{\tau c_* \delta\beta})^2 + \frac{1}{2}\tau\beta g_2(z_1 + \frac{z_2}{\tau c_* \delta\beta})z_1 \\ &+ \frac{1}{2}\tau g_3 \beta^2 z_1^2, \\ \tilde{F}_2^2 =& \frac{1}{2}\tau g_4(z_1 - z_2 + \frac{z_2}{\tau c_* \delta\beta})^2 + \frac{1}{2}\tau\beta g_5(z_1 - z_2 + \frac{z_2}{\tau c_* \delta\beta})(z_1 - z_2) \\ &+ \frac{1}{2}\tau\beta^2 g_6(z_1 - z_2)^2 + \frac{1}{2}\tau\beta g_7(z_1 - z_2 + \frac{z_2}{\tau c_* \delta\beta})z_1 + \frac{1}{2}\tau\beta^2 g_8(z_1 - z_2)z_1. \end{aligned}$$

On the center manifold, system (3.1) can be written as

$$\begin{cases} \dot{z_1} = z_2 + m_1\tilde{F}_2^1 + m_2\tilde{F}_2^2, \\ \dot{z_2} = -\beta n\tilde{F}_2^1 + n\tilde{F}_2^2. \end{cases} \tag{3.10}$$

Following the computation of the normal forms introduced by Faria and Magalh$\tilde{a}$es [8], we can get the normal form with versal unfolding on the center manifold:

$$\begin{cases} \dot{z_1} = z_2 + h.o.t, \\ \dot{z_2} = \lambda_1 z_1 + \lambda_2 z_2 + \eta_1 z_1^2 + \eta_2 z_1 z_2 + h.o.t. \end{cases} \tag{3.11}$$

where

$$\begin{aligned} \lambda_1 =& -\frac{n\beta\tau}{x_*}\mu_1, \\ \lambda_2 =& -\frac{n\beta}{c_*}\mu_2 - \frac{n\beta + c_*\delta\beta\tau m_1}{x_* c_* \delta\beta}\mu_1, \\ \eta_1 =& \frac{1}{2}\tau[-n\beta(g_1 + \beta g_2 + \beta^2 g_3) + n(g_4 + \beta g_5 + \beta^2 g_6 + \beta g_7 + \beta^2 g_8)], \\ \eta_2 =& -\frac{n(2g_1 + \beta g_2)}{2c_*\delta} + \tau n g_4(\frac{1}{\tau c_* \delta\beta} - 1) + \frac{1}{2}\tau\beta n g_5(\frac{1}{\tau c_* \delta\beta} - 2) \\ &- \tau\beta^2 n g_6 + \frac{1}{2}\tau\beta n g_7(\frac{1}{\tau c_* \delta\beta} - 1) + \frac{1}{2}\tau\beta^2 n g_8 \\ &- \tau[m_1(g_1 + \beta g_2 + \beta^2 g_3) + m_2(g_4 + \beta g_5 + \beta^2 g_6 + \beta g_7 + \beta^2 g_8)]. \end{aligned}$$

When $\eta_1\eta_2 \neq 0$, the dynamics of system (3.11) in the small neighborhood of Bogdanov- Takens bifurcation point $(\mu_1, \mu_2) = (0, 0)$ can be determined by the normal form truncated to the second order. We assume that $\eta_1 \neq 0,\ \eta_2 \neq 0$. After time rescaling and coordinate transformation given by

$$t \to \frac{\eta_2}{\eta_1}t,\ z_1 \to \frac{\eta_1}{\eta_2^2}(z_1 - \frac{\eta_2^2}{2\eta_1^2}\lambda_1),\ z_2 \to \frac{\eta_1^2}{\eta_2^3}z_2,$$

the normal form (3.11) up to the second order terms becomes

$$\begin{cases} \dot{z_1} = z_2, \\ \dot{z_2} = u_1 + u_2 z_2 + z_1^2 + z_1 z_2, \end{cases} \tag{3.12}$$

where $u_1 = -\frac{\eta_2^4}{4\eta_1^4}\lambda_1^2$, $u_2 = (\frac{\eta_1}{\eta_2}\lambda_2 - \frac{\lambda_1}{2})\frac{\eta_2^2}{\eta_1^2}$.

The complete bifurcation diagrams of system (3.12) can be found in many references and monographs [6, 9, 27]. Here, we briefly list the results for small u_1, u_2 as follows:

(i) System (3.12) undergoes a saddle-node bifurcation on the curve

$$SN = \{(u_1, u_2) : u_1 = 0,\ u_2 \neq 0\},$$

(ii) system (3.12) undergoes a Hopf bifurcation on the curve

$$H = \{(u_1, u_2) : u_1 = -u_2^2,\ u_2 > 0\},$$

(iii) system (3.12) undergoes a homoclinic bifurcation on the curve

$$HL = \{(u_1, u_2) : u_1 = -\frac{49}{25}u_2^2 + o(|u_2|^{5/2}),\ u_2 > 0\}.$$

Applying the above results and using the parameters of μ_1, μ_2, we obtain the following result.

Theorem 3.1. *Suppose that $(H1)$ holds.*

(i) System (3.1) undergoes a saddle-node bifurcation on the curve

$$SN = \{(\mu_1, \mu_2) : \mu_1 = 0, \mu_2 < 0\},$$

(ii) system (3.1) undergoes a Hopf bifurcation on the curve

$$H = \{(\mu_1, \mu_2) : \mu_1 < 0, \mu_2 = c_*(\frac{\eta_2\tau}{\eta_1 x_*} - \frac{\tau c_*\delta m_1 + n}{x_* c_*\delta\beta n})\mu_1\},$$

(iii) system (3.1) undergoes a homoclinic bifurcation on the curve

$$HL = \{(\mu_1, \mu_2) : \mu_1 < 0, \mu_2 = \frac{-p_1 + \sqrt{p_1^2 - 4p_2}}{2}\mu_1\},$$

where $p_1 = \frac{2(n+\tau c_*\delta m_1)}{n\beta\delta x_*} - \frac{\tau c_*\eta_2}{x_*\eta_1}$, $p_2 = \frac{(n+\tau c_*\delta m_1)^2}{n^2\beta^2\delta^2 x_*^2} - \frac{\tau c_*\eta_2(n+\tau c_*\delta m_1)}{n\beta\delta x_*^2\eta_1} + \frac{6\tau^2 c_*^2\eta_2^2}{49x_*^2\eta_1^2}$, *and we suppose that* $\frac{\eta_2\tau}{\eta_1 x_*} - \frac{\tau c_*\delta m_1 + n}{x_* c_*\delta\beta n} > 0$, $\frac{-p_1+\sqrt{p_1^2-4p_2}}{2} > 0$.

The bifurcation diagram of system (3.1) in the plane of the perturbation parameters μ_1 and μ_2 is sketched in Fig.1. The $\mu_1 - \mu_2$ plane for $\mu_1 < 0, \mu_2 < 0$ is divided into three regions ①, ②, ③ by the Hopf bifurcation line H and the homoclinic bifurcation line HL. On the line SN, there is only one equilibrium, which is a saddle-node. In the region ① there are two equilibria, the left one is a saddle and the right one is a stable focus. For fixed μ_1, when μ_2 goes up through the line H, Hopf bifurcation occurs. In the region ②, there is a stable periodic solution and the focus changes its stability. On the curve HL, there is a homoclinic orbit connecting the left equilibrium to itself and the periodic solution vanishes. In the region ③ there are a unstable focus and a saddle.

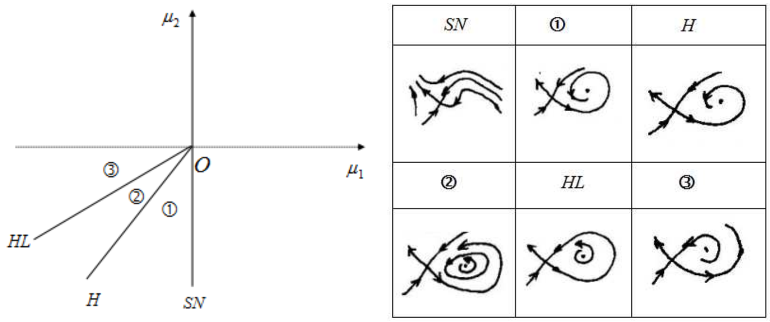

Figure 1. The B-T bifurcation diagram and phase portraits of system (3.1) in the $\mu_1 - \mu_2$ plane for $\mu_1 < 0, \mu_2 < 0$.

4. Numerical simulations

In this section, we present numerical simulations of (1.3) to support the analytical results obtained above.

According to the center manifold theory and the method of the normal form for FDEs developed in [12, 27], the dynamics of the original system (1.3) near the positive equilibrium $E_*(x_*, y_*)$ are topologically equivalent to that of the associated normal form (3.1) on the center manifold. Therefore, for given r, K, c, d, m and f, we can get the Bogdanov-Takens point $h_*,\ c_*$. Let $h = h_* + \mu_1, c = c_* + \mu_2$, when the perturbation parameters (μ_1, μ_2) vary small, the local representations of the bifurcation curves can be determined by the normal form (3.1) on the center manifold. The dynamical classification of the original system (1.3) near the positive equilibrium $E_*(x_*, y_*)$ is obtained when the parameters (h, c) vary in a small neighborhood of the Bogdanov-Takens point (h_*, c_*).

To demonstrate the main results obtained in the previous sections, we choose $r = 1,\ K = 1,\ d = 1.14159,\ m = 1,\ f = 2, \tau = 1.5$. Then we can compute the B-T point: $c_* = mf = 2,\ h_* = \frac{K}{4r}(r - \frac{c_*\beta}{1+m\beta})^2 = 0.005$, and $E_* = (0.0708, 0.0532)$.

To investigate the dynamics of system (1.3), we fix $\mu_1 = -0.001 < 0$ and change the perturbation parameter μ_2 from c_*. For fixed $\mu_1 = -0.001, \mu_2 = -0.4$, there is two positive equilibria $E_1 = (0.2999, 0.2255),\ E_2 = (0.0134, 0.0101)$ bifurcating from the positive equilibrium E_*, where the positive equilibrium E_2 is a saddle, and E_1 is a stable focus(see Fig.2). E_2 is always a saddle and the stability of E_1 will be changed when Hopf bifurcations occur with the increasing of the parameter μ_2. Fig.3 is a numerical simulation of system (1.3) with $\mu_1 = -0.001, \mu_2 = -0.145$ when (μ_1, μ_2) increases across the Hopf bifurcation line $H = \{(\mu_1, \mu_2) : \mu_1 < 0, \mu_2 = 147.9977\mu_1\}$ into ②, the bifurcating periodic solution occurs, where $E_1 = (0.1818, 0.1367),\ E_2 = (0.0221, 0.0166)$. When (μ_1, μ_2) continuously increases closing to the homoclinic bifurcation line $HL = \{(\mu_1, \mu_2) : \mu_1 < 0, \mu_2 = 34.0576\mu_1\}$, the bifurcating periodic solution is near the homoclinic orbit , as shown in Fig.4 when $\mu_1 = -0.001, \mu_2 = -0.035$ and $E_1 = (0.1243, 0.0935),\ E_2 = (0.0323, 0.0243)$. When (μ_1, μ_2) crosses the line HL entering the region ③, the periodic solutions disappear and E_1 becomes unstable.

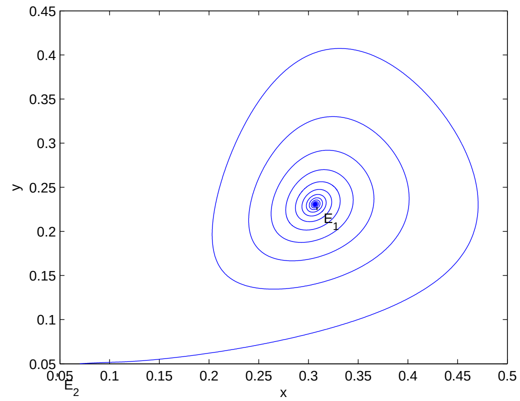

Figure 2. Numerical simulations for system (1.3) near the Bogdanov-Takens bifurcation point $(h_*,\ c_*)$ for $(\mu_1, \mu_2) = (-0.001, -0.4)$, $(\mu_1, \mu_2) \in$ ①, where the positive equilibrium E_1 is stable.

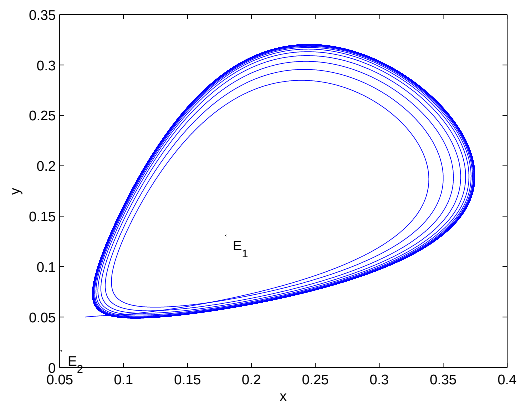

Figure 3. Numerical simulations for system (1.3) near the Bogdanov-Takens bifurcation point $(h_*,\ c_*)$ for $(\mu_1, \mu_2) = (-0.001, -0.145)$, $(\mu_1, \mu_2) \in$ ② sufficiently close to the line H. E_1 is unstable and a stable bifurcating periodic solution occurs.

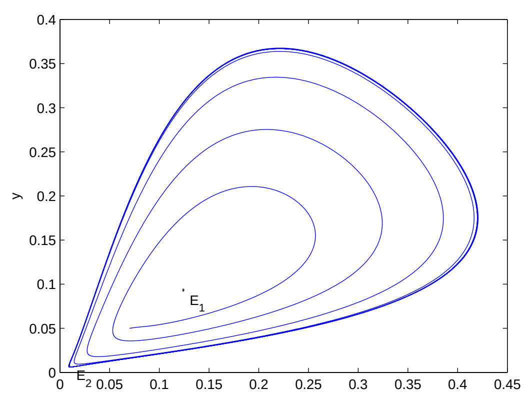

Figure 4. E_1 is unstable for $(\mu_1, \mu_2) = (-0.001, -0.035)$, $(\mu_1, \mu_2) \in$ ② sufficiently closing to the line HL. The orbit of the periodic solution sufficiently closes to the homoclinic orbit.

5. Conclusion

In this paper, we study a delayed ratio-dependent predator-prey model with prey harvesting. Using h and c as bifurcating parameters, we find that h, the rate of the harvesting of prey, has an important effect on the dynamics of the model. For the Bogdanov-Takens singularity at the parameters h_* and c_*, the model has a positive equilibrium at which the corresponding characteristic equation has two zero eigenvalues. Considering the normal form on the two dimensional center manifold, the small neighbourhood of the Bogdanov-Takens singularity in $h < h_*, c < c_*$ can be divided into three dynamical regions by two bifurcation lines: Hopf bifurcation line H and homoclinic bifurcation line HL. When the prey harvesting parameter lies in some specific interval, there exists a stable limit cycle which means that when the initial population of the prey and predator are near this cycle, both the prey and predator populations oscillate about this cyclic state. Our results show that the introduction of the prey harvesting plays an important role for the oscillation period. When the parameters (h, c) near the line HL in the region ②, the oscillation period will become more longer. We show that even though the prey harvesting parameter h is small, as long as it is not zero, the dynamical behaviors of the model are quite different from which of $h = 0$ considered in [3,30], the saddle-node bifurcation, Hopf bifurcation and homoclinic bifurcation occur from the Bogdanov-Takens bifurcation.

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