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# MULTIDIMENSIONAL STABILITY OF PLANAR WAVES FOR DELAYED REACTION-DIFFUSION EQUATION WITH NONLOCAL DIFFUSION*

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**Abstract** In this paper, we consider the multidimensional stability of planar waves for a class of nonlocal dispersal equation in $n$–dimensional space with time delay. We prove that all noncritical planar waves are exponentially stable in $L^\infty(\mathbb{R}^n)$ in the form of $\mathrm{e}^{-\mu_\tau t}$ for some constant $\mu_\tau = \mu(\tau) > 0$( $\tau > 0$ is the time delay) by using comparison principle and Fourier transform. It is also realized that, the effect of time delay essentially causes the decay rate of the solution slowly down. While, for the critical planar waves, we prove that they are asymptotically stable by establishing some estimates in weighted $L^1(\mathbb{R}^n)$ space and $H^k(\mathbb{R}^n)(k \geq [\frac{n+1}{2}])$ space.

**Keywords** Multidimensional stability, planar waves, nonlocal diffusion, weighted energy, Fourier transform.

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## 1. Introduction

The theory of traveling wave solutions of reaction-diffusion equations has attracted much attention due to its significant nature in biology, chemistry, epidemiology and physics (see, [3,7,11,21,23,26] ). Among the basic problems in the theory of traveling wave solutions, the stability of traveling wave solutions is an extremely important one. Recently, a great interest has been drawn to the study of the multidimensional stability of traveling wave solutions. Levermore and Xin [11] first considered the following bistable reaction-diffusion equation,

$$u_t(x,t) = \Delta u(x,t) + f(u(x,t)), \quad x \in \mathbb{R}^n, t > 0, \tag{1.1}$$

where $f(u) = u(1-u)(u-\theta)$ for some $\theta \in (0, 1/2)$. They proved that the planar traveling wave solutions of (1.1) are stable in $L^2_{loc}(\mathbb{R}^n)$ with the small initial perturbation by using the maximum principle and energy methods. Xin [25] investigated the multidimensional stability of planar traveling wave solutions of (1.1) via an application of linear semigroup theory. He showed that if the perturbation of a planar traveling wave solution is small enough in $H^m(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)(m \geq n+1, n \geq 4)$,

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then the solution of the initial value problem converges to the planar wave solution in $H^m(\mathbb{R}^n)$ as $t$ goes to infinity with rate $O(t^{-\frac{n-1}{4}})$. Matano et al. [16] obtained that planar waves of (1.1) are asymptotically stable under almost periodic perturbation or under any possibly large initial perturbations which decay at space infinity. Furthermore, they also found a special solution that oscillates permanently between two planar waves, which implies that planar waves are not asymptotically stable under more general perturbations. We can study more works of the multidimensional stability of traveling waves by referring to [1, 2, 9, 15, 16, 22, 24, 27] and references therein for more details.

Mei and Wang [20] considered the following Fisher-KPP type reaction-diffusion equation

$$\frac{\partial u(x,t)}{\partial t} = D\Delta u(x,t) - d(u(x,t)) + \int_{\mathbb{R}^n} f_\alpha(y) b(u(x-y,t-\tau))\mathrm{d}y, \tag{1.2}$$

where $D > 0$ denotes the diffusion rate and $d(u), b(u)$ are nonnegative nonlinear functions. They obtained that all noncritical planar traveling waves are exponentially stable and critical planar traveling waves are algebraically stable in the form $t^{-\frac{n}{2}}$ by using weighted energy method and Fourier transform. Huang et al. [8] extended the results in [20] to the nonlocal diffusion equations.

Very recently, Faye [6] studied the multidimensional stability of planar waves of the following nonlocal Allen-Cahn equations by using semigroup estimates,

$$u_t(x,t) = \int_{\mathbb{R}^n} J(x-y)u(y,t)\mathrm{d}y - u(x,t) + f(u(x,t)), \quad x \in \mathbb{R}^n, t > 0, \tag{1.3}$$

where $J(x)$ is the kernel function and $f$ is a smooth function with bistable type. They showed that if the traveling wave is spectrally stable in one-dimensional space, then it is stable in $n$–dimensional space under some special perturbations of planar waves.

Motivated above, in this paper, we consider the multidimensional stability of planar waves for the following class of nonlocal diffusion equation with nonlocal time-delayed response term

$$\frac{\partial u(x,t)}{\partial t} = d\int_{\mathbb{R}^n} J(y)(u(x-y,t)-u(x,t))\mathrm{d}y + f(u(x,t), k*u(x,t-\tau)), x \in \mathbb{R}^n, t \geq 0, \tag{1.4}$$

with the initial data

$$u(x,s) = u_0(x,s), \quad x \in \mathbb{R}^n, s \in [-\tau, 0], \tag{1.5}$$

where

$$k * u(x,t-\tau) = \int_{\mathbb{R}^n} k(y)u(x-y,t-\tau)\mathrm{d}y.$$

Here, we give the following assumptions throughout this paper.

**(H1)** There exist $u_- = 0$ and $u_+ > 0$ such that $f(0,0) = f(u_+,u_+) = 0$, $f \in C^2([0,u_+]^2, \mathbb{R})$, $f(u,u) > 0$ for all $u \in (0,u_+)$;

**(H2)** $\partial_1 f(0,0) \leq 0$ and $\partial_1 f(u_+,u_+) + \partial_2 f(u_+,u_+) < 0$;

**(H3)** $\partial_2 f(u,v) \geq 0, \partial_{11} f(0,0) < 0$ and $\partial_{ij} f(u,v) \leq 0 (i,j = 1,2)$ for all $(u,v) \in [0,u_+]^2$;

**(H4)** $J \geq 0, J(x) = J(-x), \int_{\mathbb{R}^n} J(x)\mathrm{d}x = 1$, and $\int_{\mathbb{R}^n} J(x)\mathrm{e}^{-\lambda x}\mathrm{d}x < +\infty, \forall \lambda \geq 0$;

**(H5)** $k \geq 0, k(x) = k(-x), \int_{\mathbb{R}^n} k(x)\mathrm{d}x = 1$, and $\int_{\mathbb{R}^n} k(x)\mathrm{e}^{-\lambda x}\mathrm{d}x < +\infty, \forall \lambda \geq 0$.

From (H1), it can be verified that both $u_- = 0$ and $u_+ > 0$ are constant equilibria of (1.4). In the biological environment, the kernel functions $J(x)$ and $k(x)$ can be chosen in the form of $J(x) = k(x) = \frac{1}{\sqrt{4\pi\alpha}}\mathrm{e}^{-\|x\|^2/4\alpha}$. It is not difficult to see that the functions $J(x)$ and $k(x)$ satisfy assumptions (H4) and (H5). Moreover, the response term $f(u, k * u)$ can be chosen in the form of

$$f(u, k * u) = -d(u(x,t)) + \int_{\mathbb{R}^n} k(x)b(u(x-y, t-\tau))\mathrm{d}y,$$

where $d(u) = -\delta u^2$ is the death rate, and $b(u)$ can be chosen in the so-called Nicholson's birth rate function $b(u) = pu\mathrm{e}^{-au^q}(p > 0, a > 0, q > 0)$. It is easy to see that the function $f$ satisfies assumptions (H1)–(H3). So, (1.4) with the assumptions (H1)–(H5) includes many models.

A planar wave of (1.4) is a special solution in the form of $u(x,t) = \phi(\nu \cdot x + ct)$ (where $\nu \in \mathbb{R}^n$ is a fixed unit vector) with $\phi(\pm\infty) = u_\pm$, where $c$ is the wave speed. In one-dimensional space, the existence, uniqueness and stability of traveling waves of (1.4) have been discussed in [12, 26]. To the best of our knowledge, there is no any results for the multidimensional stability of planar waves of (1.4). The main purpose of this paper is to investigate the multidimensional stability of the planar wave $\phi(\nu \cdot x + ct)$ of (1.4), including the case of the critical planar wave $\phi(\nu \cdot x + c^* t)$. In one-dimensional space, lots of investigations has been done concerning the stability of traveling waves by using the spectral analysis method, the squeezing technique, the weighted energy method. We can refer to [3, 4, 12, 17, 18, 21, 23] and the references therein for more results on the study of the stability of traveling waves for monostable equations in one-dimensional space.

In this paper, we will prove that all noncritical planar waves are exponentially stability by using the comparison principle and Fourier transform. But for the critical planar waves, the expecting optimal convergence rate $O(t^{-\frac{n}{2}})$ is unable to estimate at this moment mainly because of the effect of the nonlocal diffusion, which has the essential difference with the classical Laplacian operator (see [20] for more details). Here, we not only obtain the $H^k$-estimate for $v(\xi, t)$ through the $L^1$-energy estimate, but also obtain the estimate for $\frac{\partial v(\xi,t)}{\partial t}$(see Lemmas 4.2 and 4.3 for more details). Fortunately, we finally obtain the asymptotic stability of the critical planar waves. But the optimal convergence rate result to the critical planar waves with the effect of the nonlocal diffusion is still an open question.

The rest of this paper is organized as follows. In section 2, we introduce some necessary notations and present main result. In section 3, we main prove the global multidimensional stability of the noncritical planar waves. In section 4, asymptotic stability of the critical planar waves is obtained by some energy estimates in weighted $L^1$ space and $H^k$ space.

## 2. Preliminaries and main results

First, we introduce some necessary notations throughout this paper. $C > 0$ denotes a generic constant and $C_i (i = 1, 2, \cdots)$ represents a specific constant. Let $\Omega$ be a domain, typically $\Omega = \mathbb{R}^n$ and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ be a multi-index with nonnegative integers $\alpha_i \geq 0 (i = 1, 2, \cdots, n)$. The derivatives for function $f(x)$ are denoted as $\partial^\alpha f(x) = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} f(x)$. $W^{k,p}(\Omega)(k \geq 0, p \geq 1)$ is the Sobolev space in which the function $f(x)$ is defined on $\Omega$ and its weak derivatives $\partial^\alpha f(x)(|\alpha| \leq k)$ also belong to $L^p(\Omega)$, and in particular, we denote $W^{k,2}(\Omega)$ as $H^k(\Omega)$. Further, $L^p_w(\Omega)$ denotes the weighted $L^p(\Omega)$ space for a weighted function $w(x) > 0$. Its norm is defined by

$$\|f\|_{L^p_w(\Omega)} = \left( \int_\Omega w(x) |f(x)|^p \mathrm{d}x \right)^{1/p}.$$

$W^{k,p}_w(\Omega)$ is the weighted Sobolev space with the norm

$$\|f\|_{W^{k,p}_w(\Omega)} = \left( \sum_{|\alpha| \leq k} \int_\Omega w(x) \left| \partial^\alpha f(x) \right|^p \mathrm{d}x \right)^{1/p}.$$

Fourier transform is defined as

$$\mathcal{F}[f](\eta) = \hat{f}(\eta) := \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}x\cdot\eta} f(x) \mathrm{d}x,$$

and the inverse Fourier transform is given by

$$\mathcal{F}^{-1}[\hat{f}](x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot\eta} \hat{f}(\eta) \mathrm{d}\eta.$$

If we look for a planar wave solution $u(x,t) = \phi(\zeta)$ (where $\zeta = \nu \cdot x + ct, \nu \in \mathbb{R}^n$ is a fixed unit vector, here we set $\nu = e_1 = (1, 0, \cdots, 0)$ for simplicity) of the equation (1.4), then $\phi$ has to satisfy the following nonlinear nonlocal equation on the line

$$c\phi'(\zeta) = d \int_{-\infty}^{+\infty} \tilde{J}(y)(\phi(\zeta - y) - \phi(\zeta)) \mathrm{d}y + f(\phi(\zeta), \tilde{k} * \phi(\zeta - c\tau)),\ \zeta \in \mathbb{R}, \quad (2.1)$$

where

$$\tilde{J}(y) = \int_{\mathbb{R}^{n-1}} J(y, x_2, \cdots, x_n) \mathrm{d}x_2 \cdots \mathrm{d}x_n,$$

and

$$\tilde{k}(y) = \int_{\mathbb{R}^{n-1}} k(y, x_2, \cdots, x_n) \mathrm{d}x_2 \cdots \mathrm{d}x_n.$$

To obtain the existence and stability of planar wave solutions, we consider the following function

$$\Delta(\lambda, c) = d \int_{\mathbb{R}^n} J(x)(\mathrm{e}^{-\lambda x_1} - 1) \mathrm{d}x - c\lambda + \partial_1 f(0,0) + \partial_2 f(0,0) \int_{\mathbb{R}^n} k(y) \mathrm{e}^{-\lambda(y_1 + c\tau)} \mathrm{d}y.$$

By the properties of function $\Delta(\lambda, c)$ (see [13, Lemma 2.2] for more details), we have the following lemma.

**Lemma 2.1.** *Under the conditions (H1)-(H5), there exist $\lambda^* > 0$ and $c^* > 0$ such that*

$$\Delta(\lambda^*, c^*) = 0, \frac{\partial \Delta(\lambda, c)}{\partial \lambda}\Big|_{(\lambda^*, c^*)} = 0.$$

*Furthermore,*

- *if* $0 < c < c^*$, *we have* $\Delta(\lambda, c) > 0$ *for all* $\lambda > 0$;
- *if* $c > c^*$, *the equation* $\Delta(\lambda, c) = 0$ *has two positive real roots* $\lambda_i = \lambda_i(c)(i = 1, 2)$ *with* $0 < \lambda_1 < \lambda^* < \lambda_2 < +\infty$, *and*

$$\Delta(\lambda, c) \begin{cases} < 0, \ \lambda \in (\lambda_1, \lambda_2), \\ > 0, \ \lambda \in (0, \lambda_1) \cup (\lambda_2, +\infty). \end{cases}$$

The existence of planar waves of (1.4) is guaranteed by the following Theorem 2.1. In [26], Yu and Yuan proved the existence of solution of (2.1) by using the upper-lower solutions and Schauder's fixed point theorem.

**Theorem 2.1** (Existence of Planar Waves). *Assume that (H1)–(H5) hold and* $c \geq c^*$. *Then* (1.4) *admits a nondecreasing positive planar wave* $u(x, t) = \phi(x \cdot e_1 + ct)$ *satisfying* (2.1) *with* $\phi(\pm\infty) = u_\pm$.

By the properties of the monotone semiflow [5] or using the similar method in [12, Lemma 2.3], we have the following comparison principle.

**Lemma 2.2** (Comparison Principle). *Assume that* $u_1$ *and* $u_2$ *are continuous functions on* $\mathbb{R}^n \times [0, +\infty)$, *such that* $0 \leq u_i \leq u_+ (i = 1, 2)$ *on* $\mathbb{R}^n \times [0, +\infty)$ *and* $u_1 \geq u_2$ *on* $\mathbb{R}^n \times [-\tau, 0]$. *Furthermore,* $u_1$ *and* $u_2$ *satisfy*

$$\frac{\partial u_1(x, t)}{\partial t} - F[u_1](x, t) \geq \frac{\partial u_2(x, t)}{\partial t} - F[u_2](x, t), \quad x \in \mathbb{R}^n, t \geq 0, \tag{2.2}$$

*where*

$$F[u](x, t) = d \int_{\mathbb{R}^n} J(y)[u(x - y, t) - u(x, t)]\mathrm{d}y + f(u(x, t), k * u(x, t - \tau)).$$

*Then* $u_1 \geq u_2$ *on* $\mathbb{R}^n \times [0, +\infty)$.

Now let us introduce the solution formula for linear delayed ODEs which will be used in Section 3.

**Lemma 2.3** ( [10, 20]). *Let* $z(t)$ *be the solution to the following linear time-delayed ODE with time delay* $\tau > 0$,

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} z(t) + k_1 z(t) = k_2 z(t - \tau), \\ z(s) = z_0(s), \qquad s \in [-\tau, 0]. \end{cases}$$

*Then*

$$z(t) = \mathrm{e}^{-k_1(t+\tau)} \mathrm{e}_{\tau}^{\bar{k}_2 t} z_0(-\tau) + \int_{-\tau}^{0} \mathrm{e}^{-k_1(t-s)} \mathrm{e}_{\tau}^{\bar{k}_2(t-s-\tau)} [z_0'(s) + k_1 z_0(s)]\mathrm{d}s,$$

*where* $\bar{k}_2 = k_2 \mathrm{e}^{k_1\tau}$ *and* $\mathrm{e}_\tau^{\bar{k}_2 t}$ *is the so-called delayed exponential function in the form*

$$\mathrm{e}_\tau^{\bar{k}_2 t} = \begin{cases} 0, & -\infty < t < -\tau, \\ 1, & -\tau \le t < 0, \\ 1 + \bar{k}_2 t, & 0 \le t < \tau, \\ 1 + \bar{k}_2 t + \dfrac{\bar{k}_2^2}{2!}(t-\tau)^2, & \tau \le t < 2\tau, \\ \quad\vdots & \quad\vdots \\ 1 + \bar{k}_2 t + \dfrac{\bar{k}_2^2}{2!}(t-\tau)^2 + \cdots + \dfrac{\bar{k}_2^m}{m!}[t-(m-1)\tau]^m, & (m-1)\tau \le t < m\tau, \\ \quad\vdots & \quad\vdots \end{cases}$$

*Furthermore, if* $k_1 \ge k_2 \ge 0$*, there exists a constant* $\varepsilon = \varepsilon(\tau)$ *with* $\varepsilon \in (0,1)$ *for* $\tau > 0$*, and* $\lim_{\tau\to 0}\varepsilon(\tau) = 1, \lim_{\tau\to+\infty}\varepsilon(\tau) = 0$ *such that*

$$\mathrm{e}^{-k_1 t}\mathrm{e}_\tau^{\bar{k}_2 t} \le C\mathrm{e}^{-\varepsilon(k_1-k_2)t}, \qquad t > 0.$$

By the standard energy method and continuity extension method (see [19]) or the theory of abstract functional differential equations in [14], we can obtain the global existence and uniqueness of the solution for (1.4) and (1.5).

**Proposition 2.1** (Global Existence and Uniqueness)**.** *Assume that (H1)-(H5) hold. For any given planar wave* $\phi(x\cdot e_1 + ct)$ *of* (2.1) *with* $c \ge c^*$ *and* $\phi(\pm\infty) = u_\pm$*, if the initial data satisfies*

$$0 = u_- \le u_0(x,s) \le u_+, \quad (x,s) \in \mathbb{R}^n \times [-\tau, 0],$$

*and the initial perturbation* $u_0(\cdot,s) - \phi(\cdot + cs) \in C^1([-\tau,0], H_w^1(\mathbb{R}^n) \cap H^1(\mathbb{R}^n))$*, where the weighted function* $w(\zeta)$ *is defined as*

$$w(\zeta) = \begin{cases} \mathrm{e}^{-\lambda^*(\zeta-\zeta_0)}, & \zeta \le \zeta_0, \\ 1, & \zeta > \zeta_0, \end{cases}$$

*where* $\zeta_0$ *is a very large constant and* $\lambda^*$ *is given in Lemma 2.1, then the solution* $u(x,t)$ *of* (1.4) *and* (1.5) *satisfies*

$$0 = u_- \le u(x,t) \le u_+, \quad \forall (x,t) \in \mathbb{R}_+^n \times \mathbb{R},$$
$$u(\cdot,t) - \phi(\cdot + ct) \in C^1([0,+\infty), H_w^1(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)).$$

Now, we present the main results of this paper.

**Theorem 2.2** (Exponential Stability of Noncritical Planar waves)**.** *If the conditions in Proposition 2.1 are satisfied and* $\mathcal{F}(u_0 - \phi) \in C^1([-\tau,0]; W^{1,1}(\mathbb{R}^n))$*, then for any* $c > c^*$*, there exists a positive number* $\mu = \mu(\tau)$ *such that the solution* $u(x,t)$ *converges to the noncritical planar wave* $\phi(x\cdot e_1 + ct)$ *exponentially*

$$\sup_{x\in\mathbb{R}^n} |u(x,t) - \phi(x\cdot e_1 + ct)| \le C\mathrm{e}^{-\mu t}, \quad t > 0.$$

**Remark 2.1.** Theorem 2.2 not only shows the convergence rate $\mu$ to the noncritical planar waves, but also tell us how the time delay $\tau$ effects the convergence rate $\mu$ from the proof of the Theorem 2.2. In fact, from the proof of the Lemmas 3.1 and 3.2, we have

$$0 < \mu < \min\{-\varepsilon(\tau)\Delta(\lambda^*, c), \delta\},$$

where $\delta = -(\partial_1 f(u_+, u_+) + \partial_2 f(u_+, u_+)) > 0$, $\varepsilon(\tau) \to 0$ as $\tau \to +\infty$ and $\varepsilon(\tau) \to 1$ as $\tau \to 0$. The effect of the time delay $\tau$ will make the decay rate $\mu$ of the solution slow down. That is, $\mu$ becomes the smallest 0 as $\tau \to +\infty$ and $\mu$ tends to biggest as $\tau \to 0$.

**Remark 2.2.** To overcome the effect of the nonlocal diffusion, we make the condition $\mathcal{F}(u_0 - \phi) \in C^1([-\tau, 0]; W^{1,1}(\mathbb{R}^n))$. However, this condition holds easily. In fact, we can choose $u_0$ in the form $u_0(x, s) = \phi(x \cdot e_1 + cs) + \varepsilon(x)$, where $\varepsilon(x) = \mathrm{e}^{-\|x\|}$ or $\mathrm{e}^{-\|x\|^2}$.

**Theorem 2.3** (Asymptotic Stability of Critical Planar Waves)**.** *Let $f \in C^{k,k}([0, u_+]^2, \mathbb{R})(k \geq [\frac{n+1}{2}])$ and the conditions in Proposition 2.1 be satisfied. Then, the solution $u(x, t)$ converges to the critical planar wave $\phi(x \cdot e_1 + c^* t)$ time–asymptotically,*

$$\lim_{t \to +\infty} \sup_{x \in \mathbb{R}^n} |u(x, t) - \phi(x \cdot e_1 + c^* t)| = 0.$$

**Remark 2.3.** From the proof the Theorem 2.2, we can obtain that the decay rate $\mu \to 0$ as $c \searrow c^*$, which implies that the method of proving the stability of noncritical planar waves is invalid for the critical ones. Thus, we first establish some energy estimates in weighted $L^1$ space, then build up the energy estimates in $H^k$, and further obtain the time-asymptotically stability for critical planar waves.

## 3. Nonlinear stability of noncritical planar waves

In this section, we mainly concentrate on proving the stability for all noncritical planar waves to (1.4) with an exponential convergence rate. Assume that the conditions in Theorem 2.2 hold throughout this section.

Let $c \geq c^*$ and define

$$\begin{cases} \widetilde{U}_0^+(x, s) = \max\{u_0(x, s), \phi(x \cdot e_1 + cs)\}, \\ \widetilde{U}_0^-(x, s) = \min\{u_0(x, s), \phi(x \cdot e_1 + cs)\}, \end{cases} \quad (x, s) \in \mathbb{R}^n \times [-\tau, 0],$$

which implies

$$\begin{aligned} &0 = u_- \leq \widetilde{U}_0^-(x, s) \leq u_0(x, s) \leq \widetilde{U}_0^+(x, s) \leq u_+, \\ &0 = u_- \leq \widetilde{U}_0^-(x, s) \leq \phi(x \cdot e_1 + cs) \leq \widetilde{U}_0^+(x, s) \leq u_+, \end{aligned} \quad (x, s) \in \mathbb{R}^n \times [-\tau, 0].$$

Clearly, the initial data $\widetilde{U}_0^{\pm}(x, s)$ are piecewise continuous and have a poor regularity, which may also cause the absence of regularity for the corresponding solutions. In order to overcome such a shortcoming, instead of these initial data, we choose smooth functions $U_0^{\pm}(x, s)$ as the new initial data and $U_0^{\pm}(x, s)$ satisfy

$$0 = u_- \leq U_0^-(x, s) \leq \widetilde{U}_0^-(x, s) \leq u_0(x, s) \leq \widetilde{U}_0^+(x, s) \leq U_0^+(x, s) \leq u_+, \tag{3.1}$$

for $(x,s) \in \mathbb{R}^n \times [-\tau, 0]$.

Let $U^{\pm}(x,t)$ be the corresponding solution of (1.4) with the initial data $U_0^{\pm}(x,s)$, that is

$$\begin{cases} \dfrac{\partial U^{\pm}}{\partial t} = D[J * U^{\pm} - U^{\pm}] + f(U^{\pm}, k * U^{\pm}(x, t-\tau)), \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}_+, \\ U^{\pm}(x,s) = U_0^{\pm}(x,s), \quad (x,s) \in \mathbb{R}^n \times [-\tau, 0]. \end{cases} \tag{3.2}$$

By the comparison principle in Lemma 2.2, we have

$$u_- \le U^-(x,t) \le u(x,t) \le U^+(x,t) \le u_+, \tag{3.3}$$

and

$$u_- \le U^-(x,t) \le \phi(x \cdot e_1 + ct) \le U^+(x,t) \le u_+, \tag{3.4}$$

for $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$.

In order to prove the stability of the noncritical planar waves, we only need to prove that $U^{\pm}(x,t)$ converge to $\phi(x \cdot e_1 + ct)$, respectively. Since the two proofs are similar in each case, here, we only prove the convergence of $U^+(x,t)$ to $\phi(x \cdot e_1 + ct)$.

For any given $c \ge c^*$, let $\xi = x + ct \cdot e_1 = (x_1 + ct, x_2, \cdots, x_n)$ and

$$v(\xi,t) := U^+(x,t) - \phi(e_1 \cdot x + ct), \quad v_0(\xi,s) := U_0^+(x,s) - \phi(e_1 \cdot x + cs). \tag{3.5}$$

It follows from (3.3) and (3.4) that $v(\xi,t) \ge 0$ and $v_0(\xi,s) \ge 0$. Then,

$$v_t + cv_{\xi_1} = d[J * v - v] + f\left(v + \phi, k * (v_\tau + \phi_\tau)\right) - f(\phi, k * \phi_\tau), \tag{3.6}$$

where $v_\tau = v(\xi - c\tau \cdot e_1, t - \tau)$ and $\phi_\tau = \phi(\xi_1 - c\tau)$. Furthermore, (3.6) can be rewritten as

$$v_t + cv_{\xi_1} = d[J * v - v] + \partial_1 f(0,0) v + \partial_2 f(0,0) k * v_\tau + Q, \tag{3.7}$$

where

$$Q = Q(\xi,t) = f(v + \phi, k * (v_\tau + \phi_\tau)) - f(\phi, k * \phi_\tau) - \partial_1 f(0,0) v - \partial_2 f(0,0) k * v_\tau.$$

By the assumptions (H1)-(H3), we can obtain $Q(\xi,t) \le \partial_{11} f(0,0) v^2 \le 0$ for the nonnegativity of $k, \phi$ and $v$.

Let $v^+(\xi,t)$ be the solution of the following Cauchy problem

$$\begin{cases} v_t + cv_{\xi_1} = d[J * v - v] + \partial_1 f(0,0) v + \partial_2 f(0,0) k * v_\tau, \quad (\xi,t) \in \mathbb{R}^n \times \mathbb{R}_+, \\ v(\xi,s) = v_0(\xi,s), \quad (\xi,s) \in \mathbb{R}^n \times [-\tau, 0]. \end{cases} \tag{3.8}$$

It follows from Lemma 2.2 that

$$0 \le v(\xi,t) \le v^+(\xi,t), \quad (\xi,t) \in \mathbb{R}^n \times \mathbb{R}_+. \tag{3.9}$$

Set $V(\xi,t) = \mathrm{e}^{-\lambda^*(\xi_1 - \zeta_0)} v^+(\xi,t)$, where $\zeta_0$ is a very large positive constant. By (3.8), we can obtain that $V(\xi,t)$ satisfy

$$V_t(\xi,t) + cV_{\xi_1}(\xi,t) = d[J_{\lambda^*} * V - V](\xi,t) + rV(\xi,t) + qk_{\lambda^*,\tau} * V(\xi, t-\tau), \tag{3.10}$$

where $(\xi,t)\in\mathbb{R}^n\times\mathbb{R}_+$, $r=\partial_1 f(0,0)-c\lambda^*$, $q=\partial_2 f(0,0)$, $J_{\lambda^*}(y)=J(y)\mathrm{e}^{-\lambda^* y_1}$, $k_{\lambda^*,\tau}=k(y)\mathrm{e}^{-\lambda^*(y_1+c\tau)}$ and

$$\begin{aligned}J_{\lambda^*}*V(\xi,t)&=\int_{\mathbb{R}^n}J(y)\mathrm{e}^{-\lambda^* y_1}V(\xi-y,t)\mathrm{d}y,\\ k_{\lambda^*,\tau}*V(\xi-c\tau\cdot e_1,t-\tau)&=\int_{\mathbb{R}^n}k(y)\mathrm{e}^{-\lambda^*(y_1+c\tau)}V(\xi-y-c\tau\cdot e_1,t-\tau)\mathrm{d}y.\end{aligned}$$

**Lemma 3.1.** *If the initial data $\hat V_0\in C^1([-\tau,0];W^{1,1}(\mathbb{R}^n))$, then there exists $\varepsilon=\varepsilon(\tau)\in(0,1)$ such that*

$$\|V(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}\le C\max_{s\in[-\tau,0]}\|\hat V_0(s)\|_{W^{1,1}(\mathbb{R}^n)}\mathrm{e}^{-\mu_1(c,\tau)t},\quad t>0,$$

*where $\mu_1(c,\tau)=-\varepsilon(\tau)\Delta(\lambda^*,c)>0$ for $c>c^*$.*

**Proof.** By taking Fourier transform to (3.10), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat V(\eta,t)+A(\eta)\hat V(\eta,t)=B(\eta)\hat V(\eta,t-\tau),\tag{3.11}$$

where

$$\begin{aligned}A(\eta)&=c\eta_1\mathrm{i}-r-d\left(\int_{\mathbb{R}^n}J(y)\mathrm{e}^{-\lambda^* y_1}\mathrm{e}^{-\mathrm{i}y\cdot\eta}\mathrm{d}y-1\right),\\ B(\eta)&=q\int_{\mathbb{R}^n}k(y)\mathrm{e}^{-\lambda^*(y_1+c\tau)}\mathrm{e}^{-\mathrm{i}(y+c\tau e_1)\cdot\eta}\mathrm{d}y\end{aligned}$$

and

$$\hat V(\eta,t)=\mathcal{F}[V](\eta,t)=\int_{\mathbb{R}^n}\mathrm{e}^{-\mathrm{i}\xi\cdot\eta}V(\xi,t)\mathrm{d}\xi.$$

By Lemma 2.3, the solution to (3.11) can be shown as

$$\begin{aligned}\hat V(\eta,t)=&\mathrm{e}^{-A(\eta)(t+\tau)}\mathrm{e}_\tau^{B_1(\eta)t}\hat V_0(\eta,-\tau)+\int_{-\tau}^0\mathrm{e}^{-A(\eta)(t-s)}\mathrm{e}_\tau^{B_1(\eta)(t-s-\tau)}[\hat V_0'(\eta,s)\\ &+A(\eta)\hat V_0(\eta,s)]\mathrm{d}s,\\ :=&I_1(\eta,t)+\int_{-\tau}^0 I_2(\eta,t-s)\mathrm{d}s,\end{aligned}\tag{3.12}$$

where $B_1(\eta)=B(\eta)\mathrm{e}^{A(\eta)\tau}$. Thus, by taking the inverse Fourier transform to (3.12), we have

$$\begin{aligned}V(\xi,t)=&\mathcal{F}^{-1}[I_1](\xi,t)+\int_{-\tau}^0\mathcal{F}^{-1}[I_2](\xi,t-s)\mathrm{d}s\\ =&\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}\xi\cdot\eta}\mathrm{e}^{-A(\eta)(t+\tau)}\mathrm{e}_\tau^{B_1(\eta)t}\hat V_0(\eta,-\tau)\mathrm{d}\eta\\ &+\int_{-\tau}^0\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}\xi\cdot\eta}\mathrm{e}^{-A(\eta)(t-s)}\mathrm{e}_\tau^{B_1(\eta)(t-s-\tau)}[\hat V_0'(\eta,s)+A(\eta)\hat V_0(\eta,s)]\mathrm{d}\eta\mathrm{d}s.\end{aligned}\tag{3.13}$$

Let

$$\alpha_1=\mathfrak{Re}A(\eta)=-d\left(\int_{\mathbb{R}^n}J(y)\mathrm{e}^{-\lambda^* y_1}\cos(y\cdot\eta)\mathrm{d}y-1\right)-r,$$

and

$$\alpha_2 = q \int_{\mathbb{R}^n} k(y)\mathrm{e}^{-\lambda^*(y_1+c\tau)}\mathrm{d}y.$$

Then, we have

$$|B_1(\eta)| = |B(\eta)\mathrm{e}^{A(\eta)\tau}| \le \alpha_2 \mathrm{e}^{\alpha_1 \tau} := \bar{\alpha}_2,$$

and

$$\begin{aligned}\alpha_2 - \alpha_1 =& d\left[\int_{\mathbb{R}^n} J(y)\mathrm{e}^{-\lambda^* y_1}\cos(y\cdot\eta)\mathrm{d}y - 1\right] + r + q\int_{\mathbb{R}} k(y)\mathrm{e}^{-\lambda^*(y_1+c\tau)}\mathrm{d}y \\ =& \Delta(\lambda^*, c) + \nu(\eta;\lambda^*) \le \Delta(\lambda^*, c) < 0, \quad c > c^*,\end{aligned}$$

where

$$\nu(\eta;\lambda^*) = d\int_{\mathbb{R}^n} J(y)\mathrm{e}^{-\lambda^* y_1}[\cos(y\cdot\eta) - 1]\mathrm{d}y.$$

Thus, by Lemma 2.3, there exists some constant $\varepsilon = \varepsilon(\tau) \in (0,1)$ such that

$$\lim_{\tau\to 0}\varepsilon(\tau) = 1, \lim_{\tau\to+\infty}\varepsilon(\tau) = 0,$$

and

$$\mathrm{e}^{-\alpha_1 t}\mathrm{e}_\tau^{\bar{\alpha}_2 t} \le C\mathrm{e}^{-\varepsilon(\alpha_1-\alpha_2)t}, \quad t > 0.$$

Then,

$$\begin{aligned}\|\mathcal{F}^{-1}[I_1](\cdot,t)\|_{L^\infty(\mathbb{R}^n)} =& \sup_{\xi\in\mathbb{R}^n}\left|\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}\xi\cdot\eta}\mathrm{e}^{-A(\eta)(t+\tau)}\mathrm{e}_\tau^{B_1(\eta)t}\hat{V}_0(\eta,-\tau)\mathrm{d}\eta\right| \\ \le& C\int_{\mathbb{R}^n}\left|\mathrm{e}^{-\alpha_1(t+\tau)}\mathrm{e}_\tau^{\bar{\alpha}_2 t}\hat{V}_0(\eta,-\tau)\right|\mathrm{d}\eta \\ \le& C\int_{\mathbb{R}^n}\left|\mathrm{e}^{-\varepsilon(\alpha_1-\alpha_2)t}\hat{V}_0(\eta,-\tau)\right|\mathrm{d}\eta \\ \le& C\mathrm{e}^{\varepsilon\Delta(\lambda^*,c)t}\|\hat{V}_0(\cdot,-\tau)\|_{L^1(\mathbb{R}^n)},\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}&\|\mathcal{F}^{-1}[I_2](\cdot,t-s)\|_{L^\infty(\mathbb{R}^n)} \\ =& \sup_{\xi\in\mathbb{R}^n}\left|\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}\xi\cdot\eta}\mathrm{e}^{-A(\eta)(t-s)}\mathrm{e}_\tau^{B_1(\eta)(t-s-\tau)}\left[\hat{V}_0'(\eta,s) + A(\eta)\hat{V}_0(\eta,s)\right]\mathrm{d}\eta\right| \\ \le& \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\mathrm{e}^{-\alpha_1(t-s)}\mathrm{e}_\tau^{\bar{\alpha}_2(t-s-\tau)}\left|\hat{V}_0'(\eta,s) + A(\eta)\hat{V}_0(\eta,s)\right|\mathrm{d}\eta \\ \le& C\mathrm{e}^{\varepsilon\Delta(\lambda^*,c)(t-s)}\|\hat{V}_0(\cdot,s)\|_{W^{1,1}(\mathbb{R}^n)}.\end{aligned} \tag{3.15}$$

Substituting (3.14) and (3.15) to (3.13), we have

$$\begin{aligned}&\|V(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \\ \le& C\mathrm{e}^{\varepsilon\Delta(\lambda^*,c)t}\|\hat{V}_0(\cdot,-\tau)\|_{L^1(\mathbb{R}^n)} + \int_{-\tau}^0 C\mathrm{e}^{\varepsilon\Delta(\lambda^*,c)(t-s)}\|\hat{V}_0(\cdot,s)\|_{W^{1,1}(\mathbb{R}^n)}\mathrm{d}s \\ \le& C\max_{s\in[-\tau,0]}\|\hat{V}_0(\cdot,s)\|_{W^{1,1}(\mathbb{R}^n)}\mathrm{e}^{-\mu_1(c,\tau)t},\end{aligned} \tag{3.16}$$

where $\mu_1(c,\tau) = -\varepsilon(\tau)\Delta(\lambda^*, c) > 0$ for $c > c^*$. □

From (3.9) and $e^{\lambda^*(\xi_1-\zeta_0)} \le 1, \xi_1 \in (-\infty, \zeta_0]$, we have

$$0 \le v(\xi,t) \le v^+(\xi,t) = e^{\lambda^*(\xi_1-\zeta_0)} V(\xi,t) \le V(\xi,t),$$

for $\xi \in \Omega_- := (-\infty, \zeta_0] \times \mathbb{R}^{n-1}, t > 0$. Thus,

$$\|v(\cdot,t)\|_{L^\infty(\Omega_-)} \le C e^{-\mu_1(c,\tau)t}, \tag{3.17}$$

for $t > 0$ and $c > c^*$.

Next, we prove the decay rate for $v(\xi,t)$ in $\Omega_+ := [\zeta_0, +\infty) \times \mathbb{R}^{n-1}$.

**Lemma 3.2.** *For any $c > c^*$, it holds that*

$$\|v(\cdot,t)\|_{L^\infty(\Omega_+)} \le C e^{-\mu t}, \quad t > 0,$$

*where $\mu > 0$ is a very small constant.*

**Proof.** (3.6) can be rewritten as

$$v_t + c v_{\xi_1} = d[J * v - v] + \partial_1 f(\phi, k * \phi_\tau) v + \partial_2 f(\phi, k * \phi_\tau) k * v_\tau + Q_1, \tag{3.18}$$

where $v_\tau = v(\xi - c\tau \cdot e_1, t - \tau), \phi_\tau = \phi(\xi_1 - c\tau)$ and

$$Q_1(\xi,t) = f(v+\phi, k*(v_\tau+\phi_\tau)) - f(\phi, k*\phi_\tau)v - \partial_1 f(\phi, k*\phi_\tau)v - \partial_2 f(\phi, k*\phi_\tau)k*v_\tau.$$

By using the mean value theorem and (H3), we can obtain $Q_1(\xi,t) \le 0$ for the nonnegativity of $k$ and $\phi, v$. Thus,

$$\begin{cases} v_t + c v_{\xi_1} \le d[J * v - v] + \partial_1 f(\phi, k * \phi_\tau) v + \partial_2 f(\phi, k * \phi_\tau) k * v_\tau, t > 0, \xi \in \Omega_+, \\ v(\xi, s) = v_0(\xi, s), \quad s \in [-\tau, 0], \xi \in \Omega_+, \\ v|_{\xi_1=\zeta_0} \le C e^{-\mu_1(c,\tau)t}, \quad t > 0, (\xi_2, \cdots, \xi_n) \in \mathbb{R}^{n-1}, \end{cases}$$

where $\mu(c) = -\varepsilon\Delta(\lambda^*, c)$.

Let $\delta = -(\partial_1 f(u_+, u_+) + \partial_2 f(u_+, u_+))$. Thus, we can choose $\mu \in (0, \min\{\mu_1(c,\tau), \delta\})$ small enough to guarantee that

$$\partial_1 f(u_+, u_+) + \partial_2 f(u_+, u_+) e^{\mu\tau} + \mu < 0.$$

Due to $\lim_{\xi \to +\infty} \phi(\xi) = u_+$, we can choose $\zeta_0$ large enough to ensure that

$$\partial_1 f(\phi(\xi_1), k * \phi(\xi_1 - c\tau)) + \partial_2 f(\phi(\xi_1), k * \phi(\xi_1 - c\tau)) e^{\mu\tau} + \mu < 0, \tag{3.19}$$

for $\xi_1 > \zeta_0$.

Let

$$\bar{V}(\xi,t) = C e^{-\mu t},$$

where $C > v_0(\xi, s)$ large enough to ensure that

$$v_0(\xi,s) \le \bar{V}(\xi,s), \quad \xi \in \mathbb{R}^n, s \in [-\tau, 0]. \tag{3.20}$$

By direct computation and (3.19), we can verify that $\bar{V}(\xi,t)$ is an upper solution in the form

$$\begin{cases}\bar{V}_t + c\bar{V}_{\xi_1} \geq d[J*\bar{V}-\bar{V}] + \partial_1 f(\phi, k*\phi_\tau)\bar{V} + \partial_2 f(\phi, k*\phi_\tau)k*\bar{V}_\tau, t>0, \xi\in\Omega_+,\\ \bar{V}(\xi,s)\geq v_0(\xi,s), \quad s\in[-\tau,0], \xi\in\Omega_+,\\ \bar{V}|_{\xi_1=\zeta_0} \geq C\mathrm{e}^{-\mu_1(c,\tau)t}, \quad t>0, (\xi_2,\cdots,\xi_n)\in\mathbb{R}^{n-1}.\end{cases}$$

Thus, we can get

$$0\leq v(\xi,t)\leq \bar{V}(\xi,t) = C\mathrm{e}^{-\mu t}, \tag{3.21}$$

for $\xi\in\Omega_+, t>0$. Thus,

$$\|v(\cdot,t)\|_{L^\infty(\Omega_+)} \leq C\mathrm{e}^{-\mu t}, \quad c>c^*,$$

for $t>0$. □

Based on (3.17) and Lemma 3.2, we immediately obtain the following lemma.

**Lemma 3.3.** *For any $c>c^*$, it holds that*

$$\sup_{x\in\mathbb{R}} |U^+(x,t) - \phi(x\cdot e_1 + ct)| \leq C\mathrm{e}^{-\mu t}, \quad t>0.$$

Similarly, we can obtain the convergence of $U^-(x,t)$ to $\phi(x\cdot e_1+ct)$.

**Lemma 3.4.** *For any $c>c^*$, there exists a positive number $\mu=\mu(\tau)$ such that*

$$\sup_{x\in\mathbb{R}^n} |U^-(x,t) - \phi(x+ct)| \leq C\mathrm{e}^{-\mu t}, \quad t>0.$$

Theorem 2.2 can be accomplished by Lemmas 3.3–3.4 and inequalities (3.3)–(3.4).

## 4. Nonlinear stability of the critical planar waves

In this section, we mainly present the proof of multidimensional asymptotic stability for the critical planar waves ($c=c^*$). Due to the difficulty caused by the unstable equilibrium, we first establish a weighted $L^1$-estimate by selecting a suitable weight function. Then using the weighted $L^1$-estimate, we further obtain the desired $H^k (k\geq [\frac{n+1}{2}])$ estimate. Finally, we get the $L^\infty$-estimate with the help of the Sobolev embedding theorem. Let $v(\xi,t)$ be defined in (3.5) with $c=c^*$ and conditions in Theorem 2.3 be satisfied throughout this section.

**Lemma 4.1.** *It holds that*

$$\|v(\cdot,t)\|_{L^1_{\hat{w}}(\mathbb{R}^n)} + \int_0^t \|v(\cdot,s)\|^2_{L^2_{\hat{w}}(\mathbb{R}^n)}\mathrm{d}s \leq C, \tag{4.1}$$

*where $\hat{w}(\xi_1) = \mathrm{e}^{-\lambda^*(\xi_1-\zeta_0)}$ ($\zeta_0$ is a large constant).*

**Proof.** If $f\in C^k([0,u_+]^2,\mathbb{R}^n)$, by the standard energy method and continuity extension method (see [19]) or the theory of abstract functional differential equations in [14], we have

$$v(\cdot,t)\in C^1([0,+\infty), H^1_{\hat{w}}(\mathbb{R}^n)\cap H^k(\mathbb{R}^n)\cap L^1_{\hat{w}}(\mathbb{R}^n)). \tag{4.2}$$

Multiplying (3.7) by $\hat{w}(\xi_1)$, we obtain

$$\frac{\partial}{\partial t}(\hat{w}v)+\frac{\partial}{\partial \xi_1}(c^*\hat{w}v)=\hat{w}[d(J*v-v)+c^*\frac{\hat{w}'}{\hat{w}}v+\partial_1 f(0,0)v+\partial_2 f(0,0)v_\tau+Q].$$

Integrating the above equation over $\mathbb{R}^n\times[0,t]$ with respect to $\xi$ and $t$, we have

$$\begin{aligned}\int_{\mathbb{R}^n}\hat{w}(\xi_1)v(\xi,t)\mathrm{d}\xi=&\|v_0(\cdot,0)\|_{L^1_{\hat{w}}(\mathbb{R}^n)}+\int_0^t\int_{\mathbb{R}^n}\hat{w}\Big[d(J*v-v)+c^*\frac{\hat{w}'}{\hat{w}}v\\&+\partial_1 f(0,0)v+\partial_2 f(0,0)k*v_\tau\Big]\mathrm{d}\xi\mathrm{d}s+\int_0^t\int_{\mathbb{R}^n}\hat{w}(\xi_1)Q(\xi,s)\mathrm{d}\xi\mathrm{d}s.\end{aligned}\tag{4.3}$$

Here, we use (4.2) to ensure that

$$\int_0^t\int_{\mathbb{R}^n}\frac{\partial}{\partial\xi_1}\left[c^*\hat{w}(\xi_1)v(\xi,s)\right]\mathrm{d}\xi\mathrm{d}s=0.$$

Because of $Q(\xi,s)\le\partial_{11}f(0,0)v^2$ and $\hat{w}(\xi_1)\ge 0$, we have

$$\int_0^t\int_{\mathbb{R}^n}\hat{w}(\xi_1)Q(\xi,s)\mathrm{d}\xi\mathrm{d}s\le\partial_{11}f(0,0)\int_0^t\|v(s)\|^2_{L^2_{\hat{w}}(\mathbb{R}^n)}\mathrm{d}s.\tag{4.4}$$

By changing variable $y\to y,\xi-y-c^*\tau\cdot e_1\to\xi, s-\tau\to s$ and using the fact

$$\int_{\mathbb{R}^n}k(y)\frac{\hat{w}(\xi_1+y_1+c^*\tau)}{\hat{w}(\xi_1)}\mathrm{d}y=\int_{\mathbb{R}^n}k(y)\mathrm{e}^{-\lambda^*(y_1+c^*\tau)}\mathrm{d}y:=k_0,$$

we obtain

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^n}\hat{w}(\xi_1)\int_{\mathbb{R}^n}k(y)v(\xi-y-c^*\tau\cdot e_1,s-\tau)\mathrm{d}y\mathrm{d}\xi\mathrm{d}s\\=&\int_{-\tau}^{t-\tau}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{w}(\xi_1+y_1+c^*\tau)k(y)v(\xi,s)\mathrm{d}y\mathrm{d}\xi\mathrm{d}s\\=&\int_{-\tau}^{t-\tau}\int_{\mathbb{R}^n}\hat{w}(\xi_1)v(\xi,s)\left[\int_{\mathbb{R}^n}k(y)\frac{\hat{w}(\xi_1+y_1+c^*\tau)}{\hat{w}(\xi_1)}\mathrm{d}y\right]\mathrm{d}\xi\mathrm{d}s\\=&\int_{-\tau}^{t-\tau}\int_{\mathbb{R}^n}\hat{w}(\xi_1)v(\xi,s)\mathrm{d}\xi\mathrm{d}s\int_{\mathbb{R}^n}k(y)\mathrm{e}^{-\lambda^*(y_1+c^*\tau)}\mathrm{d}y\\\le&k_0\left[\int_0^t\|v(\cdot,s)\|_{L^1_{\hat{w}}(\mathbb{R}^n)}\mathrm{d}s+\int_{-\tau}^0\|v_0(\cdot,s)\|_{L^1_{\hat{w}}(\mathbb{R}^n)}\mathrm{d}s\right].\end{aligned}$$

By similar method above, we obtain

$$d\int_0^t\int_{\mathbb{R}^n}\hat{w}(\xi_1)\int_{\mathbb{R}^n}J(x)v(\xi-x,s)\mathrm{d}x\mathrm{d}\xi\mathrm{d}s=d\int_{\mathbb{R}^n}J(x)\mathrm{e}^{-\lambda^*x_1}\mathrm{d}x\int_0^t\|v(\cdot,s)\|_{L^1_{\hat{w}}(\mathbb{R}^n)}\mathrm{d}s.$$

So we obtain

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^n}\hat{w}(\xi_1)\left[d(J*v-v)+c\frac{\hat{w}'}{\hat{w}}v+\partial_1 f(0,0)v+\partial_2 f(0,0)k*v_\tau\right]\mathrm{d}\xi\mathrm{d}s\\\le&\int_0^t\int_{\mathbb{R}^n}\hat{w}(\xi_1)v(\xi,s)\big[d\int_{\mathbb{R}^n}J(x)(\mathrm{e}^{-\lambda^*x_1}-1)\mathrm{d}x-c\lambda^*+\partial_1 f(0,0)\end{aligned}$$

$$+\partial_2 f(0,0)k_0\big]\mathrm{d}\xi\mathrm{d}s+\partial_2 f(0,0)k_0\int_{-\tau}^{0}\|v_0(\cdot,s)\|_{L^1_{\hat{w}}(\mathbb{R}^n)}\mathrm{d}s$$

$$=\Delta(\lambda^*,c^*)\int_0^t\|v(\cdot,s)\|_{L^1_{\hat{w}}(\mathbb{R}^n)}\mathrm{d}s+\partial_2 f(0,0)k_0\int_{-\tau}^{0}\|v_0(\cdot,s)\|_{L^1_{\hat{w}}(\mathbb{R}^n)}\mathrm{d}s. \tag{4.5}$$

Substituting (4.4) and (4.5) to (4.3) and noting $\Delta(\lambda^*,c^*)=0$, we have

$$\|v(\cdot,t)\|_{L^1_{\hat{w}}(\mathbb{R}^n)}-\partial_{11}f(0,0)\int_0^t\|v(\cdot,s)\|^2_{L^2_{\hat{w}}(\mathbb{R}^n)}\mathrm{d}s\le\mathcal{E}_0, \tag{4.6}$$

where $\mathcal{E}_0=\|v_0(\cdot,0)\|_{L^1_{\hat{w}}(\mathbb{R}^n)}+\partial_2 f(0,0)k_0\int_{-\tau}^{0}\|v_0(\cdot,s)\|_{L^1_{\hat{w}}(\mathbb{R}^n)}\mathrm{d}s$. Due to $\partial_{11}f(0,0)<0$, we obtain

$$\|v(\cdot,t)\|_{L^1_{\hat{w}}(\mathbb{R}^n)}+\int_0^t\|v(\cdot,s)\|^2_{L^2_{\hat{w}}(\mathbb{R}^n)}\mathrm{d}s\le C.$$

This completes the proof. □

**Lemma 4.2.** *It holds that*

$$\|v(\cdot,t)\|^2_{H^k(\mathbb{R}^n)}+\int_0^t\|v(\cdot,s)\|^2_{H^k(\mathbb{R}^n)}\mathrm{d}s\le C. \tag{4.7}$$

**Proof.** Since $\hat{w}(\xi_1)=\mathrm{e}^{-\lambda^*(\xi_1-\zeta_0)}\ge 1$ for $\xi_1\le\zeta_0$ and from the Lemma 4.1, we obtain

$$\int_{\Omega_-}v(\xi,t)\mathrm{d}\xi+\int_0^{t-\tau}\int_{\Omega_-}v^2(\xi,s)\mathrm{d}\xi\mathrm{d}s\le C,\quad\forall t\ge 0,$$

where $\Omega_-=(-\infty,\zeta_0)\times\mathbb{R}^n$, and in particular by taking $t=+\infty$, we obtain

$$\int_0^{+\infty}\int_{\Omega_-}v^2(\xi,s)\mathrm{d}\xi\mathrm{d}s\le C. \tag{4.8}$$

Multiplying (3.18) by $v(\xi,t)$ and integrating it over $\mathbb{R}^n\times[0,t]$ with respect to $\xi$ and $t$, then we have

$$\|v(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}\le\|v_0(\cdot,0)\|^2_{L^2(\mathbb{R}^n)}+2\int_0^t\int_{\mathbb{R}^n}d(J*v-v)v\mathrm{d}\xi\mathrm{d}s+2\int_0^t\int_{\mathbb{R}^n}\partial_1 f(\phi,k*\phi_\tau)v^2\mathrm{d}\xi\mathrm{d}s+2\int_0^t\int_{\mathbb{R}^n}\partial_2 f(\phi,k*\phi_\tau)k*v_\tau v\mathrm{d}\xi\mathrm{d}s. \tag{4.9}$$

Using the Cauchy inequality $|ab|\le\frac{\eta}{2}a^2+\frac{1}{2\eta}b^2$ for $\eta>0$, which will be specified later, we obtain

$$\begin{aligned}&2\int_0^t\int_{\mathbb{R}^n}\partial_2 f(\phi,k*\phi_\tau)\int_{\mathbb{R}^n}k(y)v(\xi-y-c^*\tau\cdot e_1,s-\tau)v(\xi,s)\mathrm{d}y\mathrm{d}\xi\mathrm{d}s\\ \le&\frac{1}{\eta}\int_0^t\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}k(y)\partial_2^2 f(\phi,k*\phi_\tau)v^2(\xi-y-c^*\tau\cdot e_1,s-\tau)\mathrm{d}y\mathrm{d}\xi\mathrm{d}s\\ &+\eta\int_0^t\int_{\mathbb{R}^n}v^2(\xi,s)\mathrm{d}\xi\mathrm{d}s.\end{aligned} \tag{4.10}$$

By changing variables $y \to y, \xi - y - c^*\tau \cdot e_1 \to \xi, s - \tau \to s$, we have

$$
\begin{aligned}
&\frac{1}{\eta}\int_0^t \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} k(y)\partial_2^2 f(\phi, k*\phi_\tau) v^2(\xi - y - c^*\tau\cdot e_1, s-\tau)\mathrm{d}y\mathrm{d}\xi\mathrm{d}s\\
=&\frac{1}{\eta}\int_{-\tau}^{t-\tau}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} k(y)\partial_2^2 f\left(\phi((\xi+y)\cdot e_1 + c^*\tau), \int_{\mathbb{R}^n} k(z)\phi((\xi+y-z)\cdot e_1)\mathrm{d}z\right)\\
&\cdot v^2(\xi,s)\mathrm{d}y\mathrm{d}\xi\mathrm{d}s\\
=&\frac{1}{\eta}\int_{-\tau}^{0}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} k(y)\partial_2^2 f\left(\phi((\xi+y)\cdot e_1 + c^*\tau), \int_{\mathbb{R}^n} k(z)\phi((\xi+y-z)\cdot e_1)\mathrm{d}z\right)\\
&\cdot v^2(\xi,s)\mathrm{d}y\mathrm{d}\xi\mathrm{d}s + \frac{1}{\eta}\int_0^{t-\tau}\left(\int_{\Omega_-} + \int_{\Omega_+}\right) k(y)\partial_2^2 f\Big(\phi((\xi+y)\cdot e_1 + c^*\tau),\\
&\int_{\mathbb{R}^n} k(z)\phi((\xi+y-z)\cdot e_1)\mathrm{d}z\Big) v^2(\xi,s)\mathrm{d}y\mathrm{d}\xi\mathrm{d}s\\
\le&\frac{1}{\eta}\partial_2^2 f(0,0)\|v_0(\cdot,0)\|^2_{L^2(\mathbb{R}^n)} + \frac{1}{\eta}\partial_2^2 f(0,0)\int_0^{+\infty}\int_{\Omega_-} v^2(\xi,s)\mathrm{d}\xi\mathrm{d}s\\
+&\frac{1}{\eta}\int_0^t\int_{\Omega_+}\left[\int_{\mathbb{R}^n} k(y)\partial_2^2 f\left(\phi((\xi+y)\cdot e_1 + c^*\tau), \int_{\mathbb{R}^n} k(z)\phi((\xi+y-z)\cdot e_1)\mathrm{d}z\right)\right.\\
&\left.\mathrm{d}y\right] v^2(\xi,s)\mathrm{d}\xi\mathrm{d}s\\
\le& C + \frac{1}{\eta}\int_0^t\int_{\Omega_+}\left[\int_{\mathbb{R}^n} k(y)\partial_2^2 f\Big(\phi((\xi+y)\cdot e_1 + c^*\tau), \int_{\mathbb{R}^n} k(z)\phi((\xi+y-z)\cdot e_1)\right.\\
&\left.\mathrm{d}z\Big)\mathrm{d}y\right] v^2(\xi,s)\mathrm{d}\xi\mathrm{d}s,
\end{aligned}
\tag{4.11}
$$

where we have used $\partial_2 f(u,v) \ge 0$ and $\partial_{ij} f(u,v) \le 0$ for all $(u,v) \in [0,u_+]^2$ and (4.8). Similarly, we obtain

$$
\begin{aligned}
&2\int_0^t\int_{\mathbb{R}^n} \partial_1 f(\phi, k*\phi_\tau) v^2(\xi,s)\mathrm{d}\xi\mathrm{d}s\\
=&2\int_0^t\int_{\Omega_-} \partial_1 f(\phi, k*\phi_\tau) v^2(\xi,s)\mathrm{d}\xi\mathrm{d}s + 2\int_0^t\int_{\Omega_+} \partial_1 f(\phi, k*\phi_\tau) v^2(\xi,s)\mathrm{d}\xi\mathrm{d}s\\
\le&2\int_0^t\int_{\Omega_+} \partial_1 f(\phi, k*\phi_\tau) v^2(\xi,s)\mathrm{d}\xi\mathrm{d}s,
\end{aligned}
\tag{4.12}
$$

and

$$
\eta\int_0^t\int_{\mathbb{R}^n} v^2(\xi,s)\mathrm{d}\xi\mathrm{d}s \le C + \eta\int_0^t\int_{\Omega_+} v^2(\xi,s)\mathrm{d}\xi\mathrm{d}s, \tag{4.13}
$$

where we use the fact that $\partial_1 f(u,v) \le 0$ and (4.8). Substituting (4.11)-(4.13) to (4.9), we obtain

$$
\|v(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} - \int_0^t\int_{\Omega_+} G(\xi) v^2(\xi,s)\mathrm{d}\xi\mathrm{d}s \le C, \tag{4.14}
$$

where

$$G(\xi) =\eta + \frac{1}{\eta}\int_{\mathbb{R}^n} k(y)\partial_2^2 f\left(\phi(\xi_1+y_1+c^*\tau), \int_{\mathbb{R}^n} k(z)\phi((\xi+y-z)\cdot e_1)\mathrm{d}z\right)\mathrm{d}y$$
$$+ 2\partial_1 f(\phi, k*\phi_\tau) \tag{4.15}$$

for $\xi \in \Omega_+$. Thus,

$$\begin{aligned}\lim_{\xi_1\to+\infty} G(\xi) =&\eta + \frac{1}{\eta}\int_{\mathbb{R}^n} k(y)\partial_2^2 f(u_+,u_+)\mathrm{d}y + 2\partial_1 f(u_+,u_+)\\ =&\eta + \frac{1}{\eta}\partial_2^2 f(u_+,u_+) + 2\partial_1 f(u_+,u_+)\\ =&\frac{1}{\eta}\left[\eta^2 + 2\eta\partial_1 f(u_+,u_+) + \partial_2^2 f(u_+,u_+)\right].\end{aligned} \tag{4.16}$$

Noting $\partial_1 f(u_+,u_+) < 0$ and $\partial_1 f(u_+,u_+) + \partial_2 f(u_+,u_+) < 0$, and using the properties of quadratic function, we can choose a suitable $\eta > 0$ such that $\lim\limits_{\xi_1\to+\infty} G(\xi) = G_\infty < 0$. Furthermore, we can choose $\zeta_0$ large enough to ensure that

$$G(\xi) < \frac{1}{2}G_\infty < 0, \xi \in \Omega_+.$$

Thus, we have

$$\|v(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} - \frac{1}{2}G_\infty\int_0^t\int_{\Omega_+} v^2(\xi,s)\mathrm{d}\xi\mathrm{d}s \le C. \tag{4.17}$$

In particular, we have

$$\int_0^t\int_{\Omega_+} v^2(\xi,s)\mathrm{d}\xi\mathrm{d}s \le C. \tag{4.18}$$

Combining (4.8) and (4.18), we have $\displaystyle\int_0^t \|v(\cdot,s)\|^2_{L^2(\mathbb{R}^n)}\mathrm{d}s \le C$. Thus, we can immediately obtain

$$\|v(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + \int_0^t \|v(\cdot,s)\|^2_{L^2(\mathbb{R}^n)}\mathrm{d}s \le C. \tag{4.19}$$

To obtain the derivatives of $v(\xi,t)$, let us differentiate (3.6) with respect to $\xi$ and multiply the resulting equation by $\partial^\alpha v(\xi,t)(|\alpha| = 1,2,\cdots,k)$ and then integrate it over $\mathbb{R}^n\times[0,t]$ with respect to $\xi$ and $t$. By the similar method above, we have

$$\|\partial^\alpha v(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + \int_0^t \|\partial^\alpha v(\cdot,s)\|^2_{L^2(\mathbb{R}^n)}\mathrm{d}s \le C, \tag{4.20}$$

for $|\alpha| = 1,2,\cdots,k$. This together with (4.19) completes the proof. □

**Lemma 4.3.** *It holds that*

$$\int_0^t \left|\frac{\mathrm{d}}{\mathrm{d}s}\|v(\cdot,s)\|^2_{H^k(\mathbb{R}^n)}\right|\mathrm{d}s \le C. \tag{4.21}$$

**Proof.** Multiplying (3.7) by $v(\xi,t)$ and integrating it over $\mathbb{R}^n$ with respect to $\xi$, then we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|v(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} =& 2d\int_{\mathbb{R}^n}(J*v-v)\mathrm{d}\xi + 2\int_{\mathbb{R}^n}\partial_1 f(0,0)v^2(\xi,t)\mathrm{d}\xi \\ &+ 2\int_{\mathbb{R}^n}\partial_2 f(0,0)k*v_\tau v(\xi,t)\mathrm{d}\xi + 2\int_{\mathbb{R}^n}Q(\xi,t)v(\xi,t)\mathrm{d}\xi. \end{aligned} \tag{4.22}$$

Noticing the fact that $Q(\xi,t)\le 0$ and $v(\xi,t)\ge 0$, we obtain

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}\|v(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}\right| \le& \left|2d\int_{\mathbb{R}^n}(J*v-v)\mathrm{d}\xi\right| + \left|2\int_{\mathbb{R}^n}\partial_1 f(0,0)v^2(\xi,t)\mathrm{d}\xi\right| \\ &+ \left|2\int_{\mathbb{R}^n}\partial_2 f(0,0)k*v_\tau v(\xi,t)\mathrm{d}\xi\right| \\ \le& C\|v(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{4.23}$$

Integrating the above inequality over $[0,t]$, we obtain

$$\int_0^t\left|\frac{\mathrm{d}}{\mathrm{d}s}\|v(\cdot,s)\|^2_{L^2(\mathbb{R}^n)}\right|\mathrm{d}s \le \int_0^t C\|v(\cdot,s)\|^2_{L^2(\mathbb{R}^n)}\mathrm{d}s.$$

Noting the result of Lemma 4.2, we have

$$\int_0^t\left|\frac{\mathrm{d}}{\mathrm{d}s}\|v(\cdot,s)\|^2_{L^2(\mathbb{R}^n)}\right|\mathrm{d}s \le C. \tag{4.24}$$

Let us differentiate (3.7) with respect to $\xi$ and multiply the resulting equation by $\partial^\alpha v(\xi,t)$, and then by the similar method above, we can obtain

$$\int_0^t\left|\frac{\mathrm{d}}{\mathrm{d}s}\|\partial^\alpha v(\cdot,s)\|^2_{L^2(\mathbb{R}^n)}\right|\mathrm{d}s \le C \tag{4.25}$$

for $|\alpha|\le k$. Combining (4.24) and (4.25), we can immediately get (4.21) and we complete the proof. □

**Lemma 4.4.** *It holds that*

$$\lim_{t\to+\infty}\sup_{x\in\mathbb{R}^n}|U^+(x,t)-\phi(x\cdot e_1+c^*t)| = 0. \tag{4.26}$$

**Proof.** Let $g(t)=\|v(\cdot,t)\|^2_{H^k(\mathbb{R}^n)}$. It follows from Lemmas 4.2 and 4.3 that

$$0\le g(t)\le C,\quad \int_0^{+\infty}g(t)\mathrm{d}t\le C,\quad \int_0^{+\infty}\left|g'(t)\right|\mathrm{d}t\le C,$$

which implies

$$\lim_{t\to+\infty}\|v(\cdot,t)\|^2_{H^k(\mathbb{R}^n)} = \lim_{t\to+\infty}g(t) = 0. \tag{4.27}$$

According to the standard Sobolev embedding inequality $H^k(\mathbb{R}^n)\hookrightarrow L^\infty(\mathbb{R}^n)(k\ge [\frac{n+1}{2}])$, we can obtain

$$\|v(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le C\|v(\cdot,t)\|_{H^k(\mathbb{R}^n)}. \tag{4.28}$$

By (4.27) and (4.28), we immediately obtain

$$\lim_{t\to+\infty}\sup_{x\in\mathbb{R}^n}|U^+(x,t)-\phi(x\cdot e_1+c^*t)| = \lim_{t\to+\infty}\|v(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} = 0.$$

□

Similarly, we have the following lemma.

**Lemma 4.5.** *It holds that*

$$\lim_{t\to+\infty}\sup_{x\in\mathbb{R}^n}|U^-(x,t)-\phi(x\cdot e_1+c^*t)| = 0.$$

Then, by Lemmas 4.4 and 4.5, we can obtain that the solution $u(x,t)$ converges to the critical planar wave $\phi(x\cdot e_1+c^*t)$ time-asymptotically

$$\lim_{t\to+\infty}\sup_{x\in\mathbb{R}^n}|u(x,t)-\phi(x\cdot e_1+c^*t)| = 0.$$

This complete the proof of Theorem 2.3.

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# References


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