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ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF CERTAIN INTEGRO–DIFFERENTIAL EQUATIONS*

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Abstract The authors present conditions under which every positive solution $x(t)$ of the integro–differential equation $x''(t) = a(t) + \int_c^t (t-s)^{\alpha-1}[e(s) + k(t,s)f(s,x(s))]ds, \quad c > 1,\ \alpha > 0$, satisfies $x(t) = O(tA(t))$ as $t \to \infty$, i.e, $\limsup_{t\to\infty} \frac{x(t)}{tA(t)} < \infty$, where $A(t) = \int_c^t a(s)ds$. From the results obtained, they derive a technique that can be applied to some related integro–differential equations that are equivalent to certain fractional differential equations of Caputo type of any order.

Keywords Asymptotic behavior, oscillation, nonoscillation, integro–differential equations, Caputo derivative, fractional differential equations.

MSC(2010) 34E10, 34A34.

1. Introduction

Consider the integro–differential equation

$$x''(t) = a(t) + \int_c^t (t-s)^{\alpha-1}[e(s) + k(t,s)f(s,x(s))]ds, \quad c > 1,\ \alpha > 0. \tag{1.1}$$

In the sequel we assume that:

(i) $a : [c,\infty] \to (0,\infty)$ is a continuous function;

(ii) $k : [c,\infty] \times [c,\infty] \to \mathbb{R}$ is a real-valued continuous function and there exists a continuous function $b : [c,\infty] \to (0,\infty)$ such that

$$|k(t,s)| \le b(t) \quad \text{for all} \quad t \ge s \ge c;$$

(iii) $e : [c,\infty] \to \mathbb{R}$ is a real-valued continuous function;

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(iv) $f:[c,\infty]\times\mathbb{R}\to\mathbb{R}$ is a real-valued continuous function and there exist a continuous function $h:[c,\infty]\to(0,\infty)$ and a real number λ with $0<\lambda\le 1$ such that

$$0\le xf(t,x)\le h(t)\,|x|^{\lambda+1}\quad \text{for}\quad x\ne 0\quad \text{and}\quad t\ge c.$$

We only consider those solutions of equation (1.1) that are continuable and nontrivial in any neighborhood of ∞. Such a solution is said to be *oscillatory* if there exists a sequence $\{t_n\}\subseteq[c,\infty)$ with $t_n\to\infty$ as $n\to\infty$ such that $x(t_n)=0$, and it is *nonoscillatory* otherwise.

In the last few decades, integral and fractional differential equations have gained considerably more attention due to their applications in many engineering and scientific disciplines as the mathematical models for systems and processes in fields of such as physics, mechanics, chemistry, aerodynamics and the electrodynamics of complex media. For more details one can refer to Băleanu et al. [3], Lakshmikantham et al. [13], Kilbas et al. [12], Medveď [16], Miller et al. [18], Ma et al. [19], Podlubny [20], Prudnikov et al. [21], and Samko et al. [22].

Oscillation and asymptotic behavior results for integral as well as integro–differential equations and fractional differential equations are scarce; some results can be found in Bohner et al. [1], Grace and Zafer [7], and Grace et al. [8–10].

It seems that there are no such results for integral equations of the type (1.1). The main objective of this paper is to establish some new criteria for the asymptotic behavior of all solutions of equation (1.1). We also investigate some new criteria on the asymptotic behavior of the nonoscillatory solutions of equation (1.1) with $a(t)$ being a polynomial of degree $n-1$, i.e., the equation

$$x''(t)=c_0+c_1(t-c)+\ldots+c_{n-1}(t-c)^{n-1}+\frac{1}{\Gamma(\alpha)}\int_c^t(t-s)^{\alpha-1}[e(s)+k(t,s)f(s,x(s))]ds, \tag{1.2}$$

where $c>1$, $\alpha\in(n-1,n)$, $n\ge 1$. This equation is equivalent to a fractional differential equation with $k(t,s)=1$ of the type

$${}^{C}D_c^{\alpha}y(t)=e(t)+f(t,x(t)),\quad c>1,\alpha\in(n-1,n),\ n\ge 1, \tag{1.3}$$

where $y(t)=r(t)x'(t)$, $r(t)$ is a positive continuous function on $[c,\infty)$,

$$c_0=\frac{y(c)}{\Gamma(1)},\ c_1=\frac{y'(c)}{\Gamma(2)},\ \ldots,\ c_{n-1}=\frac{y^{(n-1)}(c)}{\Gamma(n)},$$

and c_0, c_1,...,c_{n-1} are real constants.

We note that

$${}^{C}D_c^{\alpha}x(t):=\frac{1}{\Gamma(n-\alpha)}\int_c^t(t-s)^{n-\alpha-1}x^{(n)}(s)ds$$

is the Caputo derivative of order $\alpha\in(n-1,n)$ of a C^n scaler valued function $x(t)$ defined on the interval $[c,\infty)$, where $x^{(n)}(t)=\frac{d^nx(t)}{dt^n}$. For the case where $\alpha\in(0,1)$, this definition was given by Caputo [4]; for the definition of a Caputo derivative of order $\alpha\in(n-1,n)$, $n\ge 1$, see Băleanu et al. [3], Diethelm et al. [5], and Furati et al. [6].

Results related to those in this paper can be found in the papers of Medveď *et al.* [2, 16, 17]. Our proofs are based on a de-singularization method introduced by Medveď in [14, 15] that has proved to be quite useful in the study of problems of this type.

2. Main results

To obtain our main results in this paper, we need the following two lemmas.

Lemma 2.1 ([2]). *Let α and p be positive constants such that $p(\alpha-1)+1>0$. Then*

$$\int_0^t (t-s)^{p(\alpha-1)} e^{ps} ds \le Q e^{pt}, \quad t \ge 0,$$

where

$$Q = \frac{\Gamma\left(1+p(\alpha-1)\right)}{p^{1+p(\alpha-1)}}$$

and

$$\Gamma(x) = \int_0^\infty s^{x-1} e^{-s} ds, \quad x>0,$$

is the Euler-Gamma function.

Lemma 2.2 ([11]). *If X and Y are nonnegative and $0<\lambda<1$, then*

$$X^\lambda - (1-\lambda)Y^\lambda - \lambda X Y^{\lambda-1} \le 0,$$

where equality holds if and only if $X=Y$.

In what follows, for any $t_1 \ge c$ and continuous function $m:[c,\infty)\to(0,\infty)$, we let

$$g(t) = (1-\lambda)\lambda^{\lambda/(1-\lambda)} b(t) \int_{t_1}^t (t-s)^{\alpha-1} m^{\lambda/(\lambda-1)}(s) h^{1/(1-\lambda)}(s) ds \quad \text{for} \quad t \ge t_1, \tag{2.1}$$

$$g^*(t) = (1-\lambda)\lambda^{\lambda/(1-\lambda)} \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} m^{\lambda/(\lambda-1)}(s) h^{1/(1-\lambda)}(s) ds \quad \text{for} \quad t \ge t_1 \tag{2.2}$$

and

$$A(t) = \int_c^t a(s) ds.$$

Now we give sufficient conditions under which any positive solution $x(t)$ of equation (1.1) satisfies

$$x(t) = O(tA(t)) \quad \text{as} \quad t \to \infty.$$

Theorem 2.1. *Let conditions (i)–(iv) hold and $0<\lambda<1$. Assume there exist real numbers $p>1$ and $\alpha>0$ with $p(\alpha-1)+1>0$, real numbers $S>0$ and $\sigma>1$ such that*

$$\left(\frac{b(t)}{A(t)}\right) \le S e^{-\sigma t} \quad \textit{for} \quad t \ge t_* > c, \tag{2.3}$$

and there exists a continuous function $m:[c,\infty)\to(0,\infty)$ such that

$$\int_c^t e^{-qs} \left(sA(s)m(s)\right)^q ds < \infty, \quad \textit{where} \quad q = \frac{p}{p-1}. \tag{2.4}$$

If

$$\frac{1}{a(t)} \int_c^t (t-s)^{\alpha-1} \left|e(s)\right| ds, \quad \frac{t^{\alpha-1} b(t)}{a(t)}, \quad \textit{and} \quad \frac{g(t)}{a(t)} \tag{2.5}$$

are bounded on $[c,\infty)$, *where* $g(t)$ *is defined by* (2.1), *then any positive solution* $x(t)$ *of equation* (1.1) *satisfies*

$$\limsup_{t\to\infty}\frac{x(t)}{tA(t)}<\infty. \tag{2.6}$$

Proof. Let x be an eventually positive solution of equation (1.1), say $x(t)>0$ for $t\ge t_1$ for some $t_1\ge c$. We let $F(t)=f(t,x(t))$. Then, in view of (i)–(iv), from (1.1) we obtain

$$\begin{aligned} x''(t) &\le a(t)+\int_c^t (t-s)^{\alpha-1}e(s)ds+b(t)\int_c^{t_1}(t-s)^{\alpha-1}\left|F(s)\right|ds\\ &\quad +b(t)\int_{t_1}^t (t-s)^{\alpha-1}\left[h(s)x^{\lambda}(s)-m(s)x(s)\right]ds\\ &\quad +b(t)\int_{t_1}^t (t-s)^{\alpha-1}m(s)x(s)ds. \end{aligned}\tag{2.7}$$

Applying Lemma 2.2 to $\left[h(s)x^{\lambda}(s)-m(s)x(s)\right]$ with

$$X=(h(s))^{1/\lambda}x(s)\quad\text{and}\quad Y=\left(\frac{1}{\lambda}m(s)h^{-1/\lambda}(s)\right)^{1/(\lambda-1)},$$

we obtain

$$h(s)x^{\lambda}(s)-m(s)x(s)\le(1-\lambda)\lambda^{\lambda/(1-\lambda)}m^{\lambda/(\lambda-1)}(s)h^{1/(1-\lambda)}(s). \tag{2.8}$$

Using (2.8) in (2.7) gives

$$\begin{aligned} x''(t) &\le a(t)+\int_c^t (t-s)^{\alpha-1}\left|e(s)\right|ds+b(t)\int_c^{t_1}(t-s)^{\alpha-1}\left|F(s)\right|ds\\ &\quad +(1-\lambda)\lambda^{\lambda/(1-\lambda)}b(t)\int_{t_1}^t (t-s)^{\alpha-1}m^{\lambda/(\lambda-1)}(s)h^{1/(1-\lambda)}(s)ds\\ &\quad +b(t)\int_{t_1}^t (t-s)^{\alpha-1}m(s)x(s)ds, \end{aligned}$$

or

$$\begin{aligned} x''(t) &\le a(t)+b(t)\int_c^{t_1}(t-s)^{\alpha-1}\left|F(s)\right|ds+\int_c^t (t-s)^{\alpha-1}\left|e(s)\right|ds\\ &\quad +g(t)+b(t)\int_{t_1}^t (t-s)^{\alpha-1}m(s)x(s)ds. \end{aligned}\tag{2.9}$$

In view of (2.5), inequality (2.9) can be written as

$$x''(t)\le C_1a(t)+b(t)\int_{t_1}^t (t-s)^{\alpha-1}m(s)x(s)ds, \tag{2.10}$$

where C_1 is an upper bound of

$$1+\frac{g(t)}{a(t)}+\frac{1}{a(t)}\int_c^t (t-s)^{\alpha-1}\left|e(s)\right|ds+\frac{b(t)}{a(t)}\int_c^{t_1}(t-s)^{\alpha-1}\left|F(s)\right|ds.$$

Integrating (2.10) from t_1 to t gives

$$x'(t) \le |x'(t_1)| + C_1 A(t) + \int_{t_1}^{t} b(u) \int_{t_1}^{u} (u-s)^{\alpha-1} m(s)x(s)dsdu,$$

or

$$x'(t) \le C_2 A(t) + \int_{t_1}^{t} b(u) \int_{t_1}^{u} (u-s)^{\alpha-1} m(s)x(s)dsdu := w(t), \tag{2.11}$$

where $|x'(t_1)|/A(t) + C_1 \le |x'(t_1)|/A(t_1) + C_1 = C_2$ is a constant. Integrating (2.11) from t_1 to t and then using the fact that $w(t)$ is nondecreasing, we see that

$$x(t) \le x(t_1) + \int_{t_1}^{t} w(s)ds \le x(t_1) + tw(t),$$

or

$$\frac{x(t)}{t} \le \frac{x(t_1)}{t} + w(t) \le \frac{x(t_1)}{t} + C_2 A(t) + \int_{t_1}^{t} b(u) \int_{t_1}^{u} (u-s)^{\alpha-1} m(s)x(s)dsdu. \tag{2.12}$$

Applying Hölder's inequality and Lemma 2.1 to the integral on the far right in (2.12), we obtain

$$\begin{aligned}
&\int_{t_1}^{u} \left((u-s)^{\alpha-1} e^{s}\right) \left(e^{-s} m(s)x(s)\right) ds \\
&\le \left(\int_{t_1}^{u} (u-s)^{p(\alpha-1)} e^{sp} ds\right)^{1/p} \left(\int_{t_1}^{u} e^{-qs} m^q(s)x^q(s)ds\right)^{1/q} \\
&\le \left(\int_{0}^{u} (u-s)^{p(\alpha-1)} e^{sp} ds\right)^{1/p} \left(\int_{t_1}^{u} e^{-qs} m^q(s)x^q(s)ds\right)^{1/q} \\
&\le (Qe^{pu})^{1/p} \left(\int_{t_1}^{u} e^{-qs} m^q(s)x^q(s)ds\right)^{1/q} \\
&= Q^{1/p} e^{u} \left(\int_{t_1}^{u} e^{-qs} m^q(s)x^q(s)ds\right)^{1/q}.
\end{aligned} \tag{2.13}$$

Using (2.13) in (2.12) gives

$$\frac{x(t)}{t} \le \frac{x(t_1)}{t} + C_2 A(t) + Q^{1/p} \int_{t_1}^{t} b(u)e^{u} \left(\int_{t_1}^{u} e^{-qs} m^q(s)x^q(s)ds\right)^{1/q} du,$$

or

$$\begin{aligned}
\frac{x(t)}{tA(t)} := z(t) \le \frac{x(t_1)}{tA(t)} &+ C_2 \\
&+ Q^{1/p} \int_{t_1}^{t} \left(\frac{b(u)e^{u}}{A(u)}\right) \left(\int_{t_1}^{u} e^{-qs} \left(sA(s)m(s)\right)^q z^q(s)ds\right)^{1/q} du \\
&\le C_3 + Q^{1/p} S \int_{t_1}^{t} e^{-(\sigma-1)u} \left(\int_{t_1}^{u} e^{-qs} \left(sA(s)m(s)\right)^q z^q(s)ds\right)^{1/q} du,
\end{aligned} \tag{2.14}$$

where C_3 is a positive constant. Since the integral on the far right in (2.14) is nondecreasing in u, we have

$$z(t) \le C_3 + Q^{1/p} S \left(\int_{t_1}^{t} e^{-(\sigma-1)u} du \right) \left(\int_{t_1}^{t} e^{-qs} \left(sA(s)m(s) \right)^q z^q(s) ds \right)^{1/q}. \quad (2.15)$$

Since $\sigma > 1$, it follows from (2.15) that

$$z(t) \le 1 + C_3 + k \left(\int_{t_1}^{t} e^{-qs} \left(sA(s)m(s) \right)^q z^q(s) ds \right)^{1/q}, \quad (2.16)$$

where $k = Q^{1/p} S/(\sigma - 1)$. Applying the inequality

$$(x+y)^q \le 2^{q-1}(x^q + y^q), \quad x, y \ge 0 \quad \text{and} \quad q \ge 1,$$

to (2.16) gives

$$z^q(t) \le 2^{q-1}(1 + C_3)^q + 2^{q-1} k^q \left(\int_{t_1}^{t} e^{-qs} \left(sA(s)m(s) \right)^q z^q(s) ds \right). \quad (2.17)$$

Setting $P_1 = 2^{q-1}(1 + C_3)^q$, $Q_1 = 2^{q-1} k^q$, and $w(t) = z^q(t)$ so that $z(t) = w^{1/q}(t)$, inequality (2.17) becomes

$$w(t) \le P_1 + Q_1 \left(\int_{t_1}^{t} e^{-qs} \left(sA(s)m(s) \right)^q w(s) ds \right) \quad \text{for} \;\; t \ge t_1 \ge c.$$

Gronwall's inequality and condition (2.4) imply that $w(t)$ and hence $z(t)$ is bounded, that is,

$$\limsup_{t \to \infty} \frac{x(t)}{tA(t)} < \infty.$$

This completes the proof of the theorem. □

The following result is concerned with the linear case of equation (1.1).

Theorem 2.2. *Let conditions (i)–(iv) hold and $\lambda = 1$. Assume that there exist numbers $p > 1$ and $\alpha > 0$ with $p(\alpha - 1) + 1 > 0$, there are real numbers $S > 0$ and $\sigma > 1$ such that* (2.3) *holds, and* (2.4) *is satisfied with $m(t)$ replaced by $h(t)$. If*

$$\frac{1}{a(t)} \int_{c}^{t} (t-s)^{\alpha-1} |e(s)| \, ds \quad \text{and} \quad \frac{t^{\alpha-1} b(t)}{a(t)} \quad (2.18)$$

are bounded on $[c, \infty)$, then any positive solution $x(t)$ of equation (1.1) *satisfies* (2.6).

Proof. Let $x(t)$ be an eventually positive solution of equation (1.1), say $x(t) > 0$ for $t \ge t_1$ for some $t_1 \ge c$. As in the proof of Theorem 2.1, we let $F(t) = f(t, x(t))$. Then, in view of (i)–(iv), from (1.1) it follows that

$$\begin{aligned} x''(t) \le a(t) &+ \int_{c}^{t} (t-s)^{\alpha-1} |e(s)| \, ds + b(t) \int_{c}^{t_1} (t-s)^{\alpha-1} |F(s)| \, ds \\ &+ b(t) \int_{t_1}^{t} (t-s)^{\alpha-1} h(s) x(s) ds. \end{aligned}$$

The remainder of the proof is similar to that of Theorem 2.1 and so we omit the details. □

Next, we have the following result for equation (1.2).

Theorem 2.3. *Let conditions (ii)–(iv) hold,* $0 < \lambda < 1$, *there exist real numbers* $p > 1$ *and* $\alpha > 0$ *with* $p(\alpha - 1) + 1 > 0$, *there are real numbers* $S > 0$ *and* $\sigma > 1$ *such that*

$$\left(\frac{b(t)}{t^n}\right) \leq Se^{-\sigma t}, \tag{2.19}$$

and there is a continuous function $m : [c, \infty) \to (0, \infty)$ *such that*

$$\int_c^t e^{-qs} \left(s^{n+1} m(s)\right)^q ds < \infty, \quad \text{where} \quad q = \frac{p}{p-1}. \tag{2.20}$$

If

$$\frac{1}{t^{n-1}} \int_c^t (t-s)^{\alpha-1} |e(s)|\, ds, \quad t^{\alpha-n} b(t), \quad \text{and} \quad \frac{g(t)}{t^{n-1}} \tag{2.21}$$

are bounded on $[c, \infty)$, *where* $g(t)$ *is defined by* (2.1), *then any positive solution* $x(t)$ *of equation* (1.2) *satisfies*

$$\limsup_{t\to\infty} \frac{x(t)}{t^{n+1}} < \infty. \tag{2.22}$$

Proof. Let $x(t)$ be an eventually positive solution of equation (1.2) with $x(t) > 0$ for $t \geq t_1$ for some $t_1 \geq c$. There exists a constant $M_1 > 0$ such that

$$x''(t) \leq M_1 t^{n-1} + \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} |e(s)|\, ds + \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} k(t,s) f(s, x(s)) ds. \tag{2.23}$$

Letting $F(t) = f(t, x(t))$, then in view of (ii)–(iv), inequality (2.23) can be written as

$$\begin{aligned} x''(t) \leq{} & M_1 t^{n-1} + \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} |e(s)|\, ds + \frac{b(t)}{\Gamma(\alpha)} \int_c^{t_1} (t-s)^{\alpha-1} |F(s)|\, ds \\ & + \frac{b(t)}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} \left[h(s) x^\lambda(s) - m(s) x(s)\right] ds \\ & + \frac{b(t)}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} m(s) x(s) ds. \end{aligned} \tag{2.24}$$

Proceeding exactly as in the proof of Theorem 2.1, we again see that (2.8) holds. Next, using (2.8) in (2.24) gives

$$\begin{aligned} x''(t) \leq{} & M_1 t^{n-1} + \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} |e(s)|\, ds + \frac{b(t)}{\Gamma(\alpha)} \int_c^{t_1} (t-s)^{\alpha-1} |F(s)|\, ds \\ & + \frac{(1-\lambda)\lambda^{\lambda/(1-\lambda)}}{\Gamma(\alpha)} b(t) \int_{t_1}^t (t-s)^{\alpha-1} m^{\lambda/(\lambda-1)}(s) h^{1/(1-\lambda)}(s) ds \\ & + \frac{b(t)}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} m(s) x(s) ds, \end{aligned}$$

or

$$\begin{aligned} x''(t) \leq{} & M_1 t^{n-1} + \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} |e(s)|\, ds + \frac{b(t)}{\Gamma(\alpha)} \int_c^{t_1} (t-s)^{\alpha-1} |F(s)|\, ds \\ & + \frac{g(t)}{\Gamma(\alpha)} + \frac{b(t)}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} m(s) x(s) ds, \end{aligned}$$

$$\leq M_2 t^{n-1} + \frac{b(t)}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} m(s)x(s)ds, \tag{2.25}$$

where M_2 is an upper bound for

$$M_1 + \frac{1}{t^{n-1}\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} |e(s)|\, ds + \frac{b(t)}{\Gamma(\alpha)t^{n-1}} \int_c^{t_1} (t-s)^{\alpha-1} |F(s)|\, ds + \frac{g(t)}{t^{n-1}\Gamma(\alpha)}.$$

Integrating inequality (2.25) from t_1 to t, we obtain

$$x'(t) \leq |x'(t_1)| + M_2 \left(\frac{t^n - t_1^n}{n} \right) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} b(u) \int_{t_1}^{u} (u-s)^{\alpha-1} m(s)x(s)dsdu,$$

from which it follows that

$$x'(t) \leq M_3 t^n + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} b(u) \int_{t_1}^{u} (u-s)^{\alpha-1} m(s)x(s)dsdu := w(t), \tag{2.26}$$

for some positive constant M_3. An integration of (2.26) from t_1 to t yields

$$x(t) \leq x(t_1) + \int_{t_1}^{t} w(s)ds \leq x(t_1) + tw(t)$$

since $w(t)$ is nondecreasing, so

$$\begin{aligned} \frac{x(t)}{t} &\leq \frac{x(t_1)}{t} + w(t) \\ &\leq \frac{x(t_1)}{t} + M_3 t^n + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} b(u) \int_{t_1}^{u} (u-s)^{\alpha-1} m(s)x(s)dsdu. \end{aligned} \tag{2.27}$$

As in Theorem 2.1, applying Hölder's inequality and Lemma 2.1 to the integral on the far right in (2.27), we see that (2.13) holds. Using (2.13) in (2.27), we obtain

$$\frac{x(t)}{t} \leq \frac{x(t_1)}{t} + M_3 t^n + \frac{Q^{1/p}}{\Gamma(\alpha)} \int_{t_1}^{t} b(u)e^u \left(\int_{t_1}^{u} e^{-qs} m^q(s) x^q(s) ds \right)^{1/q} du,$$

or

$$\begin{aligned} \frac{x(t)}{t^{n+1}} := z(t) &\leq M_4 + \frac{Q^{1/p}}{\Gamma(\alpha)} \int_{t_1}^{t} \left(\frac{b(u)e^u}{u^n} \right) \left(\int_{t_1}^{u} e^{-qs} \left(s^{n+1} m(s) \right)^q z^q(s) ds \right)^{1/q} du \\ &\leq M_4 + \frac{Q^{1/p} S}{\Gamma(\alpha)} \int_{t_1}^{t} e^{-(\sigma-1)u} \left(\int_{t_1}^{u} e^{-qs} \left(s^{n+1} m(s) \right)^q z^q(s) ds \right)^{1/q} du, \end{aligned}$$

where $M_4 = x(t_1)/t_1^{n+1} + M_3$. As in the proof of Theorem 2.1, since $\sigma > 1$, we have the estimate

$$z(t) \leq 1 + M_4 + k \left(\int_{t_1}^{t} e^{-qs} \left(s^{n+1} m(s) \right)^q z^q(s) ds \right)^{1/q},$$

where $k = Q^{1/p} S/(\sigma - 1)\Gamma(\alpha)$. The rest of the proof is similar to that of Theorem 2.1. □

Similar to the sublinear case, one can easily prove the following result.

Theorem 2.4. *Let conditions (ii)–(iv) hold, $\lambda = 1$, and assume there exist real numbers $p > 1$ and $\alpha > 0$ with $p(\alpha - 1) + 1 > 0$, $S > 0$ and $\sigma > 1$ such that* (2.19) *holds, and let* (2.20) *be satisfied with $m(t)$ replaced by $h(t)$. If*

$$\frac{1}{t^{n-1}} \int_c^t (t-s)^{\alpha-1} |e(s)| \, ds \quad \text{and} \quad t^{\alpha-n} b(t) \tag{2.28}$$

are bounded on $[c, \infty)$, then any positive solution $x(t)$ of equation (1.2) *satisfies* (2.22).

Proof. Let $x(t)$ be an eventually positive solution of equation (1.2) with $x(t) > 0$ for $t \geq t_1$ for some $t_1 \geq c$. Now there exists a constant $M_1 > 0$ such that

$$x''(t) \leq M_1 t^{n-1} + \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} |e(s)| \, ds + \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} k(t,s) f(s, x(s)) ds. \tag{2.29}$$

With $F(t) = f(t, x(t))$, in view of (ii)–(iv), (2.29) can be written as

$$x''(t) \leq M_1 t^{n-1} + \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} |e(s)| \, ds + \frac{b(t)}{\Gamma(\alpha)} \int_c^{t_1} (t-s)^{\alpha-1} |F(s)| \, ds$$
$$+ \frac{b(t)}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} h(s) x(s) ds.$$

The remainder of the proof is similar to that of Theorem 2.3 and hence is omitted. □

The following results are concerned with the asymptotic behavior of equation (1.3).

Theorem 2.5. *Let conditions (ii)–(iv) hold, $0 < \lambda < 1$, there exist real numbers $p > 1$ and $\alpha > 0$ with $p(\alpha - 1) + 1 > 0$, there are numbers $S > 0$ and $\sigma > 1$ such that*

$$\left(\frac{1}{t^n r(t)} \right) \leq S e^{-\sigma t}, \tag{2.30}$$

and there is a continuous function $m : [c, \infty) \to (0, \infty)$ such that

$$\int_c^t e^{-qs} \left(s^n m(s) \right)^q ds < \infty, \quad \text{where} \quad q = \frac{p}{p-1}. \tag{2.31}$$

If

$$\frac{1}{t^{n-1} r(t)} \int_c^t (t-s)^{\alpha-1} |e(s)| \, ds, \quad \frac{t^{\alpha-n}}{r(t)}, \quad \text{and} \quad \frac{g^*(t)}{r(t) t^{n-1}} \tag{2.32}$$

are bounded on $[c, \infty)$, where $g^*(t)$ is defined by (2.2), *then any nonoscillatory solution $x(t)$ of equation* (1.3) *satisfies*

$$\limsup_{t \to \infty} \frac{|x(t)|}{t^n} < \infty. \tag{2.33}$$

Proof. Let $x(t)$ be an eventually positive solution of equation (1.3), say $x(t) > 0$ for $t \geq t_1 \geq c$. Then, there exists a constant $N_1 >$ such that

$$r(t)x'(t) \leq N_1 t^{n-1} + \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} |e(s)|\, ds + \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} f(s, x(s)) ds. \tag{2.34}$$

Again letting $F(t) = f(t, x(t))$, in view of (ii)–(iv), (2.34) can be written as

$$\begin{aligned} r(t)x'(t) \leq{}& N_1 t^{n-1} + \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} |e(s)|\, ds + \frac{1}{\Gamma(\alpha)} \int_c^{t_1} (t-s)^{\alpha-1} |F(s)|\, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} \left[h(s)x^{\lambda}(s) - m(s)x(s)\right] ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} m(s)x(s) ds. \end{aligned} \tag{2.35}$$

Proceeding exactly as in the proof of Theorem 2.1, we again see that (2.8) holds. Next, using (2.8) in (2.35) gives

$$\begin{aligned} r(t)x'(t) \leq{}& N_1 t^{n-1} + \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} |e(s)|\, ds + \frac{1}{\Gamma(\alpha)} \int_c^{t_1} (t-s)^{\alpha-1} |F(s)|\, ds \\ &+ \frac{(1-\lambda)\lambda^{\lambda/(1-\lambda)}}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} m^{\lambda/(\lambda-1)}(s) h^{1/(1-\lambda)}(s) ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} m(s)x(s) ds, \end{aligned}$$

or

$$x'(t) \leq N_2 t^{n-1} + \frac{1}{\Gamma(\alpha) r(t)} \int_{t_1}^t (t-s)^{\alpha-1} m(s)x(s) ds, \tag{2.36}$$

where N_2 is an upper bound for

$$\begin{aligned} &\frac{N_1}{r(t)} + \frac{1}{t^{n-1} r(t)\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} |e(s)|\, ds \\ &\qquad + \frac{1}{t^{n-1} r(t)\Gamma(\alpha)} \int_c^{t_1} (t-s)^{\alpha-1} |F(s)|\, ds + \frac{g^*(t)}{r(t)t^{n-1}}. \end{aligned}$$

Integrating inequality (2.36) from t_1 to t yields

$$x(t) \leq x(t_1) + N_2 \left(\frac{t^n - t_1^n}{n}\right) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \frac{1}{r(u)} \int_{t_1}^u (u-s)^{\alpha-1} m(s)x(s) ds du,$$

from which we see that

$$x(t) \leq N_3 t^n + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \frac{1}{r(u)} \int_{t_1}^u (u-s)^{\alpha-1} m(s)x(s) ds du, \tag{2.37}$$

for some positive constant N_3. As in Theorem 2.1, applying Hölder's inequality and Lemma 2.1 to the integral on the far right in (2.37), we again see that (2.13) holds. Using (2.13) in (2.37), we obtain

$$x(t) \leq N_3 t^n + \frac{Q^{1/p}}{\Gamma(\alpha)} \int_{t_1}^t \frac{e^u}{r(u)} \left(\int_{t_1}^u e^{-qs} m^q(s) x^q(s) ds\right)^{1/q} du,$$

or

$$\frac{x(t)}{t^n} := z(t) \le N_3 + \frac{Q^{1/p}}{\Gamma(\alpha)} \int_{t_1}^{t} \frac{e^u}{r(u)u^n} \left(\int_{t_1}^{u} e^{-qs} \left(s^n m(s)\right)^q z^q(s) ds \right)^{1/q} du$$
$$\le N_3 + \frac{Q^{1/p} S}{\Gamma(\alpha)} \int_{t_1}^{t} e^{-(\sigma-1)u} \left(\int_{t_1}^{u} e^{-qs} \left(s^n m(s)\right)^q z^q(s) ds \right)^{1/q} du.$$

Since $\sigma > 1$, as in the proof of Theorem 2.1, we have the estimate

$$z(t) \le 1 + N_3 + k \left(\int_{t_1}^{t} e^{-qs} \left(s^n m(s)\right)^q z^q(s) ds \right)^{1/q}, \tag{2.38}$$

where $k = Q^{1/p} S/(\sigma - 1)\Gamma(\alpha)$. The rest of the proof is similar to that of Theorem 2.1 and so we omit the details. □

Similarly we have the following result.

Theorem 2.6. *Let conditions (ii)–(iv) hold, $\lambda = 1$, there exist $p > 1$ and $\alpha > 0$ with $p(\alpha - 1) + 1 > 0$, there are numbers $S > 0$ and $\sigma > 1$ such that* (2.30) *holds, and* (2.31) *is satisfied with $m(t)$ replaced by $h(t)$. If*

$$\frac{1}{t^{n-1} r(t)} \int_{c}^{t} (t-s)^{\alpha-1} |e(s)| \, ds \quad \text{and} \quad \frac{t^{\alpha-n}}{r(t)} \tag{2.39}$$

are bounded on $[c, \infty)$, then any nonoscillatory solution $x(t)$ of equation (1.3) *satisfies* (2.33).

Example 2.1. Consider the integro–differential equation

$$x''(t) = c_0 + c_1(t-2) + \frac{1}{\Gamma(\alpha)} \int_{2}^{t} (t-s)^{\alpha-1} \left[\frac{1}{s^2} + k(t,s)h(s) \left|x(s)\right|^{\lambda-1} x(s) \right] ds, \tag{2.40}$$

with $0 < \lambda < 1$. Here we have $c = 2$, $n = 2$, $e(t) = \frac{1}{t^2}$, $f(t, x(t)) = h(t) \left|x(t)\right|^{\lambda-1} x(t)$, and we take $k(t,s) = \frac{e^{-6t}}{1+s+t^2}$, $b(t) = \frac{e^{-5t}}{t^2}$, and $h(t) = t$. Then, it is easy to see that conditions (i)–(iv) hold. Letting $p = \frac{3}{2}$ and $\alpha = 2 - \frac{1}{p} = \frac{4}{3} \in (1,2)$, we see that $q = 3$ and $p(\alpha - 1) + 1 = \frac{3}{2}$. With $\sigma = 5$, $S = 1$, and $m(t) = h(t) = t$, conditions (2.19) and (2.20) become

$$\left(\frac{b(t)}{t^n} \right) = \frac{e^{-5t}}{t^4} \le e^{-5t},$$

and

$$\int_{c}^{t} e^{-qs} \left(s^{n+1} m(s)\right)^q ds = \int_{2}^{t} e^{-3s} \left(s^3 \times s\right)^3 ds = \int_{2}^{t} \frac{s^{12}}{e^{3s}} ds < \infty,$$

respectively. Since

$$\frac{1}{t^{n-1}} \int_{c}^{t} (t-s)^{\alpha-1} |e(s)| \, ds = \frac{1}{t} \int_{2}^{t} (t-s)^{1/3} \frac{1}{s^2} ds$$
$$\le \frac{(t-2)^{1/3}}{t} \int_{2}^{t} \frac{1}{s^2} ds \le \frac{1}{2t^{2/3}} < \infty,$$

$$t^{\alpha-n} b(t) = \frac{1}{t^{8/3} e^{5t}} < \infty,$$

and with $m(t) = h(t) = t$

$$\begin{aligned}\frac{g(t)}{t^{n-1}} &= (1-\lambda)\lambda^{\lambda/(1-\lambda)}\frac{e^{-5t}}{t^3}\int_{t_1}^{t}(t-s)^{\alpha-1}m(s)ds\\ &= (1-\lambda)\lambda^{\lambda/(1-\lambda)}\frac{e^{-5t}}{t^3}\int_{2}^{t}(t-s)^{1/3}sds\\ &\le (1-\lambda)\lambda^{\lambda/(1-\lambda)}\frac{e^{-5t}t}{t^3}\int_{2}^{t}(t-s)^{1/3}ds\\ &\le (1-\lambda)\lambda^{\lambda/(1-\lambda)}\frac{3}{4e^{5t}t^{2/3}} < \infty,\end{aligned}$$

condition (2.21) holds. All conditions of Theorem 2.3 are satisfied and so every positive solution $x(t)$ of equation (2.40) satisfies

$$\limsup_{t\to\infty}\frac{x(t)}{t^3} < \infty.$$

Example 2.2. Consider the integro–differential equation

$$x''(t) = 2t + \int_2^t (t-s)^{\alpha-1}\left[\sin s + k(t,s)h(s)\left|x(s)\right|^{\lambda-1}x(s)\right]ds, \tag{2.41}$$

with $0 < \lambda < 1$. Here we have $a(t) = 2t$, $c = 2$, $e(t) = \sin t$, $f(t, x(t)) = h(t)\left|x(t)\right|^{\lambda-1}x(t)$, and we take $k(t,s) = \frac{2e^{-2t}}{1+s^2}$, $b(t) = 2e^{-2t}$, and $h(t) = t$. Then, it is easy to see that conditions (i)–(iv) hold. Letting $p = 3/2$ and $\alpha = 1/2$, we see that $q = 3$ and $p(\alpha-1)+1 = 1/4 > 0$. Since

$$A(t) = \int_c^t a(s)ds = \int_2^t 2sds = t^2 - 4,$$

we see that, for $t \ge 2c = 4$, i.e., for $t \ge 2\times 2$,

$$\frac{t}{2} \ge 2,$$

so

$$t^2 - 4 \ge \frac{3t^2}{4},$$

and thus

$$A(t) \ge \frac{3t^2}{4}.$$

Therefore, with $\sigma = 2$, and $S = 2/3$, condition (2.3) becomes

$$\left(\frac{b(t)}{A(t)}\right) \le \frac{8e^{-2t}}{3t^2} \le \frac{2e^{-2t}}{3},$$

i.e, condition (2.3) holds.

With $m(t) = h(t) = t$, condition (2.4) becomes

$$\int_c^t e^{-qs}\left(sA(s)m(s)\right)^q ds = \int_2^t e^{-3s}\left(s\times(s^2-4)\times s\right)^3 ds \le \int_2^t \frac{s^{12}}{e^{3s}}ds < \infty,$$

i.e, condition (2.4) holds.

Since

$$\begin{aligned}\frac{1}{a(t)}\int_c^t (t-s)^{\alpha-1}\,|e(s)|\,ds &= \frac{1}{2t}\int_2^t (t-s)^{-1/2}\,|\sin s|\,ds\\ &\le \frac{1}{2t}\int_2^t (t-s)^{-1/2}ds\\ &\le \frac{1}{t^{1/2}} < \infty,\end{aligned}$$

$$\frac{t^{\alpha-1}b(t)}{a(t)} = \frac{1}{t^{3/2}e^{2t}} < \infty,$$

and with $m(t) = h(t)$

$$\begin{aligned}\frac{g(t)}{a(t)} &= (1-\lambda)\lambda^{\lambda/(1-\lambda)}\frac{e^{-2t}}{t}\int_2^t (t-s)^{-1/2} \times s ds\\ &\le (1-\lambda)\lambda^{\lambda/(1-\lambda)}\frac{e^{-2t}\times t}{t}\int_2^t (t-s)^{-1/2}ds\\ &\le (1-\lambda)\lambda^{\lambda/(1-\lambda)}\frac{2t^{1/2}}{e^{2t}} < \infty,\end{aligned}$$

condition (2.5) holds. All conditions of Theorem 2.1 are satisfied and so every positive solution $x(t)$ of equation (2.41) satisfies

$$\limsup_{t\to\infty}\frac{x(t)}{t^3} < \infty.$$

We end our paper by noting that it would be of interest to study equations (1.1)–(1.3) for the case where f satisfies condition (iv) with $\lambda > 1$.

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