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NUMERICAL RESOLUTION OF AN EXACT HEAT CONDUCTION MODEL WITH A DELAY TERM*

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Abstract In this paper we analyze, from the numerical point of view, a dynamic thermoelastic problem. Here, the so-called exact heat conduction model with a delay term is used to obtain the heat evolution. Thus, the thermomechanical problem is written as a coupled system of partial differential equations, and its variational formulation leads to a system written in terms of the velocity and the temperature fields. An existence and uniqueness result is recalled. Then, fully discrete approximations are introduced by using the classical finite element method to approximate the spatial variable and the implicit Euler scheme to discretize the time derivatives. A priori error estimates are proved, from which the linear convergence of the algorithm could be derived under suitable additional regularity conditions. Finally, a two-dimensional numerical example is solved to show the accuracy of the approximation and the decay of the discrete energy.

Keywords Thermoelasticity, exact heat condution, delay parameter, finite elements, a priori error estimates.

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1. Introduction

The classical linear theory of heat conduction, based on Fourier's law, implies that thermal perturbations will be felt instantly at all the points of the body. That is, the thermal waves propagate with infinity speed. It is physically unrealistic because of causality's fundamental role in modern physics. Different heat conduction theories have been put forth over the course of the 20th century and the present century (see Chandrasekharaiah [2], Hetnarski and Ignaczak [10,11] and the references cited therein). In the books [12,26], several studies concerning applicability of nonclassical thermoelastic theories are considered.

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In 1995, Tzou [24, 25] suggested a theory where the heat flux and the gradient of the temperature have a delay in the constitutive equations. The constitutive equations proposed by Tzou are given by

$$q_i(\boldsymbol{x}, t+\tau_1) = -\kappa\theta_{,i}(\boldsymbol{x}, t+\tau_2), \qquad \kappa > 0, \tag{1.1}$$

where τ_1 and τ_2 are the delay parameters which are assumed to be positive.This equation says that the temperature gradient established across a material volume at position $\boldsymbol{x}$ and time $t+\tau_2$ results in a heat flux to flow at a different time $t+\tau_1$. The delays can be understood in terms of the microstructure of the material. This theory has several derivations when the heat flux and the gradients of the temperature and the thermal displacement are replaced by Taylor approximations. More recently, Choudhuri [3] proposed a constitutive law with three-phase-lag which is an extension of Tzou's proposition. The equation is

$$q_i(\boldsymbol{x}, t+\tau_1) = -k_1\alpha_{,i}(\boldsymbol{x}, t+\tau_3) - k_2\theta_{,i}(\boldsymbol{x}, t+\tau_2), \tag{1.2}$$

where $\dot{\alpha} = \theta$. The variable α is called the *thermal displacement*. The parameter τ_3 is another delay parameter. It seems that the aim of Choudhuri was to establish a mathematical model that includes phase-lags in the heat flux vector, in the temperature gradient and in the thermal displacement gradient. Moreover, if Taylor approximation is introduced in the equation, the Green and Naghdi models are obtained [7, 8].

These two proposals lead to *ill-posed* problems in the sense of Hadamard. In fact, it can be shown that combining equation (1.1) (or (1.2)) with the energy equation

$$c\,\dot{\theta} + \operatorname{div}\mathbf{q} = 0, \quad (c > 0), \tag{1.3}$$

leads to the existence of a family of elements in the point spectrum such that its real part tends to infinity [5]. It has been also showed that the Tzou's theory is not compatible with the basic axioms of the thermomechanics [6]. Therefore, it is difficult to accept these proposals either from a mathematical point of view or from the thermodynamical perspective.

In a recent note [21] it was proved that when $\tau_3 < \tau_1 = \tau_2$, equation (1.2) combined with the energy equation (1.3) defines a well posed problem. That is, a well posed problem in the context of the exact three-dual-phase theory. This was a non-trivial case with this property. We could accept it as a possible exact phase-lag constitutive equation to describe the heat conduction which is different from the ones proposed in [19, 20].

As it was used in the theories proposed by Choudhuri and Tzou, we could replace equation (1.2) by truncated Taylor expansions. If we consider a second order approximation we find the heat equation:

$$\ddot{\nu} - \frac{k^*\tau^2}{2}\Delta\ddot{\nu} = (k_1 + k^*\tau)\Delta\theta + k^*\Delta\nu,$$

where now we assume that $\tau = \tau_3 - \tau_1 (= \tau_2) > 0$. This heat conduction equation has been complemented with other equations to obtain a thermoelastic problem and several contributions has been dedicated to study it. We recall some of them [13–18, 22].

2. The mechanical and variational problems: existence and uniqueness

In this section, we present a brief description of the model and we obtain its mechanical and variational formulations (details can be found in [22]). We also recall an existence and uniqueness result.

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, be the domain and denote by $[0, T]$, $T > 0$, the time interval of interest. The boundary of the body $\Gamma = \partial\Omega$ is assumed to be Lipschitz. Moreover, let $\boldsymbol{x} \in \Omega$ and $t \in [0, T]$ be the spatial and time variables, respectively. In order to simplify the writing, we do not indicate the dependence of the functions on $\boldsymbol{x} = (x_j)_{j=1}^d$ and t, and a subscript after a comma under a variable represents its spatial derivative with respect to the prescribed variable, i.e. $f_{i,j} = \dfrac{\partial f_i}{\partial x_j}$. The time derivatives are represented as a dot for the first order and two points for the second order over each variable. Finally, as usual the repeated index notation is used for the summation.

We denote by $\boldsymbol{u} = (u_i)_{i=1}^d$ and ν the displacement field and the thermal displacement, respectively. We note that the temperature θ is then obtained as $\theta = \dot{\nu}$.

Assuming that the material is isotropic and homogeneous and following the work by Quintanilla [22], the model is written as follows, for $i, j = 1, \dots, d$, in $\Omega \times (0, T)$,

$$\begin{aligned} &\rho \ddot{u}_i - \mu u_{i,jj} - (\lambda + \mu) u_{j,ji} - \beta \theta_{,i} = 0, \\ &c\ddot{\nu} - \frac{\tau^2}{2} k^* \Delta \ddot{\nu} = \beta \dot{u}_{i,i} + k^* \Delta \nu + (k_1 + \tau k^*) \Delta \dot{\nu}. \end{aligned} \tag{2.1}$$

In the above equations, constants ρ and k^* denote the mass density and the thermal diffusion coefficient, respectively, and λ and μ represent the Lamé's coefficients. Moreover, β is a thermal expansion coefficient and τ is the delay parameter.

As boundary conditions, we assume, for $i = 1, \dots, d$,

$$u_i(\boldsymbol{x}, t) = \theta(\boldsymbol{x}, t) = 0 \quad \text{for } (\boldsymbol{x}, t) \in \partial\Omega \times (0, T). \tag{2.2}$$

We point out that other boundary conditions could be used but we restrict ourselves to this case for the sake of simplicity.

In order to complete the definition of the mechanical problem we impose the following initial conditions for $\boldsymbol{x} \in \Omega$:

$$\begin{aligned} &u_i(\boldsymbol{x}, 0) = u_i^0(\boldsymbol{x}), \quad \dot{u}_i(\boldsymbol{x}, 0) = v_i^0(\boldsymbol{x}), \quad \nu(\boldsymbol{x}, 0) = \nu^0(\boldsymbol{x}), \\ &\dot{\nu}(\boldsymbol{x}, 0) = \theta^0(\boldsymbol{x}), \end{aligned} \tag{2.3}$$

where $\boldsymbol{u}^0 = (u_i^0)_{i=1}^d$, $\boldsymbol{v}^0 = (v_i^0)_{i=1}^d$, ν^0 and θ^0 are prescribed functions.

Therefore, the thermo-mechanical problem modelling the deformation of a thermoelastic body with an exact heat conduction model with a delay is the following (see [22] for details).

Problem P. *Find the displacement field $\boldsymbol{u} = (u_i)_{i=1}^d : \overline{\Omega} \times [0, T] \to \mathbb{R}^d$ and the thermal displacement $\nu : \overline{\Omega} \times [0, T] \to \mathbb{R}$ such that equations (2.1), boundary conditions (2.2) and initial conditions (2.3) are fulfilled.*

Now, in order to obtain the variational formulation of Problem P, let $Y = L^2(\Omega)$, $H = [L^2(\Omega)]^d$ and $Q = [L^2(\Omega)]^{d\times d}$ and denote by $(\cdot,\cdot)_Y$, $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_Q$ the respective scalar products in these spaces, with corresponding norms $\|\cdot\|_Y$, $\|\cdot\|_H$ and $\|\cdot\|_Q$. Moreover, let us define the variational spaces V and E as follows,

$$V = \{\boldsymbol{z} \in [H^1(\Omega)]^d\,;\ \boldsymbol{z} = \boldsymbol{0} \quad \text{on} \quad \Gamma\},$$
$$E = \{r \in H^1(\Omega)\,;\ r = 0 \quad \text{on} \quad \Gamma\},$$

with respective scalar products $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_E$, and norms $\|\cdot\|_V$ and $\|\cdot\|_E$.

By using Green's formula and boundary conditions (2.2), we write the variational formulation of Problem P in terms of the velocity field $\boldsymbol{v} = \dot{\boldsymbol{u}}$ and the temperature $\theta = \dot{\nu}$.

Problem VP. *Find the velocity field $\boldsymbol{v} : [0,T] \to V$ and the temperature $\theta : [0,T] \to E$ such that $\boldsymbol{v}(0) = \boldsymbol{v}^0$, $\theta(0) = \theta^0$, and, for a.e. $t \in (0,T)$ and for all $\boldsymbol{w} \in V$, $r \in E$,*

$$\begin{aligned}\rho(\dot{\boldsymbol{v}}(t), \boldsymbol{w})_H + (\lambda+\mu)(\operatorname{div}\boldsymbol{u}(t), \operatorname{div}\boldsymbol{w})_Y + \mu(\nabla\boldsymbol{u}(t), \nabla\boldsymbol{w})_Q&\\ -\beta(\nabla\theta(t), \boldsymbol{w})_H = 0,&\end{aligned} \tag{2.4}$$

$$\begin{aligned}c(\dot{\theta}(t), r)_Y + \frac{\tau^2}{2}k^*(\nabla\dot{\theta}(t), \nabla r)_H + k^*(\nabla\nu(t), \nabla r)_H&\\ +(k_1+\tau k^*)(\nabla\theta(t), \nabla r)_H = \beta(\operatorname{div}\boldsymbol{v}(t), r)_Y,&\end{aligned} \tag{2.5}$$

where the displacement field and the thermal displacement are then recovered from the relations

$$\boldsymbol{u}(t) = \int_0^t \boldsymbol{v}(s)\,ds + \boldsymbol{u}^0, \quad \nu(t) = \int_0^t \theta(s)\,ds + \theta^0, \tag{2.6}$$

and we note that div *represents the classical divergence operator.*

In [22] it has been proved the following existence and uniqueness result.

Theorem 2.1. *Let the following conditions on the constitutive coefficients hold:*

$$\rho > 0, \quad \mu > 0, \quad \lambda > 0, \quad c > 0, \quad \tau > 0, \quad k^* > 0, \quad k_1 > 0.$$

If the initial conditions satisfy:

$$\boldsymbol{u}^0, \boldsymbol{v}^0 \in V, \quad \nu^0, \theta^0 \in E,$$

then there exists a unique solution to Problem VP with the following regularity:

$$\boldsymbol{u} \in C^1([0,T]; V) \cap C^2([0,T]; H), \quad \nu \in C^2([0,T]; E).$$

3. Fully discrete approximations: an a priori error analysis

In this section, we now consider a fully discrete approximation of Problem VP. This is done in two steps. First, we assume that the domain $\overline{\Omega}$ is polyhedral and we denote by $\mathcal{T}^h$ a regular triangulation in the sense of [4]. Thus, we construct the finite dimensional spaces $V^h \subset V$ and $E^h \subset E$ given by

$$V^h = \{\boldsymbol{w}^h \in [C(\overline{\Omega})]^d\,;\ \boldsymbol{w}^h_{|Tr} \in [P_1(Tr)]^d \quad \forall Tr \in \mathcal{T}^h, \quad \boldsymbol{w}^h = \boldsymbol{0} \quad \text{on} \quad \Gamma\}, \tag{3.1}$$

$$E^h = \{r^h \in C(\overline{\Omega})\,;\ r^h_{|Tr} \in P_1(Tr) \quad \forall Tr \in \mathcal{T}^h, \quad r^h = 0 \quad \text{on} \quad \Gamma\}, \tag{3.2}$$

where $P_1(Tr)$ represents the space of polynomials of degree less or equal to one in the element Tr, i.e. the finite element spaces V^h and E^h are composed of continuous and piecewise affine functions. Here, $h > 0$ denotes the spatial discretization parameter. Moreover, we assume that the discrete initial conditions, denoted by $\boldsymbol{u}^{0h}$, $\boldsymbol{v}^{0h}$, θ^{0h} and ν^{0h}, are given by

$$\boldsymbol{u}^{0h} = \mathcal{P}_1^h \boldsymbol{u}^0, \ \boldsymbol{v}^{0h} = \mathcal{P}_1^h \boldsymbol{v}^0, \ \theta^{0h} = \mathcal{P}_2^h \theta^0, \ \nu^{0h} = \mathcal{P}_2^h \nu^0, \tag{3.3}$$

where $\mathcal{P}_1^h$ and $\mathcal{P}_2^h$ are the classical finite element interpolation operators over V^h and E^h, respectively (see, e.g., [4]).

Secondly, we consider a partition of the time interval $[0, T]$, denoted by $0 = t_0 < t_1 < \cdots < t_N = T$. In this case, we use a uniform partition with step size $k = T/N$ and nodes $t_n = n\,k$ for $n = 0, 1, \dots, N$. For a continuous function $z(t)$, we use the notation $z_n = z(t_n)$ and, for the sequence $\{z_n\}_{n=0}^N$, we denote by $\delta z_n = (z_n - z_{n-1})/k$ its corresponding divided differences.

Therefore, using the backward Euler scheme, the fully discrete approximations are considered as follows.

Problem VP^{hk}. *Find the discrete velocity $\boldsymbol{v}^{hk} = \{\boldsymbol{v}_n^{hk}\}_{n=0}^N \subset V^h$ and the discrete temperature $\theta^{hk} = \{\theta_n^{hk}\}_{n=0}^N \subset E^h$ such that $\boldsymbol{v}_0^{hk} = \boldsymbol{v}^{0h}$, $\theta_0^{hk} = \theta^{0h}$, and, for $n = 1, \dots, N$ and for all $\boldsymbol{w}^h \in V^h$ and $r^h \in E^h$,*

$$\begin{aligned} &\rho(\delta \boldsymbol{v}_n^{hk}, \boldsymbol{w}^h)_H + (\lambda + \mu)(\operatorname{div} \boldsymbol{u}_n^{hk}, \operatorname{div} \boldsymbol{w}^h)_Y + \mu(\nabla \boldsymbol{u}_n^{hk}, \nabla \boldsymbol{w}^h)_Q \\ &\qquad -\beta(\nabla \theta_n^{hk}, \boldsymbol{w}^h)_H = 0, \end{aligned} \tag{3.4}$$

$$\begin{aligned} &c(\delta \theta_n^{hk}, r^h)_Y + \frac{\tau^2}{2} k^*(\nabla \delta \theta_n^{hk}, \nabla r^h)_H + k^*(\nabla \nu_n^{hk}, \nabla r^h)_H \\ &\qquad +(k_1 + \tau k^*)(\nabla \theta_n^{hk}, \nabla r^h)_H = \beta(\operatorname{div} \boldsymbol{v}_n^{hk}, r^h)_Y, \end{aligned} \tag{3.5}$$

where the discrete displacement field and the discrete thermal displacement are then recovered from the relations

$$\boldsymbol{u}_n^{hk} = k \sum_{j=1}^n \boldsymbol{v}_j^{hk} + \boldsymbol{u}^{0h}, \quad \nu_n^{hk} = k \sum_{j=1}^n \theta_j^{hk} + \nu^{0h}. \tag{3.6}$$

We note that the existence of a unique discrete solution to Problem VP^{hk} is obtained in a straightforward way using the classical Lax-Milgram lemma.

Now, we will find some a priori error estimates on the numerical errors $\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}$ and $\theta_n - \theta_n^{hk}$. We have the following.

Theorem 3.1. *Under the assumptions of Theorem 2.1, if we denote by $(\boldsymbol{u}, \boldsymbol{v}, \theta, \nu)$ the solution to Problem VP and by $(\boldsymbol{u}^{hk}, \boldsymbol{v}^{hk}, \theta^{hk}, \nu^{hk})$ the solution to Problem VP^{hk}, then we have the following a priori error estimates, for all $\boldsymbol{w}^h = \{\boldsymbol{w}_j^h\}_{j=0}^N \subset V^h$ and $r^h = \{r_j^h\}_{j=0}^N \subset E^h$,*

$$\begin{aligned} &\max_{0 \le n \le N} \Big\{ \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + \|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Q^2 + \|\operatorname{div}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y^2 \\ &\qquad + \|\theta_n - \theta_n^{hk}\|_Y^2 + \|\nabla(\theta_n - \theta_n^{hk})\|_H^2 + \|\nabla(\nu_n - \nu_n^{hk})\|_H^2 \Big\} \\ &\le Ck \sum_{j=1}^N \Big(\|\dot{\boldsymbol{v}}_j - \delta \boldsymbol{v}_j\|_H^2 + \|\boldsymbol{v}_j - \boldsymbol{w}_j^h\|_V^2 + \|\dot{\boldsymbol{u}}_j - \delta \boldsymbol{u}_j\|_V^2 \end{aligned}$$

$$
\begin{aligned}
&+\|\dot\theta_j-\delta\theta_j\|_E^2+\|\nabla(\dot\nu_j-\delta\nu_j)\|_H^2+\|\theta_j-r_j^h\|_E^2\Big)\\
&+C\max_{0\le n\le N}\|\boldsymbol{v}_n-\boldsymbol{w}_n^h\|_H^2+C\max_{0\le n\le N}\|\theta_n-r_n^h\|_Y^2\\
&+\frac{C}{k}\sum_{j=1}^{N-1}\|\theta_j-r_j^h-(\theta_{j+1}-r_{j+1}^h)\|_Y^2\\
&+\frac{C}{k}\sum_{j=1}^{N-1}\|\boldsymbol{v}_j-\boldsymbol{w}_j^h-(\boldsymbol{v}_{j+1}-\boldsymbol{w}_{j+1}^h)\|_H^2+C\Big(\|\boldsymbol{v}^0-\boldsymbol{v}^{0h}\|_H^2\\
&+\|\boldsymbol{u}^0-\boldsymbol{u}^{0h}\|_V^2+\|\theta^0-\theta^{0h}\|_E^2+\|\nabla(\nu^0-\nu^{0h})\|_H^2\Big),
\end{aligned}\tag{3.7}
$$

where $C>0$ is a positive constant which is independent of the discretization parameters h and k, but depending on the continuous solution, and $\delta\boldsymbol{v}_j=(\boldsymbol{v}_j-\boldsymbol{v}_{j-1})/k$, $\delta\boldsymbol{u}_j=(\boldsymbol{u}_j-\boldsymbol{u}_{j-1})/k$, $\delta\theta_j=(\theta_j-\theta_{j-1})/k$ and $\delta\nu_j=(\nu_j-\nu_{j-1})/k$.

Proof. First, we obtain some estimates for the velocity field. Then, we subtract variational equation (2.4) at time $t=t_n$ for a test function $\boldsymbol{w}=\boldsymbol{w}^h\in V^h\subset V$ and discrete variational equation (3.4) to obtain, for all $\boldsymbol{w}^h\in V^h$,

$$
\begin{aligned}
&\rho(\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n^{hk},\boldsymbol{w}^h)_H+(\lambda+\mu)(\operatorname{div}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\operatorname{div}\boldsymbol{w}^h)_Y\\
&\quad+\mu(\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\nabla\boldsymbol{w}^h)_Q-\beta(\nabla(\theta_n-\theta_n^{hk}),\boldsymbol{w}^h)_H=0,
\end{aligned}
$$

and so, we have, for all $\boldsymbol{w}^h\in V^h$,

$$
\begin{aligned}
&\rho(\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})_H+(\lambda+\mu)(\operatorname{div}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\operatorname{div}(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}))_Y\\
&\quad+\mu(\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\nabla(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}))_Q-\beta(\nabla(\theta_n-\theta_n^{hk}),(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}))_H\\
&=\rho(\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{w}^h)_H+(\lambda+\mu)(\operatorname{div}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\operatorname{div}(\boldsymbol{v}_n-\boldsymbol{w}^h))_Y\\
&\quad+\mu(\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\nabla(\boldsymbol{v}_n-\boldsymbol{w}^h))_Q-\beta(\nabla(\theta_n-\theta_n^{hk}),\boldsymbol{v}_n-\boldsymbol{w}^h)_H=0.
\end{aligned}
$$

Taking into account that

$$
\begin{aligned}
(\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})_H\ge&(\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n,\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})_H\\
&+\frac{1}{2k}\left\{\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2-\|\boldsymbol{v}_{n-1}-\boldsymbol{v}_{n-1}^{hk}\|_H^2\right\},\\
(\operatorname{div}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\operatorname{div}(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}))_Y\ge&(\operatorname{div}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\operatorname{div}(\dot{\boldsymbol{u}}_n-\delta\boldsymbol{u}_n))_Y\\
&+\frac{1}{2k}\{\|\operatorname{div}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Y^2-\|\operatorname{div}(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{hk})\|_Y^2\},\\
(\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\nabla(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}))_Q\ge&(\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\nabla(\dot{\boldsymbol{u}}_n-\delta\boldsymbol{u}_n))_Q\\
&+\frac{1}{2k}\left\{\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q^2-\|\nabla(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{hk})\|_Q^2\right\},\\
-\beta(\nabla(\theta_n-\theta_n^{hk}),\boldsymbol{v}_n-\boldsymbol{w}^h)_H=&\beta(\theta_n-\theta_n^{hk},\operatorname{div}(\boldsymbol{v}_n-\boldsymbol{w}^h))_Y,
\end{aligned}
$$

using again Cauchy-Schwarz inequality and Young's inequality it follows that, for all $\boldsymbol{w}^h \in V^h$,

$$
\begin{aligned}
&\frac{\rho}{2k}\left\{\|\boldsymbol{v}_n \ast \boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}\|_H^2\right\} - \beta(\nabla(\xi_n - \xi_n^{hk}), \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_H \\
&\quad + \frac{\lambda+\mu}{2k}\left\{\|\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y^2 - \|\mathrm{div}\,(\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk})\|_Y^2\right\} \\
&\quad + \frac{\mu}{2k}\left\{\|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Q^2 - \|\nabla(\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk})\|_Q^2\right\} \\
&\le C\Big(\|\dot{\boldsymbol{v}}_n - \delta\boldsymbol{v}_n\|_H^2 + \|\boldsymbol{v}_n - \boldsymbol{w}^h\|_V^2 + \|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Q^2 + \|\dot{\boldsymbol{u}}_n - \delta\boldsymbol{u}_n\|_V^2 \\
&\quad + \|\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y^2 + \|\theta_n - \theta_n^{hk}\|_Y^2 + \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 \\
&\quad + (\delta\boldsymbol{v}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{w}^h)_H\Big).
\end{aligned}
$$

Now, we obtain the error estimates on the temperature. Then, we subtract variational equation (2.5) at time $t = t_n$ for a test function $r = r^h \in E^h \subset E$ and discrete variational equation (3.5) to obtain, for all $r^h \in E^h$,

$$
\begin{aligned}
&c(\dot{\theta}_n - \delta\theta_n^{hk}, r^h)_Y + \frac{\tau^2}{2}k^*(\nabla(\delta\theta_n - \delta\theta_n^{hk}), \nabla r^h)_H \\
&\quad + (k_1 + \tau k^*)(\nabla(\theta_n - \theta_n^{hk}), \nabla r^h)_H \\
&\quad + k^*(\nabla(\nu_n - \nu_n^{hk}), \nabla r^h)_H = \beta(\mathrm{div}\,(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), r^h)_Y,
\end{aligned}
$$

and so we have, for all $r^h \in E^h$,

$$
\begin{aligned}
&c(\dot{\theta}_n - \delta\theta_n^{hk}, \theta_n - \theta_n^{hk})_Y + k^*(\nabla(\nu_n - \nu_n^{hk}), \nabla(\theta_n - \theta_n^{hk}))_H \\
&+ \frac{\tau^2}{2}k^*(\nabla(\delta\theta_n - \delta\theta_n^{hk}), \nabla(\theta_n - \theta_n^{hk}))_H \\
&+ (k_1 + \tau k^*)(\nabla(\theta_n - \theta_n^{hk}), \nabla(\theta_n - \theta_n^{hk}))_H \\
&- \beta(\mathrm{div}\,(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), \theta_n - \theta_n^{hk})_Y \\
=&c(\dot{\theta}_n - \delta\theta_n^{hk}, \theta_n - r^h)_Y + k^*(\nabla(\nu_n - \nu_n^{hk}), \nabla(\theta_n - r^h))_H \\
&+ \frac{\tau^2}{2}k^*(\nabla(\delta\theta_n - \delta\theta_n^{hk}), \nabla(\theta_n - r^h))_H \\
&+ (k_1 + \tau k^*)(\nabla(\theta_n - \theta_n^{hk}), \nabla(\theta_n - r^h))_H \\
&- \beta(\mathrm{div}\,(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), \theta_n - r^h)_Y.
\end{aligned}
$$

Keeping in mind that

$$
\begin{aligned}
(\dot{\theta}_n - \delta\theta_n^{hk}, \theta_n - \theta_n^{hk})_Y \ge&(\dot{\theta}_n - \delta\theta_n, \theta_n - \theta_n^{hk})_Y \\
&+ \frac{1}{2k}\left\{\|\theta_n - \theta_n^{hk}\|_Y^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|_Y^2\right\}, \\
(\nabla(\dot{\theta}_n - \delta\theta_n^{hk}), \nabla(\theta_n - \theta_n^{hk}))_H \ge&(\nabla(\dot{\theta}_n - \delta\theta_n), \nabla(\theta_n - \theta_n^{hk}))_H \\
&+ \frac{1}{2k}\left\{\|\nabla(\theta_n - \theta_n^{hk})\|_H^2 - \|\nabla(\theta_{n-1} - \theta_{n-1}^{hk})\|_H^2\right\}, \\
(\nabla(\nu_n - \nu_n^{hk}), \nabla(\theta_n - \theta_n^{hk}))_H \ge&(\nabla(\nu_n - \nu_n^{hk}), \nabla(\dot{\nu}_n - \delta\nu_n))_H \\
&+ \frac{1}{2k}\left\{\|\nabla(\nu_n - \nu_n^{hk})\|_H^2 - \|\nabla(\nu_{n-1} - \nu_{n-1}^{hk})\|_H^2\right\}, \\
-\beta(\mathrm{div}\,(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), \theta_n - \theta_n^{hk})_Y =&\beta(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}, \nabla(\theta_n - \theta_n^{hk}))_H,
\end{aligned}
$$

using several times Cauchy-Schwarz and Young inequalities we find that, for all $r^h \in E^h$,

$$\begin{aligned}
&\frac{1}{2k}\left\{\|\theta_n-\theta_n^{hk}\|_Y^2-\|\theta_{n-1}-\theta_{n-1}^{hk}\|_Y^2\right\}+\beta(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk},\nabla(\theta_n-\theta_n^{hk}))_H\\
&+\frac{1}{2k}\left\{\|\nabla(\theta_n-\theta_n^{hk})\|_H^2-\|\nabla(\theta_{n-1}-\theta_{n-1}^{hk})\|_H^2\right\}\\
&+\frac{1}{2k}\left\{\|\nabla(\nu_n-\nu_n^{hk})\|_H^2-\|\nabla(\nu_{n-1}-\nu_{n-1}^{hk})\|_H^2\right\}\\
\leq{}& C\Big(\|\dot\theta_n-\delta\theta_n\|_E^2+\|\theta_n-r^h\|_E^2+\|\theta_n-\theta_n^{hk}\|_Y^2\\
&+\|\nabla(\theta_n-\theta_n^{hk})\|_H^2+\|\nabla(\dot\nu_n-\delta\nu_n)\|_H^2+\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2\\
&+(\delta\theta_n-\delta\theta_n^{hk},\theta_n-r^h)_Y+\|\nabla(\nu_n-\nu_n^{hk})\|_H^2\Big).
\end{aligned}$$

Combining now these estimates we find that

$$\begin{aligned}
&\frac{\rho}{2k}\left\{\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2-\|\boldsymbol{v}_{n-1}-\boldsymbol{v}_{n-1}^{hk}\|_H^2\right\}\\
&+\frac{\mu}{2k}\left\{\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q^2-\|\nabla(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{hk})\|_Q^2\right\}\\
&+\frac{\lambda+\mu}{2k}\left\{\|\mathrm{div}\,(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Y^2-\|\mathrm{div}\,(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{hk})\|_Y^2\right\}\\
&+\frac{1}{2k}\left\{\|\theta_n-\theta_n^{hk}\|_Y^2-\|\theta_{n-1}-\theta_{n-1}^{hk}\|_Y^2\right\}\\
&+\frac{1}{2k}\left\{\|\nabla(\theta_n-\theta_n^{hk})\|_H^2-\|\nabla(\theta_{n-1}-\theta_{n-1}^{hk})\|_H^2\right\}\\
&+\frac{1}{2k}\left\{\|\nabla(\nu_n-\nu_n^{hk})\|_H^2-\|\nabla(\nu_{n-1}-\nu_{n-1}^{hk})\|_H^2\right\}\\
\leq{}& C\Big(\|\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n\|_H^2+\|\boldsymbol{v}_n-\boldsymbol{w}^h\|_V^2+\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q^2\\
&+\|\dot{\boldsymbol{u}}_n-\delta\boldsymbol{u}_n\|_V^2+\|\mathrm{div}\,(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Y^2+\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2\\
&+\|\dot\theta_n-\delta\theta_n\|_E^2+\|\theta_n-r^h\|_E^2+(\delta\boldsymbol{v}_n-\delta\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{w}^h)_H\\
&+(\delta\theta_n-\delta\theta_n^{hk},\theta_n-r^h)_Y+\|\theta_n-\theta_n^{hk}\|_Y^2+\|\nabla(\theta_n-\theta_n^{hk})\|_H^2\\
&+\|\nabla(\dot\nu_n-\delta\nu_n)\|_H^2+\|\nabla(\nu_n-\nu_n^{hk})\|_H^2\Big).
\end{aligned}$$

Multiplying the previous estimates by k and summing up the resulting equation, using the estimates on the temperature fields given above we have

$$\begin{aligned}
&\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q^2+\|\mathrm{div}\,(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Y^2\\
&+\|\theta_n-\theta_n^{hk}\|_Y^2+\|\nabla(\theta_n-\theta_n^{hk})\|_H^2+\|\nabla(\nu_n-\nu_n^{hk})\|_H^2\\
\leq{}& Ck\sum_{j=1}^{n}\Big(\|\dot{\boldsymbol{v}}_j-\delta\boldsymbol{v}_j\|_H^2+\|\boldsymbol{v}_j-\boldsymbol{w}_j^h\|_V^2+\|\nabla(\boldsymbol{u}_j-\boldsymbol{u}_j^{hk})\|_Q^2\\
&+\|\dot{\boldsymbol{u}}_j-\delta\boldsymbol{u}_j\|_V^2+\|\mathrm{div}\,(\boldsymbol{u}_j-\boldsymbol{u}_j^{hk})\|_Y^2+\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_H^2\\
&+\|\dot\theta_j-\delta\theta_j\|_E^2+\|\theta_j-r_j^h\|_E^2+(\delta\boldsymbol{v}_j-\delta\boldsymbol{v}_j^{hk},\boldsymbol{v}_j-\boldsymbol{w}_j^h)_H\\
&+(\delta\theta_j-\delta\theta_j^{hk},\theta_j-r_j^h)_Y+\|\theta_j-\theta_j^{hk}\|_Y^2+\|\nabla(\dot\nu_j-\delta\nu_j)\|_H^2\\
&+\|\nabla(\theta_j-\theta_j^{hk})\|_H^2+\|\nabla(\nu_j-\nu_j^{hk})\|_H^2\Big)+C\Big(\|\boldsymbol{v}^0-\boldsymbol{v}^{0h}\|_H^2\\
&+\|\boldsymbol{u}^0-\boldsymbol{u}^{0h}\|_V^2+\|\theta^0-\theta^{0h}\|_E^2+\|\nabla(\nu^0-\nu^{0h})\|_H^2\Big).
\end{aligned}$$

Finally, taking into account that

$$
\begin{aligned}
k\sum_{j=1}^{n}(\delta\theta_j-\delta\theta_j^{hk},\theta_j-r_j^h)_Y &= (\theta_n-\theta_n^{hk},\theta_n-r_n^h)_Y+(\theta^{0h}-\theta^0,\theta_1-r_1^h)_Y\\
&\quad+\sum_{j=1}^{n-1}(\theta_j-\theta_j^{hk},\theta_j-r_j^h-(\theta_{j+1}-r_{j+1}^h))_Y,\\
k\sum_{j=1}^{n}(\delta\boldsymbol{v}_j-\delta\boldsymbol{v}_j^{hk},\boldsymbol{v}_j-\boldsymbol{w}_j^h)_H &= (\boldsymbol{v}_n-\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{w}_n^h)_H+(\boldsymbol{v}^{0h}-\boldsymbol{v}^0,\boldsymbol{v}_1-\boldsymbol{w}_1^h)_H\\
&\quad+\sum_{j=1}^{n-1}(\boldsymbol{v}_j-\boldsymbol{v}_j^{hk},\boldsymbol{v}_j-\boldsymbol{w}_j^h-(\boldsymbol{v}_{j+1}-\boldsymbol{w}_{j+1}^h))_H,
\end{aligned}
$$

using the above estimates and a discrete version of Gronwall's inequality (see [1]) we conclude the proof. □

Remark 3.1. We note that error estimates (3.7) are the basis to get the convergence order of the approximations given by Problem VP^{hk}. Therefore, as an example, under the following additional regularity condition:

$$
\begin{aligned}
&\boldsymbol{u}\in C^1([0,T];[H^2(\Omega)]^d)\cap H^3(0,T;H)\cap H^2(0,T;V),\\
&\nu\in C^1([0,T];H^2(\Omega))\cap H^3(0,T;Y),
\end{aligned}
$$

using the classical results on the approximation by finite elements and the regularities of the initial conditions (see, for instance, [4]), it follows that the approximations obtained by Problem VP^{hk} are linearly convergent.

4. Numerical results

In this final section, we describe the numerical scheme implemented in the well-known finite element code FreeFem++ for solving Problem VP^{hk} (see [9] for details), and we show a numerical example to demonstrate the accuracy of the approximations and the decay of the discrete energy.

4.1. Numerical scheme

Given the solution $\boldsymbol{u}_{n-1}^{hk},\boldsymbol{v}_{n-1}^{hk},\theta_{n-1}^{hk}$ and ν_{n-1}^{hk} at time t_{n-1}, the velocity and the temperature are obtained by solving the following discrete nonsymmetric linear system, for all $\boldsymbol{w}^h\in V^h$ and $r^h\in E^h$,

$$
\begin{aligned}
&\rho(\boldsymbol{v}_n^{hk},\boldsymbol{w}^h)_H+k^2(\lambda+\mu)(\operatorname{div}\boldsymbol{v}_n^{hk},\operatorname{div}\boldsymbol{w}^h)_Y\\
&+k^2\mu(\nabla\boldsymbol{v}_n^{hk},\nabla\boldsymbol{w}^h)_Q-k\beta(\nabla\theta_n^{hk},\boldsymbol{w}^h)_H\\
=\;&\rho(\boldsymbol{v}_{n-1}^{hk},\boldsymbol{w}^h)_H-k(\lambda+\mu)(\operatorname{div}\boldsymbol{u}_{n-1}^{hk},\operatorname{div}\boldsymbol{w}^h)_Y\\
&-k\mu(\nabla\boldsymbol{u}_{n-1}^{hk},\nabla\boldsymbol{w}^h)_Q+k(\boldsymbol{H}_n,\boldsymbol{w}^h)_H,\\
&c(\theta_n^{hk},r^h)_Y+\frac{\tau^2}{2}k^*(\nabla\theta_n^{hk},\nabla r^h)_H+k^2k^*(\nabla\theta_n^{hk},\nabla r^h)_H\\
&+k(k_1+\tau k^*)(\nabla\theta_n^{hk},\nabla r^h)_H-\beta(\operatorname{div}\boldsymbol{v}_n^{hk},r^h)_Y
\end{aligned}
$$

$$= c(\theta_{n-1}^{hk}, r^h)_Y + \frac{\tau^2}{2} k^* (\nabla \theta_{n-1}^{hk}, \nabla r^h)_H - k k^* (\nabla \nu_{n-1}^{hk}, \nabla r^h)_H \\ + k(P_n, r^h)_Y,$$

where the discrete displacements and the discrete thermal displacement are then recovered from the relations

$$\boldsymbol{u}_n^{hk} = k \boldsymbol{v}_n^{hk} + \boldsymbol{u}_{n-1}^{hk}, \quad \nu_n^{hk} = k\theta_n^{hk} + \nu_{n-1}^{hk}.$$

Here, for the sake of generality we added body forces $\boldsymbol{H}$ and a heat supply P.

4.2. Numerical example: convergence and energy decay

We will consider the following academic problem:

Problem P^{ex}. *Find the displacements $\boldsymbol{u} : [0,1] \times [0,1] \times [0,1] \to \mathbb{R}^2$ and the temperature $\theta : [0,1] \times [0,1] \times [0,1] \to \mathbb{R}$ such that*

$$\begin{aligned}
&\ddot{u}_i - u_{i,jj} - 2u_{j,ji} - \theta_{,i} = H_i \ \textit{in}\ [0,1] \times [0,1] \times [0,1], \\
&\dot{\theta} - \frac{1}{2}\dot{\theta}_{,ii} - 2\theta_{,ii} - \nu_{,ii} - \beta \dot{u}_{i,i} = P \ \textit{in}\ [0,1] \times [0,1] \times [0,1], \\
&u_i(x,y,t) = \theta(x,y,t) = 0 \quad \textit{for} \quad i = 1,2 \\
&\qquad \textit{and}\ (x,y,t) \in \partial([0,1] \times [0,1]) \times [0,1], \\
&u_i(x,y,0) = (xy(1-x)(1-y), xy(1-x)(1-y)) \\
&\qquad \textit{for} \quad (x,y) \in [0,1] \times [0,1], \\
&\dot{u}_i(x,y,0) = (xy(1-x)(1-y), xy(1-x)(1-y)) \\
&\qquad \textit{for} \quad (x,y) \in [0,1] \times [0,1], \\
&\theta(x,y,0) = xy(1-x)(1-y) \quad \textit{for} \quad (x,y) \in [0,1] \times [0,1], \\
&\nu(x,y,0) = xy(1-x)(1-y) \quad \textit{for} \quad (x,y) \in [0,1] \times [0,1],
\end{aligned}$$

where the body forces $\boldsymbol{H}$ and the heat supply P are given by

$$\begin{aligned}
\boldsymbol{H}(x,y,t) &= e^t(x^2y^2 - x^2y - 2x^2 - 3xy^2 - 5xy + 6x - 5y^2 + 9y - 2, \\
&\qquad x^2y^2 - 3x^2y - 5x^2 - xy^2 - 5xy + 9x - 2y^2 + 6y - 2), \\
P(x,y,t) &= e^t\left(x^2y^2 - 3x^2y - 2x^2 - 3xy^2 + 5xy + 2x - 2y^2 + 2y\right).
\end{aligned}$$

We note that Problem P^{ex} corresponds to Problem P with the following data:

$$\begin{aligned}
&\Omega = (0,1) \times (0,1), \quad T = 1, \quad \rho = 1, \quad \mu = \lambda = 1, \quad \beta = 1, \\
&\tau = 1, \quad c = k^* = k_1 = 1,
\end{aligned}$$

and the initial conditions, for $(x,y) \in (0,1) \times (0,1)$,

$$\begin{aligned}
&\boldsymbol{u}^0(x,y) = \boldsymbol{v}^0(x,y) = (xy(1-x)(1-y), xy(1-x)(1-y)), \\
&\theta^0(x,y) = \nu^0(x,y) = xy(1-x)(1-y).
\end{aligned}$$

Table 1. Numerical errors (×10) for some nd and k.

$nd \downarrow k \rightarrow$	0.02	0.01	0.005	0.001
16	0.125254	0.088769	0.076804	0.082744
32	0.080294	0.045405	0.028722	0.019348
64	0.070355	0.035439	0.018915	0.006389
128	0.070268	0.033910	0.016746	0.003876
256	0.070462	0.034058	0.016683	0.003329

The exact solution to Problem P^{ex} is the following one, for $(x, y, t) \in [0,1] \times [0,1] \times [0,1]$ and $i = 1, 2$,

$$u_i(x, y, t) = \theta(x, y, t) = xy(1-x)(1-y)e^t.$$

Our aim here is to show the numerical convergence of the finite element scheme. Therefore, several uniform partitions for the time interval and the domain, dividing $\Omega = [0,1] \times [0,1]$ into $2(nd)^2$ triangles, have been performed. Note that the number of degrees of freedom is $3(nd+1)^2$.

In Table 1 the numerical errors given by

$$\max_{0 \le n \le N} \Big\{ \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + \|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Q^2 + \|\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y^2 \\ + \|\theta_n - \theta_n^{hk}\|_Y^2 + \|\nabla(\theta_n - \theta_n^{hk})\|_H^2 + \|\nabla(\nu_n - \nu_n^{hk})\|_H^2 \Big\},$$

and obtained for some discretization parameters nd and k, are shown, and the convergence of the numerical scheme clearly observed. The evolution of the error with respect to the parameter $h + k$ is plotted in Figure 1.

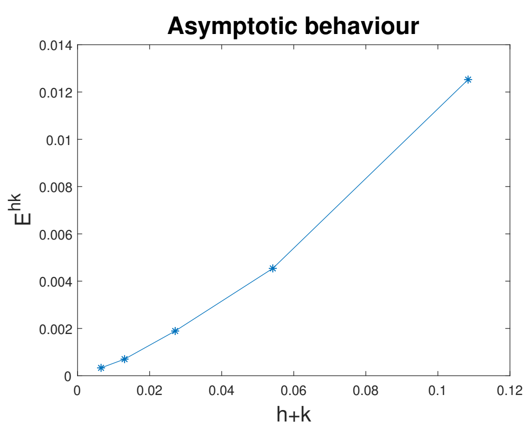

Figure 1. Asymptotic behaviour of the numerical scheme.

If we assume now that there are not volume forces, and we use the final time $T = 100$ s, and the same data and initial conditions than in the previous simulations, taking the discretization parameters $h = 0.01$ and $k = 0.001$, the evolution in time of the discrete energy E_n^{hk}, defined by (see [22])

$$E_n^{hk} = \rho\|\boldsymbol{v}_n^{hk}\|_H^2 + \mu\|\nabla \boldsymbol{u}_n^{hk}\|_Q^2 + (\lambda+\mu)\|\mathrm{div}\, \boldsymbol{u}_n^{hk}\|_Y^2 + c\|\theta_n^{hk}\|_Y^2 + \frac{\tau^2}{2}k^*\|\nabla\theta_n^{hk}\|_H^2 + k^*\|\nabla\nu_n^{hk}\|_H^2$$

is plotted in Figure 2 in both natural and semi-log scales. We observe that an exponential energy decay has been achieved. Anyway, we note that in [22] it has been proved that such behaviour is not found in the continuous case.

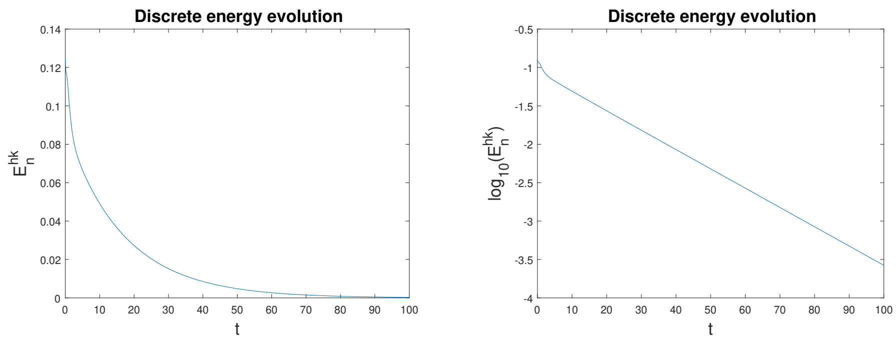

Figure 2. Discrete energy evolution in natural and semi-log scales.

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References

[1] M. Campo, J. R. Fernández, K. L. Kuttler, M. Shillor and J. M. Viaño, *Numerical analysis and simulations of a dynamic frictionless contact problem with damage*, Comput. Methods Appl. Mech. Engrg., 2006, 196(1–3), 476–488.

[2] D. Chandrasekharaiah, *Hyperbolic thermoelasticity: A review of recent literature*, ASME. Appl. Mech. Rev., 1998, 51(12), 705–729.

[3] S. K. R. Choudhuri, *On a thermoelastic three-phase-lag model*, J. Thermal Stresses, 2007, 30, 231–238.

[4] P. G. Ciarlet, *Basic error estimates for elliptic problems. In: Handbook of Numerical Analysis*, P.G. Ciarlet and J.L. Lions eds., 1993, vol I, 17–351.

[5] M. Dreher, R. Quintanilla and R. Racke, *Ill-posed problems in thermomechanics*, Appl. Math. Letters, 2009, 22, 1374–1379.

[6] M. Fabrizio and F. Franchi, *Delayed thermal models: stability and thermodynamics*, J. Thermal Stresses, 2014, 37, 160–173.

[7] A. E. Green and P. M. Naghdi, *On undamped heat waves in an elastic solid*, J. Thermal Stresses, 1992, 15, 253–264.

[8] A. E. Green and P. M. Naghdi, *Thermoelasticity without energy dissipation*, J. Elasticity, 1993, 31, 189–208.

[9] F. Hecht, *New development in FreeFem++*, J. Numer. Math., 2012, 20(3–4), 251–265.

[10] R. B. Hetnarski and J. Ignaczak, *Generalized thermoelasticity*, J. Thermal Stresses, 1999, 22, 451–470.

[11] R. B. Hetnarski and J. Ignaczak, *Nonclassical dynamical thermoelasticity*, Int. J. Solids Struct. 2000, 37, 215–224.

[12] J. Ignaczak and M. Ostoja-Starzewski, *Thermoelasticity with Finite Wave Speeds*, Oxford Mathematical Monographs, Oxford, 2010.

[13] S. Kant, M. Gupta, O. N. Shivay and S. Mukhopadhyay, *An investigation on a two-dimensional problem of Mode-I crack in a thermoelastic medium*, Z. Ang. Math. Phys., 2018, 69, 21.

[14] A. Kumar and S. Mukhopadhyay, *Investigation on the effects of temperature dependency of material parameters on a thermoelastic loading problem*, Z. Ang. Math. Phys., 2017, 68, 98.

[15] B. Kumari, A. Kumar and S. Mukhopadhyay, *Investigation of harmonic plane waves: detailed analysis of recent thermoplastic model with single delay term*, Math. Mech. Solids, 2018. DOI: 10.1177/1081286518755232.

[16] B. Kumari and S. Mukhopadhyay, *Fundamental solutions of thermoelasticity with a recent heat conduction model with a single delay term*, J. Thermal Stresses, 2017, 40, 866–878.

[17] B. Kumari, and S. Mukhopadhyay, *Some theorems on linear theory of thermoelasticity for an anisotropic medium under an exact heat conduction model with a delay*, Math. Mech. Solids, 2017, 22, 1177–1189.

[18] M. C. Leseduarte and R. Quintanilla, *Phragmen-Lindelof alternative for an exact heat conduction equation with delay*, Comm. Pure Appl. Anal., 2013, 12, 1221–1235.

[19] R. Quintanilla, *A well-posed problem for the Dual-Phase-Lag heat conduction*, J. Thermal Stresses, 2008, 31, 260–269.

[20] R. Quintanilla, *A well-posed problem for the Three-Dual-Phase-Lag heat conduction*, J. Thermal Stresses, 2009, 32, 1270–1278.

[21] R. Quintanilla, *Some solutions for a family of exact phase-lag heat conduction problems*, Mech. Res. Commun., 2011, 38, 355–360.

[22] R. Quintanilla, *On uniqueness and stability for a thermoelastic theory*, Math. Mech. Solids, 2017, 22(6), 1387–396.

[23] O. N. Shivay and S. Mukhopadhyay, *Some basic theorems on a recent model of linear thermoelasticity for a homogeneous and isotropic medium*, Math. Mech. Solids, 2018. DOI: 10.1177/1081286518762612.

[24] D. Y. Tzou, *A unified approach for heat conduction from macro to micro-scales*, ASME J. Heat Transfer, 1995, 117, 8–16.

[25] D. Y. Tzou, *The generalized lagging response in small-scale and high-rate heating*, Int. J. Heat Mass Transfer, 1995, 38, 3231–3240.

[26] L. Wang, X. Zhou and X. Wei, *Heat Conduction: Mathematical Models and Analyticial solutions*, Springer, 2008.