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LIMIT CYCLES FOR TWO CLASSES OF PLANAR POLYNOMIAL DIFFERENTIAL SYSTEMS WITH UNIFORM ISOCHRONOUS CENTERS*

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Abstract In this article, we study the maximum number of limit cycles for two classes of planar polynomial differential systems with uniform isochronous centers. Using the first-order averaging method, we analyze how many limit cycles can bifurcate from the period solutions surrounding the centers of the considered systems when they are perturbed inside the class of homogeneous polynomial differential systems of the same degree. We show that the maximum number of limit cycles, m and $m+1$, that can bifurcate from the period solutions surrounding the centers for the two classes of differential systems of degree $2m$ and degree $2m+1$, respectively. Both of the bounds can be reached for all m.

Keywords Averaging method, homogeneous polynomial, limit cycle, period solutions, uniform isochronous center.

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1. Introduction

Recall that the second part of the 16th Hilbert's problem [11, 13] asks about the maximal number and relative configurations of limit cycles for planar polynomial differential systems of degree n:

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y). \tag{1.1}$$

Many interesting and profound results have been obtained, see [7–9, 19, 20, 30, 31] and the references therein. Nevertheless, Hilbert's problem is still open even in the case $n = 2$.

There are several methods for studying the bifurcation of limit cycles. One of the methods is based on perturbation of system (1.1) with centers via Poincaré

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bifurcation, by means of which limit cycles can bifurcate from the period solutions of the centers. In the last three decades extensive work about the bifurcation of limit cycles for planar differential systems with uniform isochronous centers has been reported in the literature, see [5,6,16–18] for instance. Isochronicity is closely related to the uniqueness and existence of solutions for boundary value problems and has important applications in physics. Moreover, the interest in this problem has also been revived due to the proliferation of powerful computerized methods, where special attention has been dedicated mainly to polynomial differential systems, see [2,3] for instance.

Let O be a center of system (1.1), without loss of generality we can assume that O is the origin of the coordinates. We say that O is an isochronous center if it is a center having a neighborhood such that all the periodic solutions in this neighborhood have the same period. Moreover, we have the following definition.

Definition 1.1. We say that O is a uniform isochronous center of system (1.1), if it is a center and, in polar coordinates $x = r\cos\theta, y = r\sin\theta$, (1.1) takes the form $\dot{r} = F(r,\theta), \dot{\theta} = k, k \in \mathbb{R} \setminus \{0\}$.

For more details on this definition see [3, 4]. The next result on the uniform isochronous planar centers is well-known, a proof of it can be found in [14].

Proposition 1.1. *Assume that system* (1.1) *has a center at the origin O. Then O is a uniform isochronous center if and only if by doing a linear change of variables and a rescaling of time the system can be written as*

$$\dot{x} = -y + xf(x,y), \quad \dot{y} = x + yf(x,y),$$

where $f(x,y)$ is a polynomial in x and y of degree $n-1$, and $f(0,0) = 0$.

In this paper we provide lower bounds for the maximum number of limit cycles that can bifurcate from the periodic solutions of a polynomial differential uniform isochronous center of degree $2n+2$ or $2n+3$ when it is perturbed inside the class of homogeneous polynomial differential systems of the same degree. The main result is based on the first-order averaging method. For more details about the averaging method see the book of Sanders, Verhulst and Murdock [28] and Verhulst [29]. We remark that the Melnikov method is a good tool for studying the number of limit cycles which bifurcate from the periodic orbits surrounding a center, see [10] for a relation of the averaging method and the Melnikov method.

More precisely, we consider the following two classes of planar polynomial differential systems

$$\begin{cases} \dot{x} = -y + xy(x^2+y^2)^n, \\ \dot{y} = x + y^2(x^2+y^2)^n, \end{cases} \tag{1.2}$$

and

$$\begin{cases} \dot{x} = -y + x^2y(x^2+y^2)^n, \\ \dot{y} = x + xy^2(x^2+y^2)^n, \end{cases} \tag{1.3}$$

of degree $2n+2$ and degree $2n+3$ with $n \geq 0$, having uniform isochronous centers at the origin of coordinates.

We recall that the perturbations of the periodic solutions of the uniform isochronous centers (1.2) and (1.3) have been considered by several papers, see [12, 15–18, 23–26] for instance. However, all these results focused on differential systems of lower degree for the bifurcation of limit cycles with specific n. As far as we know, for the integrable systems of higher degree, it seems to be very complicated to study the bifurcation of limit cycles from these systems via the averaging approaches. The main difficulty exists in the technical and cumbersome computations of the averaged function, which in some cases are out of reach with the present state of knowledge. Hence, it is necessary and also challenging to study the bifurcation of limit cycles for some differential systems of higher degree under any small perturbation.

In the present paper, we focus our attention on the study of differential systems of (1.2) and (1.3) in the general cases. In other words, using the averaging method we bound the maximum number of limit cycles that can bifurcate from the periodic solutions surrounding the uniform isochronous centers at the origin of the systems of (1.2) and (1.3) of respective degrees $2n+2$ and $2n+3$, when the systems are perturbed inside the classes of homogeneous polynomial differential systems of respective degrees $2n+2$ and $2n+3$. More concretely, our purpose is to provide lower bounds for the maximum number of limit cycles of the following polynomial differential systems:

$$\begin{cases} \dot{x} = -y + xy(x^2+y^2)^n + \varepsilon \displaystyle\sum_{i+j=2n+2} a_{i,j} x^i y^j, \\ \dot{y} = x + y^2(x^2+y^2)^n + \varepsilon \displaystyle\sum_{i+j=2n+2} b_{i,j} x^i y^j, \end{cases} \tag{1.4}$$

and

$$\begin{cases} \dot{x} = -y + x^2y(x^2+y^2)^n + \varepsilon \displaystyle\sum_{i+j=2n+3} c_{i,j} x^i y^j, \\ \dot{y} = x + xy^2(x^2+y^2)^n + \varepsilon \displaystyle\sum_{i+j=2n+3} d_{i,j} x^i y^j, \end{cases} \tag{1.5}$$

where ε is a small parameter.

The main result of this paper is stated as follows.

Theorem 1.1. *For $|\varepsilon| \neq 0$ sufficiently small using the first-order averaging method we obtain that*

(a) system (1.4) has up to $n+1$ limit cycles bifurcating from periodic solutions of the unperturbed system, and this number can be reached for all n;

(b) system (1.5) has up to $n+2$ limit cycles bifurcating from periodic solutions of the unperturbed system, and this number can be reached for all n.

Remark 1.1. The results of Theorem 1.1 for systems (1.4) and (1.5) with $n = 0, 1$ have been already contained in the papers [23–26]. In this work we extend these previous results to the general cases. More importantly, Theorem 1.1 gives the exact lower bounds for the maximum number of limit cycles bifurcated from the period solutions of the unperturbed systems, which is far from being trivial.

The organization of this paper is as follows. In Section 2, we introduce the basic results on the averaging method. Sections 3 and 4 are dedicated to the proof of Theorem 1.1 by exploring the maximum number of simple zeros of the obtained averaged functions associated to systems (1.4) and (1.5).

2. Preliminary Results

In this section we present some basic results which are the basis of the averaging method and which will be used for proofs in later sections.

We consider the system

$$x'(t) = F_0(t, x), \tag{2.1}$$

with $F_0 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ a $\mathcal{C}^2$ function, T-periodic in the first variable, and Ω is an open subset of $\mathbb{R}^n$. We assume that system (2.1) has a submanifold of periodic solutions.

Let ε be sufficiently small and we consider a perturbation of system (2.1) of the form

$$x'(t) = F_0(t, x) + \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x, \varepsilon), \tag{2.2}$$

with $F_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^2$ functions, T-periodic in the first variable, and Ω is an open subset of $\mathbb{R}^n$. A solution of this problem is given using the averaging method.

Let $x(t, z)$ be the periodic solution of the unperturbed system (2.1) satisfying the initial condition $x(0, z) = z$. Now we consider the linearization of the system (2.1) along the solution $x(t, z)$, namely

$$y' = D_x F_0(t, x(t, z))y, \tag{2.3}$$

and let $M_z(t, z)$ be a fundamental matrix of this linear system satisfying that $M_z(0, z)$ is the identity matrix.

We assume that there exists an open set V with $Cl(V) \subset \Omega$ such that for each $z \in Cl(V), x(t, z)$ is T-periodic, where $x(t, z)$ denotes the solution of the unperturbed system (2.1) with $x(0, z) = z$. The set $Cl(V)$ is isochronous for the system (2.1); i.e. it is a set formed only by periodic orbits, all of them having the same period. Then, an answer to the problem of the bifurcation of T-periodic solutions from the periodic solutions $x(t, z)$ contained in $Cl(V)$ is given in the following result.

Theorem 2.1 (Perturbations of an isochronous set)**.** *Assume that there exists an open and bounded set V with $Cl(V) \subset \Omega$ such that for each $z \in Cl(V)$, the solution $x(t, z)$ is T-periodic, then we consider the function $\mathcal{F} : Cl(V) \to \mathbb{R}^n$*

$$\mathcal{F}(z) = \int_0^T M_z^{-1}(t, z) F_1(t, x(t, z)) dt. \tag{2.4}$$

If there exists $a \in V$ with $\mathcal{F}(a) = 0$ and $det((d\mathcal{F}/dz)(a)) \neq 0$, then there exists a T-periodic solution $\varphi(t, \varepsilon)$ of system (2.2) *such that $\varphi(0, \varepsilon) \to a$ as $\varepsilon \to 0$.*

Theorem 2.1 goes back to [22] and [27], for a shorter proof see [1].

In order to study the number of zeros of the averaged function (2.4) we will use the following result proved in [21].

Let $\Delta \subset \mathbb{R}$ be an interval and let $f_1, f_2, \ldots, f_n : \Delta \to \mathbb{R}$. We say that $f_1, \ldots, f_n$ are linearly independent functions if and only if we have that

$$\sum_{i=1}^n \alpha_i f_i(\delta) = 0 \ \text{ for all } \ \delta \in \Delta \quad \Longrightarrow \quad \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

Proposition 2.1. *(See [21].) If $f_1, f_2, \dots, f_n : \Delta \to \mathbb{R}$ are linearly independent then there exist $\delta_1, \dots, \delta_{n-1} \in \Delta$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ such that for every $i \in \{1, \dots, n-1\}$*

$$\sum_{k=1}^{n} \alpha_k f_k(\delta_i) = 0.$$

3. Proof of Theorem 1.1 (a)

This section is devoted to the proof of statement (a) of Theorem 1.1 by using Theorem 2.1.

First, in polar coordinates (θ, r) defined by $x = r\cos\theta, y = r\sin\theta$, system (1.4) can be written as

$$\begin{aligned}\frac{dr}{d\theta} &= r \cdot \frac{x\dot{x} + y\dot{y}}{\dot{y}x - y\dot{x}} \\ &= F_0(\theta, r) + \varepsilon F_1(\theta, r) + O(\varepsilon^2),\end{aligned} \tag{3.1}$$

where $F_0(\theta, r) = r^{2n+2}\sin\theta$ and

$$\begin{aligned}F_1(\theta, r) = r^{2n+2} &\sum_{i+j=2n+2} \left(a_{i,j}\cos^{i+1}\theta\sin^j\theta + b_{i,j}\cos^i\theta\sin^{j+1}\theta\right) \\ &+ r^{4n+3}\sum_{i+j=2n+2}\left(a_{i,j}\cos^i\theta\sin^{j+2}\theta - b_{i,j}\cos^{i+1}\theta\sin^{j+1}\theta\right).\end{aligned}$$

A direct computation shows that equation $(3.1)_{\varepsilon=0}$ has the periodic solutions

$$r(\theta, z) = \left((2n+1)(\cos\theta - 1) + \frac{1}{z^{2n+1}}\right)^{-\frac{1}{2n+1}},$$

satisfying $r(0, z) = z$ for $0 < z < (2(2n+1))^{-\frac{1}{2n+1}}$. According to the averaging method described in Sec. 2, we solve the variational differential equation

$$\frac{dM}{d\theta} = \frac{\partial}{\partial r}F_0(\theta, r(\theta, z))M,$$

with $M_z(0, z) = 1$ and get the fundamental solution

$$M_z(\theta, z) = \left[(2n+1)z^{2n+1}(\cos\theta - 1) + 1\right]^{-\frac{2n+2}{2n+1}}.$$

Since all the assumptions of Theorem 2.1 are satisfied, we must study the maximum number of zeros of the function $\mathcal{F}(z)$. More precisely, we have

$$\begin{aligned}\mathcal{F}(z) &= \int_0^{2\pi} M_z^{-1}(\theta, z) \cdot F_1(\theta, r)d\theta \\ &= \int_0^{2\pi} z^{2n+2}\left[(2n+1)(\cos\theta + \eta)\right]^{\frac{2n+2}{2n+1}}\left\{r^{2n+2}\sum_{i+j=2n+2}\Big(a_{i,j}\cos^{i+1}\theta\sin^j\theta\right. \\ &\quad + b_{i,j}\cos^i\theta\sin^{j+1}\theta\Big) + r^{4n+3}\sum_{i+j=2n+2}\Big(a_{i,j}\cos^i\theta\sin^{j+2}\theta\end{aligned}$$

$$
\begin{aligned}
&\quad - b_{i,j}\cos^{i+1}\theta\sin^{j+1}\theta\Big)\Bigg\}\Bigg|_{r=\left[(2n+1)(\cos\theta+\eta)\right]^{-\frac{1}{2n+1}}} d\theta \\
&= z^{2n+2}\Bigg\{\int_0^{2\pi}\sum_{i+j=2n+2}\Big(a_{i,j}\cos^{i+1}\theta\sin^j\theta + b_{i,j}\cos^i\theta\sin^{j+1}\theta\Big)d\theta \\
&\quad + \int_0^{2\pi}\sum_{i+j=2n+2}\frac{a_{i,j}\cos^i\theta\sin^{j+2}\theta - b_{i,j}\cos^{i+1}\theta\sin^{j+1}\theta}{(2n+1)(\cos\theta+\eta)}d\theta\Bigg\} \\
&= \frac{z^{2n+2}}{2n+1}\Bigg[\sum_{k=1}^{n+1}\Big(a_{2k,2n-2k+2} - b_{2k-1,2n-2k+3}\Big)\int_0^{2\pi}\frac{\cos^{2k}\theta\sin^{2n-2k+4}\theta}{\cos\theta+\eta}d\theta \\
&\quad + a_{0,2n+2}\int_0^{2\pi}\frac{\sin^{2n+4}\theta}{\cos\theta+\eta}d\theta\Bigg], \qquad (3.2)
\end{aligned}
$$

where $\eta = -1 + \left[(2n+1)z^{2n+1}\right]^{-1} \in (1, +\infty)$, in the last equality we have used the equalities

$$\int_0^{2\pi}\cos^{q+1}\theta\sin^{2n-q+2}\theta d\theta = 0, \quad \int_0^{2\pi}\cos^q\theta\sin^{2n-q+3}\theta d\theta = 0$$

and

$$\int_0^{2\pi}\frac{\cos^{2q+1}\theta\sin^{2n-2q+3}\theta}{\cos\theta+\eta}d\theta = 0,$$

for any nonnegative integer number q. Then we reduce the problem of analyzing the number of zeros of $\mathcal{F}(z)$ to the problem of studying the number of zeros of

$$G(\eta) = \sum_{k=1}^{n+1}(a_{2k,2n-2k+2} - b_{2k-1,2n-2k+3})\cdot L_{2k,2n-2k+4} + a_{0,2n+2}\cdot L_{0,2n+4}, \quad (3.3)$$

where

$$L_{2m,2s} = \int_0^{2\pi}\frac{\cos^{2m}\theta\sin^{2s}\theta}{\cos\theta+\eta}d\theta, \quad (3.4)$$

for $\eta \in (1, +\infty)$ and $m, s \in N$. First, we state and prove some lemmas.

Lemma 3.1. *For $m, s \geq 0$*

$$L_{2m,2s} = \sum_{r=0}^{s}\binom{s}{r}(-1)^r L_{2m+2r,0}. \quad (3.5)$$

Proof. It is direct that for $m, s \geq 0$,

$$
\begin{aligned}
L_{2m,2s} &= \int_0^{2\pi}\frac{\cos^{2m}\theta(1-\cos^2\theta)^s}{\cos\theta+\eta}d\theta \\
&= \sum_{r=0}^{s}\binom{s}{r}(-1)^r\int_0^{2\pi}\frac{\cos^{2m+2r}\theta}{\cos\theta+\eta}d\theta \\
&= \sum_{r=0}^{s}\binom{s}{r}(-1)^r L_{2m+2r,0}.
\end{aligned}
$$

□

Lemma 3.2. *For $m \geq 0$, the following statement holds*

$$L_{2m,0} = 2\pi\left(\frac{\eta^{2m}}{\sqrt{\eta^2-1}} - \eta^{2m-1} + \sum_{r=0}^{m-2} A_{2r+1}^{(m)}\eta^{2r+1}\right), \tag{3.6}$$

where $A_{2r+1}^{(m)} = -\frac{(2m-2r-3)!!}{(2m-2r-2)!!}$.

Proof. Note that

$$\begin{aligned}
L_{2m,0} &= \int_0^{2\pi} \frac{\cos^{2m}\theta}{\cos\theta+\eta}d\theta = \int_0^{2\pi} \frac{(\cos\theta+\eta-\eta)^{2m}}{\cos\theta+\eta}d\theta \\
&= \sum_{k=0}^{2m}\binom{2m}{k}(-\eta)^k \int_0^{2\pi}(\cos\theta+\eta)^{2m-k-1}d\theta \\
&= \frac{2\pi\eta^{2m}}{\sqrt{\eta^2-1}} - 4\pi m\eta^{2m-1} + \sum_{k=0}^{2m-2}\binom{2m}{k}(-1)^k \\
&\quad \cdot\left(\sum_{t=0}^{2m-k-1}\binom{2m-k-1}{t}\eta^{2m-t-1}\int_0^{2\pi}\cos^t\theta d\theta\right) \\
&= \frac{2\pi\eta^{2m}}{\sqrt{\eta^2-1}} - 2\pi\eta^{2m-1} + \sum_{k=0}^{2m-2}\binom{2m}{k}(-1)^k \\
&\quad \cdot\left(\sum_{\tilde{t}=1}^{\lfloor\frac{2m-k-1}{2}\rfloor}\binom{2m-k-1}{2\tilde{t}}\eta^{2m-2\tilde{t}-1}\int_0^{2\pi}\cos^{2\tilde{t}}\theta d\theta\right).
\end{aligned}$$

Hence, by using the well-known formula

$$\int_0^{2\pi}\cos^{2\tilde{t}}\theta d\theta = 2\pi\frac{(2\tilde{t}-1)!!}{(2\tilde{t})!!}, \tag{3.7}$$

we perform the computation and obtain

$$L_{2m,0} = 2\pi\left(\frac{\eta^{2m}}{\sqrt{\eta^2-1}} - \eta^{2m-1} + \sum_{r=0}^{m-2} A_{2r+1}^{(m)}\eta^{2r+1}\right),$$

where $A_{2r+1}^{(m)} = \sum_{k=0}^{2r+1}\binom{2m}{k}(-1)^k\binom{2m-k-1}{2m-2r-2}\frac{(2m-2r-3)!!}{(2m-2r-2)!!} = -\frac{(2m-2r-3)!!}{(2m-2r-2)!!}$. Hence the desired result follows. □

Remark 3.1. Some explicit expressions of $L_{2m,0}$ are:

$$\begin{aligned}
L_{0,0} &= \frac{2\pi}{\sqrt{\eta^2-1}}, \\
L_{2,0} &= \frac{2\pi\eta^2}{\sqrt{\eta^2-1}} - 2\pi\eta, \\
L_{4,0} &= \frac{2\pi\eta^4}{\sqrt{\eta^2-1}} - 2\pi\eta^3 - \pi\eta, \\
L_{6,0} &= \frac{2\pi\eta^6}{\sqrt{\eta^2-1}} - 2\pi\eta^5 - \pi\eta^3 - \frac{3}{4}\pi\eta, \\
L_{8,0} &= \frac{2\pi\eta^8}{\sqrt{\eta^2-1}} - 2\pi\eta^7 - \pi\eta^5 - \frac{3}{4}\pi\eta^3 - \frac{5}{8}\pi\eta.
\end{aligned}$$

By using the above lemmas and (3.3), we have

$$
\begin{aligned}
G(\eta) &= \sum_{k=1}^{n+1} (a_{2k,2n-2k+2} - b_{2k-1,2n-2k+3}) \cdot L_{2k,2n-2k+4} + a_{0,2n+2} \cdot L_{0,2n+4} \\
&= \sum_{k=1}^{n+1} (a_{2k,2n-2k+2} - b_{2k-1,2n-2k+3}) \cdot \left(\sum_{r=0}^{n-k+2} \binom{n-k+2}{r} (-1)^r L_{2k+2r,0} \right) \\
&\quad + a_{0,2n+2} \cdot \left(\sum_{r=0}^{n+2} \binom{n+2}{r} (-1)^r L_{2r,0} \right) \\
&= A_{0,0} \cdot L_{0,0} + A_{2,0} \cdot L_{2,0} + A_{4,0} \cdot L_{4,0} + \ldots + A_{2n+4,0} \cdot L_{2n+4,0} \\
&= 2\pi \left(\frac{A_{0,0} + A_{2,0}\eta^2 + A_{4,0}\eta^4 + \ldots + A_{2n+4,0}\eta^{2n+4}}{\sqrt{\eta^2 - 1}} + \sum_{i=1}^{n+1} B_{2i-1} \cdot \eta^{2i-1} \right. \\
&\quad \left. - A_{2n+4,0} \cdot \eta^{2n+3} \right),
\end{aligned} \tag{3.8}
$$

where the coefficients $A_{2m,0}$ are the followings:

$$
\begin{aligned}
A_{0,0} &= a_{0,2n+2}, \quad m = 0, \\
A_{2m,0} &= \sum_{k=1}^{m} (a_{2k,2n-2k+2} - b_{2k-1,2n-2k+3}) \binom{n-k+2}{m-k} (-1)^{m-k} \\
&\quad + a_{0,2n+2} \binom{n+2}{m} (-1)^m, \quad 1 \le m \le n+1, \\
A_{2n+4,0} &= \sum_{k=1}^{n+1} (a_{2k,2n-2k+2} - b_{2k-1,2n-2k+3}) (-1)^{n-k+2} \\
&\quad + a_{0,2n+2} (-1)^{n+2}, \quad m = n+2
\end{aligned}
$$

and $B_{2i-1} = \sum_{m=i+1}^{n+2} A_{2i-1}^{(m)} \cdot A_{2m,0} - A_{2i,0}$ $(1 \le i \le n+1)$, in which the coefficients $A_{2i-1}^{(m)}$ appearing in the equality (3.6). In order to simplify the expression (3.8), we give the following lemmas.

Lemma 3.3. *The coefficients of the polynomial* $\sum_{i=0}^{n+2} A_{2i,0}\eta^{2i}$ *satisfy the following equality*

$$
\sum_{i=0}^{n+2} A_{2i,0} = 0. \tag{3.9}
$$

Proof. According to (3.3) and the penultimate equality in (3.8), we have

$$
\begin{aligned}
&\sum_{k=1}^{n+1} (a_{2k,2n-2k+2} - b_{2k-1,2n-2k+3}) \cdot \frac{\cos^{2k}\theta \sin^{2n-2k+4}\theta}{\cos\theta + \eta} + a_{0,2n+2} \cdot \frac{\sin^{2n+4}\theta}{\cos\theta + \eta} \\
&= \frac{A_{0,0} + A_{2,0}\cos^2\theta + A_{4,0}\cos^4\theta + \ldots + A_{2n+4,0}\cos^{2n+4}\theta}{\cos\theta + \eta},
\end{aligned} \tag{3.10}
$$

let $\cos\theta = 1$ in the above expression, we can obtain $\sum_{i=0}^{n+2} A_{2i,0} = 0$. $\square$

We first estimate the lower bound of the number of zeros of the function $G(\eta)$. Using (3.8) and Lemma 3.3, we have

$$\begin{aligned} G(\eta) &= A_{0,0}\cdot L_{0,0} + A_{2,0}\cdot L_{2,0} + A_{4,0}\cdot L_{4,0} + \ldots + A_{2n+4,0}\cdot L_{2n+4,0} \\ &= -[A_{0,0}(L_{2n+4,0}-L_{0,0}) + A_{2,0}(L_{2n+4,0}-L_{2,0}) + \ldots \\ &\quad + A_{2n+2,0}(L_{2n+4,0}-L_{2n+2,0})]. \end{aligned} \tag{3.11}$$

It is worth to notice that the determinant of the Jacobian matrix

$$\left|\frac{\partial(A_{0,0}, A_{2,0}, \ldots, A_{2n+2,0})}{\partial(a_{0,2n+2}, a_{2,2n}, \ldots, a_{2n+2,0})}\right| = 1 \neq 0.$$

This implies that the coefficients $A_{0,0}, A_{2,0}, \ldots, A_{2n+2,0}$ are independent. In order to identify that $L_{2n+4,0}-L_{0,0}, L_{2n+4,0}-L_{2,0}, \ldots, L_{2n+4,0}-L_{2n+2,0}$ are linearly independent functions, we carry out Taylor expansions in the variable η around $\eta=+\infty$ for the functions $L_{2n+4,0}-L_{2m,0}(m=0,1,\ldots,n+1)$:

$$\begin{aligned} L_{2n+4,0}-L_{0,0} &= 2\pi\Bigg[\left(\frac{(2n+3)!!}{(2n+4)!!}-1\right)\frac{1}{\eta} + \left(\frac{(2n+5)!!}{(2n+6)!!}-\frac{1!!}{2!!}\right)\frac{1}{\eta^3} \\ &\quad + \ldots + \left(\frac{(4n+5)!!}{(4n+6)!!}-\frac{(2n+1)!!}{(2n+2)!!}\right)\frac{1}{\eta^{2n+3}}\Bigg] + \mathcal{O}\left(\frac{1}{\eta^{2n+5}}\right), \\ L_{2n+4,0}-L_{2m,0} &= 2\pi\Bigg[\left(\frac{(2n+3)!!}{(2n+4)!!}-\frac{(2m-1)!!}{(2m)!!}\right)\frac{1}{\eta} + \left(\frac{(2n+5)!!}{(2n+6)!!}-\frac{(2m+1)!!}{(2m+2)!!}\right)\frac{1}{\eta^3} \\ &\quad + \ldots + \left(\frac{(4n+5)!!}{(4n+6)!!}-\frac{(2n+2m+1)!!}{(2n+2m+2)!!}\right)\frac{1}{\eta^{2n+3}}\Bigg] \\ &\quad + \mathcal{O}\left(\frac{1}{\eta^{2n+5}}\right), \quad 1\le m\le n+1. \end{aligned} \tag{3.12}$$

Now computing the determinant of the coefficient matrix of the variables $\frac{1}{\eta}, \frac{1}{\eta^3}, \ldots, \frac{1}{\eta^{2n+3}}$, we obtain

$$\bar{D} = (2\pi)^{n+2}\begin{vmatrix} \frac{(2n+3)!!}{(2n+4)!!}-1 & \frac{(2n+5)!!}{(2n+6)!!}-\frac{1!!}{2!!} & \cdots & \frac{(4n+5)!!}{(4n+6)!!}-\frac{(2n+1)!!}{(2n+2)!!} \\ \frac{(2n+3)!!}{(2n+4)!!}-\frac{1!!}{2!!} & \frac{(2n+5)!!}{(2n+6)!!}-\frac{3!!}{4!!} & \cdots & \frac{(4n+5)!!}{(4n+6)!!}-\frac{(2n+3)!!}{(2n+4)!!} \\ \frac{(2n+3)!!}{(2n+4)!!}-\frac{3!!}{4!!} & \frac{(2n+5)!!}{(2n+6)!!}-\frac{5!!}{6!!} & \cdots & \frac{(4n+5)!!}{(4n+6)!!}-\frac{(2n+5)!!}{(2n+6)!!} \\ \frac{(2n+3)!!}{(2n+4)!!}-\frac{5!!}{6!!} & \frac{(2n+5)!!}{(2n+6)!!}-\frac{7!!}{8!!} & \cdots & \frac{(4n+5)!!}{(4n+6)!!}-\frac{(2n+7)!!}{(2n+8)!!} \\ \vdots & \vdots & & \vdots \\ \frac{(2n+3)!!}{(2n+4)!!}-\frac{(2n+1)!!}{(2n+2)!!} & \frac{(2n+5)!!}{(2n+6)!!}-\frac{(2n+3)!!}{(2n+4)!!} & \cdots & \frac{(4n+5)!!}{(4n+6)!!}-\frac{(4n+3)!!}{(4n+4)!!} \end{vmatrix}$$

$$
= (2\pi)^{n+2}\begin{vmatrix}
1 & 1 & \frac{1!!}{2!!} & \cdots & \frac{(2n+1)!!}{(2n+2)!!} \\
1 & \frac{1!!}{2!!} & \frac{3!!}{4!!} & \cdots & \frac{(2n+3)!!}{(2n+4)!!} \\
1 & \frac{3!!}{4!!} & \frac{5!!}{6!!} & \cdots & \frac{(2n+5)!!}{(2n+6)!!} \\
1 & \frac{5!!}{6!!} & \frac{7!!}{8!!} & \cdots & \frac{(2n+7)!!}{(2n+8)!!} \\
\vdots & \vdots & \vdots & & \vdots \\
1 & \frac{(2n+1)!!}{(2n+2)!!} & \frac{(2n+3)!!}{(2n+4)!!} & \cdots & \frac{(4n+3)!!}{(4n+4)!!} \\
1 & \frac{(2n+3)!!}{(2n+4)!!} & \frac{(2n+5)!!}{(2n+6)!!} & \cdots & \frac{(4n+5)!!}{(4n+6)!!}
\end{vmatrix}
$$

$$
= (2\pi)^{n+2}\begin{vmatrix}
1 & 1 & \frac{1!!}{2!!} & \cdots & \frac{(2n+1)!!}{(2n+2)!!} \\
0 & -\frac{1!!}{2!!} & -\frac{1!!}{4!!} & \cdots & -\frac{(2n+1)!!}{(2n+4)!!} \\
0 & -\frac{1!!}{4!!} & -\frac{3!!}{6!!} & \cdots & -\frac{(2n+3)!!}{(2n+6)!!} \\
0 & -\frac{3!!}{6!!} & -\frac{5!!}{8!!} & \cdots & -\frac{(2n+5)!!}{(2n+8)!!} \\
\vdots & \vdots & \vdots & & \vdots \\
0 & -\frac{(2n-1)!!}{(2n+2)!!} & -\frac{(2n+1)!!}{(2n+4)!!} & \cdots & -\frac{(4n+1)!!}{(4n+4)!!} \\
0 & -\frac{(2n+1)!!}{(2n+4)!!} & -\frac{(2n+3)!!}{(2n+6)!!} & \cdots & -\frac{(4n+3)!!}{(4n+6)!!}
\end{vmatrix}
$$

$$
= (-2\pi)^{n+2}\begin{vmatrix}
\frac{1!!}{2!!} & \frac{1!!}{4!!} & \cdots & \frac{(2n+1)!!}{(2n+4)!!} \\
\frac{1!!}{4!!} & \frac{3!!}{6!!} & \cdots & \frac{(2n+3)!!}{(2n+6)!!} \\
\frac{3!!}{6!!} & \frac{5!!}{8!!} & \cdots & \frac{(2n+5)!!}{(2n+8)!!} \\
\vdots & \vdots & & \vdots \\
\frac{(2n-1)!!}{(2n+2)!!} & \frac{(2n+1)!!}{(2n+4)!!} & \cdots & \frac{(4n+1)!!}{(4n+4)!!} \\
\frac{(2n+1)!!}{(2n+4)!!} & \frac{(2n+3)!!}{(2n+6)!!} & \cdots & \frac{(4n+3)!!}{(4n+6)!!}
\end{vmatrix}
$$

$$
= K_0\begin{vmatrix}
1 & 1 & \cdots & 1 \\
\frac{1}{4} & \frac{3}{6} & \cdots & \frac{2n+3}{2n+6} \\
\frac{3\cdot 1}{6\cdot 4} & \frac{5\cdot 3\cdot 1}{8\cdot 6} & \cdots & \frac{(2n+5)(2n+3)}{(2n+8)(2n+6)} \\
\vdots & \vdots & & \vdots \\
\frac{(2n-1)!!}{(2n+2)\cdots 6\cdot 4} & \frac{(2n+1)!!}{(2n+4)\cdots 8\cdot 6} & \cdots & \frac{(4n+1)\cdots(2n+3)}{(4n+4)\cdots(2n+6)} \\
\frac{(2n+1)!!}{(2n+4)\cdots 6\cdot 4} & \frac{(2n+3)!!}{(2n+6)\cdots 8\cdot 6} & \cdots & \frac{(4n+3)\cdots(2n+3)}{(4n+6)\cdots(2n+6)}
\end{vmatrix}
$$

$$
= K \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \frac{1}{4} & \frac{1}{6} & \cdots & \frac{1}{2n+6} \\ \frac{1}{6} & \frac{1}{8} & \cdots & \frac{1}{2n+8} \\ \vdots & \vdots & & \vdots \\ \frac{1}{2n+2} & \frac{1}{2n+4} & \cdots & \frac{1}{4n+4} \\ \frac{1}{2n+4} & \frac{1}{2n+6} & \cdots & \frac{1}{4n+6} \end{vmatrix} = \left(\frac{1}{2}\right)^{n+1} K \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+3} \\ \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+4} \\ \vdots & \vdots & & \vdots \\ \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n+2} \\ \frac{1}{n+2} & \frac{1}{n+3} & \cdots & \frac{1}{2n+3} \end{vmatrix}
$$

where

$$
\begin{aligned}
K_0 &= (-2\pi)^{n+2} \cdot \frac{1}{2} \cdot \frac{1}{4!!} \cdot \frac{3!!}{6!!} \cdots \frac{(2n-1)!!}{(2n+2)!!} \cdot \frac{(2n+1)!!}{(2n+4)!!}, \\
K &= K_0 (-3)^{n+1} \prod_{k=3}^{n+2} \left(\frac{2k-1}{2k-4}\right)^{n-k+3}.
\end{aligned}
$$

It is clear that $K \neq 0$. Hence, $\bar{D} \neq 0$. So we have that the set of $n+2$ functions given by $\{L_{2n+4,0} - L_{0,0}, L_{2n+4,0} - L_{2,0}, \ldots, L_{2n+4,0} - L_{2n+2,0}\}$ is linearly independent. By Proposition 2.1, it is easy to know that $G(\eta)$ has at least $n+1$ zeros in $(1, +\infty)$, which means that system (1.4) has at least $n+1$ limit cycles bifurcating from the period solutions of the unperturbed one.

In what follows, we estimate the sharp upper bound of the number of zeros of $G(\eta)$. We first recall some definitions and properties of *symmetric polynomial*.

Given a real polynomial

$$
S(x) = s_n x^n + s_{n-1} x^{n-1} + \ldots + s_0 = s_n \prod_{i=1}^{n} (x - x_i),
$$

let us denote by $S^*(x)$ the reciprocated polynomial of $S(x)$, namely,

$$
S^*(x) = x^n S(x^{-1}) = s_n + s_{n-1} x + \ldots + s_0 x^n.
$$

Obviously, the zeros of $S^*(x)$ are the inverses x_i^{-1} of the zeros of $S(x)$.

A polynomial $S_k(x)$ of degree k is called a symmetric (or mirror) polynomial if $S_k^*(x) = S_k(x)$. Therefore, $S_k(x)$ is symmetric if and only if

$$
S_k(x) = \sum_{i=0}^{k} s_i x^i, \quad s_i = s_{k-i},\ i = 0, 1, \ldots, k.
$$

The following result, which will be used later, is based on Lemma 3.3.

Lemma 3.4. *Let*

$$
h(\omega^2) = \sum_{i=0}^{n+2} A_{2i,0} (1+\omega^2)^{2i} (1-\omega^2)^{2n-2i+4},
$$

then the leading coefficient and the constant term of $h(\omega^2)$ are both zeros, i.e. $h(\omega^2) = \omega^2 R^*_{2n+2}(\omega^2)$, where $R^*_{2n+2}(\omega^2)$ denotes a symmetric polynomial of degree $2n+2$ with respect to ω^2.

Proof. Note that $h(\omega^2)$ is a symmetric polynomial with respect to ω^2, whose constant term is $\sum_{i=0}^{n+2} A_{2i,0}$. It follows directly from Lemma 3.3 that $h(\omega^2) = \omega^2 R^*_{2n+2}(\omega^2)$, the symmetry of coefficients of $R^*_{2n+2}(\omega^2)$ follows directly from $h(\omega^2)$. This completes the proof. $\square$

Using the above results and making the transformation $\eta = \frac{1+\omega^2}{1-\omega^2}$ for $0 < \omega < 1$, (3.8) can be changed to

$$\begin{aligned}
G(\eta) &= 2\pi\left(\frac{A_{0,0} + A_{2,0}\eta^2 + A_{4,0}\eta^4 + \ldots + A_{2n+4,0}\eta^{2n+4}}{\sqrt{\eta^2-1}} + \sum_{i=1}^{n+1} B_{2i-1}\cdot\eta^{2i-1}\right.\\
&\qquad \left. - A_{2n+4,0}\cdot\eta^{2n+3}\right)\\
&= 2\pi\left\{\frac{1-\omega^2}{2\omega}\left[A_{0,0} + A_{2,0}\left(\frac{1+\omega^2}{1-\omega^2}\right)^2 + A_{4,0}\left(\frac{1+\omega^2}{1-\omega^2}\right)^4 + \ldots\right.\right.\\
&\qquad \left. + A_{2n+4,0}\left(\frac{1+\omega^2}{1-\omega^2}\right)^{2n+4}\right] + \sum_{i=1}^{n+1} B_{2i-1}\left(\frac{1+\omega^2}{1-\omega^2}\right)^{2i-1}\\
&\qquad \left. - A_{2n+4,0}\left(\frac{1+\omega^2}{1-\omega^2}\right)^{2n+3}\right\}\\
&= 2\pi\left[\frac{h(\omega^2)}{2\omega(1-\omega^2)^{2n+3}} + \frac{1}{(1-\omega^2)^{2n+3}}\cdot\left(\sum_{i=1}^{n+1} B_{2i-1}(1+\omega^2)^{2i-1}(1-\omega^2)^{2n-2i+4}\right.\right.\\
&\qquad \left.\left. - A_{2n+4,0}(1+\omega^2)^{2n+3}\right)\right]\\
&= \frac{\pi}{(1-\omega^2)^{2n+3}}\left[\omega\cdot R^*_{2n+2}(\omega^2) + R_{2n+3}(\omega^2)\right]\\
&= \tilde{G}(\omega),
\end{aligned} \tag{3.13}$$

where

$$R_{2n+3}(\omega^2) = 2\left(\sum_{i=1}^{n+1} B_{2i-1}(1+\omega^2)^{2i-1}(1-\omega^2)^{2n-2i+4} - A_{2n+4,0}(1+\omega^2)^{2n+3}\right)$$

denotes a symmetric polynomial of degree $2n+3$ with respect to ω^2. Moreover, we have the following result for $\tilde{G}(\omega)$.

Lemma 3.5. *The function $\tilde{G}(\omega)$ can be expressed as*

$$\tilde{G}(\omega) = \frac{\pi(1-\omega)}{(1+\omega)^{2n+3}}\cdot g(\omega),$$

where $g(\omega)$ is a symmetric polynomial of degree $2n+2$. Then, $g(\omega)$ has at most $n+1$ simple zeros in $\omega \in (0,1)$, which means that system (1.4) has at most $n+1$ limit cycles bifurcating from the period solutions of the unperturbed one.

Proof. Note the fact that

$$\lim_{z\to 0}\int_0^{2\pi} \frac{A_{0,0} + A_{2,0}\cos^2\theta + A_{4,0}\cos^4\theta + \ldots + A_{2n+4,0}\cos^{2n+4}\theta}{\cos\theta - 1 + \frac{1}{(2n+1)z^{2n+1}}}d\theta$$

$$= \lim_{z\to 0}\int_0^{2\pi} \frac{(2n+1)z^{2n+1}\left(A_{0,0}+A_{2,0}\cos^2\theta+\ldots+A_{2n+4,0}\cos^{2n+4}\theta\right)}{(2n+1)z^{2n+1}(\cos\theta-1)+1}d\theta$$
$$=0,$$

which implies $\tilde{G}(\omega=1)=0$. Hence

$$\tilde{G}(\omega)=\frac{\pi}{(1-\omega^2)^{2n+3}}\left[\omega\cdot R^*_{2n+2}(\omega^2)+R_{2n+3}(\omega^2)\right]=\frac{\pi(1-\omega)}{(1+\omega)^{2n+3}}\cdot g(\omega).$$

Recalling the properties that $R^*_{2n+2}(\omega^2)$ and $R_{2n+3}(\omega^2)$ are both symmetric polynomials with respect to ω^2, we conclude that $g(\omega)$ is a symmetric polynomial of degree $2n+2$ with respect to ω, then we know that if $\omega_0\neq 0$ is one root of $g(\omega)=0$, so is $1/\omega_0$. Hence, the function $\tilde{G}(\omega)$ has at most $n+1$ roots in $(0,1)$, which means that system (1.4) has at most $n+1$ limit cycles bifurcating from the period solutions of the unperturbed one. $\square$

Up to now, we see that system (1.4) has at most $n+1$ limit cycles, and the upper bound can be reached for all n. The proof of statement (a) of Theorem 1.1 is finished.

4. Proof of Theorem 1.1 (b)

The goal of this section is to investigate the number of limit cycles of system (1.5) which bifurcate from the period solutions of the unperturbed system.

Under the polar coordinate transformation, system (1.5) can be changed to

$$\frac{dr}{d\theta}=F_0(\theta,r)+\varepsilon F_1(\theta,r)+O(\varepsilon^2), \tag{4.1}$$

where $F_0(\theta,r)=r^{2n+3}\cos\theta\sin\theta$ and

$$F_1(\theta,r)=r^{2n+3}\sum_{i+j=2n+3}\left(c_{i,j}\cos^{i+1}\theta\sin^j\theta+d_{i,j}\cos^i\theta\sin^{j+1}\theta\right)$$
$$+r^{4n+5}\sum_{i+j=2n+3}\left(c_{i,j}\cos^{i+1}\theta\sin^{j+2}\theta-d_{i,j}\cos^{i+2}\theta\sin^{j+1}\theta\right).$$

Equation $(4.1)_{\varepsilon=0}$ has the periodic solutions $r(\theta,z)=\left((n+1)(\cos^2\theta-1)+\frac{1}{z^{2n+2}}\right)^{-\frac{1}{2n+2}}$ satisfying $r(0,z)=z$ for $0<z<(n+1)^{-\frac{1}{2n+2}}$. The corresponding variational differential equation

$$\frac{dM}{d\theta}=\frac{\partial}{\partial r}F_0(\theta,r(\theta,z))M,$$

with $M_z(0,z)=1$ and has the fundamental solution

$$M_z(\theta,z)=\left[(n+1)z^{2n+2}(\cos^2\theta-1)+1\right]^{-\frac{2n+3}{2n+2}}.$$

Next, a straightforward calculation leads to

$$\mathcal{F}(z)=\int_0^{2\pi}M_z^{-1}(\theta,z)\cdot F_1(\theta,r)d\theta \tag{4.2}$$

$$= z^{2n+3}\Bigg\{\int_0^{2\pi}\sum_{i+j=2n+3}\left(c_{i,j}\cos^{i+1}\theta\sin^j\theta + d_{i,j}\cos^i\theta\sin^{j+1}\theta\right)d\theta$$

$$+\int_0^{2\pi}\sum_{i+j=2n+3}\frac{c_{i,j}\cos^{i+1}\theta\sin^{j+2}\theta - d_{i,j}\cos^{i+2}\theta\sin^{j+1}\theta}{(n+1)(\cos^2\theta+\lambda)}d\theta\Bigg\}$$

$$= z^{2n+3}\left(\overline{A}_1 + \frac{\overline{A}_2}{n+1}\right), \tag{4.3}$$

where $\lambda = -1 + \left[(n+1)z^{2n+2}\right]^{-1} \in (0, +\infty)$, and

$$\overline{A}_1 = c_{2n+3,0}I_{2n+4,0} + d_{0,2n+3}I_{0,2n+4} + \sum_{k=1}^{n+1}(c_{2k-1,2n-2k+4} + d_{2k,2n-2k+3})I_{2k,2n-2k+4},$$

$$\overline{A}_2 = \sum_{k=1}^{n+2}(c_{2k-1,2n-2k+4} - d_{2k-2,2n-2k+5})J_{2k,2n-2k+6} \tag{4.4}$$

with

$$I_{2i,2j} = \int_0^{2\pi}\cos^{2i}\theta\sin^{2j}\theta d\theta$$

for $i = 0, 1, \ldots, n+2; j = n-i+2$, and

$$J_{2i,2j} = \int_0^{2\pi}\frac{\cos^{2i}\theta\sin^{2j}\theta}{\cos^2\theta+\lambda}d\theta$$

for $i = 1, 2, \ldots, n+2; j = n-i+3$.

Then the problem of the number of zeros of $\mathcal{F}(z)$ is equivalent to the number of zeros of $\overline{A}_1 + \frac{\overline{A}_2}{n+1}$. In the following, we list an important lemma which will be used in the derivation of the formulas $\overline{A}_1$ and $\overline{A}_2$.

Lemma 4.1. *For the above integrals* $I_{2i,2j}$ $(i = 0, 1, \ldots, n+2; j = n-i+2)$ *and* $J_{2i,2j}$ $(i = 1, 2, \ldots, n+2; j = n-i+3)$*, we have*

$$I_{2i,2j} = \sum_{r=0}^{j}\binom{j}{r}(-1)^r I_{2i+2r,0}, \quad J_{2i,2j} = \sum_{r=0}^{j}\binom{j}{r}(-1)^r J_{2i+2r,0}. \tag{4.5}$$

Moreover,

$$J_{2m,0} = 2\pi\left(\frac{(-\lambda)^m}{\sqrt{\lambda^2+\lambda}} + (-\lambda)^{m-1} + \sum_{r=0}^{m-2}D_r^{(m)}\lambda^r\right), \quad m \geq 1, \tag{4.6}$$

where $D_r^{(m)} = (-1)^r\frac{(2m-2r-3)!!}{(2m-2r-2)!!}$.

Proof. Here we only prove the formula (4.6), the discussions for $I_{2i,2j}$ and $J_{2i,2j}$ are closely similar to Lemma 3.1, so we omit them.

Note that

$$\begin{aligned}
J_{2m,0} &= \int_0^{2\pi} \frac{(\cos^2\theta+\lambda-\lambda)^m}{\cos^2\theta+\lambda} d\theta \\
&= \sum_{k=0}^{m} \binom{m}{k} (-\lambda)^k \int_0^{2\pi} (\cos^2\theta+\lambda)^{m-k-1} d\theta \\
&= \frac{2\pi(-\lambda)^m}{\sqrt{\lambda^2+\lambda}} + 2\pi m(-\lambda)^{m-1} + \sum_{k=0}^{m-2} \binom{m}{k} (-1)^k \cdot \left(\sum_{t=0}^{m-k-1} \binom{m-k-1}{t} \lambda^{m-t-1} I_{2t,0} \right) \\
&= \frac{2\pi(-\lambda)^m}{\sqrt{\lambda^2+\lambda}} + 2\pi(-\lambda)^{m-1} + \sum_{k=0}^{m-2} \binom{m}{k} (-1)^k \cdot \left(\sum_{t=1}^{m-k-1} \binom{m-k-1}{t} \lambda^{m-t-1} I_{2t,0} \right).
\end{aligned}$$

By using the well-known formula $I_{2t,0} = 2\pi\frac{(2t-1)!!}{(2t)!!}$, we perform the computation and obtain

$$J_{2m,0} = 2\pi \left(\frac{(-\lambda)^m}{\sqrt{\lambda^2+\lambda}} + (-\lambda)^{m-1} + \sum_{r=0}^{m-2} D_r^{(m)} \lambda^r \right),$$

where $D_r^{(m)} = \sum_{k=0}^{r} (-1)^k \binom{m}{k} \binom{m-k-1}{m-r-1} \frac{(2m-2r-3)!!}{(2m-2r-2)!!} = (-1)^r \frac{(2m-2r-3)!!}{(2m-2r-2)!!}$. Hence the proof of the formula (4.6) is finished. □

Remark 4.1. Some explicit expressions of $J_{2m,0}$ are:

$$\begin{aligned}
J_{2,0} &= -\frac{2\pi\lambda}{\sqrt{\lambda^2+\lambda}} + 2\pi, \\
J_{4,0} &= \frac{2\pi\lambda^2}{\sqrt{\lambda^2+\lambda}} - 2\pi\lambda + \pi, \\
J_{6,0} &= -\frac{2\pi\lambda^3}{\sqrt{\lambda^2+\lambda}} + 2\pi\lambda^2 - \pi\lambda + \frac{3}{4}\pi, \\
J_{8,0} &= \frac{2\pi\lambda^4}{\sqrt{\lambda^2+\lambda}} - 2\pi\lambda^3 + \pi\lambda^2 - \frac{3}{4}\pi\lambda + \frac{5}{8}\pi.
\end{aligned}$$

It follows from Lemma 4.1 and (4.4) that

$$\begin{aligned}
\overline{A}_1 &= c_{2n+3,0} I_{2n+4,0} + d_{0,2n+3} I_{0,2n+4} + \sum_{k=1}^{n+1} (c_{2k-1,2n-2k+4} + d_{2k,2n-2k+3}) I_{2k,2n-2k+4} \\
&= c_{2n+3,0} I_{2n+4,0} + d_{0,2n+3} \cdot \left(\sum_{r=0}^{n+2} (-1)^r \binom{n+2}{r} I_{2r,0} \right) \\
&\quad + \sum_{k=1}^{n+1} (c_{2k-1,2n-2k+4} + d_{2k,2n-2k+3}) \cdot \left(\sum_{r=0}^{n-k+2} (-1)^r \binom{n-k+2}{r} I_{2k+2r,0} \right) \\
&= H_{0,0} \cdot I_{0,0} + H_{2,0} \cdot I_{2,0} + H_{4,0} \cdot I_{4,0} + \ldots + H_{2n+4,0} \cdot I_{2n+4,0} \\
&= 2\pi \left(H_{0,0} + \sum_{k=1}^{n+2} H_{2k,0} \cdot \frac{(2k-1)!!}{(2k)!!} \right),
\end{aligned} \tag{4.7}$$

where the coefficients $H_{2m,0}$ are the followings:

$$H_{0,0} = d_{0,2n+3}, \quad m = 0,$$

$$\begin{aligned}
H_{2m,0} &= \sum_{k=1}^{m}(c_{2k-1,2n-2k+4}+d_{2k,2n-2k+3})\binom{n-k+2}{m-k}(-1)^{m-k}\\
&\quad + d_{0,2n+3}\binom{n+2}{m}(-1)^{m}, \quad 1\le m\le n+1,\\
H_{2n+4,0} &= \sum_{k=1}^{n+1}(c_{2k-1,2n-2k+4}+d_{2k,2n-2k+3})(-1)^{n-k+2}\\
&\quad + d_{0,2n+3}(-1)^{n+2}+c_{2n+3,0}, \quad m=n+2.
\end{aligned}$$

Based on Lemma 4.1 and (4.4), we can further obtain that

$$\begin{aligned}
\overline{A}_2 &= \sum_{k=1}^{n+2}(c_{2k-1,2n-2k+4}-d_{2k-2,2n-2k+5})J_{2k,2n-2k+6}\\
&= \sum_{k=1}^{n+2}(c_{2k-1,2n-2k+4}-d_{2k-2,2n-2k+5})\left(\sum_{r=0}^{n-k+3}(-1)^r\binom{n-k+3}{r}J_{2k+2r,0}\right)\\
&= K_{2,0}\cdot J_{2,0}+K_{4,0}\cdot J_{4,0}+\ldots+K_{2n+6,0}\cdot J_{2n+6,0}\\
&= 2\pi\left(\frac{\overline{K}_1\lambda+\overline{K}_2\lambda^2+\ldots+\overline{K}_{n+3}\lambda^{n+3}}{\sqrt{\lambda^2+\lambda}}+\sum_{i=0}^{n+1}E_i\lambda^i+F\lambda^{n+2}\right),
\end{aligned} \tag{4.8}$$

where the coefficients $K_{2m,0}$ are the followings:

$$\begin{aligned}
K_{2m,0} &= \sum_{k=1}^{m}(c_{2k-1,2n-2k+4}-d_{2k-2,2n-2k+5})\binom{n-k+3}{m-k}(-1)^{m-k},\ 1\le m\le n+2,\\
K_{2n+6,0} &= \sum_{k=1}^{n+2}(c_{2k-1,2n-2k+4}-d_{2k-2,2n-2k+5})(-1)^{n-k+3},\ m=n+3
\end{aligned}$$

and $\overline{K}_i = K_{2i,0}(-1)^i$ $(i=1,\ldots,n+3)$, and $E_i = K_{2i+2,0}(-1)^i+\sum_{m=i+2}^{n+3}K_{2m,0}D_i^{(m)}$ $(i=0,1,\ldots,n+1)$, and $F=K_{2n+6,0}(-1)^{n+2}$.

In order to simplify the expression (4.8), we give the following lemmas.

Lemma 4.2. *The coefficients of the polynomial* $\sum_{i=1}^{n+3}\overline{K}_i\lambda^i$ *satisfy the following equality*

$$\sum_{i=1}^{n+3}(-1)^{n-i+3}\overline{K}_i = 0. \tag{4.9}$$

Proof. According to the last equality in (4.4) and the penultimate equality in (4.8), we have

$$\begin{aligned}
&\sum_{k=1}^{n+2}(c_{2k-1,2n-2k+4}-b_{2k-2,2n-2k+5})\cdot\frac{\cos^{2k}\theta\sin^{2n-2k+6}\theta}{\cos^2\theta+\lambda}\\
=&\frac{K_{2,0}\cos^2\theta+K_{4,0}\cos^4\theta+\ldots+K_{2n+6,0}\cos^{2n+6}\theta}{\cos^2\theta+\lambda},
\end{aligned} \tag{4.10}$$

let $\cos\theta=1$ in the above expression, one can obtain $\sum_{i=1}^{n+3}K_{2i,0}=0$. Hence, $\sum_{i=1}^{n+3}(-1)^{n-i+3}\overline{K}_i=(-1)^{n+3}\sum_{i=1}^{n+3}K_{2i,0}=0$. □

Lemma 4.3. *Let*

$$\psi(\omega^2) = \sum_{i=1}^{n+3} \overline{K}_i \cdot (\omega^2)^{i-1}(1-\omega^2)^{n-i+3},$$

then the leading coefficient of $\psi(\omega^2)$ *is zero, i.e.* $\psi(\omega^2) = R^*_{n+1}(\omega^2)$, *where* $R^*_{n+1}(\omega^2)$ *denotes a polynomial of degree* $n+1$ *with respect to* ω^2.

Proof. The conclusion follows directly from Lemma 4.2. □

Next, we first study the number of zeros of $\overline{A}_2$. In order to make the computation easier we need to make the transformation $\lambda = \frac{\omega^2}{1-\omega^2}$ for $0 < \omega < 1$. Using (4.8) and Lemma 4.3, we have

$$\begin{aligned}
\overline{A}_2 &= 2\pi\left(\frac{\overline{K}_1\lambda + \overline{K}_2\lambda^2 + \ldots + \overline{K}_{n+3}\lambda^{n+3}}{\sqrt{\lambda^2+\lambda}} + \sum_{i=0}^{n+1} E_i\lambda^i + F\lambda^{n+2}\right) \\
&= 2\pi\left\{\frac{1-\omega^2}{\omega}\left[\overline{K}_1\left(\frac{\omega^2}{1-\omega^2}\right) + \overline{K}_2\left(\frac{\omega^2}{1-\omega^2}\right)^2 + \ldots + \overline{K}_{n+3}\left(\frac{\omega^2}{1-\omega^2}\right)^{n+3}\right]\right. \\
&\quad \left. + E_0 + E_1\left(\frac{\omega^2}{1-\omega^2}\right) + \ldots + E_{n+1}\left(\frac{\omega^2}{1-\omega^2}\right)^{n+1} + F\left(\frac{\omega^2}{1-\omega^2}\right)^{n+2}\right\} \\
&= \frac{2\pi}{(1-\omega^2)^{n+2}}\left(\omega\cdot\psi(\omega^2) + \sum_{i=0}^{n+1} E_i\cdot(\omega^2)^i(1-\omega^2)^{n-i+2} + F\cdot(\omega^2)^{n+2}\right) \\
&= \frac{2\pi}{(1-\omega^2)^{n+2}}\left(\omega\cdot R^*_{n+1}(\omega^2) + R_{n+2}(\omega^2)\right),
\end{aligned} \tag{4.11}$$

where $R_{n+2}(\omega^2)$ denotes a polynomial of degree $n+2$ with respect to ω^2. Moreover, we can obtain the following result.

Lemma 4.4. *The formula* $\overline{A}_2$ *can be expressed as*

$$\overline{A}_2 = \frac{2\pi(1-\omega)}{(1+\omega)^{n+2}}\cdot\tilde{A}(\omega), \tag{4.12}$$

where $\tilde{A}(\omega)$ *denotes a polynomial of degree* $n+1$ *with respect to* ω.

Proof. Refer to (4.8) and note that

$$\begin{aligned}
&\lim_{z\to 0}\int_0^{2\pi} \frac{K_{2,0}\cos^2\theta + K_{4,0}\cos^4\theta + \ldots + K_{2n+6,0}\cos^{2n+6}\theta}{\cos^2\theta - 1 + \frac{1}{(n+1)z^{2n+2}}}d\theta \\
=&\lim_{z\to 0}\int_0^{2\pi} \frac{(n+1)z^{2n+2}\left(K_{2,0}\cos^2\theta + K_{4,0}\cos^4\theta + \ldots + K_{2n+6,0}\cos^{2n+6}\theta\right)}{(n+1)z^{2n+2}(\cos^2\theta - 1) + 1}d\theta \\
=&0.
\end{aligned}$$

This implies that $\overline{A}_2(\omega = 1) = 0$. Hence

$$\overline{A}_2 = \frac{2\pi}{(1-\omega^2)^{n+2}}\left(\omega\cdot R^*_{n+1}(\omega^2) + R_{n+2}(\omega^2)\right) = \frac{2\pi(1-\omega)}{(1+\omega)^{n+2}}\cdot\tilde{A}(\omega),$$

where $\tilde{A}(\omega)$ denotes a polynomial of degree $n+1$ with respect to ω. □

Proof of Theorem 1.1 (b). It follows from (4.7) and Lemma 4.4 that

$$
\begin{aligned}
&\overline{A}_1+\frac{\overline{A}_2}{n+1}=2\pi\left(H_{0,0}+\sum_{k=1}^{n+2}H_{2k,0}\cdot\frac{(2k-1)!!}{(2k)!!}\right)+\frac{2\pi(1-\omega)}{(n+1)(1+\omega)^{n+2}}\cdot\tilde{A}(\omega)\\
&=\frac{2\pi}{(n+1)(1+\omega)^{n+2}}\left[(n+1)\left(H_{0,0}+\sum_{k=1}^{n+2}H_{2k,0}\cdot\frac{(2k-1)!!}{(2k)!!}\right)(1+\omega)^{n+2}+(1-\omega)\tilde{A}(\omega)\right]\\
&=\frac{2\pi}{(n+1)(1+\omega)^{n+2}}\cdot A^*(\omega),
\end{aligned}\tag{4.13}
$$

where $A^*(\omega)$ denotes a polynomial of degree $n+2$ with respect to ω. Hence, $\overline{A}_1+\frac{\overline{A}_2}{n+1}$ has at most $n+2$ zeros in $\omega\in(0,1)$.

On the other hand, it is obvious that the set of monomials $\{1,\omega,\omega^2,\dots,\omega^{n+2}\}$ is linearly independent. By Proposition 2.1, we know that $A^*(\omega)$ has at least $n+2$ zeros in $(0,1)$. Hence, statement (b) of Theorem 1.1 is proved.

Herewith, we complete the proof of Theorem 1.1. □

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