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MIXED MONOTONE ITERATIVE TECHNIQUE FOR SEMILINEAR IMPULSIVE FRACTIONAL EVOLUTION EQUATIONS*

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Abstract In this paper, we deals with the existence of mild L-quasi-solutions to the boundary value problem for a class of semilinear impulsive fractional evolution equations in an ordered Banach space E. Under a new concept of upper and lower solutions, a new monotone iterative technique on the initial value problem of impulsive fractional evolution equations has been established. The results improve and extend some relevant results in ordinary differential equations and partial differential equations. As some application that illustrate our results, An example is also given.

Keywords Monotone iterative technique, coupled L-quasi-upper and lower solutions, impulsive fractional evolution equation, C_0-semigroup.

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1. Introduction

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, and they have been emerging as an important area of investigation in the last few decades; see [1–6, 10–12, 14, 15, 25, 31, 44].

The theory of impulsive differential equations is a new and important branch of differential equation theory, which has an extensive physical, population dynamics, ecology, chemical, biological systems, and engineering background. Therefore, it has been an object of intensive investigation in recent years, some basic results on impulsive differential equations have been obtained and applications to different areas have been considered by many authors, see [20,37–40]. Particularly, the theory of impulsive evolution equations has become more important in resent years because of its wide applicability in control, mechanics, electrical engineering, biological and medical fields. There has been a significant development in impulsive evolution equations in Banach spaces. For more details on this theory and its applications, we refer to the Refs. [7, 8, 27–29].

The monotone iterative method based on lower and upper solutions is an effective and flexible mechanism. It yields monotone sequences of lower and upper

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approximate solutions that converge to the minimal and maximal solutions between the lower and upper solutions. Early on, Du and Lakshmikantham [17], Sun and Zhao [32] investigated the existence of extremal solutions to initial value problem of ordinary differential equation without impulse by using the method of lower and upper solutions and the monotone iterative technique. Later on, Guo and Liu [22], Li and Liu [25] developed the monotone iterative method for impulsive integro-differential equations. Lately, the monotone iterative method has been extended to evolution equations in ordered Banach spaces by Li [26]. Moreover, Wang et al. [41] and EI-Gebeily et al. [18] for evolution equations with classical initial conditions, and Chen and Mu [7] and Chen and Li [8] for impulsive evolution equations with classical initial conditions.

Periodic boundary problems for fractional differential equations serve as a class of important models to study the dynamics of processes that are subject to periodic changes in their initial state and final state. There are some papers discussing periodic (or anti-periodic) boundary problems for fractional differential equations in finite dimensional spaces. However, there are few results on the theory on periodic boundary problems for fractional evolution equations in infinite dimensional spaces. Since the unbounded operator is involved in the fractional evolution equations, it is obvious that periodic boundary problems for fractional evolution equations are much more difficult than the same problems for fractional differential equations.

In [28], Mu et al. use the monotone iterative technique to investigate the existence and uniqueness of mild solutions of the impulsive fractional evolution equations in an order Banach space E:

$$\begin{cases} {}^cD_t^\alpha u(t) + Au(t) = f(t, u(t)), \quad t \in J,\ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \quad k = 1, 2, \ldots, m, \\ u(0) = x_0 \in E, \end{cases}$$

and

$$\begin{cases} {}^cD_t^\alpha u(t) + Au(t) = f(t, u(t)), \quad t \in J,\ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \quad k = 1, 2, \ldots, m, \\ u(0) + g(u) = x_0 \in E, \end{cases}$$

where ${}^cD_t^\alpha$ is the Caputo fractional derivative of order $\alpha \in (0,1)$, $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generates a C_0-semigroup $T(t)(t \geq 0)$.

In [27], Li and Gou used a monotone iterative method in the presence of lower and upper solutions to discuss the existence and uniqueness of mild solutions for the boundary value problem of impulsive evolution equation in an ordered Banach space E:

$$\begin{cases} u'(t) + Au(t) = f(t, u(t), Fu(t), Gu(t)), \quad t \in J,\ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \quad k = 1, 2, \ldots, m, \\ u(0) = u(\omega), \end{cases}$$

where $A : D(A) \subset E \to E$ is a closed linear operator and $-A$ generates a C_0-semigroup $T(t)(t \geq 0)$ in E. Under wide monotonicity conditions and the noncompactness measure condition of the nonlinearity f, we obtain the existence of extremal mild solutions and a unique mild solution between lower and upper solutions requiring only that $-A$ generates a C_0-semigroup.

However, to the best of our knowledge, the theory of periodic boundary value problems for nonlinear impulsive fractional evolution equations is still in the initial stages and many aspects of this theory need to be explored, motivated by the above discussion, in this paper, we use a monotone iterative method in the presence of lower and upper L-quasi-solutions to discuss the existence of mild solutions for the periodic boundary value problem (PBVP) of impulsive fractional evolution equations in an ordered Banach space E:

$$\begin{cases} {}^cD_{0^+}^{\alpha}u(t) + Au(t) = f(t, u(t), u(t)), \quad t \in J,\ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k), u(t_k)), \quad k = 1, 2, \ldots, m, \\ u(0) = u(\omega), \end{cases} \tag{1.1}$$

where ${}^cD_{0^+}^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (0, 1]$ with the lower limit zero, $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generates a C_0-semigroup $T(t)(t \geq 0)$ in E; $f \in C(J \times E \times E, E)$, $I_k \in C(E, E)$ is an impulsive function, $k = 1, 2, \ldots, m$; $J = [0, \omega]$, $J' = J \setminus \{t_1, t_2, \ldots, t_m\}$, $J_0 = [0, t_1]$, $J_k = (t_k, t_{k+1}]$, the $\{t_k\}$ satisfy $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = \omega, m \in N$; $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$ respectively.

In this paper, we improve and extend the above mentioned results and obtain the existence of the coupled minimal and maximal L-quasi-solution, and the mild solutions between the coupled minimal and maximal mild L-quasi-solution of the BVP (1.1) through the mixed monotone iterative about the coupled lower and upper quasi-solutions.

2. Preliminaries

In this section, we briefly recall some basic known results which will be used in the sequel.

Let E be an ordered Banach space with the norm $\|\cdot\|$ and partial order $\leq$, whose positive cone $P = \{x \in E : x \geq 0\}$ is normal with normal constant N. Let $C(J, E)$ denote the Banach space of all continuous E-value functions on interval J with the norm $\|u\|_C = \max_{t\in J} \|u(t)\|$. Evidently, $C(J, E)$ is also an ordered Banach space induced by the convex cone $P' = \{u \in E | u(t) \geq 0, t \in J\}$, which is also a normal cone.

Let $PC(J, E) = \{u : J \to E, u(t)$ is continuous at $t \neq t_k$, and left continuous at $t = t_k$, and $u(t_k^+)$ exists, $k = 1, 2, \ldots, m\}$. Evidently, $PC(J, E)$ is a Banach space with the norm $\|u\|_{PC} = \sup_{t\in J} \|u(t)\|$. We use E_1 to denote the Banach space $D(A)$ with the graph norm $\|\cdot\|_1 = \|\cdot\| + \|A\cdot\|$. An abstract function $u \in PC(J, E) \cap C(J', E_1)$ is called a solution of the problem (1.1) if $u(t)$ satisfies all the equalities of (1.1). Let $\alpha(\cdot)$ denote the Kuratowski measure of noncompactness of the bounded set. For the details of the definition and properties of the measure of noncompactness, see [16]. For any $B \subset C(J, E)$ and $t \in J$, set $B(t) = \{u(t) : u \in B\} \subset E$. If B is bounded in $C(J, E)$, then $B(t)$ is bounded in E, and $\alpha(B(t)) \leq \alpha(B)$.

For completeness we recall the definition of the Caputo derivative of fractional order.

Definition 2.1. The fractional integral of order γ of a function $f:[0,\infty)\to\mathbb{R}$ is defined as

$$I_{0+}^{\gamma}f(t)=\frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}f(s)ds,\ t>0,\gamma>0,$$

provided the right side is point-wise defined on $(0,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. The Riemann-Liouville derivative of order γ with the lower limit zero for a function $f:[0,\infty)\to\mathbb{R}$ can be written as

$$D_{0+}^{\gamma}f(t)=\frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}}ds,\quad t>0, n-1<\gamma<n.$$

Definition 2.3. The Caputo fractional derivative of order γ for a function $f:[0,\infty)\to\mathbb{R}$ can be written as

$${}^cD_{0+}^{\gamma}f(t)=D_{0+}^{\gamma}\Big[f(t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\Big],\ t>0,\ n-1<\gamma<n,$$

where $n=[\gamma]+1$ and $[\gamma]$ denotes the integer part of γ.

Remark 2.1. In the case $f(t)\in C^n[0,\infty)$, then

$${}^cD_{0+}^{\gamma}f(t)=\frac{1}{\Gamma(n-\gamma)}\int_0^t (t-s)^{n-\gamma-1}f^{(n)}(s)ds=I_{0+}^{n-\gamma}f^{n}(t),\ t>0,\ n-1<\gamma<n.$$

Remark 2.2. If u is an abstract function with values in E, then the integrals which appear in Definitions 2.2 and 2.3 are taken in Bochner's sense.

Lemma 2.1 ([5]). *For $\gamma>0$, the general solution of the fractional differential equation ${}^cD_{0+}^{\gamma}u(t)=0$ is given by*

$$u(t)=c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1},$$

where $c_i\in\mathbb{R}, i=0,1,\ldots,n-1, n=[\gamma]+1$ and $[\gamma]$ denotes the integer part of the real number γ.

We will give the following lemmas to be used in proving our main results.

Lemma 2.2 ([13]). *Let E be a Banach space, and let $B\subset E$ be bounded. Then there exists a countable set $B_0\subset B$, such that $\alpha(B)\le 2\alpha(B_0)$.*

Lemma 2.3 ([26]). *Let E be a Banach space, and let $B\subset C(J,E)$ is equicontinuous and bounded, then $\alpha(B(t))$ is continuous on J, and $\alpha(B)=\max_{t\in J}\alpha(B(t))$.*

Lemma 2.4 ([23]). *Let $B=\{u_n\}\subset PC(J,E)$ be a bounded and countable set. Then $\alpha(B(t))$ is Lebesgue integral on J, and*

$$\alpha\Big(\Big\{\int_J u_n(t)dt : n\in\mathbb{N}\Big\}\Big)\le 2\int_J \alpha(B(t))dt.$$

Lemma 2.5 (Sadovskii's fixed point theorem)**.** *Let E be a Banach space and Ω_0 be a nonempty bounded convex closed set in E. If $Q : \Omega_0 \to \Omega_0$ is a condensing mapping, then Q has a fixed point in Ω_0.*

Let $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generates a C_0-semigroup $T(t)(t \geq 0)$ in E. Then there exist constants $C > 0$ and $\delta \in \mathbb{R}$ such that

$$\|T(t)\| \leq Ce^{\delta t}, \quad t \geq 0.$$

We consider the initial value problem of linear impulsive fractional evolution equations

$$\begin{cases} {}^cD_{0^+}^{\alpha} u(t) + Au(t) = h(t), \quad t \in J', \\ \Delta u|_{t=t_k} = y_k, \quad k = 1, 2, \ldots, m, \\ u(0) = u_0, \end{cases} \tag{2.1}$$

where $h \in C(J, E), u_0 \in D(A)$, $y_k \in E, k = 1, 2, \ldots, m$.

Now, we are ready to construct a mild solution for the impulsive system (2.1). It is different from the method of the paper [42].

Lemma 2.6. *Let E be a Banach space, $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generate a C_0-semigroup $T(t)(t \geq 0)$ in E. For any $h \in PC(J, E)$, $u_0 \in E$ and $y_k \in E, k = 1, 2, \ldots, m$, then the problem (2.1) has a unique mild solution $u \in PC(J, E)$ given by*

$$u(t) = \begin{cases} \mathscr{T}_{\alpha}(t)u_0 + \int_0^t \mathscr{S}_{\alpha}(t-s)h(s)ds, \quad t \in [0, t_1], \\ \mathscr{T}_{\alpha}(t)u_0 + \mathscr{T}_{\alpha}(t-t_1)y_1 + \int_0^t \mathscr{S}_{\alpha}(t-s)h(s)ds, \quad t \in (t_1, t_2], \\ \vdots \\ \mathscr{T}_{\alpha}(t)u_0 + \sum\limits_{i=1}^{m} \mathscr{T}_{\alpha}(t-t_i)y_i + \int_0^t \mathscr{S}_{\alpha}(t-s)h(s)ds, \quad t \in (t_m, \omega], \end{cases} \tag{2.2}$$

where

$$\mathscr{T}_{\alpha}(t) = \int_0^{\infty} \xi_{\alpha}(\sigma)T(t^{\alpha}\sigma)d\sigma = E_{\alpha,1}(At^{\alpha}),$$
$$\mathscr{S}_{\alpha}(t) = \alpha \int_0^{\infty} \sigma t^{\alpha-1}\xi_{\alpha}(\sigma)T(t^{\alpha}\sigma)d\sigma = t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha}),$$
$$\xi_{\alpha}(\sigma) = \frac{1}{\pi\alpha}\sum_{n=1}^{\infty}(-\sigma)^{n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \quad \sigma \in (0, \infty)$$

are the functions of Wright type defined on $(0, \infty)$ which satisfies

$$\xi_{\alpha}(\sigma) \geq 0, \quad s \in (0, \infty), \qquad \int_0^{\infty} \xi_{\alpha}(\sigma)d\sigma = 1,$$

and

$$\int_0^{\infty} \sigma^{v}\xi_{\alpha}(\sigma)d\sigma = \frac{\Gamma(1+v)}{\Gamma(1+\alpha v)}, \quad v \in [0, 1].$$

Proof. With Lemma 2.1, a general solution u of the equation (2.1) on each interval $(t_k, t_{k+1}]$ $(k = 0, 1, \dots, m)$ is given by

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[-Au(s) + h(s) \Big] ds + c_k,$$

where $t_0 = 0, t_{m+1} = \omega$. From $u(0) = u_0$ and $\Delta u(t_k) = y_k$, we get $c_0 = u_0$ and

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)} \int_0^{t_k} (t_k-s)^{\alpha-1} \Big[-Au(s) + h(s) \Big] ds + c_k \\ &- \Big(\frac{1}{\Gamma(\alpha)} \int_0^{t_k} (t_k-s)^{\alpha-1} \Big[-Au(s) + h(s) \Big] ds + c_{k-1} \Big) \\ =& y_k. \end{aligned}$$

This implies that

$$c_k = c_{k-1} + y_k, \quad k = 1, 2, \dots, m, \tag{2.3}$$

which by (2.3) imply

$$c_k = u_0 + \sum_{i=1}^{k} y_i, \quad k = 1, 2, \dots, m.$$

Hence for $k = 1, 2, \dots, m$ and (2.3), we get

$$u(t) = u_0 + \sum_{i=1}^{k} y_i + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[-Au(t) + h(s) \Big] ds, \ t \in J.$$

In general, the above equation can be expressed as

$$u(t) = u_0 + \sum_{i=1}^{k} \chi_i(t) y_i + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[-Au(t) + h(s) \Big] ds, \ t \in J, \tag{2.4}$$

where

$$\chi_i(t) = \begin{cases} 0, & t \le t_i, \\ 1, & t > t_i. \end{cases}$$

We adopt the idea used in [42] and taking the Laplace Transformation

$$\widehat{u}(\lambda) = \int_0^{\infty} e^{-\lambda t} u(t) dt, \quad v(\lambda) = \int_0^{\infty} e^{-\lambda t} h(s) dt$$

to the (2.4) on both sides, we have

$$\begin{aligned} \widehat{u}(\lambda) &= \frac{1}{\lambda} u_0 + \sum_{i=1}^{k} \frac{e^{-t_i \lambda}}{\lambda} y_i - \frac{1}{\lambda^\alpha} A \widehat{u}(\lambda) + \frac{1}{\lambda^\alpha} v(\lambda) \\ &= \frac{\lambda^{\alpha-1}}{(\lambda^\alpha I + A)} u_0 + \frac{\lambda^{\alpha-1}}{(\lambda^\alpha I + A)} \sum_{i=1}^{k} e^{-t_i \lambda} y_i \end{aligned}$$

$$+\frac{1}{(\lambda^\alpha I+A)}v(\lambda).$$

Thus

$$\begin{aligned}\widehat{u}(\lambda) =&\lambda^{\alpha-1}(\lambda^\alpha I+A)^{-1}u_0\\ &+\lambda^{\alpha-1}(\lambda^\alpha I+A)^{-1}\sum_{i=1}^{k}e^{-t_i\lambda}y_i\\ &+(\lambda^\alpha I+A)^{-1}v(\lambda),\end{aligned}\tag{2.5}$$

where I is the identity operator defined on E.

Taking the inverse Laplace transformations on both sides of the equation (2.5), we obtain

$$\begin{aligned}u(t) =&E_{\alpha,1}(At^\alpha)u_0+\sum_{i=1}^{k}\chi_i(t)E_{\alpha,1}(A(t-t_i)^\alpha)y_i\\ &+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)h(s)ds.\end{aligned}\tag{2.6}$$

Setting $\mathscr{T}_\alpha(t)=E_{\alpha,1}(At^\alpha), \mathscr{S}_\alpha(t)=t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha)$ in the above formula, we get

$$u(t)=\mathscr{T}_\alpha(t)u_0+\sum_{i=1}^{k}\chi_i(t)\mathscr{T}_\alpha(t-t_i)y_i+\int_0^t\mathscr{S}_\alpha(t-s)h(s)ds.\tag{2.7}$$

Conversely, assume that u satisfies (2.2). If $t\in(0,t_1]$ then $u(0)=u_0$ and using the fact that ${}^cD_t^\alpha$ is the left inverse of I_t^α we get (2.1). If $t\in(t_k,t_{k+1}], k=1,2,\ldots,m$ and using the fact of the Caputo derivative of a constant is equal to zero, for all $t\in(t_k,t_{k+1}]$, by Lemma 3.3, [35], we obtain

$$\begin{aligned}{}^cD_{0+}^\alpha u(t) =&{}^c D_{0+}^\alpha\Big(\mathscr{T}_\alpha(0^+)u_0+\sum_{i=1}^{k}\mathscr{T}_\alpha(t-t_i)y_i+\int_0^t\mathscr{S}_\alpha(t-s)h(s)ds\Big)\\ &=A\mathscr{T}_\alpha(0^+)u_0+A\sum_{i=1}^{k}\mathscr{S}_\alpha(t-t_i)y_i+A\int_0^t\mathscr{S}_\alpha(t-s)h(s)ds+h(s)\\ &=Au(t)+h(s).\end{aligned}$$

For $t=0, u(0)=\mathscr{T}_\alpha(t)u_0+\int_0^0(0-s)^{\alpha-1}\mathscr{S}_\alpha(0-s)h(s)ds$. Moreover,

$$\begin{aligned}\Delta u(t_k)&=u(t_k^+)-u(t_k^-)\\ &=\sum_{k=1}^{m}\mathscr{T}_\alpha(t-t_i)y_i-\sum_{k=1}^{m-1}\mathscr{T}_\alpha(t-t_i)y_i\\ &=\mathscr{T}_\alpha(t_k-t_k)y_k\\ &=\mathscr{T}_\alpha(0)y_k\\ &=y_k.\end{aligned}$$

It is easy to see that expresses (2.2) is a solution of the linear impulsive fractional differential equation (2.1). This completes the proof. □

Definition 2.4. By a mild solution of the initial value problem (2.1) has a unique mild solution $u \in PC(J,E)$ given by (2.2).

We will give the following lemmas to be used in proving our main results, which can be found in [46].

Lemma 2.7. *The operators $\mathscr{T}_\alpha(t)$ and $\mathscr{S}_\alpha(t)(t \geq 0)$ have the following properties:*

(i) For any fixed $t \geq 0, \mathscr{T}_\alpha(t)$ and $\mathscr{S}_\alpha(t)$ are linear and bounded operators, i.e., for any $u \in E$,

$$\|\mathscr{T}_\alpha(t)u\| \leq M\|u\|, \qquad \|\mathscr{S}_\alpha(t)u\| \leq \frac{Mt^{\alpha-1}}{\Gamma(\alpha)}\|u\|,$$

where $M = \sup_{t\in J} \|T(t)\|$, which is a finite number.

(ii) For every $u \in E,\ t \to \mathscr{T}_\alpha(t)u$ and $t \to \mathscr{S}_\alpha(t)u$ are continuous functions from $[0,\infty)$ into E.

(iii) The operators $\mathscr{T}_\alpha(t)$ and $\mathscr{S}_\alpha(t)$ are strongly continuous for all $t \geq 0$.

(iv) If $T(t)(t \geq 0)$ is an equicontinuous semigroup, $\mathscr{T}_\alpha(t)$ and $\mathscr{S}_\alpha(t)$ are equicontinuous in E for $t > 0$.

(v) For every $t > 0$, $\mathscr{T}_\alpha(t)$ and $\mathscr{S}_\alpha(t)$ are compact operators if $T(t)$ is compact.

Suppose that here the bounded operator $B : E \to E$ exists given by

$$B = [I - \mathscr{T}_\alpha(\omega)]^{-1}, \tag{2.8}$$

and $M^* = \|B\|$.

We present sufficient conditions for the existence and boundedness of the operator B.

Lemma 2.8 (see Theorem 3.3 and Remark 3.4 [43])**.** *The operator B defined in (2.8) exists and is bounded, if one of the following three conditions holds:*

(i) $T(t)$ is compact for each $t > 0$ and the homogeneous linear nonlocal problem

$$\begin{cases} {}^cD^\alpha_{0+}u(t) = Au(t), \quad t \in J, \\ u(0) = u(\omega), \end{cases}$$

has no non-trivial mild solutions.

(ii) If $\|\mathscr{T}_\alpha(\omega)\| < 1$, then the operator $I-\mathscr{T}_\alpha(\omega)$ is invertible and $[I-\mathscr{T}_\alpha(\omega)]^{-1} \in L_b(E)$.

(iii) If $\|T(t)\| < 1$ for $t \in (0,\omega]$, then $\mathscr{T}_\alpha(n\omega) \to 0$ as $n \to \infty$ and the operator $I - \mathscr{T}_\alpha(\omega)$ is invertible and $[I - \mathscr{T}_\alpha(\omega)]^{-1} \in L_b(E)$, where $L_b(E)$ denote the space of bounded linear operators from E to E.

Definition 2.5. An abstract function $u \in PC(J,E) \cap C(J', E_1)$ is called a solution of the PBVP (1.1) if $u(t)$ satisfies all the equalities of (1.1).

Lemma 2.9. *Let $T(t)(t \geq 0)$ be a compact C_0-semigroup in E generated by $-A$, then the boundary value problem of linear impulsive fractional evolution equations*

$$\begin{cases} {}^cD^\alpha_{0+}u(t) + Au(t) = h(t), \quad t \in J', \\ \Delta u|_{t=t_k} = y_k, \quad k = 1,2,\dots,m, \\ u(0) = u(\omega), \end{cases} \tag{2.9}$$

has a unique mild solution $u \in PC(J, E)$ given by

$$u(t) = \begin{cases} \mathscr{T}_\alpha(t)R(h) + \int_0^t \mathscr{S}_\alpha(t-s)h(s)ds, & t \in [0, t_1], \\ \mathscr{T}_\alpha(t)R(h) + \mathscr{T}_\alpha(t-t_1)y_1 + \int_0^t \mathscr{S}_\alpha(t-s)h(s)ds, & t \in (t_1, t_2], \\ \vdots \\ \mathscr{T}_\alpha(t)R(h) + \sum\limits_{i=1}^{m} \mathscr{T}_\alpha(t-t_i)y_i + \int_0^t \mathscr{S}_\alpha(t-s)h(s)ds, & t \in (t_m, \omega], \end{cases} \tag{2.10}$$

where

$$R(h) = \begin{cases} B\Big[\int_0^\omega \mathscr{S}_\alpha(\omega-s)h(s)ds\Big], & t \in [0, t_1], \\ B\Big[\int_0^\omega \mathscr{S}_\alpha(\omega-s)h(s)ds + \mathscr{T}_\alpha(\omega-t_1)y_1\Big], & t \in (t_1, t_2], \\ \vdots \\ B\Big[\int_0^\omega \mathscr{S}_\alpha(\omega-s)h(s)ds + \sum\limits_{i=1}^{m} \mathscr{T}_\alpha(\omega-t_i)y_i\Big], & t \in (t_m, \omega], \end{cases}$$

and $\mathscr{T}_\alpha(t), \mathscr{S}_\alpha(t)(t > 0)$ are given by (2.2).

Proof. For any $u \in PC(J, E)$, by Definition 2.5 and Lemma 2.6, we know easily that the initial value problem of impulsive fractional evolution equation (2.1) has a unique mild solution $u \in PC(J, E)$ given by (2.2).

We show that the PBVP (2.9) has a unique mild solution $u \in PC(J, E)$ given by (2.10). If a function $u \in PC(J, E)$ defined by (2.10) is a solutions of the PBVP (2.9) and $u_0 = u(\omega)$, then

$$[I - \mathscr{T}_\alpha(\omega)]u_0 = \int_0^\omega \mathscr{S}_\alpha(\omega-s)h(s)ds + \sum_{i=1}^{m} \mathscr{T}_\alpha(\omega-t_i)y_i. \tag{2.11}$$

By (v) of Lemma 2.7, $\mathscr{T}_\alpha(u)$ is a compact operator. By the Fredholm alternative theorem, $[I - \mathscr{T}_\alpha(\omega)]^{-1}$ exists and is bounded. Since the periodic boundary value problem

$$\begin{cases} {}^cD_{0+}^\alpha u(t) + Au(t) = 0, & t \in J', \\ \Delta u|_{t=t_k} = y_k, & k = 1, 2, \ldots, m, \\ u(0) = u(\omega), \end{cases}$$

has no non-trivial mild solution, then the operator equation (2.11) has an unique solution. Hence we choose

$$u_0 = B\Big[\int_0^\omega \mathscr{S}_\alpha(\omega-s)h(s)ds + \sum_{i=1}^{m} \mathscr{T}_\alpha(\omega-t_i)y_i\Big] \triangleq R(h).$$

Then u_0 is the unique initial value of the problem (2.1) in E, which satisfies $u(0) = u_0 = u(\omega)$. It follows that the mild solution u of the problem (2.1) corresponding to initial value $u(0) = u_0 = R(h)$ is just the mild solution of the PBVP (2.9). Therefore, the conclusion of Lemma 2.9 holds. □

Remark 2.3. By Lemma 2.8, we can replace the assumption of $\{T(t), t \geq 0\}$ being compact by $\|T(t)\| < 1$ for $t \in (0, \omega]$ or $\|\mathscr{T}_\alpha(\omega)\| \leq 1$ directly. It is obvious that all the results in Lemma 2.9 also hold.

Definition 2.6. Let $L \geq 0$ be a constant. If functions $v_0, w_0 \in PC(J, E) \cap C(J', E_1)$ satisfies

$$\begin{aligned} {}^cD^\alpha_{0+} v_0(t) + Av_0(t) \leq f(t, v_0(t), w_0(t)) + L(v_0(t) - w_0(t)), \quad t \in J', \\ \Delta v_0|_{t=t_k} \leq I_k(v_0(t_k), w_0(t_k)), \quad k = 1, 2, \dots, m, \\ v_0(0) \leq v_0(\omega), \end{aligned} \tag{2.12}$$

$$\begin{aligned} {}^cD^\alpha_{0+} w_0(t) + Aw_0(t) \geq f(t, w_0(t), v_0(t)) + L(w_0(t) - v_0(t)), \quad t \in J', \\ \Delta w_0|_{t=t_k} \geq I_k(w_0(t_k), v_0(t_k)), \quad k = 1, 2, \dots, m, \\ w_0(0) \geq w_0(\omega), \end{aligned} \tag{2.13}$$

we call v_0, w_0 coupled lower and upper L-quasi-solution of the PBVP (1.1). Only choosing $=$ in (2.12) and (2.13), we call (v_0, w_0) coupled L-quasi-solution pair of the PBVP (1.1). Furthermore, if $u_0 := v_0 = w_0$, we call u_0 a solution of the PBVP (1.1).

Definition 2.7. A C_0-semigroup $T(t)(t \geq 0)$ in E is called to be positive, if order inequality $T(t)x \geq \theta$ holds for each $x \geq \theta, x \in E$ and $t \geq 0$.

Remark 2.4. It is easy to see that for any $C \geq 0$, $-(A + CI)$ also generates a C_0-semigroup $S(t) = e^{-Ct}T(t)(t \geq 0)$ in E. And $S(t)(t \geq 0)$ is a positive C_0-semigroup if $T(t)(t \geq 0)$ is a positive C_0-semigroup (about the positive C_0-semigroup, see [26, 30]).

Lemma 2.10 ([45])**.** *Suppose $\beta > 0, a(t)$ is a nonnegative function locally integrable on $0 \leq t \leq T$ (some $T \leq +\infty$) and $g(t)$ is a nonnegative, nondecreasing continuous function defined on $0 \leq t < T, g(t) \leq M$, and suppose $u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with*

$$u(t) \leq a(t) + g(t) \int_0^t (t - s)^{\beta - 1} u(s) ds$$

on this interval. Then

$$u(t) \leq a(t) + \int_0^t \Big[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)} (t - s)^{n\beta - 1} a(s) \Big] ds, \; 0 \leq t < T.$$

Evidently, $PC(J, E)$ is also an ordered Banach space with the partial order $\leq$ induced by the positive cone $K_{PC} = \{u \in PC(J, E) : u(t) \geq 0, t \in J\}$, which is also normal with the same normal constant N. For $v, w \in PC(J, E)$ with $v \leq w$, we use $[v, w]$ to denote the order interval $\{u \in PC(J, E) | v \leq u \leq w\}$ in $PC(J, E)$, and $[v(t), w(t)]$ to denote the order interval $\{u \in E | v(t) \leq u(t) \leq w(t), t \in J\}$ in E. From Lemma 9, if $T(t)(t \geq 0)$ is a positive C_0-semigroup, $h \geq \theta, u_0 \geq \theta$ and $y_k \geq 0, k = 1, 2, \dots, m$, then the mild solution $u \in PC(J, E)$ of the PBVP (2.9) satisfies $u \geq 0$.

3. Main results

In this section, we will present some main results.

Theorem 3.1. *Let E be an ordered Banach space, whose positive cone P is normal, $A : D(A) \subset E \to E$ be a closed linear operator, the positive C_0-semigroup $T(t)(t \geq 0)$ generated by $-A$ is compact in $E, f \in C(J \times E \times E, E)$ and $I_k \in C(E, E)$, $k = 1, 2, \ldots, m$. Assume that PBVP (1.1) has coupled lower and upper L-quasi-solutions v_0 and w_0 with $v_0 \leq w_0$. Suppose also that the following conditions are satisfied:*

(H1) There exist a constant $C > 0$ and $L \geq 0$ such that

$$f(t, u_2, v_2) - f(t, u_1, v_1) \geq -C(u_2 - u_1) - L(v_1 - v_2),$$

for any $t \in J$, and $v_0(t) \leq u_1 \leq u_2 \leq w_0(t)$, $v_0(t) \leq v_2 \leq v_1 \leq w_0(t)$.

(H2) The impulsive function $I_k(\cdot, \cdot)$ satisfies

$$I_k(u_1, v_1) \leq I_k(u_2, v_2), \quad k = 1, 2, \ldots, m,$$

for any $t \in J$, and $v_0(t) \leq u_1 \leq u_2 \leq w_0(t)$, $v_0(t) \leq v_2 \leq v_1 \leq w_0(t)$.

Then the PBVP (1.1) has minimal and maximal coupled mild L-quasi-solutions $\underline{u}$ and $\overline{u}$ between v_0 and w_0.

Proof. Let $C > \delta_0$, it is easy to see that $-(A + CI)$ generates an exponentially stable, positive C_0-semigroup $S(t) = e^{-Ct}T(t)(t \geq 0)$. Also, it is compact. Let $\Phi(t) = \int_0^\infty \xi_\alpha(\sigma)S(t^\alpha\sigma)d\sigma$, $\Psi(t) = \alpha \int_0^\infty \sigma t^{\alpha-1}\xi_\alpha(\sigma)S(t^\alpha\sigma)d\sigma$. By Remark 2.4 and Lemma 2.7, the operators $\Phi(t)$ and $\Psi(t)$ are also positive and comapct for all $t \geq 0$. By Lemma 2.7, we have that

$$\|\Phi(t)\| \leq M, \quad \|\Psi(t)\| \leq \frac{Mt^{\alpha-1}}{\Gamma(\alpha)}.$$

Let $J_0 = [t_0, t_1] = [0, t_1], J_k = (t_k, t_{k+1}], k = 1, 2, \ldots, m$, we define the mapping $Q : [v_0, w_0] \times [v_0, w_0] \to PC(J, E)$ given by

$$Q(u,v)(t) = \begin{cases} \Phi(t)R(u,v) + \int_0^t \Psi(t-s)[f(s,u(s),v(s)) \\ +(C+L)u(s) - Lv(s)]ds, \qquad t \in J_0, \\ \Phi(t)R(u,v) + \Phi(t-t_1)I_1(u(t_1),v(t_1)) \\ + \int_0^t \Psi(t-s)[f(s,u(s),v(s)) \\ +(C+L)u(s) - Lv(s)]ds, \qquad t \in J_1, \\ \vdots \\ \Phi(t)R(u,v) + \sum\limits_{i=1}^{m} \Phi(t-t_i)I_i(u(t_i),v(t_i)) \\ + \int_0^t \Psi(t-s)[f(s,u(s),v(s)) \\ +(C+L)u(s) - Lv(s)]ds, \qquad t \in J_m, \end{cases} \tag{3.1}$$

where

$$R(u,v)=\begin{cases}(I-\Phi(\omega))^{-1}\Big[\int_0^\omega\Psi(\omega-s)[f(s,u(s),v(s))\\ +(C+L)u(s)-Lv(s)]ds\Big], \qquad t\in J_0,\\ (I-\Phi(\omega))^{-1}\Big[\int_0^\omega\Psi(\omega-s)[f(s,u(s),v(s))\\ +(C+L)u(s)-Lv(s)]ds\\ +\Phi(\omega-t_1)I_1(u(t_1),v(t_1))\Big], \qquad t\in J_1,\\ \vdots\\ (I-\Phi(\omega))^{-1}\Big[\int_0^\omega\Psi(\omega-s)[f(s,u(s),v(s))\\ +(C+L)u(s)-Lv(s)]ds+\sum_{i=1}^{m}\\ \Phi(\omega-t_i)I_i(u(t_i),v(t_i)))\Big], \qquad t\in J_m.\end{cases} \tag{3.2}$$

Clearly, $Q:[v_0,w_0]\times[v_0,w_0]\to PC(J,E)$ is continuous. And by Lemma 2.9, the coupled mild L-quasi-solutions of the PBVP (1.1) are equivalent to the coupled fixed points of operator Q.

Next, we show $Q:[v_0,w_0]\times[v_0,w_0]\to PC(J,E)$ is a mixed monotone operator, and $v_0\le Q(v_0,w_0),Q(w_0,v_0)\le w_0$. In fact, for $\forall t\in J, v_0(t)\le u_1(t)\le u_2(t)\le w_0, v_0(t)\le v_2(t)\le v_1(t)\le w_0(t)$, from the assumptions (H1) and (H2), we have

$$f(t,u_1(t),v_1(t))+(C+L)u_1(t)-Lv_1(t)\le f(t,u_2(t),v_2(t))+(C+L)u_2(t)-Lv_2(t),$$

$$I_k(u_1(t_k),v_1(t_k))\le I_k(u_2(t_k),v_2(t_k)),\quad k=1,2,\dots,m.$$

By the positivity of operators $\Phi(t)$ and $\Psi(t)$, it follows that $(I-\Phi(\omega))^{-1}=\sum_{n=1}^{\infty}\Phi(n\omega)$ is a positive operator. Then $R(u_1,v_1)\le R(u_2,v_2)$. So

$$\int_0^t\Psi(t-s)[f(t,u_1(t),v_1(t))+(C+L)u_1(t)-Lv_1(t)]ds$$
$$\le\int_0^t\Psi(t-s)[f(t,u_2(t),v_2(t))+(C+L)u_2(t)-Lv_2(t)],$$

$$\sum_{0<t_k<t}\Phi(t-t_k)I_k(u_1(t_k),v_1(t_k))\le\sum_{0<t_k<t}\Phi(t-t_k)I_k(u_2(t_k),v_2(t_k)),\ k=1,2,\dots,m.$$

Hence from (3.1) we see that $Q(u_1,v_1)\le Q(u_2,v_2)$, which means that Q is a mixed monotone operator.

Now, we show that $v_0\le Q(v_0,w_0),Q(w_0,v_0)\le w_0$. Let $h(t)={}^cD_{0+}^{\alpha}v_0(t)+Av_0(t)+Cv_0(t)$, by (2.11), $h\in PC(J,E)$ and $h(t)\le f(t,v_0,w_0)+(C+L)v_0-Lw_0, t\in J$. By Lemma 2.9, the positivity of operator $\Phi(t)$ and $\Psi(t)$, for $t\in J_0$, we have that

$$v_0(t)=\Phi(t)v_0(0)+\int_0^t\Psi(t-s)h(s)ds$$
$$\le\Phi(t)v_0(0)+\int_0^t\Psi(t-s)[f(s,v_0(s),w_0(s))+(C+L)v_0(s)-Lw_0(s)]ds.$$

Especially, we have

$$v_0(\omega) \le \Phi(\omega)v_0(0) + \int_0^\omega \Psi(\omega - s)[f(s, v_0(s), w_0(s)) \\ + (C+L)v_0(s) - Lw_0(s)]ds.$$

Combining this inequality with $v_0(0) = v_0(\omega)$, it follows that

$$v_0(0) \le (I - \Phi(\omega))^{-1}\Big[\int_0^\omega \Psi(\omega - s)[f(s, v_0(s), w_0(s)) \\ + (C+L)v_0(s) - Lw_0(s)]ds\Big] \triangleq R(v_0, w_0).$$

On the other hand, from (3.1), we have

$$Q(v_0, w_0)(t) = \Phi(t)R(v_0, w_0) + \int_0^t \Psi(t - s)[f(s, v_0(s), w_0(s)) \\ + (C+L)v_0(s) - Lw_0(s)]ds, \ t \in J_0.$$

Therefore, $Q(v_0, w_0)(t) - v_0(t) \ge \Phi(t)(R(v_0, w_0) - v_0(0)) \ge 0$ for all $t \in J_0$. It implies that $v_0 \le Q(v_0, w_0)$.

For $t \in J_1$, we have that

$$\begin{aligned} v_0(t) &= \Phi(t)v_0(0) + \int_0^t \Psi(t-s)h(s)ds + \Phi(t - t_1)\Delta v_0|_{t=t_1} \\ &\le \Phi(t)v_0(0) + \int_0^t \Psi(t-s)[f(s, v_0(s), w_0(s)) + (C+L)v_0(s) - Lw_0(s)]ds \\ &\quad + \Phi(t - t_1)I_1(v_0(t_1), w_0(t_1)). \end{aligned}$$

Especially, we have

$$\begin{aligned} v_0(\omega) \le& \Phi(\omega)v_0(0) + \int_0^\omega \Psi(\omega - s)[f(s, v_0(s), w_0(s)) + (C+L)v_0(s) - Lw_0(s)]ds \\ &+ \Phi(\omega - t_1)I_1(v_0(t_1), w_0(t_1)). \end{aligned}$$

Combining this inequality with $v_0(0) = v_0(\omega)$, it follows that

$$\begin{aligned} v_0(0) \le& (I - \Phi(\omega))^{-1}\Big[\int_0^\omega \Psi(\omega - s)[f(s, v_0(s), w_0(s)) + (C+L)v_0(s) - Lw_0(s)]ds \\ &+ \Phi(\omega - t_1)I_1(v_0(t_1), w_0(t_1))\Big] \triangleq R(v_0, w_0). \end{aligned}$$

On the other hand, from (3.1), we have

$$\begin{aligned} Q(v_0, w_0)(t) =& \Phi(t)R(v_0, w_0) + \int_0^t \Psi(t - s)[f(s, v_0(s), w_0(s)) \\ &+ (C+L)v_0(s) - Lw_0(s)]ds + \Phi(t - t_1)I_1(v_0(t_1), w_0(t_1)), t \in J_1. \end{aligned}$$

Therefore, $Q(v_0, w_0)(t) - v_0(t) \ge \Phi(t)(R(v_0, w_0) - v_0(0)) \ge 0$ for all $t \in J_1$. It implies that $v_0 \le Q(v_0, w_0)$.

Continuing such a process interval by interval to J_m, by (3.1), we obtain that $v_0 \leq Q(v_0, w_0)$.

Similarly, it can be shown that $Q(w_0, v_0) \leq w_0$. So, $Q : [v_0, w_0] \times [v_0, w_0] \to [v_0, w_0]$ is continuous mixed monotone operator.

Next, we show that $Q : [v_0, w_0] \times [v_0, w_0] \to [v_0, w_0]$ is completely continuous. Let

$$\begin{aligned} W(u,v)(t) &= \int_0^t \Psi(t-s)[f(s,u(s),v(s)) + (C+L)u(s) - Lv(s)]ds, \\ V(u,v)(t) &= \sum_{0<t_k<t} \Phi(t-t_k) I_k(u(t_k), v(t_k)), \quad u \in [v_0, w_0]. \end{aligned} \tag{3.3}$$

On the one hand, we prove that for any $0 < t \leq \omega$, $Y(t) = \{W(u,v)(t) : u, v \in [v_0, w_0]\}$ is precompact in E. For $0 < \epsilon < t$ and $u, v \in [v_0, w_0]$,

$$\begin{aligned} (W_\epsilon)(u,v)(t) &= \int_0^{t-\epsilon} \Psi(t-s)[f(s,u(s),v(s)) + (C+L)u(s) - Lv(s)]ds \\ &= S(\epsilon^\alpha \delta)\Big[\alpha \int_0^{t-\epsilon} \int_\delta^\infty \eta (t-s)^{\alpha-1} \xi_\alpha(\eta) S((t-s)^\alpha \eta - \epsilon^\alpha \delta) d\eta \\ &\quad \times [f(s,u(s),v(s)) + (C+L)u(s) - Lv(s)]ds\Big]. \end{aligned} \tag{3.4}$$

For any $u \in [v_0, w_0]$, by assumption (H1), we have

$$\begin{aligned} f(t, v_0(t), w_0(t)) + (C+L)v_0(t) - Lw_0(t) &\leq f(t, u(t), v(t)) + (C+L)u(t) - Lv(t) \\ &\leq f(t, w_0(t), v_0(t)) + (C+L)w_0(t) - Lv_0(t). \end{aligned}$$

By the normality of the cone P, there exists $\overline{M_1} > 0$ such that

$$\|f(t,u(t),v(t)) + (C+L)u(t) - Lv(t)\| \leq \overline{M_1}, \quad u, v \in [v_0, w_0].$$

By the compactness of $S(\epsilon)$, $Y_\epsilon(t) = \{(W_\epsilon(u,v)(t) : u, v \in [v_0, w_0]\}$ is precompact in E. Since

$$\begin{aligned} \|W(u,v)(t) - W_\epsilon(u,v)(t)\| &\leq \int_{t-\epsilon}^t \|\Psi(t-s)\| \cdot \|f(t,u(t),v(t)) \\ &\quad + (C+L)u(t) - Lv(t)\| ds \\ &\leq \frac{M\overline{M_1}\epsilon^\alpha}{\Gamma(1+\alpha)}, \end{aligned}$$

the set $Y(t)$ is totally bounded in E. Furthermore, $Y(t)$ is precompact in E.

On the other hand, for any $0 \leq t_1 \leq t_2 \leq \omega$, we have

$$\begin{aligned} &\|W(u,v)(t_2) - W(u,v)(t_1)\| \\ &= \Big\| \int_0^{t_1} (\Psi(t_2-s) - \Psi(t_1-s))[f(t,u(t),v(t)) + (C+L)u(t) - Lv(t)]ds \\ &\quad + \int_{t_1}^{t_2} \Psi(t_2-s)[f(t,u(t),v(t)) + (C+L)u(t) - Lv(t)]ds \Big\| \\ &\leq \overline{M_1} \int_0^{t_1} \|\Psi(t_2-s) - \Psi(t_1-s)\| ds + \frac{M\overline{M_1}}{\Gamma(1+\alpha)}(t_2-t_1)^\alpha \end{aligned}$$

$$\leq \overline{M_1}\int_0^{\omega}\|\Psi(t_2-t_1+s)-\Psi(s)\|ds+\frac{M\overline{M_1}}{\Gamma(1+\alpha)}(t_2-t_1)^{\alpha}. \tag{3.5}$$

The right side of (3.5) depends on t_2-t_1, but is independen of u,v. As $S(\cdot)$ is compact, $\Psi(\cdot)$ is also compact and therefore $\Psi(t)$ is continuous in the uniform operator topology for $t>0$. So, the right side of (3.5) tends to zero as $t_2-t_1\to 0$. Hence $W([v_0,w_0]\times[v_0,w_0])$ is equicontinuous function of cluster in Y.

The same idea can be used to prove the compactness of V.

For $0\leq t\leq\omega$, since $\{Q(u,v)(t): u\in[v_0,w_0]\}=\{\Phi(t)R(u,v)+W(u,v)(t)+V(u,v)(t): u,v\in[v_0,w_0]\}$, and $Q(u,v)(0)=R(u,v)=u(\omega)$ is precompact in E. Hence, by the Arzela-Ascoli theorem, Q is precompact. So $Q:[v_0,w_0]\times[v_0,w_0]\to[v_0,w_0]$ is completely continuous.

Hence, by Theorem 1 in [21], Q has minimal and maximal coupled fixed points $\underline{u}$ and $\overline{u}$ in $[v_0,w_0]$, and therefore, they are the minimal and maximal coupled mild L-quasi-solutions of the PBVP (1.1) in $[v_0,w_0]$, respectively. □

Remark 3.1. If $f(t,u,u)=f(t,u)$ and $v_0=w_0:=u_0$, then Theorem 3.1 in this paper is Theorem 3.1 in [29].

Remark 3.2. By Lemma 2.8, we can replace the assumption of $\{T(t)\}_{t\geq 0}$ being compact by $\|T(t)\|<1$ for $t\in(0,\omega]$ or $\|\Phi(\omega)\|<1$ directly. It is obvious that all results in Theorem 3.1 also hold.

Theorem 3.2. *Let E be an ordered Banach space, whose positive cone P is normal, $A: D(A)\subset E\to E$ be a closed linear operator and $-A$ generates a positive C_0-semigroup $T(t)(t\geq 0)$ in E and $\|T(t)\|<1$ for $t\in(0,\omega]$, $f\in C(J\times E\times E,E)$ and $I_k\in C(E,E)$, $k=1,2,\ldots,m$. If the PBVP (1.1) has coupled lower and upper L-quasi-solution v,w_0 with $v_0\leq w_0$, conditions* (H1) *and* (H2) *hold, and satisfy*

(H3) There exist a constant $L_1>0$ such that for all $t\in J$,

$$\alpha(\{f(t,u_n,v_n)\})\leq L_1(\alpha(\{u_n\})+\alpha(\{v_n\})),$$

and increasing or decreasing sequences $\{u_n\}\subset[v_0(t),w_0(t)]$, $\{v_n\}\subset[v_0(t),w_0(t)]$.

(H4) The sequences $v_n(0)$ and $w_n(0)$ are convergent, where $v_n=Q(v_{n-1},w_{n-1})$, $w_n=Q(w_{n-1},v_{n-1})$, $n=1,2,\ldots$.

Then the PBVP (1.1) has minimal and maximal coupled mild L-quasi-solutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0 respectively.

Proof. From Theorem 3.1, we know that $Q:[v_0,w_0]\times[v_0,w_0]\to[v_0,w_0]$ is a continuously mixed monotone operator. Now, we define two sequences $\{v_n\}$ and $\{w_n\}$ in $[v_0,w_0]$ by the iterative scheme

$$v_n=Q(v_{n-1},w_{n-1}),\quad w_n=Q(w_{n-1},v_{n-1}),\quad n=1,2,\ldots. \tag{3.6}$$

Then from the monotonicity of Q, it follows that

$$v_0\leq v_1\leq v_2\leq\cdots\leq v_n\leq\cdots\leq w_n\leq\cdots\leq w_2\leq w_1\leq w_0. \tag{3.7}$$

We prove that $\{v_n\}$ and $\{w_n\}$ are convergent in J.

For convenience, we denote $B=\{v_n: n\in\mathbb{N}\}+\{w_n: n\in\mathbb{N}\}$ and $B_1=\{v_n: n\in\mathbb{N}\}$, $B_2=\{w_n: n\in\mathbb{N}\}$, $B_{10}=\{v_{n-1}: n\in\mathbb{N}\}$, $B_{20}=\{w_{n-1}: n\in\mathbb{N}\}$.

Then $B_1 = Q(B_{10}, B_{20})$ and $B_2 = Q(B_{20}, B_{10})$. Let $J'_1 = [0, t_1]$, $J'_k = (t_k, t_{k+1}]$, $k = 1, 2, 3, \dots m$. From $B_{10} = B_1 \bigcup \{v_0\}$ and $B_{20} = B_2 \bigcup \{w_0\}$ it follows that $\alpha(B_{10}(t)) = \alpha(B_1(t))$ and $\alpha(B_{20}(t)) = \alpha(B_2(t))$ for $t \in J$. Let $\varphi(t) := \alpha(B(t)), t \in J$, going from J'_1 to J'_m interval by interval we show that $\varphi(t) \equiv 0$ in J.

Since $\|T(t)\| < 1$, so $\|\Phi(t)\| < 1, \|\Psi(t)\| < \frac{t^{\alpha-1}}{\Gamma(\alpha)}, t \in J$. For $t \in J'_1$, by (3.1), Lemma 2.2 and the positivity of operator $\Phi(t), \Psi(t)$, and assumption (H3) and (H4), we have

$$\begin{aligned}
\varphi(t) &= \alpha(B(t)) = \alpha(B_1(t) + B_2(t)) = \alpha(Q(B_{10}, B_{20})(t) + Q(B_{20}, B_{10})(t)) \\
&= \alpha\Big(\Big\{\Phi(t)R(v_{n-1}, w_{n-1}) + \int_0^t \Psi(t-s)[f(s, v_{n-1}(s), w_{n-1}(s)) \\
&\quad + (C+L)v_{n-1}(s) - Lw_{n-1}(s)]ds \\
&\quad + \Phi(t)R(w_{n-1}, v_{n-1}) + \int_0^t \Psi(t-s)[f(s, w_{n-1}(s), v_{n-1}(s)) \\
&\quad + (C+L)w_{n-1}(s) - Lv_{n-1}(s)]ds\Big\}\Big) \\
&\le \alpha\Big(\Big\{\Phi(t)v_n(0)\Big\}\Big) + \alpha\Big(\Big\{\Phi(t)w_n(0)\Big\}\Big) \\
&\quad + \frac{2}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\alpha\Big(\Big\{f(s, v_{n-1}(s), w_{n-1}(s)) + f(s, w_{n-1}(s), v_{n-1}(s)) \\
&\quad + C(v_{n-1}(s) + w_{n-1})\Big\}\Big)ds \\
&\le \frac{2(2L_1 + C)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\alpha(B_{10}(s)) + \alpha(B_{20}(s)))ds \\
&\le \frac{4(2L_1 + C)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi(s)ds.
\end{aligned}$$

Hence by Lemma 2.10, $\varphi(t) = 0$, a.e. $t \in J'_1$. So $\int_0^t \varphi(s)ds \equiv 0$, by the above inequality, $\varphi(t) \le 0$, combing this with the property of noncompactness, $\varphi(t) \equiv 0, t \in J'_1$. In particular, $\alpha(B_{10}(t_1)) = 0, \alpha(B_{20}(t_1)) = 0$, this implies that $B_{10}(t_1)$ and $B_{20}(t_1)$ are precompact in E. Thus $I_1(B_{10}(t_1), B_{20}(t_1))$ and $I_1(B_{20}(t_1), B_{10}(t_1))$ are precompact in E, and $\alpha(I_1(B_{10}(t_1), B_{20}(t_1))) = 0$, $\alpha(I_1(B_{20}(t_1), B_{10}(t_1))) = 0$.

Now, for $t \in J'_2$, by (3.1) and the above argument for $t \in J'_1$, we have

$$\begin{aligned}
\varphi(t) &= \alpha(B(t)) = \alpha(B_1(t) + B_2(t)) = \alpha(Q(B_{10}, B_{20})(t) + Q(B_{20}, B_{10})(t)) \\
&= \alpha\Big(\Big\{\Phi(t)R(v_{n-1}, w_{n-1}) + \int_0^t \Psi(t-s)[f(s, v_{n-1}(s), w_{n-1}(s)) \\
&\quad + (C+L)v_{n-1}(s) - Lw_{n-1}(s)]ds \\
&\quad + \Phi(t)R(w_{n-1}, v_{n-1}) + \int_0^t \Psi(t-s)[f(s, w_{n-1}(s), v_{n-1}(s)) \\
&\quad + (C+L)w_{n-1}(s) - Lv_{n-1}(s)]ds \\
&\quad + \Phi(t-t_1)I_1(v_{n-1}(t_1), w_{n-1}(t_1)) + \Phi(t-t_1)I_1(w_{n-1}(t_1), v_{n-1}(t_1))\Big\}\Big) \\
&\le \alpha\Big(\Big\{\Phi(t)v_n(0)\Big\}\Big) + \alpha\Big(\Big\{\Phi(t)w_n(0)\Big\}\Big) \\
&\quad + \frac{2}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\alpha\Big(\Big\{f(s, v_{n-1}(s), w_{n-1}(s)) + f(s, w_{n-1}(s), v_{n-1}(s))
\end{aligned}$$

$$
\begin{aligned}
&\quad + C(v_{n-1}(s) + w_{n-1})\Big\}\Big) \\
&\quad + \alpha(I_1(B_{10}(t_1), B_{20}(t_1))) + \alpha(I_1(B_{20}(t_1), B_{10}(t_1))) \\
&\leq \frac{2(2L_1 + C)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (\alpha(B_{10}(s)) + \alpha(B_{20}(s)))ds \\
&\leq \frac{4(2L_1 + C)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s)ds.
\end{aligned}
$$

Again by Lemma 2.10, $\varphi(t) \equiv 0$ in J_2', from which we obtain that $\alpha(B_{10}(t_2)) = 0, \alpha(B_{20}(t_2)) = 0$ and $\alpha(I_2(B_{10}(t_2), B_{20}(t_2))) = 0, \alpha(I_2(B_{20}(t_2), B_{10}(t_2))) = 0$.

Continuing such a process interval by interval up to J_m', we can prove that $\varphi(t) \equiv 0$ in every $J_k', k = 1, 2, \dots, m$. Hence, for any $t \in J, \{v_n(t)\} + \{w_n(t)\}$ is precompact. So $\{v_n(t)\}, \{w_n(t)\}$ are precompact. Combing this with the monotonicity (3.7), we easily prove that $\{v_n(t)\}$ and $\{w_n(t)\}$ are convergent, i.e., $\lim_{n\to\infty} v_n(t) = \underline{u}(t)$, $t \in J$. Similarly, $\lim_{n\to\infty} w_n(t) = \overline{u}(t)$, $t \in J$.

Evidently $\{v_n(t)\}, \{w_n(t)\} \in PC(J, E)$, so $\underline{u}(t), \overline{u}(t)$ are bounded integrable in J. Since for any $t \in J$, we have

$$
\begin{aligned}
v_n(t) &= Q(v_{n-1}, w_{n-1})(t) \\
&= \Phi(t)R(v_{n-1}, w_{n-1}) + \int_0^t \Psi(t-s)(f(s, v_{n-1}(s), w_{n-1}(s)) \\
&\quad + (C+L)v_{n-1}(s) - Lw_{n-1}(s))ds + \sum_{0<t_i<t} \Phi(t-t_i) I_i(v_{n-1}(t_i), w_{n-1}(t_i)),
\end{aligned}
$$

and

$$
\begin{aligned}
w_n(t) &= Q(w_{n-1}, v_{n-1})(t) \\
&= \Phi(t)R(w_{n-1}, v_{n-1}) + \int_0^t \Psi(t-s)(f(s, w_{n-1}(s), v_{n-1}(s)) \\
&\quad + (C+L)w_{n-1}(s) - Lv_{n-1}(s))ds + \sum_{0<t_i<t} \Phi(t-t_i) I_i(w_{n-1}(t_i), v_{n-1}(t_i)),
\end{aligned}
$$

letting $n \to \infty$, by the Lebesgue dominated convergence theorem, we have

$$
\begin{aligned}
\underline{u}(t) &= \Phi(t)R(\underline{u}, \overline{u}) + \int_0^t \Psi(t-s)[f(s, \underline{u}(s), \overline{u}(s)) \\
&\quad + (C+L)\underline{u}(s) - L\overline{u}(s)]ds + \sum_{0<t_k<t} \Phi(t-t_k) I_k(\underline{u}(t_k), \overline{u}(t_k)).
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{u}(t) &= \Phi(t)R(\overline{u}, \underline{u}) + \int_0^t \Psi(t-s)[f(s, \overline{u}(s), \underline{u}(s)) \\
&\quad + (C+L)\overline{u}(s) - L\underline{u}(s)]ds + \sum_{0<t_k<t} \Phi(t-t_k) I_k(\overline{u}(t_k), \underline{u}(t_k)).
\end{aligned}
$$

Therefore, $\underline{u}(t), \overline{u}(t) \in PC(J, E)$, and $\underline{u} = Q\underline{u}, \overline{u} = Q\overline{u}$. Combing this with monotonicity (3.7), we see that $v_0 \leq \underline{u} \leq \overline{u} \leq w_0$. By the monotonicity of Q, it is easy to see that $\underline{u}$ and $\overline{u}$ are the minimal and maximal coupled fixed points of Q in $[v_0, w_0]$.

Therefore, $\underline{u}$ and $\overline{u}$ are the minimal and maximal coupled mild L-quasi-solutions of the PBVP (1.1) in $[v_0, w_0]$, respectively. □

In Theorem 3.2, if E is weakly sequentially complete, the condition (H3) holds automatically. In fact, by Theorem 2.2 in [19], any monotonic and order-bounded sequence is precompact. Let $\{u_n\}$ and $\{v_n\}$ be increasing or decreasing sequences obeying condition (H3), then by condition (H1), $\{f(t, u_n, v_n) + Cu_n - Lv_n\}$ is a monotone and order-bounded sequence. By the property of measure of noncompactness, we have

$$\alpha(\{f(t, u_n, v_n)\}) \le \alpha(\{f(t, u_n, v_n) + Cu_n - Lv_n\}) + C\alpha(\{u_n\}) + L\alpha(\{v_n\}) = 0.$$

Hence, condition (H3) holds. From Theorem 3.2, we obtain the following corollary.

Corollary 3.1. *Let E be an ordered and weakly sequentially complete Banach space, whose positive cone P is normal, $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generates a positive C_0-semigroup $T(t)(t \ge 0)$ in E and $\|T(t)\| < 1$ for $t \in (0, \omega]$, $f \in C(J \times E \times E, E)$ and $I_k \in C(E, E)$, $k = 1, 2, \ldots, m$. If the PBVP (1.1) has coupled lower and upper L-quasi-solution v, w_0 with $v_0 \le w_0$, conditions* (H1) *and* (H2) *hold. Then the PBVP (1.1) has minimal and maximal coupled mild L-quasi-solutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0 respectively.*

Now, we discuss the existence of the mild solution to the PBVP (1.1) between the minimal and maximal coupled mild L-quasi-solutions $\underline{u}$ and $\overline{u}$. If we replace the assumptions (H2) and (H3) by the following assumptions:

$(H2)^*$ The impulsive function $I_k(\cdot, \cdot)$ satisfies

$$I_k(u_1, v_1) \le I_k(u_2, v_2), \quad k = 1, 2, \ldots, m,$$

for any $t \in J$, and $v_0(t) \le u_1 \le u_2 \le w_0(t), v_0(t) \le v_2 \le v_1 \le w_0(t)$, and there exist $M_k > 0, \sum_{k=1}^{m} M_k < \frac{\Gamma(1+\alpha) - 4(M^*+1)(L_1+C)\omega^\alpha}{4(M^*+1)\Gamma(1+\alpha)}$, such that

$$\alpha(I_k(\{u_n(t_k)\} \times \{v_n(t_k)\})) \le M_k[\alpha(\{u_n(t_k)\}) + \alpha(\{v_n(t_k)\})],$$

for any countable sets $\{u_n\}$ and $\{v_n\}$ in $[v_0(t), w_0(t)]$.

$(H3)^*$ There exists a constant $L_1 > 0$ such that

$$\alpha(f, D_1 \times D_2) \le L_1(\alpha(D_1) + \alpha(D_2)),$$

for any $t \in J$, where $D_1 = \{v_n\}$ and $D_2 = \{w_n\}$ are countable sets in $[v_0(t), w_0(t)]$.

We have the following existence result.

Theorem 3.3. *Let E be an ordered Banach space, whose positive cone P is normal, $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generates a positive equicontinuous C_0-semigroup $T(t)(t \ge 0)$ in E and $\|T(t)\| < 1$ for $t \in (0, \omega]$, $f \in C(J \times E \times E, E)$ and $I_k \in C(E \times E, E), k = 1, 2 \ldots, m$. If the PBVP (1.1) has coupled lower and upper L-quasi-solutions v_0 and w_0 with $v_0 \le w_0$, such that assumptions (H1), and $(H3)^*$ hold, then the PBVP (1.1) has minimal and maximal coupled mild L-quasi-solutions u and u between v_0 and w_0, and at least has one mild solution between $\underline{u}$ and $\overline{u}$.*

Proof. We can easily see that $(H2)^* \Rightarrow$ (H2), $(H3)^* \Rightarrow$ (H3). Hence, by Theorem 3.2, the PBVP (1.1) has minimal and maximal coupled mild L-quasi-solutions $\underline{u}$

and $\overline{u}$ between v_0 and w_0. Next, we prove the existence of the mild solution of the equation between $\underline{u}$ and $\overline{u}$. Let $Tu = Q(u,u)$, clearly, $T : [v_0, w_0] \to [v_0, w_0]$ is continuous and the mild solution of the PBVP (1.1) is equivalent to the fixed point of operator T . For any $D \subset [v_0, w_0]$, by the proof of Theorem 3.1, $T(D)$ is bounded and equicontinuous. So, by Lemma 2.2, there exists a countable set $D_0 = \{u_n\}$, such that

$$\alpha(T(D)) \leq 2\alpha(T(D_0)).$$

Since $\|T(t)\| < 1$, so $\|\Phi(t)\| < 1, \|\Psi(t)\| < \frac{t^{\alpha-1}}{\Gamma(\alpha)}, t \in J$. Let $M^* = \|[I - \Phi(\omega)]^{-1}\|$.

For $t \in J_0 = [0, t_1]$, by assumptions $(H2)^*$, $(H3)^*$ and Lemma 2.1, we have

$$\begin{aligned}
\alpha(T(D_0(t))) &= \alpha\Big(\Big\{\Phi(t)R(u_n) + \int_0^t \Psi(t-s)(f(s, u_n(s), u_n(s) + Cu_n(s)))ds\Big\}\Big)\\
&\leq \alpha\Big(\Big\{R(u_n)\Big\}\Big) + \frac{2}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\alpha\Big(\Big\{(f(s, D_0(s), D_0(s)) + CD_0(s))\Big\}\Big)ds\\
&\leq M^*\Big[\frac{2(2L_1+C)}{\Gamma(\alpha)}\int_0^\omega (\omega-s)^{\alpha-1}\alpha(D_0(s))ds\Big]\\
&+ \frac{2(2L_1+C)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\alpha\Big(D_0(s)\Big)ds\\
&\leq (M^*+1)\frac{2(2L_1+C)\omega^\alpha}{\Gamma(1+\alpha)}\alpha(D).
\end{aligned}$$

For $t \in J_1 = (t_1, t_2]$, by assumptions $(H2)^*$, $(H3)^*$ and Lemma 2.1, we have

$$\begin{aligned}
\alpha(T(D_0(t))) &= \alpha\Big(\Big\{\Phi(t)R(u_n) + \int_0^t \Psi(t-s)(f(s, u_n(s), u_n(s)\\
&\quad + Cu_n(s)))ds + \Phi(t-t_1)I_1(u_n(t_1))\Big\}\Big)\\
&\leq \alpha\Big(\Big\{R(u_n)\Big\}\Big) + \frac{2}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\alpha\Big(\Big\{(f(s, D_0(s), D_0(s)\\
&\quad + Cu_n(s)))\Big\}\Big)ds + \alpha(I_1(D_0(t_1), D_0(t_1)))\\
&\leq M^*\Big[\frac{2(2L_1+C)}{\Gamma(\alpha)}\int_0^\omega (\omega-s)^{\alpha-1}\alpha(D_0(s))ds + 2M_1\alpha(D_0(t_1))\Big]\\
&\quad + \frac{2(2L_1+C)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\alpha\Big(D_0(s)\Big)ds + 2M_1\alpha(D_0(t_1))\\
&\leq (M^*+1)\Big[\frac{2(L_1+C)\omega^\alpha}{\Gamma(1+\alpha)} + 2M_1\Big]\alpha(D).
\end{aligned}$$

For $t \in J_m = (t_m, t_{m+1}]$, by assumptions $(H2)^*$, $(H3)^*$ and Lemma 2.1, we have

$$\begin{aligned}
\alpha(T(D_0(t))) &= \alpha\Big(\Big\{\Phi(t)R(u_n) + \int_0^t \Psi(t-s)(f(s, u_n(s), u_n(s)\\
&\quad + Cu_n(s)))ds + \sum_{k=1}^m \Phi(t-t_k)I_k(u_n(t_k))\Big\}\Big)\\
&\leq \alpha\Big(\Big\{R(u_n)\Big\}\Big) + \frac{2}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\alpha\Big(\Big\{(f(s, D_0(s), D_0(s)
\end{aligned}$$

$$
\begin{aligned}
&+ Cu_n(s)))\Big\}\Big)ds + \sum_{k=1}^{m} \alpha(I_k(D_0(t_k), D_0(t_k))) \\
&\leq M^*\Big[\frac{2(2L_1+C)}{\Gamma(\alpha)}\int_0^{\omega}(\omega-s)^{\alpha-1}\alpha(D_0(s))ds + 2\sum_{k=1}^{m} M_k\alpha(D_0(t_k))\Big] \\
&\quad + \frac{2(2L_1+C)}{\Gamma(\alpha)}\int_0^{t}(t-s)^{\alpha-1}\alpha\Big(D_0(s)\Big)ds + 2\sum_{k=1}^{m} M_k\alpha(D_0(t_k)) \\
&\leq (M^*+1)\Big[\frac{2(L_1+C)\omega^{\alpha}}{\Gamma(1+\alpha)} + 2\sum_{k=1}^{m} M_k\Big]\alpha(D).
\end{aligned}
$$

Since $T(D_0)$ is bounded and equicontinuous, by Lemma 2.3, we have

$$
\alpha(T(D)) \leq 2(M^*+1)\Big[\frac{2(L_1+C)\omega^{\alpha}}{\Gamma(1+\alpha)} + 2\sum_{k=1}^{m} M_k\Big]\alpha(D) \leq \alpha(D).
$$

(i) If $2(M^*+1)\Big[\frac{2(L_1+C)\omega^{\alpha}}{\Gamma(1+\alpha)} + 2\sum_{k=1}^{m} M_k\Big] < 1$, then the operator $T : [v_0, w_0] \to [v_0, w_0]$ is condensing, by Lemma 2.5, T has fixed point u in $[v_0, w_0]$, so u is the mild solution of the PBVP (1.1) in $[v_0, w_0]$.

(ii) If $2(M^*+1)\Big[\frac{2(L_1+C)\omega^{\alpha}}{\Gamma(1+\alpha)} + 2\sum_{k=1}^{m} M_k\Big] \geq 1$, divide $J = [0, \omega]$ into n equal parts, let $\Delta_n : 0 = t'_0 < t'_1 < \ldots < t'_n = \omega$ and $t'_i (i = 1, 2, \ldots, n-1)$ not be the impulsive points, such that

$$
2(M^*+1)\Big[\frac{2(L_1+C)\|\Delta_n\|^{\alpha}}{\Gamma(1+\alpha)} + 2\sum_{k=1}^{m} M_k\Big] < 1.
$$

By (i) and (ii), the PBVP (1.1) has mild solution $u_1(t)$ in $[0, t'_1]$; Again by (i) and (ii), if Eq. (1) with $u(t'_1) = u_1(t'_1)$ as initial value, then it has mild solution $u_2(t)$ in $[t'_1, t'_2]$ and satisfies $u_2(t'_1) = u_1(t'_1)$. Thus, the mild solution of the equation continuously extend from $[0, t'_1]$ to $[0, t'_2]$; Continuing such a process, the mild solution of the equation can be continuously extended to J. So, we obtain a mild solution $u \in PC(J, E)$ of PBVP (1.1), which satisfies $u(t) = u_i(t), t'_{i-1} \leq t \leq t'_i, i = 1, 2, \ldots, n$.

Finally, since $u = Tu = Q(u, u), v_0 \leq u \leq w_0$, by the mixed monotonicity of $Qv_1 = Q(v_0, w_0) \leq Q(u, u) \leq Q(w_0, v_0) = w_1$. Similarly, $v_2 \leq u \leq w_2$, in general, $v_n \leq u \leq w_n$, letting $n \to \infty$, we get $\underline{u} \leq u \leq \overline{u}$. Therefore, the PBVP (1.1) at least has one mild solution between $\underline{u}$ and $\overline{u}$. □

Remark 3.3. Analytic semigroup and differentiable semigroup are equicontinuous semigroup [30]. In the application of partial differential equations, such as parabolic and strongly damped wave equations, the corresponding solution semigroup are analytic semigroup. So, Theorem 3.3 in this paper has extensive applicability.

Now we discuss the uniqueness of the mild solution to PBVP (1.1) in $[v_0, w_0]$. If we replace the assumption (H3) by the assumption:

(H5) There exist positive constants $\overline{C}$, $\overline{L}$ such that

$$
f(t, u_2, v_2) - f(t, u_1, v_1) \leq \overline{C}(u_2 - u_1) + \overline{L}(v_1 - v_2),
$$

for any $t \in J$, and $v_0(t) \leq u_1 \leq u_2 \leq w_0(t)$, $v_0(t) \leq v_2 \leq v_1 \leq w_0(t)$.

We have the following unique existence result.

Theorem 3.4. *Let E be an ordered Banach space, whose positive cone P is normal. $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generates a positive C_0-semigroup $T(t)(t \geq 0)$ in E and $\|T(t)\| < 1$ for $t \in (0, \omega]$, $f \in C(J \times E \times E, E)$ and $I_k \in C(E, E)$, $k = 1, 2, \ldots, m$. If the PBVP (1.1) has has coupled lower and upper L-quasi-solutions v_0 and w_0 with $v_0 \leq w_0$, and conditions (H1), (H2),(H4) and (H5) hold, then the PBVP (1) has a unique mild solution between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 or w_0.*

Proof. We firstly prove that (H1) and (H5) imply (H3). For $t \in J$, let $\{u_n\} \subset [v_0(t), w_0(t)]$ be an increasing sequence and $\{v_n\} \subset [v_0(t), w_0(t)]$ be decreasing sequences. For $m, n \in \mathbb{N}$ with $m > n$, by (H1) and (H5),

$$\begin{aligned} \theta &\leq f(t, u_m, v_m) - f(t, u_n, v_n) + C(u_m - u_n) + L(v_n - v_m) \\ &\leq (C + \overline{C})(u_m - u_n) + (L + \overline{L})(v_n - v_m). \end{aligned}$$

By this and the normality of cone P, we have

$$\begin{aligned} &\|f(t, u_m, v_m) - f(t, u_n, v_n)\| \\ &\leq N(C + \overline{C})(u_m - u_n) + (L + \overline{L})(v_n - v_m) + C\|u_m - u_n\| + L\|v_n - v_m\| \\ &\leq (N(C + \overline{C}) + C)\|u_m - u_n\| + [N(L + \overline{L}) + L]\|v_n - v_m\|. \end{aligned}$$

From this inequality and the definition of the measure of noncompactness, it follows that

$$\begin{aligned} \alpha(\{f(t, u_n, v_n)\}) &\leq (N(C + \overline{C}) + C)\alpha(\{u_n\}) + [N(L + \overline{L}) + L]\alpha(\{v_n\}) \\ &\leq L_1(\alpha(\{u_n\}) + \alpha(\{v_n\})), \end{aligned}$$

where $L_1 = N(C + \overline{C}) + L + \overline{L}) + C + L$. If $\{u_n\}$ is a decreasing sequence and $\{v_n\}$ is an increasing sequences, the above inequality is also valid. Hence (H3) holds. Therefore, by Theorem 3.2, the PBVP (1.1) has minimal and maximal coupled mild L-quasi-solutions $\underline{u}$ and$\overline{u}$ between v_0 and w_0. By the proof of Theorem 3.2, (3.6) and (3.7) are valid. Going from J_1' to J_m' interval by interval we show that $\underline{u}(t) \equiv \overline{u}(t)$ in every J_k'.

Since $\|T(t)\| < 1$, so $\|\Phi(t)\| < 1, \|\Psi(t)\| < \frac{t^{\alpha-1}}{\Gamma(\alpha)}, t \in J$. For $t \in J_1'$, by (3.1) and assumption (H5), we have

$$\begin{aligned} \theta &\leq \overline{u}(t) - \underline{u}(t) = Q(\overline{u}, \underline{u})(t) - Q(\underline{u}, \overline{u})(t) \\ &= \int_0^t \Psi(t - s)\big[f(s, \overline{u}(s), \underline{u}(s)) - f(s, \underline{u}(s), \overline{u}(s)) + (C + 2L)(\overline{u}(s) - \underline{u}(s))\big]ds \\ &\leq \frac{C + 2L + \overline{C} + \overline{L}}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}(\overline{u}(s) - \underline{u}(s))ds. \end{aligned}$$

From this and the normality of cone P it follows that

$$\|\overline{u}(t) - \underline{u}(t)\| \leq \frac{N(C + 2L + \overline{C} + \overline{L})}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}\|\overline{u}(s) - \underline{u}(s)\|ds.$$

By this and Lemma 2.10, we obtained that $\underline{u}(t) \equiv \overline{u}(t)$ in J_1'.

For $t \in J_2'$, since $I_1(\overline{u}(t_1), \underline{u}(t_1)) = I_1(\underline{u}(t_1), \overline{u}(t_1))$, using (3.1) and completely the same argument as above for $t \in J_1'$, we can prove that

$$\|\overline{u}(t) - \underline{u}(t)\| \leq \frac{N(C + 2L + \overline{C} + \overline{L})}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|\overline{u}(s) - \underline{u}(s)\| ds.$$

And by Lemma 2.10, we obtain that $\underline{u}(t) \equiv \overline{u}(t)$ in J_2'.

Continuing such a process interval by interval up to J_m', we see that $\underline{u}(t) \equiv \overline{u}(t)$ over the whole of J. Hence, $\widetilde{u} := \underline{u} = \overline{u}$ is the unique mild solution of the PBVP (1.1) in $[v_0, w_0]$, which can be obtained by the monotone iterative procedure (3.7) starting from v_0 or w_0. □

Remark 3.4. The condition (H4) is easily to be verified in applications. So, application of Theorem 3.4 is very convenient in applications.

4. Example

In this section, we give an example to demonstrate how to utilize our results.

Example 4.1. Consider the impulsive fractional parabolic partial differential equation

$$\begin{cases} \frac{\partial^\alpha u}{\partial t} - \nabla^2 u = g(x, t, u, u), \quad x \in \Omega, t \in J, \ t \neq t_k, \\ \Delta u|_{t=t_k} = J_k(u(x, t_k), u(x, t_k)), \quad x \in \Omega, k = 1, 2, \ldots, m, \\ u|_{\partial\Omega} = 0, \\ u(x, 0) = u(x, 2\pi), \quad x \in \Omega, \end{cases} \tag{4.1}$$

where $\frac{\partial^\alpha u}{\partial t}$ is the Caputo fractional partial derivative of order $0 < \alpha < 1$, ∇^2 is the Laplace operator, $J = [0, 2\pi]$, $0 < t_1 < t_2 < \cdots < t_m < 2\pi$, $J' = J \backslash \{t_1, t_2, \ldots, t_m\}$, $J'' = J \backslash \{0, t_1, t_2, \ldots, t_m\}$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$, $g : \overline{\Omega} \times J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, $J_k : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are also continuous, $k = 1, 2, \ldots, m$.

Let $E = L^p(\Omega)$ with $p > N + 2$, $P = \{u \in L^p(\Omega) : u(x) \geq 0, a.e.\ x \in \Omega\}$, and define the operator $A : D(A) \subset E \to E$ as follows:

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega), \quad Au = -\nabla^2 u.$$

Then E is a Banach space, P is a regular cone of E, and $-A$ generates a positive and analytic C_0-semigroup $T(t) (t \geq 0)$ in E, which is equicontinuous and $M = 1$. Moreover, $T(\cdot)$ is also compact and $\|T(t)\| \leq e^{-t} \leq 1, t \geq 0$. By the Fredholm alternative theorem, $[I - \mathscr{T}_\alpha(1)]^{-1}$ exists and is bounded where $\mathscr{T}_\alpha(\cdot)$ is defined in Section 2.

Let $u(t) = u(\cdot, t)$, $f(t, u, u) = g(\cdot, t, u(\cdot, t), u(\cdot, t))$, $I_k(u(t_k), u(t_k)) = I_k(u(\cdot, t_k), u(\cdot, t_k))$. Then the problem (4.1) can be rewritten in the abstract form of problem (1.1). In order to solve the PBVP (4.1), we also need the following assumptions:

(a) There exist $a \geq 0, h \in PC(\Omega \times J) \cap C^1(\overline{\Omega} \times J'), h(x, t) \geq 0$ and $y_k \in H^2(\Omega) \cap H_0^1(\Omega), y_k(x) \geq 0, k = 1, 2, \ldots, m$, such that for any $u \in L^2(\Omega), u \geq 0$, we have

$$-au - h(x, t) \leq g(x, t, -u, u) \leq g(x, t, u, -u) \leq au + h(x, t), x \in \Omega, t \in J',$$

$$-y_k \leq J_k(-u,u) \leq J_k(u,-u) \leq y_k, x \in \Omega, k = 1,2,\dots,m.$$

(b) The partial derivative $g'_u(x,t,u,v)$ is continuous on any bounded domain and $g'_v(x,t,u,v)$ has upper bound.

(c) For any u_1, u_2, v_1, v_2 in any bounded and ordered interval, and $u_1 \leq u_2, v_2 \leq v_1$, we have

$$J_k(u_1(x,t_k), v_1(x,t_k)) \leq J_k(u_2(x,t_k), v_2(x,t_k)), x \in \Omega, k = 1,2,\dots,m.$$

Theorem 4.1. *If the assumptions (a)-(c) are satisfied, then the PBVP (4.1) has a unique mild solution.*

Proof. First, we consider the following PBVP of linear impulsive parabolic partial differential equation

$$\begin{cases} \frac{\partial^\alpha u}{\partial t} - \nabla^2 u - (a+2L)u = h(x,t), \quad (x,t) \in \Omega \times J', \\ \Delta u|_{t=t_k} = y_k, \quad x \in \Omega,\ k = 1,2,\dots,m, \\ u|_{\partial\Omega} = 0, \\ u(x,0) = u(x,2\pi), \quad x \in \Omega, \end{cases} \tag{4.2}$$

where $L = \sup_{(x,t)\in\Omega\times J} |g'_v(x,t,u,v)|$. From the above discussion, the problem (4.2) can be transformed into the following abstract problem

$$\begin{cases} \frac{\partial^\alpha u}{\partial t} + Au(t) - (a+2L)u(t) = \widetilde{h}(t), t \in J', \\ \Delta u|_{t=t_k} = y_k, k = 1,2,\dots,m, \\ u(0) = u(2\pi), \end{cases} \tag{4.3}$$

where $\widetilde{h}(t) = h(\cdot,t)$. Since $-A + (a+2L)I$ generate a positive C_0-semigroup $S(t) = e^{(a+2L)t}T(t)(t \geq 0)$ in E, by Lemma 2.9 we know that PBIVP (4.3) exist a unique positive classical solution $u \in PC(J,E) \cap C^1(J',E) \cap C(J',E_1)$. Let $v_0 = u, w_0 = u$, by assumption (a), it is easy to see that v_0, w_0 are coupled L-quasi-solutions of PBVP (1.1). From assumptions (b) and (c), it is easy to verify that conditions (H1), (H2), (H4) and (H5) are satisfied. So, from Theorem 3.4, we know that PBVP (4.1) has a unique mild solution. □

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