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A SEARCH FOR LUMP SOLUTIONS TO A COMBINED FOURTH-ORDER NONLINEAR PDE IN (2+1)-DIMENSIONS*

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Abstract We aim to explore new (2+1)-dimensional nonlinear equations which possess lump solutions. Through the Hirota bilinear method, we formulate a combined fourth-order nonlinear equation while guaranteeing the existence of lump solutions. The class of lump solutions is constructed explicitly in terms of the coefficients of the combined nonlinear equation via symbolic computations. Specific examples are discussed to show the richness of the considered combined nonlinear equation. Three dimensional plots and contour plots of specific lump solutions to two specially chosen cases of the equation are made to shed light on the presented lump solutions.

Keywords Lump solution, Hirota bilinear method, integrable equation, symbolic computation, soliton.

MSC(2010) 35Q51, 35Q53, 37K40.

1. Introduction

Explicitly solvable differential equations include constant-coefficient and linear differential equations, but usually, it is difficult to construct exact solutions to variable-coefficient or nonlinear equations. Soliton theory provides a few working methods to solve nonlinear partial differential equations [1, 39]. Among them is the Hirota bilinear approach to soliton solutions, historically developed for integrable equations [2, 11].

Soliton solutions are analytic and exponentially localized. Let a polynomial P

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determine a Hirota bilinear differential equation

$$P(D_x, D_y, D_t)f \cdot f = 0$$

in (2+1)-dimensions, D_x, D_y and D_t being Hirota's bilinear derivatives. The corresponding partial differential equation with a dependent variable u is defined often by one of the logarithmical transformations: $u = 2(\ln f)_x$ or $u = 2(\ln f)_{xx}$. Within the Hirota bilinear formulation, soliton solutions are formulated as follows:

$$f = \sum_{\mu=0,1} \exp(\sum_{i=1}^{N} \mu_i \xi_i + \sum_{i<j} \mu_i \mu_j a_{ij}),$$

where $\sum_{\mu=0,1}$ denotes the sum over all possibilities for $\mu_1, \mu_2, \cdots, \mu_N$ taking either 0 or 1, and the wave variables and the phase shifts are given by

$$\xi_i = k_i x + l_i y - \omega_i t + \xi_{i,0},\ 1 \le i \le N,$$

and

$$\mathrm{e}^{a_{ij}} = -\frac{P(k_i - k_j, l_i - l_j, \omega_j - \omega_i)}{P(k_i + k_j, l_i + l_j, \omega_j + \omega_i)},\ 1 \le i < j \le N,$$

in which k_i, l_i and ω_i, $1 \le i \le N$, satisfy the corresponding dispersion relation and $\xi_{i,0}$, $1 \le i \le N$, are arbitrary phase shifts.

Lump solutions are a class of rational function solutions which are localized in all directions in space, originated from solving integrable equations in (2+1)-dimensions (see, e.g., [35, 36, 42]). Long wave limits of N-soliton solutions generate specific lumps as envelope solutions [40]. Various (2+1)-dimensional integrable equations exhibit the remarkable richness of lump solutions (see, e.g., [35,36]), which can be used to describe various wave phenomena in sciences. The KPI equation possesses abundant lump solutions (see, e.g., [23]), among which are special lump solutions derived from N-soliton solutions [37]. Other integrable equations which possess lump solutions include the three-dimensional three-wave resonant interaction [14], the BKP equation [8, 51], the Davey-Stewartson II equation [40], the Ishimori-I equation [13] and the KP equation with a self-consistent source [57]. An important step in the process of computing lumps is to find positive quadratic function solutions to Hirota bilinear equations [35]. Then one presents lump solutions to nonlinear equations through the logarithmical transformations mentioned above.

In this paper, we would like to consider a combined fourth-order nonlinear equation in (2+1)-dimensions which has abundant lump solution structures. The Hirota bilinear form is the key element that our discussion is based on (see, e.g., [22, 35, 36, 68] for other equations). We will formulate a combined fourth-order nonlinear equation including all second-order linear terms while guaranteeing the existence of lump solutions. We will present the simplified expressions for lump solutions with Maple symbolic computations. For two specially chosen cases of the nonlinear equation, three dimensional plots and contour plots will be made for two specific lump solutions via the Maple plot tool, to shed light on the presented lump solutions. Some concluding remarks will be given in the final section.

2. A combined fourth-order nonlinear PDE and its lump solutions

2.1. A combined fourth-order nonlinear model

We would like to consider a general combined fourth-order nonlinear equation as follows:

$$\begin{aligned} P(u) &= \alpha[3(u_xu_t)_x + u_{xxxt}] + \beta[3(u_xu_y)_x + u_{xxxy}] \\ &\quad + \gamma_1u_{yt} + \gamma_2u_{xx} + \gamma_3u_{xt} + \gamma_4u_{xy} + \gamma_5u_{yy} + \gamma_6u_{tt} = 0, \end{aligned} \tag{2.1}$$

where the constants α and β satisfy $\alpha^2 + \beta^2 \neq 0$, but the constants $\gamma_i,\ 1 \le i \le 6$, are all arbitrary. The equation contains all linear second-order derivative terms.

When $\alpha = 1,\ \beta = 0,\ \gamma_1 = \gamma_2 = 1$ and the other γ_i's are zero, we obtain the Hirota-Satsuma-Ito (HSI) equation in (2+1)-dimensions [10], an integrable $(2+1)$-dimensional extension of the Hirota-Satsuma equation [11]:

$$3(u_xu_t)_x + u_{xxxt} + u_{yt} + u_{xx} = 0, \tag{2.2}$$

which passes the Hirota three-soliton test and has a bilinear form under the logarithmic transformation $u = 2(\ln f)_x$:

$$(D_x^3D_t + D_yD_t + D_x^2)f \cdot f = 0. \tag{2.3}$$

This equation is called the bilinear HSI equation.

When $\alpha = 0,\ \beta = 1,\ \gamma_3 = \gamma_5 = 1$ and the other γ_i's are zero, we obtain a generalized Calogero-Bogoyavlenskii-Schiff equation [3]:

$$3(u_xu_y)_x + u_{xxxy} + u_{xt} + u_{yy} = 0, \tag{2.4}$$

which possesses a Hirota bilinear form

$$(D_x^3D_y + D_xD_t + D_y^2)f \cdot f = 0, \tag{2.5}$$

under $u = 2(\ln f)_x$, and has lump solutions [3].

Generally, the combined nonlinear equation (2.1) has a bilinear form

$$\begin{aligned} B(f) &= (\alpha D_x^3D_t + \beta D_x^3D_y + \gamma_1D_yD_t + \gamma_2D_x^2 \\ &\quad + \gamma_3D_xD_t + \gamma_4D_xD_y + \gamma_5D_y^2 + \gamma_6D_t^2)f \cdot f = 0, \end{aligned} \tag{2.6}$$

under the logarithmic transformation

$$u = 2(\ln f)_x = \frac{2f_x}{f}. \tag{2.7}$$

Precisely, we have the relation $P(u) = (\frac{B(f)}{f^2})_x$, where u and f satisfy the link (2.7).

2.2. Lump solutions

In the subsequent discussion, we would like to determine lump solutions to the combined fourth-order (2+1)-dimensional nonlinear equation (2.1), through symbolic computations with Maple.

We start to search for positive quadratic solutions to the combined bilinear equation (2.6):

$$f = (a_1x + a_2y + a_3t + a_4)^2 + (a_5x + a_6y + a_7t + a_8)^2 + a_9, \tag{2.8}$$

where the constant parameters a_i, $1 \le i \le 9$, are to be determined, in order to generate lump solutions to the combined fourth-order nonlinear equation (2.1).

We first consider the case of $\gamma_6 = 0$ for the combined nonlinear equation (2.1). A direct symbolic computation gives a set of solutions for the parameters, where

$$\begin{cases} a_3 = -\dfrac{b_1}{(a_2\gamma_1 + a_1\gamma_3)^2 + (a_6\gamma_1 + a_5\gamma_3)^2}, \\ a_7 = -\dfrac{b_2}{(a_2\gamma_1 + a_1\gamma_3)^2 + (a_6\gamma_1 + a_5\gamma_3)^2}, \\ a_9 = \dfrac{3\,(a_1^2 + a_5^2)(\alpha b_3 - \beta b_4)}{(a_1a_6 - a_2a_5)^2(\gamma_1^2\gamma_2 - \gamma_1\gamma_3\gamma_4 + \gamma_3^2\gamma_5)}, \end{cases} \tag{2.9}$$

and all other a_i's are arbitrary. The above involved four constants are defined as follows:

$$\begin{cases} b_1 = [(a_1^2a_2 + 2\,a_1a_5a_6 - a_2a_5^2)\gamma_2 + a_1(a_2^2 + a_6^2)\gamma_4 + a_2(a_2^2 + a_6^2)\gamma_5]\gamma_1 \\ \qquad +[a_1(a_1^2 + a_5^2)\gamma_2 + a_2(a_1^2 + a_5^2)\gamma_4 + (a_1a_2^2 + 2\,a_2a_5a_6 - a_1a_6^2)\gamma_5]\gamma_3, \\ b_2 = [(-a_1^2a_6 + 2\,a_1a_2a_5 + a_5^2a_6)\gamma_2 + a_5(a_2^2 + a_6^2)\gamma_4 + a_6(a_2^2 + a_6^2)\gamma_5]\gamma_1 \\ \qquad +[a_5(a_1^2 + a_5^2)\gamma_2 + a_6(a_1^2 + a_5^2)\gamma_4 + (-a_2^2a_5 + 2\,a_1a_2a_6 + a_5a_6^2)\gamma_5]\gamma_3, \\ b_3 = (a_1^2 + a_5^2)(a_1a_2 + a_5a_6)(\gamma_1\gamma_2 + \gamma_3\gamma_4) + (a_1^2 + a_5^2)(a_2^2 + a_6^2)\gamma_1\gamma_4 \\ \qquad +(a_1^2 + a_5^2)^2\gamma_2\gamma_3 + (a_2^2 + a_6^2)(a_1a_2 + a_5a_6)\gamma_1\gamma_5 \\ \qquad +[(a_1a_2 + a_5a_6)^2 - (a_1a_6 - a_2a_5)^2]\gamma_3\gamma_5, \\ b_4 = (a_1a_2 + a_5a_6)[(a_2\gamma_1 + a_1\gamma_3)^2 + (a_6\gamma_1 + a_5\gamma_3)^2]. \end{cases} \tag{2.10}$$

We secondly consider the case of $\gamma_5 = 0$ for the combined nonlinear equation (2.1). A similar direct symbolic computation provides us with a set of solutions for the parameters, where

$$\begin{cases} a_2 = -\dfrac{c_1}{(a_3\gamma_1 + a_1\gamma_4)^2 + (a_7\gamma_1 + a_5\gamma_4)^2}, \\ a_6 = -\dfrac{c_2}{(a_3\gamma_1 + a_1\gamma_4)^2 + (a_7\gamma_1 + a_5\gamma_4)^2}, \\ a_9 = \dfrac{3\,(a_1^2 + a_5^2)(\beta c_3 - \alpha c_4)}{(a_1a_7 - a_3a_5)^2(\gamma_1^2\gamma_2 - \gamma_1\gamma_3\gamma_4 + \gamma_4^2\gamma_6)}, \end{cases} \tag{2.11}$$

and all other a_i's are arbitrary. The above involved four constants are defined as follows:

$$\begin{cases} c_1 = [(a_1^2a_3 + 2\,a_1a_5a_7 - a_3a_5^2)\gamma_2 + a_1(a_3^2 + a_7^2)\gamma_3 + a_3(a_3^2 + a_7^2)\gamma_6]\gamma_1 \\ \qquad +[a_1(a_1^2 + a_5^2)\gamma_2 + a_3(a_1^2 + a_5^2)\gamma_3 + (a_1a_3^2 + 2\,a_3a_5a_7 - a_1a_7^2)\gamma_6]\gamma_4, \\ c_2 = [(-a_1^2a_7 + 2\,a_1a_3a_5 + a_5^2a_7)\gamma_2 + a_5(a_3^2 + a_7^2)\gamma_3 + a_7(a_3^2 + a_7^2)\gamma_6]\gamma_1 \\ \qquad +[a_5(a_1^2 + a_5^2)\gamma_2 + a_7(a_1^2 + a_5^2)\gamma_3 + (-a_3^2a_5 + 2\,a_1a_3a_7 + a_5a_7^2)\gamma_6]\gamma_4, \\ c_3 = (a_1^2 + a_5^2)(a_1a_3 + a_5a_7)(\gamma_1\gamma_2 + \gamma_3\gamma_4) + (a_1^2 + a_5^2)(a_3^2 + a_7^2)\gamma_1\gamma_3 \\ \qquad +(a_1^2 + a_5^2)^2\gamma_2\gamma_4 + (a_3^2 + a_7^2)(a_1a_3 + a_5a_7)\gamma_1\gamma_6 \\ \qquad +[(a_1a_3 + a_5a_7)^2 - (a_1a_7 - a_3a_5)^2]\gamma_4\gamma_6, \\ c_4 = (a_1a_3 + a_5a_7)[(a_3\gamma_1 + a_1\gamma_4)^2 + (a_7\gamma_1 + a_5\gamma_4)^2]. \end{cases} \tag{2.12}$$

All the above formulas in (2.9), (2.10), (2.11) and (2.12) were obtained under a certain simplification process with Maple.

For the case of $\gamma_5 = 0$, we need to check when the set of the resulting parameters presents lumps, and so, we compute

$$\begin{aligned} & a_1a_6 - a_2a_5 \\ & = \frac{(a_1a_7 - a_3a_5)[(a_1^2 + a_5^2)(\gamma_1\gamma_2 - \gamma_3\gamma_4) - (a_3^2 + a_7^2)\gamma_1\gamma_6 - 2(a_1a_3 + a_5a_7)\gamma_4\gamma_6]}{(a_3\gamma_1 + a_1\gamma_4)^2 + (a_7\gamma_1 + a_5\gamma_4)^2}. \end{aligned}$$

It then follows that $a_1a_6 - a_2a_5 \neq 0$ if and only if

$$\begin{cases} a_1a_7 - a_3a_5 \neq 0, \ \gamma_1^2 + \gamma_4^2 \neq 0, \\ (a_1^2 + a_5^2)(\gamma_1\gamma_2 - \gamma_3\gamma_4) - (a_3^2 + a_7^2)\gamma_1\gamma_6 - 2(a_1a_3 + a_5a_7)\gamma_4\gamma_6 \neq 0, \end{cases} \tag{2.13}$$

which, together with $a_9 > 0$, guarantees that the corresponding set of the parameters will present lump solutions.

2.3. Diverse examples of the combined equation

We enumerate diverse examples of the considered combined nonlinear equation, based on the presented lump solutions above in the two solution situations.

2.3.1. The case of $\gamma_1 = \gamma_2 = 1$

When $\gamma_1 = \gamma_2 = 1$ and the other γ_i's are zero, the combined bilinear equation (2.6) reduces to

$$(\alpha D_x^3D_t + \beta D_x^3D_y + D_yD_t + D_x^2)f \cdot f = 0. \tag{2.14}$$

The subcase of $\alpha = 0$ and $\beta = 1$ gives the dimensionally reduced Jimbo-Miwa equation with $z = x$ [25].

The subcase of $\alpha = 1$ and $\beta = 0$ gives us the original HSI equation in (2+1)-dimensions (2.2). Moreover, the function f by (2.8) with (2.9) and (2.10) presents a class of lump solutions to the HSI equation (2.2), where

$$\begin{cases} a_3 = -\dfrac{a_1^2 a_2 + 2a_1 a_5 a_6 - a_2 a_5^2}{a_2^2 + a_6^2}, \\ a_7 = \dfrac{a_1^2 a_6 - 2a_1 a_2 a_5 - a_5^2 a_6}{a_2^2 + a_6^2}, \\ a_9 = \dfrac{3(a_1^2 + a_5^2)^2 (a_1 a_2 + a_5 a_6)}{(a_1 a_6 - a_2 a_5)^2}, \end{cases} \tag{2.15}$$

and all other a_i's are arbitrary; and the function f by (2.8) with (2.11) and (2.12) presents another class of lump solutions to the HSI equation (2.2), where [69]

$$\begin{cases} a_2 = -\dfrac{a_1^2 a_3 + 2a_1 a_5 a_7 - a_3 a_5^2}{a_3^2 + a_7^2}, \\ a_6 = \dfrac{a_1^2 a_7 - 2a_1 a_3 a_5 - a_5^2 a_7}{a_3^2 + a_7^2}, \\ a_9 = -\dfrac{3(a_1^2 + a_5^2)(a_3^2 + a_7^2)(a_1 a_3 + a_5 a_7)}{(a_1 a_7 - a_3 a_5)^2}. \end{cases} \tag{2.16}$$

It is easy to see that

$$a_1 a_3 + a_5 a_7 = -\frac{(a_1^2 + a_5^2)(a_1 a_2 + a_5 a_6)}{a_2^2 + a_6^2}, \tag{2.17}$$

and

$$a_1 a_6 - a_2 a_5 = \frac{(a_1^2 + a_5^2)(a_1 a_7 - a_3 a_5)}{a_3^2 + a_7^2}. \tag{2.18}$$

It then follows that the conditions of

$$a_1 a_2 + a_5 a_6 > 0, \ a_1 a_6 - a_2 a_5 \neq 0, \tag{2.19}$$

under which f by (2.8) with (2.15) will present lump solutions to (2.2), are equivalent to the conditions of

$$a_1 a_3 + a_5 a_7 < 0, \ a_1 a_7 - a_3 a_5 \neq 0, \tag{2.20}$$

under which f by (2.8) with (2.16) will present lump solutions to (2.2). Therefore, the two classes of lump solutions above are the same and just have different representations.

2.3.2. The case of $\gamma_3 = \gamma_5 = 1$

When $\gamma_3 = \gamma_5 = 1$ and the other γ_i's are zero, the combined bilinear equation (2.6) reduces to

$$(\alpha D_x^3 D_t + \beta D_x^3 D_y + D_x D_t + D_y^2) f \cdot f = 0. \tag{2.21}$$

The subcase of $\alpha = 1$ and $\beta = 0$ gives us a new model, for which a specific lump will be presented later. The subcase of $\alpha = 0$ and $\beta = 1$ is the generalized Calogero-Bogoyavlenskii-Schiff equation discussed in [3].

2.3.3. The case of $\gamma_4 = \gamma_6 = 1$

When $\gamma_4 = \gamma_6 = 1$ and the other γ_i's are zero, the combined bilinear equation (2.6) reduces to

$$(\alpha D_x^3 D_t + \beta D_x^3 D_y + D_x D_y + D_t^2) f \cdot f = 0. \tag{2.22}$$

The lump condition $a_1a_6 - a_2a_5 \neq 0$ requires

$$a_1a_7 - a_3a_5 \neq 0, \;\; a_1a_3 + a_5a_7 \neq 0; \tag{2.23}$$

and thus, $a_9 > 0$ requires

$$\alpha(a_1a_3 + a_5a_7)(a_1^2 + a_5^2) + \beta(a_1a_7 - a_3a_5)^2 < 0. \tag{2.24}$$

It then follows that if $\alpha = 0$, one needs to require $\beta < 0$ to have lumps. A specific lump in the subcase of $\alpha = 1$ and $\beta = 1$ will be presented and plotted later.

2.3.4. The case of $\gamma_1 = \gamma_3 = \gamma_4 = 1$

When $\gamma_1 = \gamma_3 = \gamma_4 = 1$ and the other γ_i's are zero, the combined bilinear equation (2.6) reduces to

$$(\alpha D_x^3 D_t + \beta D_x^3 D_y + D_y D_t + D_x D_t + D_x D_y) f \cdot f = 0, \tag{2.25}$$

which provides two new nonlinear equations possessing lump solutions, when $\alpha = 1$ and $\beta = 0$ or when $\alpha = 0$ and $\beta = 1$.

2.3.5. The case of $\gamma_1 = \gamma_3 = \gamma_5 = 1$

When $\gamma_1 = \gamma_3 = \gamma_5 = 1$ and the other γ_i's are zero, the combined bilinear equation (2.6) reduces to

$$(\alpha D_x^3 D_t + \beta D_x^3 D_y + D_y D_t + D_x D_t + D_y^2) f \cdot f = 0. \tag{2.26}$$

Note that $a_1a_6 - a_2a_5 \neq 0$ leads to $(a_1 + a_2)^2 + (a_5 + a_6)^2 \neq 0$, which guarantees a_3 and a_7 in (2.9) are well defined. Therefore, besides $a_1a_6 - a_2a_5 \neq 0$, the condition for guaranteeing lumps is

$$\begin{aligned}&\alpha\{(a_2^2 + a_6^2)(a_1a_2 + a_5a_6) + [(a_1a_2 + a_5a_6)^2 - (a_1a_6 - a_2a_5)^2]\} \\ &- \beta\{(a_1a_2 + a_5a_6)[(a_1 + a_2)^2 + (a_5 + a_6)^2]\} > 0,\end{aligned} \tag{2.27}$$

which guarantees that a_9 defined in (2.9) is positive.

2.3.6. The case of $\gamma_1 = \gamma_4 = \gamma_6 = 1$

When $\gamma_1 = \gamma_4 = \gamma_6 = 1$ and the other γ_i's are zero, the combined bilinear equation (2.6) reduces to

$$(\alpha D_x^3 D_t + \beta D_x^3 D_y + D_y D_t + D_x D_y + D_t^2) f \cdot f = 0. \tag{2.28}$$

Note that $a_1a_7 - a_3a_5 \neq 0$ leads to $(a_1 + a_3)^2 + (a_5 + a_7)^2 \neq 0$, which guarantees a_2 and a_6 in (2.11) are well defined. Also, it is easily seen that in this case, we have

$$a_1a_6 - a_2a_5 = \frac{(a_1a_7 - a_3a_5)[(a_3^2 + a_7^2) + 2(a_1a_3 + a_5a_7)]}{(a_1 + a_3)^2 + (a_5 + a_7)^2}.$$

Therefore, the conditions for guaranteeing lumps are

$$a_1a_7 - a_3a_5 \neq 0,\ (a_3^2 + a_7^2) + 2(a_1a_3 + a_5a_7) \neq 0, \tag{2.29}$$

and

$$\begin{aligned}&\alpha\{(a_1a_3 + a_5a_7)[(a_1 + a_3)^2 + (a_5 + a_7)^2]\}\\ &\ - \beta\{(a_3^2 + a_7^2)(a_1a_3 + a_5a_7) + [(a_1a_3 + a_5a_7)^2 - (a_1a_7 - a_3a_5)^2]\} < 0,\end{aligned} \tag{2.30}$$

which, together with the first condition in (2.29), guarantees that a_9 defined in (2.11) is positive.

2.4. Profiles of two specific lumps

First, particularly taking

$$\alpha = 1,\ \beta = 0,\ \gamma_3 = \gamma_5 = 1,\ \gamma_1 = \gamma_2 = \gamma_4 = \gamma_6 = 0, \tag{2.31}$$

we obtain a special fourth-order nonlinear equation as follows:

$$u_{xxxt} + 3(u_xu_t)_x + u_{xt} + u_{yy} = 0, \tag{2.32}$$

which has a Hirota bilinear form

$$(D_x^3D_t + D_xD_t + D_y^2)f \cdot f = 0,$$

under the logarithmic transformation (2.7). Associated with

$$a_1 = 1,\ a_2 = -2,\ a_4 = 3,\ a_5 = -1,\ a_6 = 1, a_8 = -7, \tag{2.33}$$

the transformation (2.7) with (2.8) provides a lump solution to the special fourth-order nonlinear equation (2.32):

$$u_1 = \frac{2(-8t + 4x - 6y - 8)}{(-\frac{7}{2}t + x - 2y + 3)^2 + (\frac{1}{2}t - x + y + 7)^2 + 48}. \tag{2.34}$$

Three three-dimensional plots and contour plots of this lump solution are made via Maple plot tools, to shed light on the characteristic of the presented lump solutions, in Figure 1.

Secondly, particularly taking

$$\alpha = 1,\ \beta = 1, \gamma_4 = \gamma_6 = 1,\ \gamma_1 = \gamma_2 = \gamma_3 = \gamma_5 = 0, \tag{2.35}$$

we obtain another special fourth-order nonlinear equation as follows:

$$u_{xxxt} + 3(u_xu_t)_x + u_{xxxy} + 3(u_xu_y)_x + u_{xy} + u_{tt} = 0, \tag{2.36}$$

which has a Hirota bilinear form

$$(D_x^3D_t + D_x^3D_y + D_xD_y + D_t^2)f \cdot f = 0,$$

under the logarithmic transformation (2.7). Associated with

$$a_1 = 2,\ a_3 = -1,\ a_4 = 6,\ a_5 = 1,\ a_7 = -1, a_8 = 10, \tag{2.37}$$

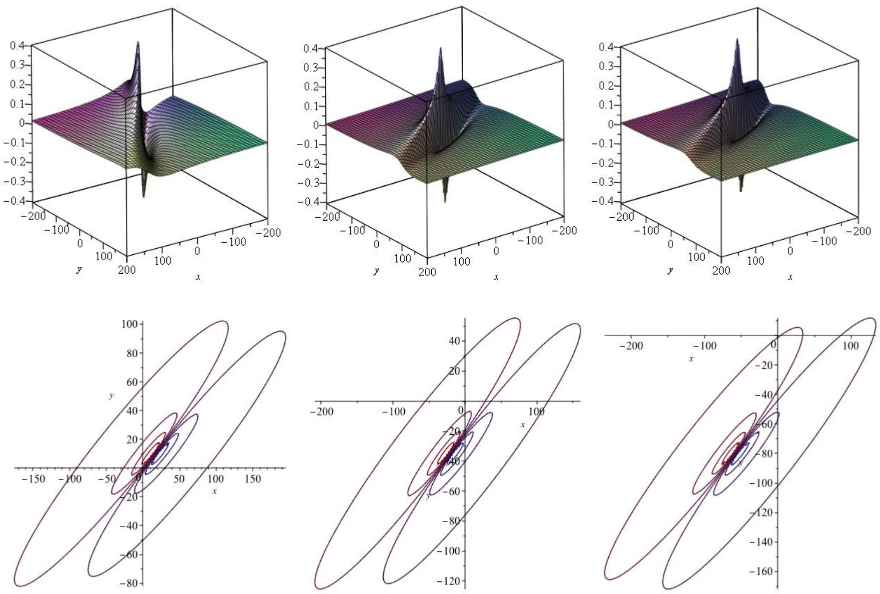

Figure 1. Profiles of u_1 when $t = 0, 15, 30$: 3d plots (top) and contour plots (bottom)

the tranformation (2.7) with (2.8) provides a lump solution to the special fourth-order nonlinear equation (2.36):

$$u_2 = \frac{2(-6t + 10x - \frac{16}{5}y + 44)}{(-t + 2x - \frac{2}{5}y + 6)^2 + (-t + x - \frac{4}{5}y + 10)^2 + 345}. \tag{2.38}$$

Three three-dimensional plots and contour plots of this lump solution are made via Maple plot tools, to shed light on the characteristic of the presented lump solutions, in Figure 2.

We point out that all the exact solutions presented above add valuable insights into the existing theories on soliton solutions and dromion-type solutions in the continuous and discrete cases, recently developed through powerful techniques including Darboux tranformations (see, e.g., [48, 50]), the generalized bilinear method (see, e.g., [21]), the Riemann-Hilbert approach (see, e.g., [27]), the Wronskian technique (see, e.g., [47]), the multiple-wave expansion method (see, e.g., [19, 28]), symmetry reductions (see, e.g., [6, 46]), and symmetry constraints (see, e.g., [16, 17, 33] for the continuous case and [5, 18, 66] for the discrete case).

3. Concluding remarks

With Maple symbolic computations, we have studied a combined fourth-order nonlinear equation in (2+1)-dimensions to explore nonlinear equations possessing lump solutions. Our analyses enrich the theory of lumps and solitons, providing new examples of (2+1)-dimensional nonlinear equations that have lump structures. Three-dimensional plots and contour plots of two lump solutions to two specially chosen cases of the equation were made by the plot tool in Maple.

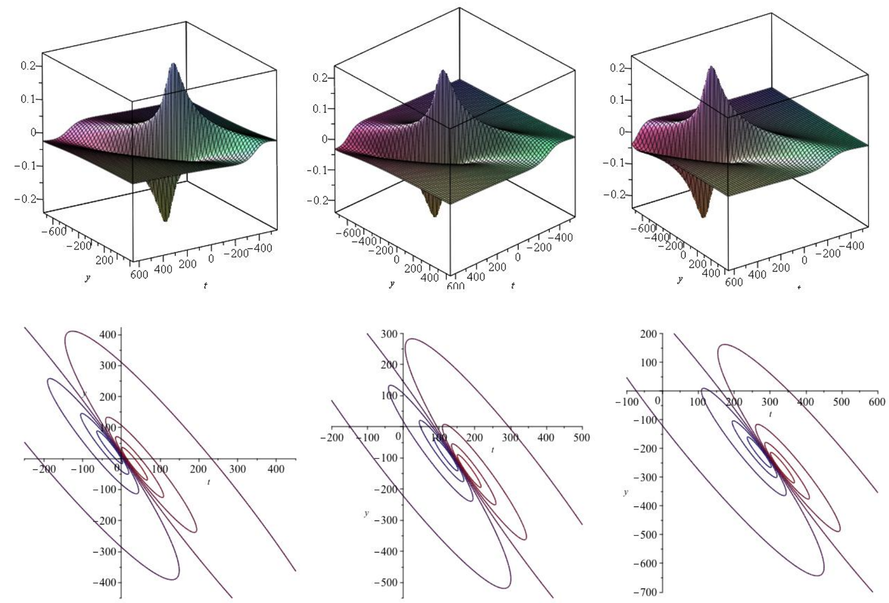

Figure 2. Profiles of u_2 when $t = 0, 50, 100$: 3d plots (top) and contour plots (bottom)

Many nonlinear equations possess lump solutions, which contain generalized KP, BKP, KP-Boussinesq, Sawada-Kotera and Bogoyavlensky-Konopelchenko equations in (2+1)-dimensions [4,20,32,34,64]. Recent studies also demonstrate the strikingly high richness of lump solutions to linear partial differential equations [28, 29] and nonlinear partial differential equations in (2+1)-dimensions (see, e.g., [31, 38] and [45,58,60,61]) and (3+1)-dimensions (see, e.g., [7,25,52] and [9,41,62,63]). Diverse lump solutions supplement the existing solutions obtained from different kinds of combinations (see, e.g., [30,44,49,59]), and can engender interesting Lie-Bäcklund symmetries, which can also be used to determine conservation laws by symmetries and adjoint symmetries [12,24,26]. Furthermore, abundant interaction solutions [34] have been reported for different integrable equations in (2+1)-dimensions, including lump-soliton interaction solutions (see, e.g., [53–55]) and lump-kink interaction solutions (see, e.g., [15,43,65,67]).

We also remark that the two nonlinear terms can be merged together into one nonlinear term in the considered nonlinear model, but the transformed linear terms will be different as well as solution situations will be changed. The combined nonlinear equation (2.1) has a symmetric characteristic for the variables y and t, and we have two situations of lump solutions. It is worth pointing out that the linear term u_{tt} has a serious effect on the determination of lump solutions. For the Hirota bilinear case [35] and the generalized bilinear cases [36], some general considerations have been made on the existence of lump solutions.

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