

# GLOBAL BEHAVIOR OF A DYNAMIC MODEL WITH BIODEGRADATION OF MICROCYSTINS\*

Keying Song<sup>1</sup>, Wanbiao Ma<sup>1,2,†</sup> and Ke Guo<sup>1</sup>

**Abstract** Considering the biodegradation pathway of Microcystins, in this paper, we propose a model described by a system of ordinary differential equations. We firstly investigate the local stability of the positive equilibrium and the existence of Hopf bifurcations. Then, the global stability of the positive equilibrium and the permanence of the model are considered. Finally, numerical simulations are carried out to illustrate the obtained results and we also consider the control strategy by changing the parameters in the model.

**Keywords** Microcystins, global stability, Hopf bifurcation, permanence.

**MSC(2010)** 92B05, 34D20.

## 1. Introduction

The problem of water eutrophication has been becoming a threat to the economic development, to the safety of residential drinking water and to the natural ecological environment [1]. One of the most common signs of water eutrophication is harmful cyanobacterial blooms [19, 35], which can be found in almost all algal blooms produced by harmful cyanobacterial blooms [20]. The explosion of cyanobacterial blooms increases the turbidity of aquatic ecosystems, smothering aquatic plants and thereby suppressing important invertebrate and fish habitats [1, 19] and some cyanobacteria produce toxins, which can cause severe and occasionally fatal human liver, digestive, neurological, and skin diseases [18].

To the best of our knowledge, microcystins (MCs) are the most common and harmful cyanotoxins [10, 30], which can be synthesized intracellularly by some cyanobacterial genera (e.g., *Microcystis*, *Planktothrix*) [12]. Intracellular MCs, via cell lysis caused by natural senescence or physical stress, are released into water bodies [23]. As a group of cyclic heptapeptides, MCs share a general structure of cyclo-(D-Ala-R<sub>1</sub>-D-isoMeAsp-R<sub>2</sub>-Adda-D-isoGlu-Mdha-), where R<sub>1</sub> and R<sub>2</sub> represent a pair of highly variable L-amino acids [4, 14]. Over 100 structural analogs of MCs have been identified mostly due to the substitution of variable L-amino acids

<sup>†</sup>the corresponding author. Email address: wanbiao\_ma@ustb.edu.cn(W. Ma)

<sup>1</sup>School of Mathematics and Physics, University of Science and Technology Beijing, Beijing 100083, China

<sup>2</sup>Beijing Key Laboratory for Magneto-Photoelectrical Composite and Interface Science, University of Science and Technology Beijing, Beijing 100083, China

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in  $R_1$  and  $R_2$  sites [5, 21]. Due to their cyclic structure, MCs are known to be stable against the general physicochemical factors (such as high temperatures, sunlight) [22]. MCs severely poison many animals, plants, and even endanger humans by inhibiting protein phosphatases (PPs) and posing oxidative stress in eukaryotic cells [2]. World Health Organization established a guideline value of  $1\mu\text{gL}^{-1}$  as the maximum concentration of MC-LR in drinking water [18]. Thus, effective removal of MCs is essential to drinking water treatment.

The limitations on physicochemical treatment methods (e.g., low efficiency, toxic by-products, high costs) have driven the exploitation for more suitable strategies to eliminate MCs from water supplies [22]. Biodegradation, which has been proven as a cost-efficient and environmentally-benign method for the degradation of MCs, has been studied heavily in recent years [15, 25]. At present, there are many experimental studies about degrading MCs by biodegradation, and many researchers are committed to seeking biodegradable bacteria from the natural water bodies [43]. Early in 1994, *Sphingomonas* sp. ACM-3962 was identified as the first bacterium capable of degrading MCs [11]. Later on, other bacteria of *Sphingomonas* sp., *Pseudomonas aeruginosa*, *Burkholderia*, *Kurthia*, *Pseudomonas*, *Bordetella*, *Stenotrophomonas* were verified that are able to degrade MCs [3]. So far, dozens of MC-degrading bacterial strains originated from diverse ecosystems have been characterized, with the majority identified as *Sphingomonas* and *Sphingopyxis* genera belonging to  $\alpha$ -Proteobacteria class [14]. These advances would lead to a better control and management for MCs in habitats.

Based on the flocculation and microorganism collection, Tai et al. first proposed a class of delayed dynamic model [31], where the backward and forward bifurcations were found and the local stability of equilibria was investigated. Guo and Ma [8] further considered the global stability of equilibria and permanence of the model. Following the study in [8], Guo et al. [9] considered a time-delayed microorganism flocculation model with saturated functional responses and they established some conditions for the global stability by using a variant of the Lyapunov-LaSalle theorem. In addition, Wang et al. [33] considered a dynamic model describing the flocculation of a microorganism that uses two different kinds of nutrients. Based on [31], Zhang et al. [39, 40] considered models with impulsive flocculant input and studied the global stability of periodic solutions and the permanence of these models. Based on the microbial flocculation and its application in wastewater treatment, Song et al. [27] proposed a class of dynamic models describing microbial flocculant with nutrient competition and metabolic products and analyzed the globally dynamic properties of these models. For the biodegradation of MCs, Song et al. consider a delay differential equation model in [28] and investigated a stochastic non-autonomous model in [29]. For the flocculation and microbial degradation of MCs, Yang et al. [36] proposed the following model:

$$\begin{cases} x_1'(t) = Da_{10} - a_{12}x_1(t)x_2(t) - a_{13}x_1(t)x_3(t) - (D + d_1)x_1(t), \\ x_2'(t) = a_{21}x_2(t)x_1(t) - a_{20}x_2(t) - (D + d_2)x_2(t), \\ x_3'(t) = a_{30}x_2(t) - a_{31}x_1(t)x_3(t) - (D + d_3)x_3(t), \end{cases} \quad (1.1)$$

where  $x_1(t)$ ,  $x_2(t)$  and  $x_3(t)$  denote the concentrations of MCs, MC-degrading bacteria and degrading enzymes produced by MC-degrading bacteria at time  $t$ , respectively.  $a_{10}$  is the input concentration of MCs;  $D$  is the washout rate;  $a_{12}$ , the consumption rate of MCs;  $a_{21}$ , the maximal growth rate of MC-degrading bacteria and  $a_{20}$ , the consumption rate of MC-degrading bacteria;  $a_{30}$ , the maximal

growth rate of degrading enzymes  $a_{13}$ , and  $a_{31}$ , the degradation rate of MCs and the consumption rate of degrading enzymes, respectively;  $d_i$ , death rates of MCs, MC-degrading bacteria, degrading enzymes,  $i = 1, 2, 3$ , respectively. Re-scaling model (1.1) by

$$\begin{aligned} x_1 &= a_{10}x, & x_2 &= y, & x_3 &= z, & t &= \frac{T}{D}, \\ D_1 &= \frac{D + d_1}{D}, & D_2 &= \frac{a_{20} + D + d_2}{D}, & D_3 &= \frac{D + d_3}{D}, \\ a_1 &= \frac{a_{12}}{D}, & a_2 &= \frac{a_{13}}{D}, & b_1 &= \frac{a_{21}a_{10}}{D}, & c_1 &= \frac{a_{30}}{D}, & c_2 &= \frac{a_{31}a_{10}}{D}, \end{aligned}$$

authors of [36] got the following dimensionless model:

$$\begin{cases} x'(t) = 1 - a_1x(t)y(t) - a_2x(t)z(t) - D_1x(t), \\ y'(t) = b_1x(t)y(t) - D_2y(t), \\ z'(t) = c_1y(t) - c_2x(t)z(t) - D_3z(t), \end{cases} \quad (1.2)$$

with the initial condition

$$x(0) = x_0 \geq 0, \quad y(0) = y_0 \geq 0, \quad z(0) = z_0 \geq 0, \quad (1.3)$$

where  $x_0$ ,  $y_0$ , and  $z_0$  denote the initial concentrations of MCs, MC-degrading bacteria and degrading enzymes produced by MC-degrading bacteria, respectively. Furthermore, it has been shown that the set

$$G = \left\{ (x, y, z) \in R_3^+ : x \leq \frac{1}{D_1}, \frac{b_1}{a_1}x + y \leq \frac{b_1}{la_1}, z \leq \frac{c_1b_1}{lD_3a_1} \right\},$$

is a positively invariant set of model (1.2) which attracts all its solutions, where  $l = \min\{D_1, D_2\}$ ,

Reference [36] showed that model (1.2) have a boundary equilibrium  $E_0 = (\frac{1}{D_1}, 0, 0)$  and a positive equilibrium  $E^* = (x^*, y^*, z^*)$  if  $R_0 (= \frac{b_1}{D_1D_2}) > 1$ , where

$$\begin{aligned} x^* &= \frac{D_2}{b_1}, & y^* &= \frac{(b_1D_3 + c_2D_2)(b_1 - D_1D_2)}{a_2b_1c_1D_2 + a_1b_1D_2D_3 + a_1c_2D_2^2}, \\ z^* &= \frac{b_1c_1(b_1 - D_1D_2)}{a_2b_1c_1D_2 + a_1b_1D_2D_3 + a_1c_2D_2^2}. \end{aligned}$$

They also obtained the following Propositions 1.1 and 1.2.

**Proposition 1.1.** *If  $R_0 < 1$ , then the boundary equilibrium  $E_0$  is globally asymptotically stable; if  $R_0 = 1$ , then the boundary equilibrium  $E_0$  is globally attractive.*

**Proposition 1.2.** *If  $R_0 > 1$  and  $D_1 + D_3 \geq D_2$ , then the positive equilibrium  $E^*$  is locally asymptotically stable.*

Please notice that (i) for the local stability of the positive equilibrium  $E^*$  in Propostion 1.2, the condition  $D_1 + D_3 \geq D_2$  is sufficient and unnecessary, (ii) the global stability of  $E^*$  is not studied in [36] and (iii) the previous studies have not discussed the effects of the diffusion even though recent studies such as [17, 38, 41] suggest that the diffusion may affect the stability of equilibrium points significantly. In this study, we will devote ourselves to address problems (i) and (ii) while leave

(iii) as a future work. More precisely, the rest of this paper is organized as follows. In Section 2, we consider the local stability of the positive equilibrium  $E^*$  and the existence of a Hopf bifurcation. The global stability of the positive equilibrium  $E^*$  and the permanence will be discussed in Section 3. Finally, Numerical simulations and conclusions are given in Section 4.

## 2. The local stability of the positive equilibrium and the existence of Hopf bifurcation

In this section, we will discuss the local stability of the positive equilibrium  $E^*$  and the existence of Hopf bifurcation.

The characteristic equation of the linearization of model (1.2) at  $E^*$  is given by

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0, \quad (2.1)$$

where

$$\begin{aligned} A &= \frac{1}{x^*} + D_3 + c_2x^* > 0, \\ B &= a_1D_2y^* + c_2D_1x^* + a_1c_2x^*y^* + \frac{D_3}{x^*} > 0, \\ C &= a_2c_1D_2y^* + a_1D_2D_3y^* + a_1c_2D_2x^*y^* > 0. \end{aligned}$$

Further, one can get

$$\begin{aligned} AB - C &= a_1b_1y^* + c_2D_1 + a_1c_2y^* + \frac{D_3}{x^{*2}} + c_2D_1D_3x^* + a_1c_2D_3x^*y^* \\ &\quad + \frac{D_3^2}{x^*} + c_2^2D_1x^{*2} + a_1c_2^2x^{*2}y^* + c_2D_3 - a_2c_1D_2y^* \\ &= y^*(a_1c_2^2x^{*2} + a_1b_1 + a_1c_2 + a_1c_2D_3x^* - a_2c_1D_2) \\ &\quad + c_2D_1 + \frac{D_3}{x^{*2}} + c_2D_1D_3x^* + \frac{D_3^2}{x^*} + c_2^2D_1x^{*2} + c_2D_3. \end{aligned}$$

In order to determine the sign of  $AB - C$ , define the following function with respect to  $a_2$ ,

$$\begin{aligned} g(a_2) &= y^*(a_2)(a_1c_2^2x^{*2} + a_1b_1 + a_1c_2 + a_1c_2D_3x^* - a_2c_1D_2) \\ &\quad + c_2D_1 + \frac{D_3}{x^{*2}} + c_2D_1D_3x^* + \frac{D_3^2}{x^*} + c_2^2D_1x^{*2} + c_2D_3. \end{aligned}$$

It is obvious that  $g(0) > 0$ , and

$$\begin{aligned} g'(a_2) &= y^{*'}(a_1c_2^2x^{*2} + a_1b_1 + a_1c_2 + a_1c_2D_3x^* - a_2c_1D_2) - c_1D_2y^* \\ &= -\frac{b_1c_1D_2(b_1D_3 + c_2D_2)(b_1 - D_1D_2)}{(a_2b_1c_1D_2 + a_1b_1D_2D_3 + a_1c_2D_2^2)^2} \left( a_1c_2^2x^{*2} + a_1b_1 + a_1c_2 \right. \\ &\quad \left. + a_1c_2D_3x^* + \frac{a_1b_1D_2D_3 + a_1c_2D_2^2}{b_1} \right) \\ &< 0, \end{aligned}$$

i.e.  $[A(a_2)B(a_2)]' < C'(a_2)$ . Further,

$$\begin{aligned} \lim_{a_2 \rightarrow +\infty} g(a_2) &= c_2 D_1 + \frac{D_3}{x^{*2}} + c_2 D_1 D_3 x^* + \frac{D_3^2}{x^*} + c_2^2 D_1 x^{*2} \\ &\quad + c_2 D_3 - \frac{(b_1 D_3 + c_2 D_2)(b_1 - D_1 D_2)}{b_1} \\ &= \delta. \end{aligned}$$

If  $\delta \geq 0$ , then  $A(a_2)B(a_2) - C(a_2) > 0$  for any  $a_2$ . Therefore,  $E^*$  is locally asymptotically stable for any  $a_2 \geq 0$ .

If  $\delta < 0$ , then there exist unique  $a_2^*$  such that  $A(a_2)B(a_2) - C(a_2) > 0$  when  $a_2 < a_2^*$ ;  $A(a_2)B(a_2) = C(a_2)$ ,  $[A(a_2)B(a_2)]' < C'(a_2)$  when  $a_2 = a_2^*$  and  $A(a_2)B(a_2) - C(a_2) < 0$  when  $a_2 > a_2^*$ .

Similar to [16, 24], we have the following results.

**Theorem 2.1.** *If  $R_0 > 1$ , then the following results hold.*

- (i) *If  $\delta \geq 0$ , the positive equilibrium  $E^*$  is locally asymptotically stable for any  $a_2 \geq 0$ .*
- (ii) *If  $\delta < 0$ , there exists a  $a_2^*$  such that*
  - (a) *when  $a_2 < a_2^*$ , the positive equilibrium  $E^*$  is locally asymptotically stable;*
  - (b) *when  $a_2 > a_2^*$ , the positive equilibrium  $E^*$  is unstable;*
  - (c) *when  $a_2 = a_2^*$ , a Hopf bifurcation occurs at the positive equilibrium  $E^*$ .*

**Remark 2.1.** It is worth mentioning that the conditions in Theorem 2.1 for the local stability of the positive equilibrium  $E^*$  are necessary and sufficient conditions. Therefore, the condition  $a_2 < a_2^*$  for the local stability of the positive equilibrium  $E^*$  is better than  $D_2 \leq D_1 + D_3$  in [36]. In addition, it is significant in mathematics and biology to have Hopf bifurcation.

### 3. The global stability of the positive equilibrium and the permanence of the model

#### 3.1. The global stability of the positive equilibrium

In the following, we consider the global properties of the positive equilibrium  $E^*$ . Let  $G' = \{(x, y, z) : (x, y, z) \in G, y > 0\}$ . It is easy to show that  $G'$  is also positively invariant with respect to model (1.2). Consider a function  $V$  as follows,

$$V = V_1 + mV_2 + pV_3,$$

where

$$V_1 = \frac{1}{2} \{(x - x^*) + \frac{a_1}{b_1} (y - y^*)\}^2, \quad V_2 = y^* \left( \frac{y}{y^*} - 1 - \ln \frac{y}{y^*} \right), \quad V_3 = \frac{1}{2} (z - z^*)^2,$$

and  $m, p$  are some undetermined constants.  $V$  is continuous on the set  $G'$  and satisfies condition (ii) of Definition 1.1 in [34] on  $\partial G = G \setminus G'$ . The derivative of  $V_1$

along solutions of model (1.2) is given by

$$\begin{aligned} V_1' &= \left\{ (x(t) - x^*) + \frac{a_1}{b_1} (y(t) - y^*) \right\} \left( x'(t) + \frac{a_1}{b_1} y'(t) \right) \\ &= -a_2 z (x(t) - x^*)^2 - D_1 (x(t) - x^*)^2 - \frac{a_1}{b_1} \frac{a_1 D_2}{b_1} (y(t) - y^*)^2 \\ &\quad + \frac{a_1 D_2}{b_1} (y^* - y(t)) (x(t) - x^*) + \frac{a_1 D_1}{b_1} (x^* - x(t)) (y(t) - y^*) \\ &\quad + \frac{a_1 a_2}{b_1} z(t) (x^* - x(t)) (y(t) - y^*) + a_2 x^* (z^* - z(t)) (x(t) - x^*) \\ &\quad + \frac{a_1}{b_1} a_2 x^* (z^* - z(t)) (y(t) - y^*), \end{aligned}$$

for  $t \geq 0$ . Similarly, the derivative of  $V_2$  along solutions of model (1.2) is given by

$$V_2' = y^* \left( \frac{y'(t)}{y^*} - \frac{y'(t)}{y(t)} \right) = b_1 (x(t) - x^*) (y(t) - y^*),$$

and the derivative of  $V_3$  along solutions of model (1.2) is given by

$$\begin{aligned} V_3' &= (z(t) - z^*) z'(t) \\ &= -(c_2 x(t) + D_3) (z^* - z(t))^2 + c_1 (y(t) - y^*) (z(t) - z^*) \\ &\quad + c_2 z^* (x^* - x(t)) (z(t) - z^*), \end{aligned}$$

for  $t \geq 0$ . Further, the derivative of  $V$  along solutions of model (1.2) is

$$\begin{aligned} V' &= V_1' + mV_2' + pV_3' \\ &= -a_2 z(t) (x(t) - x^*)^2 - D_1 (x(t) - x^*)^2 - \frac{a_1}{b_1} \frac{a_1 D_2}{b_1} (y(t) - y^*)^2 \\ &\quad - p(c_2 x(t) + D_3) (z^* - z(t))^2 + \frac{a_1 a_2}{b_1} z(t) (x^* - x(t)) (y(t) - y^*) \\ &\quad + \left( mb_1 - \frac{a_1 D_2}{b_1} - \frac{a_1 D_1}{b_1} \right) (x(t) - x^*) (y(t) - y^*) \\ &\quad + (pc_1 - \frac{a_1}{b_1} a_2 x^*) (y(t) - y^*) (z^* - z(t)) \\ &\quad + (a_2 x^* + pc_2 z^*) (x^* - x(t)) (z(t) - z^*) \end{aligned}$$

for  $t \geq 0$ . Let

$$m = \frac{a_1 D_2 + a_1 D_1}{b_1^2}, \quad p = \frac{a_1 a_2 x^*}{b_1 c_1},$$

it follows that

$$\begin{aligned} V' &\leq -D_1 (x(t) - x^*)^2 - \frac{a_1^2 D_2}{b_1^2} (y(t) - y^*)^2 - pD_3 (z^* - z(t))^2 \\ &\quad + \frac{a_1 a_2}{b_1} z(t) (x^* - x(t)) (y(t) - y^*) + (a_2 x^* + pc_2 z^*) (x^* - x(t)) (z(t) - z^*) \end{aligned}$$

for  $t \geq 0$ . Furthermore, one can get

$$\begin{aligned}
 V' &\leq -D_1(x(t) - x^*)^2 - \frac{a_1^2 D_2}{b_1^2} (y(t) - y^*)^2 - pD_3(z(t) - z^*)^2 \\
 &\quad + \frac{a_2 c_1}{lD_3} |x(t) - x^*| |y(t) - y^*| + (a_2 x^* + pc_2 z^*) |x(t) - x^*| |z(t) - z^*| \\
 &= -(|x(t) - x^*|, |y(t) - y^*|, |z(t) - z^*|) J \begin{pmatrix} |x(t) - x^*| \\ |y(t) - y^*| \\ |z(t) - z^*| \end{pmatrix},
 \end{aligned}$$

where

$$J = \begin{pmatrix} D_1 & -\frac{a_2 c_1}{2lD_3} & -\frac{a_2 x^* + pc_2 z^*}{2} \\ -\frac{a_2 c_1}{2lD_3} & \frac{a_1^2 D_2}{b_1^2} & 0 \\ -\frac{a_2 x^* + pc_2 z^*}{2} & 0 & pD_3 \end{pmatrix}.$$

It is obvious that  $V' \leq 0$  if the matrix  $J$  is positive definite. The matrix  $J$  is positive definite if all upper-left submatrices are positive definite (Sylvester's criterion), that is,  $G_1 = D_1 > 0$ ,  $G_2 = \frac{a_1^2 D_1 D_2}{b_1^2} - (\frac{a_2 c_1}{2lD_3})^2 > 0$  and  $G_3 = pD_3 G_2 - \frac{a_2 x^* + pc_2 z^*}{2} (\frac{a_2 x^* + pc_2 z^*}{2}) \frac{a_1^2 D_2}{b_1^2} > 0$ . It is clear that  $G_2 > 0$  if  $G_3 > 0$ . Therefore, we only need to verify  $G_3 > 0$ . In fact,

$$\begin{aligned}
 G_3 &= pD_3 G_2 - \frac{a_2 x^* + pc_2 z^*}{2} \left( \frac{a_2 x^* + pc_2 z^*}{2} \right) \frac{a_1^2 D_2}{b_1^2} \\
 &= \frac{a_1 a_2 x^*}{b_1} \left( \frac{a_1^2 D_1 D_2 D_3}{b_1^2 c_1} - \frac{a_2^2 c_1}{4l^2 D_3} - \frac{a_1 a_2 x^* D_2 (b_1 c_1 + a_1 c_2 z^*)^2}{4b_1^3 c_1^2} \right) \\
 &= \frac{a_1 a_2 x^*}{4b_1^3 c_1 l^2 D_3 (a_2 b_1 c_1 D_2 + a_1 b_1 D_2 D_3 + a_1 c_2 D_2^2)^2} \{ 4a_1^2 D_1 D_2 D_3^2 l^2 (a_2 b_1 c_1 D_2 \\
 &\quad + a_1 b_1 D_2 D_3 + a_1 c_2 D_2^2)^2 - a_2^2 b_1^2 c_1^2 (a_2 b_1 c_1 D_2 + a_1 b_1 D_2 D_3 + a_1 c_2 D_2^2)^2 \\
 &\quad - a_1 a_2 b_1 c_1 x^* D_2 l^2 D_3 ((a_2 b_1 c_1 D_2 + a_1 b_1 D_2 D_3 + a_1 c_2 D_2^2)^2 + a_1^2 c_2^2 (b_1 - D_1 D_2)^2 \\
 &\quad + 2a_1 c_2 (b_1 - D_1 D_2) (a_2 b_1 c_1 D_2 + a_1 b_1 D_2 D_3 + a_1 c_2 D_2^2)) \}.
 \end{aligned}$$

Define the following function with respect to  $a_2$ ,

$$\begin{aligned}
 h(a_2) &= 4a_1^2 D_1 D_2 D_3^2 l^2 (a_2 b_1 c_1 D_2 + a_1 b_1 D_2 D_3 + a_1 c_2 D_2^2)^2 \\
 &\quad - a_2^2 b_1^2 c_1^2 (a_2 b_1 c_1 D_2 + a_1 b_1 D_2 D_3 + a_1 c_2 D_2^2)^2 \\
 &\quad - a_1 a_2 b_1 c_1 x^* D_2 l^2 D_3 ((a_2 b_1 c_1 D_2 + a_1 b_1 D_2 D_3 + a_1 c_2 D_2^2)^2 + a_1^2 c_2^2 (b_1 - D_1 D_2)^2 \\
 &\quad + 2a_1 c_2 (b_1 - D_1 D_2) (a_2 b_1 c_1 D_2 + a_1 b_1 D_2 D_3 + a_1 c_2 D_2^2)),
 \end{aligned}$$

it follows

$$\begin{aligned}
h(a_2) = & -b_1^4 c_1^4 D_2^2 a_2^4 - 2b_1^3 c_1^3 D_2^2 (a_1 b_1 D_3 + a_1 c_2 D_2) a_2^3 - a_1 b_1^2 c_1^3 D_2^4 l^2 D_3 a_2^3 \\
& + 4a_1^2 b_1^2 c_1^2 D_1 D_2^3 D_3^2 l^2 a_2^2 - b_1^2 c_1^2 (a_1 b_1 D_2 D_3 + a_1 c_2 D_2^2) a_2^2 \\
& - 2a_1 b_1 c_1^2 D_2^4 l^2 D_3 (a_1 b_1 D_3 + a_1 c_2 D_2) a_2^2 \\
& - 2a_1^2 b_1 c_1^2 D_2^3 l^2 D_3 (b_1 - D_1 D_2) a_2^2 - a_1^3 c_1 D_2^2 l^2 D_3 c_2^2 (b_1 - D_1 D_2)^2 a_2 \\
& + 8a_1^2 b_1 c_1 D_1 D_2^3 D_3^2 l^2 (a_1 b_1 D_3 + a_1 c_2 D_2) a_2 \\
& - a_1 c_1 D_2^4 l^2 D_3 (a_1 b_1 D_3 + a_1 c_2 D_2)^2 a_2 \\
& - 2a_1^2 c_1 c_2 D_2^3 l^2 D_3 (b_1 - D_1 D_2) (a_1 b_1 D_3 + a_1 c_2 D_2) a_2 \\
& + 4a_1^2 D_1 D_2^3 D_3^2 l^2 (a_1 b_1 D_3 + a_1 c_2 D_2)^2.
\end{aligned}$$

It is obvious that there exists a  $a_2^{**} > 0$  such that  $h(a_2) > 0$  holds for all  $0 < a_2 < a_2^{**}$ . Furthermore,  $G_3 > 0$  for all  $0 < a_2 < a_2^{**}$ . Therefore, we have  $V' \leq 0$  when  $a_2 < a_2^{**}$ . In addition, it is easy to verify  $a_2^{**} < a_2^*$ . From Theorem 1.2 in [34], we have the following result.

**Theorem 3.1.** *If  $R_0 > 1$ , then there exist a  $a_2^{**}$  such that the positive equilibrium  $E^*$  is globally asymptotically stable when  $a_2 < a_2^{**}$  with respect to  $G'$ .*

### 3.2. Permanence

The permanence of a biological system is of profound biological significance and can be used to reveal the permanent existence of biological populations [26]. Therefore, in this section, we discuss the permanence of model (1.2). By using a similar analysis method as in [6, 7, 13, 32], For the permanence of model (1.2) with the initial condition (1.3), it has that

**Theorem 3.2.** *If  $R_0 > 1$ , then model (1.2) is permanent, and each positive solution  $(x(t), y(t), z(t))^T$  of model (1.2) with initial condition (1.3) satisfies*

$$\begin{aligned}
\liminf_{t \rightarrow +\infty} x(t) & \geq \frac{1}{b_1/l + a_2 c_1 b_1 / (l a_1 D_3) + D_1} = \nu_1, \\
\liminf_{t \rightarrow +\infty} y(t) & \geq \frac{y^*}{2e^{dD_2}} = \nu_2, \\
\liminf_{t \rightarrow +\infty} z(t) & \geq \frac{c_1 D_1 \nu_2}{c_2 + D_1 D_3} = \nu_3,
\end{aligned} \tag{3.1}$$

where  $d > 0$  and satisfies

$$q \equiv \frac{1}{a_1 y^* / 2 + a_2 c_1 b_1 / (l a_1 D_3) + D_1} > x^*, \quad x^\Delta \equiv q(1 - e^{-d/q}) > x^*.$$

**Proof.** Let us consider a positive solution  $(x(t), y(t), z(t))^T$  of model (1.2) with initial condition (1.3). From Theorem 1 in [36], it follows that for any given  $\varepsilon > 0$ , there exists some  $t_0 > 0$  such that  $y(t) \leq \frac{b_1}{l a_1} + \varepsilon$ ,  $z(t) \leq \frac{c_1 b_1}{l a_1 D_3} + \varepsilon$  for all  $t \geq t_0$ . Hence, we have from the first equation of model (1.2) that,

$$\begin{aligned}
x'(t) & = 1 - (a_1 y(t) + a_2 z(t) + D_1)x(t) \\
& \geq 1 - \left( \left( \frac{b_1}{l} + a_1 \varepsilon \right) + \frac{a_2 c_1 b_1}{l a_1 D_3} + a_2 \varepsilon + D_1 \right) x(t),
\end{aligned} \tag{3.2}$$

it follows

$$\liminf_{t \rightarrow +\infty} x(t) \geq \frac{1}{(b_1/l + a_1\varepsilon) + a_2c_1b_1/(la_1D_3) + a_2\varepsilon + D_1}.$$

Taking into account the arbitrariness of  $\varepsilon > 0$ , one has

$$\liminf_{t \rightarrow +\infty} x(t) \geq \nu_1.$$

Furthermore, let us show that  $\liminf_{t \rightarrow +\infty} y(t) \geq \nu_2$ . Notice that  $x^* < x_0 = \frac{1}{D_1}$ . Define

$$V(t) = y(t),$$

it follows

$$V'(t) = (b_1x(t) - D_2)y(t) = b_1(x(t) - x^*)y(t). \tag{3.3}$$

Claim: For all large  $t$ , it is impossible that  $y(t) \leq y^*/2$ . Assume the contrary, then there exists a  $t^* > t_0$  such that  $y(t) \leq y^*/2$  for any  $t \geq t^*$ . Thus, for any  $t \geq t^*$ , it follows from the first equation of model (1.2) that

$$\begin{aligned} x'(t) &\geq 1 - a_1y^*x(t)/2 - a_2x(t)z(t) - D_1x(t) \\ &\geq 1 - \left( a_1y^*/2 + \frac{a_2c_1b_1}{la_1D_3} + D_1 \right) x(t). \end{aligned}$$

This implies

$$\begin{aligned} x(t) &\geq (x(t^*) + qe^{q(t-t^*)} - q)e^{-\frac{1}{q}(t-t^*)} \\ &\geq q(1 - e^{-\frac{d}{q}}) \\ &= x^\Delta > x^*, \end{aligned} \tag{3.4}$$

for any  $t \geq t^* + d$ . Set

$$\underline{y} = y(t^* + d) > 0.$$

Now verify  $y(t) \geq \underline{y}$  for all  $t \geq t^* + d$ . In fact, if there exists a  $T > 0$  such that  $y(t) \geq \underline{y}$  for  $t^* + d \leq t \leq t^* + d + T$ ,  $y(t^* + d + T) = \underline{y}$  and  $y'(t^* + d + T) \leq 0$ , it follows from the second equation of model (1.2) that

$$\begin{aligned} y'(t) &= b_1x(t)y(t) - D_2y(t) \\ &\geq b_1(x^\Delta - x^*)\underline{y} > 0, \end{aligned}$$

for  $t = t^* + d + T$ . This is a contradiction to  $y'(t^* + d + T) \leq 0$ . Therefore,  $y(t) \geq \underline{y}$  for all  $t \geq t^* + d$ . Hence, for all  $t \geq t^* + d$ , it follows from

$$V'(t) = b_1(x(t) - x^*)y(t) > b_1(x^\Delta - x^*)\underline{y}$$

that  $V(t) \rightarrow +\infty$  as  $t \rightarrow +\infty$ . On the other hand, from Theorem 1 in [36], there exists a sufficiently  $\bar{T} > 0$  such that, for  $t \geq \bar{T}$ ,

$$V(t) = y(t) \leq \frac{b_1}{la_1}.$$

This is a contradiction with  $V(t) \rightarrow +\infty$  as  $t \rightarrow +\infty$ . Therefore, the claim is proved.

Next, we consider the following two cases.

(i)  $y(t) \geq y^*/2$  for all large  $t$ . (ii)  $y(t)$  oscillates about  $y^*/2$  for all large  $t$ .

Clearly, we only need to consider case (ii). Let  $t_1, t_2$  big enough such that  $t^* < t_1 < t_2$  and

$$y(t_1) = y(t_2) = y^*/2, \quad y(t) < y^*/2, \quad (t_1 < t < t_2).$$

If  $t_2 - t_1 \leq d$ , it follows from the second equation of model (1.2) that

$$y'(t) > -D_2y(t),$$

which implies

$$y(t) > y(t_1)e^{-D_2(t-t_1)},$$

for  $t \in (t_1, t_2)$ . It is obvious that

$$y(t) \geq \frac{y^*}{2e^{dD_2}} = \nu_2,$$

for  $t \in (t_1, t_2)$ . If  $t_2 - t_1 > d$ , we can easily obtain that  $y(t) \geq \nu_2$  for  $t \in [t_1, t_1 + d]$ . Then, as the proof of the above claim, we can show that  $y(t) \geq \nu_2$  for  $t_1 + d \leq t \leq t_2$ . In fact, if not, there exists a  $T^* > 0$ , such that,  $y(t) \geq \nu_2$  for  $t_1 \leq t \leq t_1 + d + T^*$ ,  $y(t_1 + d + T^*) = \nu_2$ , and  $y'(t_1 + d + T^*) \leq 0$ . On the other hand, for  $t_1 \leq t \leq t_1 + d + T^* \leq t_2$ ,

$$y(t) \leq y^*/2.$$

From the first equation of model (1.2), for  $t_1 \leq t \leq t_1 + d + T^* \leq t_2$ , it follows

$$x'(t) \geq 1 - \left( a_1y^*/2 + \frac{a_2c_1b_1}{la_1D_3} + D_1 \right) x(t).$$

This implies

$$\begin{aligned} x(t) &\geq x(t_1)e^{-(a_1y^*/2 + \frac{a_2c_1b_1}{la_1D_3} + D_1)(t-t_1)} + \int_{t_1}^t e^{-(a_1y^*/2 + \frac{a_2c_1b_1}{la_1D_3} + D_1)(t-\theta)} d\theta \\ &> \frac{1}{a_1y^*/2 + \frac{a_2c_1b_1}{la_1D_3} + D_1} \left( 1 - e^{-(a_1y^*/2 + \frac{a_2c_1b_1}{la_1D_3} + D_1)(t-t_1)} \right) \\ &\geq \frac{1}{a_1y^*/2 + \frac{a_2c_1b_1}{la_1D_3} + D_1} \left( 1 - e^{-(a_1y^*/2 + \frac{a_2c_1b_1}{la_1D_3} + D_1)d} \right) \\ &= q(1 - e^{-\frac{d}{q}}) = x^\Delta > x^* \end{aligned} \quad (3.5)$$

for  $t_1 + d \leq t \leq t_1 + d + T^* \leq t_2$ . From the second equation of model (1.2), it follows that

$$y'(t) = b_1x(t)y(t) - D_2y(t) \geq b_1(x^\Delta - x^*)\nu_2 > 0, \quad (3.6)$$

for  $t = t_1 + d + T^*$ . This is a contradiction to  $y'(t_1 + d + T^*) \leq 0$ . Therefore, we have that  $y(t) \geq \nu_2$  for  $t \in [t_1, t_2]$ . Consider the interval  $[t_1, t_2]$  is chosen in an arbitrary way, then we can conclude that  $y(t) \geq \nu_2$  for all large  $t$  for case (ii). Hence, it has that  $\liminf_{t \rightarrow +\infty} y(t) \geq \nu_2$ . From the last equation of model (1.2), it follows that

$$z'(t) = c_1y(t) - c_2x(t)z(t) - D_3z(t) \geq c_1\nu_2 - \left( \frac{c_2}{D_1} + D_3 \right) z(t).$$

Furthermore, we can get

$$\liminf_{t \rightarrow +\infty} z(t) \geq \frac{c_1 \nu_2 D_1}{c_2 + D_1 D_3} = \nu_3.$$

The proof is complete. □

**Remark 3.1.** Compared to Theorem 5 in [36], we obtain the explicit lower bounds  $\nu_1, \nu_2$  and  $\nu_3$  of  $\liminf_{t \rightarrow +\infty} x(t)$ ,  $\liminf_{t \rightarrow +\infty} y(t)$  and  $\liminf_{t \rightarrow +\infty} z(t)$ . Therefore, Theorem 3.1 gives more practical meaning whether in mathematics or in biology.

### 4. Numerical simulations and conclusions

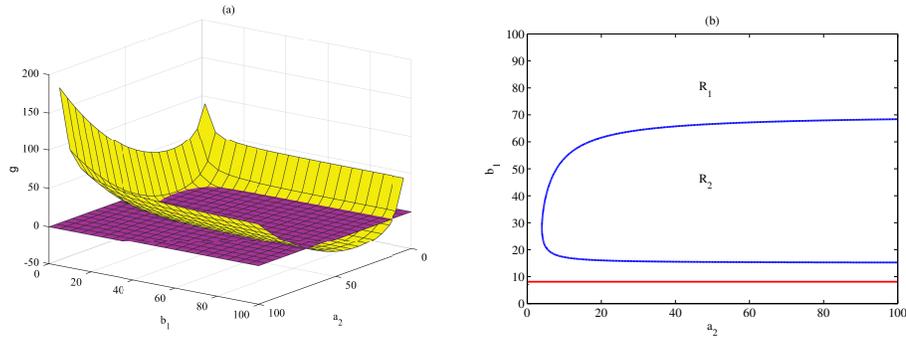
In this section, we present some numerical examples to demonstrate the theoretical results obtained in the previous sections for the positive equilibrium  $E^*$ . Choose  $a_2, b_1$  as two bifurcation parameters and fix all other parameter values:  $a_1 = 1, D_1 = 1.01, c_1 = 5, c_2 = 5, D_2 = 8, D_3 = 1.02$ . In order to find the Hopf critical point, we apply the Hurwitz condition in terms of the parameters  $a_2, b_1$ . The characteristic polynomial at  $E^*$  with the given parameter values is  $\det(\lambda - J) = \lambda^3 + A\lambda^2 + B\lambda + C = 0$ , where

$$\begin{aligned} A(a_2, b_1) &= \frac{b_1}{8} + \frac{40}{b_1} + 1.02, \\ B(a_2, b_1) &= \frac{8(40 + 1.02b_1)(-8.08 + b_1)}{(320 + 40a_2b_1 + 8.16b_1)} - \frac{200a_2(-8.08 + b_1)}{320 + 40a_2b_1 + 8.16b_1} + \frac{1}{8}(40 + 1.02b_1), \\ C(a_2, b_1) &= \frac{40a_2(40 + 1.02b_1)(-8.08 + b_1)}{(320 + 40a_2b_1 + 8.16b_1)} + \frac{8(40 + 1.02b_1)^2(-8.08 + b_1)}{b_1(320 + 40a_2b_1 + 8.16b_1)}. \end{aligned}$$

Then, we obtain

$$\begin{aligned} g(a_2, b_1) &= A(a_2, b_1)B(a_2, b_1) - C(a_2, b_1) \\ &= -\frac{323.2b_1}{320 + 40a_2b_1 + 8.16b_1} + \frac{31.7584b_1^2}{320 + 40a_2b_1 + 8.16b_1} \\ &\quad + \frac{1.02b_1^3}{320 + 40a_2b_1 + 8.16b_1} - \frac{1272.336a_2b_1}{320 + 40a_2b_1 + 8.16b_1} \\ &\quad - \frac{65.8a_2b_1^2}{320 + 40a_2b_1 + 8.16b_1} + \frac{64640.0a_2}{b_1(320 + 40a_2b_1 + 8.16b_1)} \\ &\quad + \frac{6576.32a_2}{320 + 40a_2b_1 + 8.16b_1} + \frac{200}{b_1} + 0.75505b_1 + 0.0159375b_1^2 + 10.2. \end{aligned}$$

The graph of the function  $g = g(a_2, b_1)$  (the yellow surface) is shown in Figure 1(a). It is obvious that the graph of the surface  $g = g(a_2, b_1)$  and the plane  $g = 0$  (the red plane) intersect. Figure 1(b) shows the intersection of this two faces in the  $a_2, b_1$  plane. In addition, it follows from  $R_0 > 1$  that  $b_1 > 8.08$ . The graph of  $b_1 = 8.08$  is shown in Figure 1(b) (the red line). From Figure 1(b), we can obtain that  $g(a_2, b_1) > 0$  in region  $R_1$ ;  $g(a_2, b_1) = 0$  in the blue curve and  $g(a_2, b_1) < 0$  in region  $R_2$ . Namely, in region  $R_1$ , the positive equilibrium  $E^*$  is asymptotically stable; a Hopf bifurcation occurs at the positive equilibrium  $E^*$  in the blue curve; in region  $R_2$ , the positive equilibrium  $E^*$  is unstable.



**Figure 1.** Behavior of the function  $g = g(a_2, b_1)$  with respect to  $b_1, a_2$  and the intersection of  $g = g(a_2, b_1)$  and  $g = 0$ , respectively.

In the following, to simulate the stability of the positive equilibrium  $E^*$ , we first take  $b_1 = 20$  (the dotted green line in Figure 2(a)). Next, we will calculate  $a_2^*$  in Figure 2(a). One can get

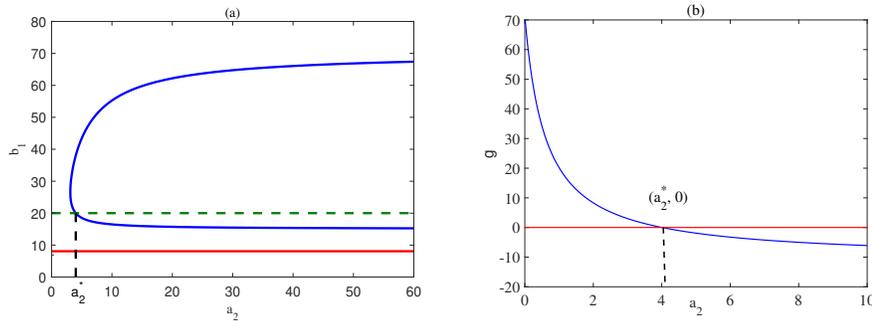
$$A = 5.52, \quad B = \frac{287.9872}{40a_2 + 24.16} - \frac{119.2a_2}{40a_2 + 24.16} + 7.55,$$

$$C = \frac{1439.936a_2}{40a_2 + 24.16} + \frac{869.721344}{40a_2 + 24.16}.$$

Furthermore, we obtain

$$g(a_2) = A(a_2)B(a_2) - C(a_2) = \frac{719.968}{40a_2 + 24.16} - \frac{2097.92a_2}{40a_2 + 24.16} + 41.676.$$

and the graph of the function  $g = g(a_2)$  is shown in Figure 2(b). It follows from



**Figure 2.** Behavior of the functions  $b_1 = b_1(a_2) = 20, g = g(a_2)$  with respect to  $a_2$ , respectively.

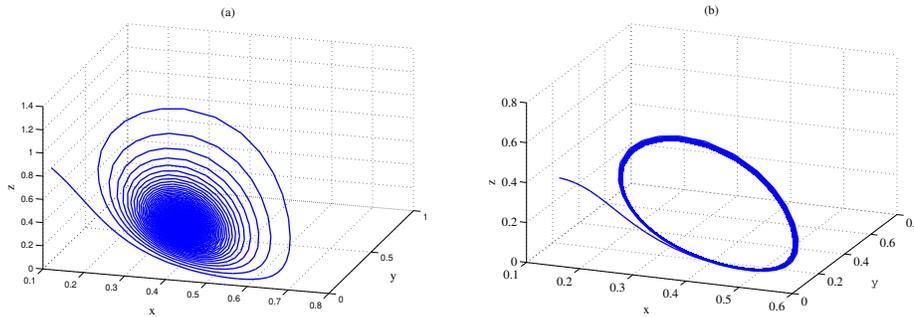
$g(a_2) = 0$  that  $a_2^* \approx 4.0078$ . This indicates that  $E^*$  is asymptotically stable for  $a_2 \in (0, a_2^*)$ . We take  $a_2 = a_2^* \approx 4.0078$ , and present a detail analysis on the Hopf bifurcation and bifurcating limit cycles. At  $a_2 = a_2^*$ , the approximate eigenvalues of the characteristic equation (2.1) are

$$\lambda \approx -5.52, \pm 2.5537i.$$

For simulations, we first take  $a_2 = 4 (< 4.0078)$ . It follows  $E^* = (0.4, 0.1955, 0.3236)$ , and the approximate eigenvalues of the characteristic equation (2.1) are

$$\lambda \approx -5.5195, \quad -0.0002 \pm 2.5538i.$$

The simulation results indicate that all three components converge to the steady-state values and the positive equilibrium  $E^*$  is stable in the region  $R_1$  (see Figure 3(a)). Next, we choose  $a_2 = 6 (> 4.0078)$ . It follows  $E^* = (0.4, 0.1364, 0.2273)$ , and



**Figure 3.** Figure a shows that the positive equilibrium  $E^*$  of model (1.2) is asymptotically stable with  $a_2 = 4$  and the initial value  $(0.1, 0.1, 0.8)$ ; Figure b shows that the positive equilibrium  $E^*$  is unstable and there is a stable periodic solution with  $a_2 = 6$  and the initial condition  $(0.1, 0.25, 0.3)$ .

the approximate eigenvalues of the characteristic equation (2.1) are

$$\lambda \approx -5.6070, \quad 0.0435 \pm 2.5335i.$$

Obviously,  $E^*$  is unstable in this case in theory. From Figure 3(b), we know that all three components are oscillatory and a periodic solution is obtained in the region  $R_2$ . In addition, from  $h(a_2) = 0$ , we obtain  $a_2^{**} = 0.0414$ . It is obvious that  $a_2^{**} < a_2^*$ . From Theorem 3.1, when  $R_0 > 1$ , we have obtain the positive equilibrium  $E^*$  is globally asymptotically stable if  $a_2 < a_2^{**}$ . However, when  $a_2^{**} < a_2 < a_2^*$ , we have not given a corresponding conclusion as to whether the positive equilibrium point is globally stable.

To sum up, in this paper, we considered a model described by a system of ordinary differential equations with the biodegradation of MCs. For the local properties of the positive equilibrium  $E^*$ , we got the Theorem 2.1. As shown in this section, for the given parameter values, the positive equilibrium  $E^*$  is asymptotically stable in region  $R_1$ ; the positive equilibrium  $E^*$  lose its stability and a Hopf bifurcation occurs in the blue curve in Figure 1(a); the positive equilibrium  $E^*$  is unstable in region  $R_2$ . For the global stability of the positive equilibrium  $E^*$ , we obtained Theorem 3.1. Furthermore, we have shown that model (1.2) is permanent when  $R_0 > 1$  from Theorem 3.2. From a biological point of view, under suitable conditions, the biodegradation of MCs is persistent. That means MCs can be degraded by adjusting certain control parameters (such as  $a_2$  in model (1.2)) in the process of degradation. From the above analysis, one can get that model (1.2) has rich dynamic behaviors, which can provide some significant information for biodegradation of MCs.

Recent studies show that the dynamics of the models may also be affected by random environment noise [29, 37, 42]. Therefore, investigating model (1.1) with stochastic perturbations will become more interesting.

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