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INPUT-TO-STATE STABILITY OF IMPULSIVE SYSTEMS WITH HYBRID DELAYED IMPULSE EFFECTS*

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Abstract The goal of this paper is to study properties of input-to-state stability (ISS) and integral input-to-state stability ($iISS$) of impulsive systems with hybrid delayed impulses, and a set of Lyapunov-based sufficient conditions ensuring $ISS/iISS$ properties are obtained. Those conditions reveal the effects of hybrid delayed impulses on $ISS/iISS$ and establish the relationship between impulsive frequency and the time delay existing in hybrid impulses. When the continuous dynamics of the system are stabilizing, the ISS property can be retained under the impulse scheme even if there exist destabilizing impulses. Conversely, when the impulse dynamics are stabilizing, but the continuous dynamics are not, the ISS property can be obtained if the interval between impulses are not overly long. Two illustrative examples are presented, with their numerical simulations, to demonstrate the effectiveness of the main results.

Keywords Impulsive systems, hybrid delayed impulses, input-to-state stability (ISS), integral input-to-state stability ($iISS$), Lyapunov method.

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1. Introduction

The concept of ISS introduced by Sontag in [36] has been proved useful in characterizing the effects of external inputs. Originally introduced for continuous-time systems, they were also studied for a variety of systems such as discrete-time systems, hybrid systems and switched systems; see [1,15–17,22,27]. Roughly speaking, the ISS property means that no matter what the size of the initial state is, the state will eventually approach a neighborhood of the origin whose size is proportional to the magnitude of the input. Some interesting results explored $ISS/iISS$ properties have been introduced in recent years. For example, in [32], the authors presented a new Lyapunov method for $ISS/iISS$ of impulsive systems and the approach was proposed on the basis of an indefinite Lyapunov function rather than a negative

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definite one; In [4], sufficient conditions which ensuring ISS and $iISS$ properties for non-autonomous time-delay systems were derived by using locally Lipschitz continuous exponential ISS Lyapunov(-Razumikhin) functions; In [33], the problem of the input to state stability (ISS) for nonlinear systems with time-delay was investigated, and a continuously differentiable Lyapunov-Krasovskii functional with indefinite derivative was introduced to derive the ISS of the systems, which generalizes the classic Lyapunov-Krasovskii functional with positive definite derivative; The ISS properties were studied in [16] for impulsive switched systems, where both types of impulses, stabilizing impulses and destabilizing impulses are considered.

Impulsive systems describe processes that combine continuous and discontinuous behavior [9, 32, 37]. The continuous behavior is typically described by differential equations, and the discontinuous behavior is instantaneous state jumps that occur at given time instants, also referred to as impulses. In recent years, impulsive control has received much attention because it is discontinuous and has a simple structure, and only discrete control is needed to obtain the desired performance; see [6, 18, 19, 38, 41]. Impulsive systems are closely related to hybrid systems [5, 23, 24, 27] and switched systems [11, 20, 25, 26, 34, 43], and a variety of applications can be found in logistics, robotics, population dynamics, etc. Especially, due to the facts that impulsive systems with external inputs arise naturally from a number of applications such as in control systems with communication constraints, control algorithms of uncertain systems and network control systems with scheduling protocol, it is important to guarantee the impulsive system to be ISS and $iISS$ when it is affected by some external inputs. Hence, it is of great practical significance to investigate the $ISS/iISS$ properties of impulsive systems and it has become one of the hot issues in control theory; see [2, 3, 6, 28, 35]. It is worth mentioning that the concepts of $ISS/iISS$ of impulsive systems were proposed in [7, 8, 12, 13, 40]. They developed the Lyapunov method to impulsive systems and established some conditions for $ISS/iISS$ properties by controlling the frequency of impulse occurrence. Two constants that are called rate coefficients were used to characterize the behaviours of ISS-Lyapunov function along the trajectories of impulsive system during continuous flows (constant c) and impulsive jumps (constant d). The positive values of rate coefficients correspond to the case of positive impact of flows/jumps onto ISS property, and vice versa, if c (or d) is negative, then it means that the corresponding flows (or jumps) play against stability.

With the development of impulsive control theory, increasing attention has been paid to the study of dynamics and controller design of impulsive systems in which the impulses involve time delays which are sometimes called delayed impulses, see [21, 30, 39, 42, 44]. Such kind of impulses describe a phenomenon where impulsive transients depend on not only their current but also historical states of the system. For example, [44] studied the input-to-state stability (ISS) and integral input-to-state stability ($iISS$) of nonlinear systems with delayed impulses and obtained some sufficient conditions ensuring $ISS/iISS$ of the addressed systems by using Lyapunov method and the technique proposed in [12]; [19] considered nonlinear differential systems with state-dependent delayed impulses, and established general and applicable results for uniform stability, uniform asymptotic stability, and exponential stability of the systems by using the impulsive control theory and some comparison arguments; [10] concerned with the problem of exponential stability for a class of impulsive switched nonlinear time-delay systems with delayed impulse effects. The derived results not only characterise the effects of delayed impulse, time delay and switching on nonlinear systems, but also removed some restriction conditions.

Motivated by the above discussion, in this paper, we further study the $ISS/iISS$ for impulsive systems with hybrid delayed impulses. First, we mainly focus on the case that the rate coefficient $c \in \mathbb{R}_+$ which implies that the continuous dynamics of the system are stabilizing and $d_k \in \mathbb{R}$ which implies that the hybrid impulses (i.e., stabilizing impulses and destabilizing impulses) are fully considered. Second, the case that the rate coefficients $c < 0, d_k \equiv d \in \mathbb{R}_+$, which means that the continuous dynamics of the system are destabilizing but the impulses are stabilizing, is considered. Then, we establish the ADT condition for the second case and the ADT condition shows that the average dwell time must not be overly long intervals between impulses. The rest of the paper is organized as follows. In Section 2, the problem is formulated and some notations and definitions are given. In Section 3, we present the main results. Examples are given in Section 4, and conclusion follows in Section 5.

2. Preliminaries

Notations. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ the set of all nonnegative real numbers, $\mathbb{Z}_+$ the set of positive integer numbers, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ the n-dimensional and $n\times m$-dimensional real spaces equipped with the Euclidean norm $|\cdot|$ respectively, and $\|\cdot\|_J$ denote the supremum norm on an interval $J \in \mathbb{R}$. Let $\alpha \vee \beta$ and $\alpha \wedge \beta$ denote the maximum and minimum value of α and β, respectively. Let $\mathcal{K}_\infty = \{\alpha \in C(\mathbb{R}_+, \mathbb{R}_+)|\alpha(0) = 0, \alpha(r)$ is strictly increasing in r, and $\alpha(r) \to \infty$ as $r \to \infty\}$, $\mathcal{KL} = \{\beta \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)|\beta(r, t)$ is in class $\mathcal{K}$ w.r.t. r for each fixed $t \geq 0$, and $\beta(r, t)$ is strictly decreasing to 0 as $t \to \infty$ for each fixed $r \geq 0\}$.

Consider the system with delayed impulses of the form

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)), t \geq t_0 \geq 0, t \neq t_k, \\ x(t) = g(x(t^- - \tau(t)), u(t^- - \tau(t))), t = t_k, k \in \mathbb{Z}_+, \end{cases} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $\dot{x}(t)$ denotes the right-hand derivative of $x(t)$, $u \in \mathcal{U}$, $\mathcal{U}$ denotes the set of measurable locally bounded functions in $\mathbb{R}^m$. $\tau(t)$ is time-varying delay with $0 \leq \tau(t) \leq l$, l is a constant. f and g :$\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are continuous and $f(0,0) = g(0,0) \equiv 0, t \geq t_0$. To prevent the occurrence of accumulation points, such as Zeno phenomenon [31], we always assume that the impulse time sequence $\{t_k, k \in \mathbb{Z}_+\}$ satisfies $0 \leq t_0 < t_1 < \cdots < t_k \to \infty$ as $k \to \infty$ (t_1 is the first impulse time). Let $\mathcal{F}_\tau$ denote the set of all impulsive time sequences satisfying $t_k - t_{k-1} \geq \tau_k$, $k \in \mathbb{Z}_+$, where $\tau_k = \tau(t_k)$. All signals in this paper (including the state x and the input u) are assumed to be right-continuous and to have left limits at all times. Given a sequence $\{t_k, k \in \mathbb{Z}_+\}$ and a pair of time (t, s) satisfying $t > s > t_0$, let $N(t, s)$ denote the number of impulse times in the semi-open interval $[s, t)$. Note that the continuity of f and g and a fact that system (2.1) is an ODE which is continuous on each interval $[t_{n-1}, t_n)$. We assume the vector field f satisfies suitable conditions so the solutions exist in relevant time intervals. These conditions can be formulated using standard conditions such as conditions $(H_1) - (H_3)$ in [29]. Denote by $x(t) \doteq x(t, t_0, x_0)$ the solution of the system (2.1).

Definition 2.1 ([14]). For the prescribed sequence $\{t_k, k \in \mathbb{Z}_+\}$, the system (2.1) is said to be *input-to-state stable (ISS)* if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for every initial condition (t_0, x_0) and input $u \in \mathcal{U}$, the corresponding solution of (2.1) satisfies

$$|x(t)| \leq \beta(|x_0|, t - t_0) + \gamma(\|u\|_{[t_0,t]}), t \geq t_0.$$

It is said to be *uniform ISS* over a given class $\mathcal{H}$ of admissible sequence of impulse times if the ISS property expressed by the above inequality holds for every sequence in $\mathcal{H}$, with functions β and γ that are independent of the choice of the sequence.

Definition 2.2 ([14]). For the prescribed sequence $\{t_k, k \in \mathbb{Z}_+\}$, the system (2.1) is said to be *integral-input-to-state stable (iISS)* if there exist functions $\beta \in \mathcal{KL}$ and $\alpha, \gamma \in \mathcal{K}_\infty$ such that for every initial condition (t_0, x_0) and input $u \in \mathcal{U}$, the corresponding solution of (2.1) satisfies

$$\alpha\big(|x(t)|\big) \leq \beta\big(|x_0|, t - t_0\big) + \int_{t_0}^{t} \gamma\big(|u(s)|\big)ds + \sum_{k \in N(t,t_0)} \gamma\big(|u({t_k}^- - \tau_k)|\big), t \geq t_0.$$

The notion of *uniform iSS* over a given class $\mathcal{S}$ of impulse time sequence is defined in the same way as for ISS.

Definition 2.3 ([8]). A function $V : \mathbb{R}^n \to \mathbb{R}_+$ is said to be an *exponential ISS-Lyapunov function* for (2.1) with rate coefficients $c, d_k \in \mathbb{R}$ if V is locally Lipschitz, positive definite, radially unbounded, and satisfies

$$\nabla V(x) \cdot f(x, u) \leq -cV(x) + \mathcal{X}(|u|), \forall x \; a.e., \forall u, \tag{2.2}$$

$$V(g_k(x, u)) \leq \exp(-d_k)V(x) + \mathcal{X}(|u|), \forall x, u, \forall k \in \mathbb{Z}_+ \tag{2.3}$$

for some $\mathcal{X} \in \mathcal{K}_\infty$.

3. Main results

In this section, we shall present some sufficient conditions for $ISS/iISS$ of system (2.1) according to the different ranges of coefficients c and d_k. The idea is inspired by the works of Dashkovskiy et al in [7, 8]. Firstly, we consider the case that the rate coefficients $c \in \mathbb{R}_+, d_k \in \mathbb{R}$.

Theorem 3.1 (Uniformly ISS). *Let V be an exponential ISS-Lyapunov function of system (2.1) with rate coefficients $c \in \mathbb{R}_+$ and $d_k \in \mathbb{R}$. If there exists a constant $M \geq 0$ such that*

$$\sum_{k=1}^{N(t,t_0)} \big(- d_k + c\tau_k\big) \leq M, \;\; \forall t \geq t_0. \tag{3.1}$$

Then system (2.1) is uniformly ISS over the class $\mathcal{F}_\tau$.

Proof. Assume that $x(t) = x(t, t_0, x_0)$ be the solution of system (2.1) with the initial value (t_0, x_0), and $u \in \mathcal{U}$ be a given input function. Choose $\varepsilon > 0, \varepsilon_k > 0$ such that $c > \bar{c} = \frac{c}{1+\varepsilon}, -d_k < \bar{d}_k = -d_k + \varepsilon_k$ and $\sum_{k \in \mathbb{Z}_+} \varepsilon_k = m < \infty$. Then using the standard argument, we have

$$\nabla V(x) \cdot f(x, u) \leq -\bar{c}V(x), \tag{3.2}$$

when $(-\bar{c}+c)V(x) \geq \mathcal{X}(|u|), \forall x\ a.e., \forall u$, and

$$V(x(t_k)) \leq \exp(\bar{d}_k)V\big(x(t_k^- - \tau_k)\big), \tag{3.3}$$

when $\big(\exp(\bar{d}_k) - \exp(-d_k)\big)V(x) \geq \mathcal{X}(|u(t_k^- - \tau_k)|), \forall x\ a.e., \forall u$.

Let $\check{t}_1 = \inf\big\{t \geq t_0 : V(x(t)) \leq \rho\mathcal{X}(\|u\|_{[t_0,t]})\big\} \leq \infty$, where $\rho = \frac{1}{c-\bar{c}} \vee \frac{1}{\exp(\bar{d}_k)-\exp(-d_k)}$. It's clear that $\check{t}_1 > t_0$. We consider two cases: $\check{t}_1 < \infty$ and $\check{t}_1 = \infty$. If $\check{t}_1 < \infty$, then let $N(\check{t}_1, t_0) = N$. When $N = 0$, it holds that

$$V(x(t)) \leq \exp\big(-\bar{c}(t-t_0)\big)V_0, \forall t \in [t_0, \check{t}_1),$$

where $V_0 = V(x_0)$. When $N > 0$, we get $V(x(t)) \leq \exp\big(-\bar{c}(t-t_0)\big)V_0, \forall t \in [t_0, t_1)$. In view of (3.3) , it gives

$$\begin{aligned} V(x(t_1)) &\leq \exp(\bar{d}_1)V\big(x(t_1-\tau_1)\big) \\ &= \exp\big(\bar{d}_1\big)\begin{cases} \exp\big(-\bar{c}(t_1-\tau_1-t_0)\big)V_0, & t_0 \leq t_1-\tau_1 < t_1 \\ V_0, & t_1-\tau_1 < t_0 \end{cases} \\ &\leq \exp\big(\bar{d}_1 + \bar{c}\tau_1 - \bar{c}(t_1-t_0)\big)V_0. \end{aligned}$$

Then it directly follows that

$$V(x(t)) \leq \exp\big(\bar{d}_1 + \bar{c}\tau_1 - \bar{c}(t-t_0)\big)V_0, \forall t \in [t_1, t_2 \wedge \check{t}_1).$$

If $t_2 > \check{t}_1$, then

$$V(x(t)) \leq \exp\big(\bar{d}_1 + \bar{c}\tau_1 - \bar{c}(t-t_0)\big)V_0, \forall t \in [t_1, \check{t}_1).$$

Or else, then

$$V(x(t)) \leq \exp\big(\bar{d}_1 + \bar{c}\tau_1 - \bar{c}(t-t_0)\big)V_0, \forall t \in [t_1, t_2).$$

Thus, in view of (3.3) and the fact $t_k - t_{k-1} \geq \tau_k$, it gives that

$$\begin{aligned} V(x(t_2)) &\leq \exp(\bar{d}_2)V\big(x(t_2-\tau_2)\big) \\ &= \exp(\bar{d}_2)\begin{cases} \exp\big(\bar{d}_1 + \bar{c}\tau_1 - \bar{c}(t_2-\tau_2-t_0)\big)V_0, & t_1 \leq t_2-\tau_2 < t_2 \\ \exp\big(-\bar{c}(t_2-\tau_2-t_0)\big)V_0, & t_0 \leq t_2-\tau_2 < t_1 \\ V_0, & t_2-\tau_2 < t_0 \end{cases} \\ &\leq \exp\big(\bar{d}_1 + \bar{d}_2 + \bar{c}\tau_1 + \bar{c}\tau_2 - \bar{c}(t_2-t_0)\big)V_0, \end{aligned}$$

which implies that

$$V(x(t)) \leq \exp\big(\bar{d}_1 + \bar{d}_2 + \bar{c}\tau_1 + \bar{c}\tau_2 - \bar{c}(t-t_0)\big)V_0, \quad \forall t \in [t_2, t_3 \wedge \check{t}_1).$$

Repeating the similar argument, we conclude that

$$V(x(t)) \leq \exp\Big(\sum_{k=0}^{N(t,t_0)}(\bar{d}_k + \bar{c}\tau_k) - \bar{c}(t-t_0)\Big)V_0, \forall t \in [t_0, \check{t}_1),$$

where $\bar{d}_0 := -\bar{c}\tau_0$.

If $\check{t}_1 = \infty$, then similar to the argument of the case that $\check{t}_1 < \infty$, one may derive that

$$V(x(t)) \leq \exp\big(\sum_{k=0}^{N(t,t_0)} (\bar{d}_k + \bar{c}\tau_k) - \bar{c}(t-t_0)\big)V_0, \forall t \in [t_0, \infty).$$

In either case, one can easily obtain that

$$V(x(t)) \leq \exp\big(\sum_{k=0}^{N(t,t_0)} (\bar{d}_k + \bar{c}\tau_k) - \bar{c}(t-t_0)\big)V_0, \forall t \in [t_0, \check{t}_1). \tag{3.4}$$

Furthermore, let $\hat{t}_1 = \inf\big\{t > \check{t}_1 : V(x(t)) \geq \rho\mathcal{X}(\|u\|_{[t_0,t]})\big\} \leq \infty$. If $\hat{t}_1 = \infty$, then it is easy to see that

$$V(x(t)) \leq \exp\big(\sum_{k=0}^{N(t,t_0)} (\bar{d}_k + \bar{c}\tau_k) - \bar{c}(t-t_0)\big)V_0 + \rho\mathcal{X}(\|u\|_{[t_0,t]}), \forall t \in [t_0, \infty).$$

Now we consider the case that $\hat{t}_1 < \infty$, which indicates that $\hat{t}_1$ must be impulse time. It follows from (2.3) that

$$V(x(\hat{t}_1)) \leq \exp(-d_{\sigma(\hat{t}_1)})V\big(x(\hat{t}_1 - \tau_{\sigma(\hat{t}_1)})\big) + \mathcal{X}(\|u\|_{[t_0,\hat{t}_1]}), \tag{3.5}$$

where $\sigma(\hat{t}_i) := \big\{m \in \mathbb{Z}_+ : \hat{t}_i = t_m\big\}$ and

$$\begin{aligned} &V\big(x(\hat{t}_1 - \tau_{\sigma(\hat{t}_1)})\big) \\ &\leq \begin{cases} \rho\mathcal{X}(\|u\|_{[t_0,\hat{t}_1-\tau_{\sigma(\hat{t}_1)}]}), & \check{t}_1 \leq \hat{t}_1 - \tau_{\sigma(\hat{t}_1)} < \hat{t}_1 \\ \exp\big(\sum_{k=1}^{N(\hat{t}_1-\tau_{\sigma(\hat{t}_1)},t_0)} (\bar{d}_k + \bar{c}\tau_k) - \bar{c}(\hat{t}_1 - \tau_{\sigma(\hat{t}_1)} - t_0)\big)V_0, & t_0 \leq \hat{t}_1 - \tau_{\sigma(\hat{t}_1)} < \check{t}_1 \\ V_0, & \hat{t}_1 - \tau_{\sigma(\hat{t}_1)} < t_0 \end{cases} \\ &\leq \exp\big(\sum_{k=0}^{N(\hat{t}_1-\tau_{\sigma(\hat{t}_1)},t_0)} (\bar{d}_k + \bar{c}\tau_k) - \bar{c}(\hat{t}_1 - \tau_{\sigma(\hat{t}_1)} - t_0)\big)V_0 + \rho\mathcal{X}(\|u\|_{[t_0,\hat{t}_1-\tau_{\sigma(\hat{t}_1)}]}). \end{aligned}$$

Combining with (3.5), it can be deduced that

$$V(x(\hat{t}_1)) \leq \exp\big(\sum_{k=0}^{N(\hat{t}_1,t_0)} (\bar{d}_k + \bar{c}\tau_k) - \bar{c}(\hat{t}_1 - t_0)\big)V_0 + \exp(\bar{d}_{\sigma(\hat{t}_1)})\rho\mathcal{X}(\|u\|_{[t_0,\hat{t}_1]}).$$

Therefore, in view of (3.4), for all $t \in [\hat{t}_1, \check{t}_2)$, we have

$$\begin{aligned} V(x(t)) &\leq \exp\big(\sum_{k=N(\hat{t}_1,t_0)}^{N(t,t_0)} (\bar{d}_k + \bar{c}\tau_k) - \bar{c}(t - \hat{t}_1)\big)V(x(\hat{t}_1)) \\ &\leq \exp\big(\sum_{k=0}^{N(t,t_0)} (\bar{d}_k + \bar{c}\tau_k) - \bar{c}(t-t_0) + (\bar{d}_{\sigma(\hat{t}_1)} + \bar{c}\tau_{\sigma(\hat{t}_1)})\big)V_0 \end{aligned}$$

$$
\begin{aligned}
&+\exp\Big(\sum_{k=N(\hat t_1,t_0)}^{N(t,t_0)}(\bar d_k+\bar c\tau_k)-\bar c(t-\hat t_1)+\bar d_{\sigma(\hat t_1)}\Big)\rho\mathcal{X}(\|u\|_{[t_0,\hat t_1]})\\
\le{}&\exp\Big(\sum_{k=0}^{N(t,t_0)}(\bar d_k+\bar c\tau_k)-\bar c(t-t_0)+(\bar d_{\sigma(\hat t_1)}+\bar c\tau_{\sigma(\hat t_1)})\Big)V_0\\
&+\exp\Big(\sum_{k=N(\hat t_1,t_0)}^{N(t,t_0)}(\bar d_k+\bar c\tau_k)-\bar c(t-\hat t_1)+(\bar d_{\sigma(\hat t_1)}+\bar c\tau_{\sigma(\hat t_1)})\Big)\rho\mathcal{X}(\|u\|_{[t_0,t]}).
\end{aligned}
$$

Repeating the similar argument as (3.5), it holds that

$$
V(x(\hat t_2))\le\exp(-d_{\sigma(\hat t_2)})V\big(x(\hat t_2-\tau_{\sigma(\hat t_2)})\big)+\mathcal{X}(\|u\|_{[t_0,\hat t_2]}),
$$

where

$$
\begin{aligned}
&V\big(x(\hat t_2-\tau_{\sigma(\hat t_2)})\big)\\
&\le\begin{cases}
\rho\mathcal{X}(\|u\|_{[t_0,\hat t_2-\tau_{\sigma(\hat t_2)}]}), & \check t_2\le\hat t_2-\tau_{\sigma(\hat t_2)}<\hat t_2,\\
\exp\big(\sum_{k=0}^{N(\hat t_2-\tau_{\sigma(\hat t_2)},t_0)}(\bar d_k+\bar c\tau_k)-\bar c(\hat t_2-\tau_{\sigma(\hat t_2)}-t_0)+(\bar d_{\sigma(\hat t_1)}+\bar c\tau_{\sigma(\hat t_1)})\big)V_0 & \\
\quad+\exp\big(\sum_{k=N(\hat t_1,t_0)}^{N(\hat t_2-\tau_{\sigma(\hat t_2)},t_0)}(\bar d_k+\bar c\tau_k)+\bar c\tau_{\sigma(\hat t_2)} & \\
\quad+(\bar d_{\sigma(\hat t_1)}+\bar c\tau_{\sigma(\hat t_1)})\big)\rho\mathcal{X}(\|u\|_{[t_0,\hat t_2-\tau_{\sigma(\hat t_2)}]}), & \hat t_1\le\hat t_2-\tau_{\sigma(\hat t_2)}<\check t_2,\\
\rho\mathcal{X}(\|u\|_{[t_0,\hat t_2-\tau_{\sigma(\hat t_2)}]}), & \check t_1\le\hat t_2-\tau_{\sigma(\hat t_2)}<\hat t_1,\\
\exp\big(\sum_{k=0}^{N(\hat t_2-\tau_{\sigma(\hat t_2)},t_0)}(\bar d_k+\bar c\tau_k)-\bar c(\hat t_2-\tau_{\sigma(\hat t_2)}-t_0)\big)V_0, & t_0\le\hat t_2-\tau_{\sigma(\hat t_2)}<\check t_1,\\
V_0, & \hat t_2-\tau_{\sigma(\hat t_2)}<t_0.
\end{cases}
\end{aligned}
$$

Let $M_{11}=\big(\sum_{k=N(\hat t_1,t_0)}^{N(\hat t_2-\tau_{\sigma(\hat t_2)},t_0)}(\bar d_k+\bar c\tau_k)\big)\vee\ 0$, then it follows that

$$
\begin{aligned}
&V\big(x(\hat t_2-\tau_{\sigma(\hat t_2)})\big)\\
&\le\begin{cases}
\rho\mathcal{X}(\|u\|_{[t_0,\hat t_2-\tau_{\sigma(\hat t_2)}]}), & \check t_2\le\hat t_2-\tau_{\sigma(\hat t_2)}<\hat t_2,\\
\exp\big(\sum_{k=0}^{N(\hat t_2-\tau_{\sigma(\hat t_2)},t_0)}(\bar d_k+\bar c\tau_k)-\bar c(\hat t_2-\tau_{\sigma(\hat t_2)}-t_0)+(\bar d_{\sigma(\hat t_1)}+\bar c\tau_{\sigma(\hat t_1)})\big)V_0 & \\
\quad+\exp\big(M_{11}+\bar c\tau_{\sigma(\hat t_2)} & \\
\quad+(\bar d_{\sigma(\hat t_1)}+\bar c\tau_{\sigma(\hat t_1)})\big)\rho\mathcal{X}(\|u\|_{[t_0,\ \hat t_2-\tau_{\sigma(\hat t_2)}]}), & \hat t_1\le\hat t_2-\tau_{\sigma(\hat t_2)}<\check t_2,\\
\rho\mathcal{X}(\|u\|_{[t_0,\hat t_2-\tau_{\sigma(\hat t_2)}]}), & \check t_1\le\hat t_2-\tau_{\sigma(\hat t_2)}<\hat t_1,\\
\exp\big(\sum_{k=0}^{N(\hat t_2-\tau_{\sigma(\hat t_2)},t_0)}(\bar d_k+\bar c\tau_k)-\bar c(\hat t_2-\tau_{\sigma(\hat t_2)}-t_0)\big)V_0, & t_0\le\hat t_2-\tau_{\sigma(\hat t_2)}<\check t_1,\\
V_0, & \hat t_2-\tau_{\sigma(\hat t_2)}<t_0,
\end{cases}\\
&\le\exp\Big(\sum_{k=0}^{N(\hat t_2-\tau_{\sigma(\hat t_2)},t_0)}(\bar d_k+\bar c\tau_k)-\bar c(\hat t_2-\tau_{\sigma(\hat t_2)}-t_0)+(\bar d_{\sigma(\hat t_1)}+\bar c\tau_{\sigma(\hat t_1)})\Big)V_0\\
&\quad+\exp\big(M_{11}+\bar c\tau_{\sigma(\hat t_2)}+(\bar d_{\sigma(\hat t_1)}+\bar c\tau_{\sigma(\hat t_1)})\big)\rho\mathcal{X}\big(\|u\|_{[t_0,\hat t_2-\tau_{\sigma(\hat t_2)}]}\big),
\end{aligned}
$$

which implies that

$$\begin{aligned}
V\big(x(\hat{t}_2)\big) &\leq \exp\Big(\sum_{k=0}^{N(\hat{t}_2,t_0)}(\bar{d}_k+\bar{c}\tau_k)-\bar{c}(\hat{t}_2-t_0)+(\bar{d}_{\sigma(\hat{t}_1)}+\bar{c}\tau_{\sigma(\hat{t}_1)})\Big)V_0\\
&\quad+\exp\big(M_{11}+(\bar{d}_{\sigma(\hat{t}_2)}+\bar{c}\tau_{\sigma(\hat{t}_2)})+(\bar{d}_{\sigma(\hat{t}_1)}+\tau_{\sigma(\hat{t}_1)})\big)\rho\mathcal{X}(\|u\|_{[t_0,\hat{t}_2]})\\
&\leq \exp\Big(\sum_{k=0}^{N(\hat{t}_2,t_0)}(\bar{d}_k+\bar{c}\tau_k)-\bar{c}(\hat{t}_2-t_0)+(\bar{d}_{\sigma(\hat{t}_1)}+\bar{c}\tau_{\sigma(\hat{t}_1)})\Big)V_0\\
&\quad+\exp\big(M_{12}+(\bar{d}_{\sigma(\hat{t}_1)}+\tau_{\sigma(\hat{t}_1)})\big)\rho\mathcal{X}(\|u\|_{[t_0,\hat{t}_2]}),
\end{aligned}$$

where $M_{12}=\big(\bar{d}_{\sigma(\hat{t}_2)}+\bar{c}\tau_{\sigma(\hat{t}_2)}\big)\vee\big(\sum_{k=N(\hat{t}_1,t_0)}^{N(\hat{t}_2,t_0)}(\bar{d}_k+\bar{c}\tau_k)\big)$. Then, for all $t\in[\hat{t}_2,\check{t}_3)$, it follows that

$$\begin{aligned}
&V(x(t))\\
&\leq \exp\Big(\sum_{k=N(\hat{t}_2,t_0)}^{N(t,t_0)}(\bar{d}_k+\bar{c}\tau_k)-\bar{c}(t-\hat{t}_2)\Big)V(x(\hat{t}_2))\\
&\leq \exp\Big(\sum_{k=0}^{N(t,t_0)}(\bar{d}_k+\bar{c}\tau_k)-\bar{c}(t-t_0)+(\bar{d}_{\sigma(\hat{t}_1)}+\bar{c}\tau_{\sigma(\hat{t}_1)})+(\bar{d}_{\sigma(\hat{t}_2)}+\bar{c}\tau_{\sigma(\hat{t}_2)})\Big)V_0\\
&\quad+\exp\Big(\sum_{k=N(\hat{t}_2,t_0)}^{N(t,t_0)}(\bar{d}_k+\bar{c}\tau_k)+M_{12}-\bar{c}(t-\hat{t}_2)+(\bar{d}_{\sigma(\hat{t}_1)}+\tau_{\sigma(\hat{t}_1)})\Big)\rho\mathcal{X}(\|u\|_{[t_0,t]})\\
&\leq \exp\Big(\sum_{k=0}^{N(t,t_0)}(\bar{d}_k+\bar{c}\tau_k)-\bar{c}(t-t_0)+(\bar{d}_{\sigma(\hat{t}_1)}+\bar{c}\tau_{\sigma(\hat{t}_1)})+(\bar{d}_{\sigma(\hat{t}_2)}+\bar{c}\tau_{\sigma(\hat{t}_2)})\Big)V_0\\
&\quad+\exp\big(M_{13}-\bar{c}(t-\hat{t}_2)+(\bar{d}_{\sigma(\hat{t}_1)}+\tau_{\sigma(\hat{t}_1)})+(\bar{d}_{\sigma(\hat{t}_2)}+\tau_{\sigma(\hat{t}_2)})\big)\rho\mathcal{X}(\|u\|_{[t_0,t]}),
\end{aligned}$$

where $M_{13}=\big(\sum_{k=N(\hat{t}_2,t_0)}^{N(t,t_0)}(\bar{d}_k+\bar{c}\tau_k))\big)\vee\big(\sum_{k=N(\hat{t}_1,t_0)}^{N(t,t_0)}(\bar{d}_k+\bar{c}\tau_k)\big)$. In the view of (3.1), one can obtain that there exist constants $T,Q<\infty$ such that

$$\begin{aligned}
T &\geq \sum_{k=N(\hat{t}_i,t_0)}^{N(t,t_0)}(\bar{d}_k+\bar{c}\tau_k), \forall t\geq\hat{t}_i, i=1,2,\cdots,n,\\
Q &\geq \sum_{k=\sigma(\hat{t}_i)}(\bar{d}_k+\bar{c}\tau_k).
\end{aligned}$$

In this way, we finally obtain the global bound

$$V(x(t))\leq\exp\big(M+m+Q-\bar{c}(t-t_0)\big)V_0+\exp(T+Q)\rho\mathcal{X}(\|u\|_{[t_0,t]}), \forall t\in[t_0,\infty). \tag{3.6}$$

Note that the function V is positive definite and radially unbounded, that is, it satisfies $\alpha_1(|x|)\leq V(x)\leq\alpha_2(|x|)$ for some $\alpha_1,\alpha_2\in\mathcal{K}$. Therefore, (3.6) implies the uniform ISS with $\beta(s,t)=\alpha_1^{-1}\big(2\exp(M+m+Q-\bar{c}(t))\alpha_2(s)\big)$ and $\gamma(s)=\alpha_1^{-1}\big(2\exp(T+Q)\rho\mathcal{X}(s)\big)$. The proof is completed. □

Remark 3.1. Note that, in [44], authors only considered the case that the rate coefficients $c > 0, d > 0$. That is, the continuous dynamics are stabilizing and the discrete dynamics play a positive effect to stability. In Theorem 2 of [3], all of the discrete dynamics governing the impulses are stabilizing impulses. But our main result of Theorem 3.1 considered the hybrid effects of both two types of impulses on the ISS property. Thus the results in this paper have wider applications than the results obtained in [44] and [3].

Corollary 3.1 (Uniformly iISS)**.** *Let all hypotheses of theorem 3.1 hold and define the same class of impulse time sequences $\mathcal{F}_\tau$. Then the system (2.1) is uniformly $iISS$ over $\mathcal{F}_\tau$.*

Proof. From Definition 2.3, it is obvious that V in (2.2) and (2.3) is upper bounded by the nonnegative solution $\upsilon(t)$ of the system with delayed impulses

$$\begin{cases} \dot{\upsilon}(t) = -c\upsilon(x(t)) + \mathcal{X}(|u(t)|), \quad t \neq t_k, \\ \upsilon(t) = \exp(-d_k)\upsilon(t^- - \tau(t)) + \mathcal{X}\big(|u(t-\tau(t))|\big), \quad t = t_k \end{cases}$$

with initial value $\upsilon_0 = \upsilon(x_0)$. Let ν be the nonnegative solution to

$$\begin{cases} \dot{\nu}(t) = \mathcal{X}(|u(t)|), \quad t \neq t_k, \\ \nu(t) = \nu(t^- - \tau(t)) + \mathcal{X}\big(|u(t-\tau(t))|\big), \quad t = t_k \end{cases}$$

with initial value $\nu_0 = 0$, which implies that

$$\nu(t) \leq \int_{t_0}^{t} \mathcal{X}(|u(s)|)ds + \sum_{k=1}^{N(t,t_0)} \mathcal{X}\big(|u(t_k^- - \tau(t_k))|\big), \quad t \geq t_0. \tag{3.7}$$

Next, define $y(t) := \upsilon(t) - \nu(t)$. Then y satisfies $y_0 = y(x_0)$ and

$$\begin{cases} \dot{y}(t) = \dot{\upsilon}(t) - \dot{\nu}(t) = -cy(t) - c\nu(t), \quad t \neq t_k, \\ y(t) = \exp(-d_k)y\big(t^- - \tau(t)\big) - \big(1 - exp(-d_k)\big)\nu\big(t^- - \tau(t)\big), \quad t = t_k, \end{cases}$$

which leads to

$$\begin{cases} \dot{y}(t) \leq -cy(t) + (-c+1)\nu(t), \quad t \neq t_k, \\ y(t) \leq \exp(-d_k)y\big(t^- - \tau(t)\big) + \big(1 - \exp(-d_k)\big)\nu\big(t^- - \tau(t)\big), \quad t = t_k. \end{cases}$$

Arguing as in the proof of Theorem 1, with y and ν playing the roles V and $\mathcal{X}(|u|)$, respectively, we obtain that the system (2.1) is $iISS$ with respect to ν with linear gain :$y(t) \leq \beta(|x_0|, t-t_0) + \eta\nu(t), t \geq t_0$ for some function $\beta \in \mathcal{KL}$ and constant $\eta > 0$. It then follows that

$$V(x(t)) \leq \upsilon(t) \leq \beta(|x_0|, t-t_0) + (\eta+1)\nu(t),$$

which together with (3.7) yields that

$$V(x(t)) \leq \beta(|x_0|, t-t_0) + \int_{t_0}^{t} (\eta+1)\nu(s)ds$$

$$+\sum_{k=1}^{N(t,t_0)}(\eta+1)\mathcal{X}\big(|u(t_k^- - \tau(t_k))|\big), \forall t\in[t_0,\infty),$$

which implies that system (2.1) is *uniformly iISS* over $\mathcal{F}_\tau$. The proof is completed. □

Next, we consider the case that the rate coefficients $c<0, d_k\equiv d>0$, which means that discrete dynamics of impulses are stabilizing but the continuous dynamics of the system are not.

Theorem 3.2. *(Uniformly ISS). Let V be an exponential ISS-Lyapunov function of system (2.1) with rate coefficients $c<0, d\in\mathbb{R}_+$. For any constants $\lambda>0$ and $\mu\in(0,d)$, let $S[\mu,\lambda]$ denote the class of impluse time sequence $\{t_k\}\in\mathcal{F}_\tau$ satisfying*

$$-dN(t,s)+c\sum_{k=0}^{N(t,t_0)}\tau_k - c(t-s)\le \mu-\lambda(t-s), \forall t\ge s\ge t_0, \forall k\in\mathbb{Z}_+, \tag{3.8}$$

and for arbitrary $k\in\mathbb{Z}_+$ such that $\mu-d+\lambda\tau_k<0$. Then the system (2.1) is uniformly ISS over $S[\mu,\lambda]$.

Proof. Assume that $x(t)=x(t,t_0,x_0)$ be the solution of system (2.1) with the initial value (t_0,x_0), and $u\in\mathcal{U}$ be a given input function. Choose $\varepsilon\in\mathbb{R}_+$ and $\delta\in(-c\varepsilon,+\infty)$ such that $d>\bar d=\frac{d}{1+\varepsilon}, c>\bar c=\frac{c-\delta}{1+\varepsilon}, \bar\lambda=\frac{\lambda-\delta}{1+\varepsilon}>0$. Adding $\delta(t-s)$ to both side of (3.8) and then dividing both side by $(1+\varepsilon)$, we conclude that

$$-\bar dN(t,s)+\bar c\Sigma_{k=0}^{N(t,s)}\tau_k-\bar c(t-s)\le\bar\mu-\bar\lambda(t-s),\quad \forall t\ge s\ge t_0,$$

where $\bar\mu=\frac{\mu}{1+\varepsilon}$. Using the similar argument as Theorem 3.1, one has

$$\nabla V(x)\cdot f(x,u)\le -\bar cV(x), \tag{3.9}$$

when $(-\bar c+c)V(x)\ge\mathcal{X}(|u|), \forall x\ a.e., \forall u$, and

$$V(x(t_k))\le\exp(-\bar d)V\big(x(t_k^- -\tau_k)\big), \tag{3.10}$$

when $\big(\exp(-\bar d)-\exp(-d)\big)V(x(t_k^- -\tau_k))\ge\mathcal{X}(|u(t_k^- -\tau_k)|), \forall x\ a.e., \forall u$.

Let $\check t_1=\inf\{t\ge t_0: V(x(t))\le\rho\mathcal{X}(\|u\|_{[t_0,t]})\}\le\infty$, where $\rho=\frac{1}{-\bar c+c}\vee\frac{1}{\exp(-\bar d)-\exp(-d)}$. It is clear that $\check t_1>t_0$. We consider two cases: $\check t_1<\infty$ and $\check t_1=\infty$. If $\check t_1<\infty$, then let $N(\check t_1,t_0)=N$. When $N=0$, it is obvious that

$$V(x(t))\le\exp\big(-\bar c(t-t_0)\big)V_0, \forall t\in[t_0,\check t_1),$$

where $V_0=V(x_0)$. When $N>0$, one has $V(x(t))\le\exp\big(-\bar c(t-t_0)\big)V_0, \forall t\in[t_0,t_1)$. In view of (3.10), it gives that

$$\begin{aligned}V(x(t_1))&\le\exp(-\bar d)V\big(x(t_1-\tau_1)\big)\\&=\exp(-\bar d)\begin{cases}\exp\big(-\bar c(t_1-\tau_1-t_0)\big)V_0, & t_0\le t_1-\tau_1<t_1\\ V_0, & t_1-\tau_1<t_0\end{cases}\\&\le\exp\big(-\bar d-\bar c(t_1-\tau_1-t_0)\big)V_0\end{aligned}$$

and $V(x(t)) \leq \exp\big(-\bar{d}+\bar{c}\tau_1-\bar{c}(t-t_0)\big)V_0, \forall t \in [t_1, t_2 \wedge \check{t}_1)$. If $t_2 > \check{t}_1$, then

$$V(x(t)) \leq \exp\big(-\bar{d}+\bar{c}\tau_1-\bar{c}(t-t_0)\big)V_0, \forall t \in [t_1, \check{t}_1).$$

Or else, then

$$V(x(t)) \leq \exp\big(-\bar{d}+\bar{c}\tau_1-\bar{c}(t-t_0)\big)V_0, \forall t \in [t_1, t_2).$$

Thus, it follows from (3.10) and the fact $t_k - t_{k-1} \geq \tau_k$ that

$$\begin{aligned} V(x(t_2)) &\leq \exp(-\bar{d})V\big(x(t_2-\tau_2)\big) \\ &= \exp(-\bar{d})\begin{cases} \exp\big(-\bar{d}+\bar{c}\tau_1-\bar{c}(t_2-\tau_2-t_0)\big)V_0, & t_1 < t_2-\tau_2 < t_2 \\ \exp\big(-\bar{c}(t_2-\tau_2-t_0)\big)V_0, & t_0 < t_2-\tau_2 < t_1 \\ V_0, & t_2-\tau_2 < t_0 \end{cases} \\ &\leq \exp\big(-2\bar{d}+\bar{c}\tau_1+\bar{c}\tau_2-\bar{c}(t_2-t_0)\big)V_0, \end{aligned}$$

which implies that

$$V(x(t)) \leq \exp\big(-2\bar{d}+\bar{c}\tau_1+\bar{c}\tau_2-\bar{c}(t-t_0)\big)V_0, \forall t \in [t_2, t_3 \wedge \check{t}_1).$$

In this way, we conclude that

$$V(x(t)) \leq \exp\big(-\bar{d}N(t,t_0)+\bar{c}\Sigma_{k=0}^{N(t,t_0)}\tau_k-\bar{c}(t-t_0)\big)V_0, \forall t \in [t_0, \check{t}_1),$$

where $\tau_0 := 0$.

If $\check{t}_1 = \infty$, then similar to the argument of the case that $\check{t}_1 < \infty$, one may derive that

$$V(x(t)) \leq \exp\big(-\bar{d}N(t,t_0)+\bar{c}\Sigma_{k=0}^{N(t,t_0)}\tau_k-\bar{c}(t-t_0)\big)V_0, \forall t \in [t_0, \infty).$$

In either case, combining with (3.8) one can easily obtain that

$$V(x(t)) \leq \exp\big(\bar{\mu}-\bar{\lambda}(t-t_0)\big)V_0, \forall t \in [t_0, \check{t}_1).$$

Furthermore, define $\hat{t}_1 = \inf\big\{t > \check{t}_1 : V(x(t)) \geq \rho\mathcal{X}(\|u\|_{[t_0,t]})\big\} \leq \infty$. If $\hat{t}_1 = \infty$, then the following ISS estimate can be derived

$$V(x(t)) \leq \exp\big(\bar{\mu}-\bar{\lambda}(t-t_0)\big)V_0 + \rho\mathcal{X}(\|u\|_{[t_0,t]}), \forall t \in [t_0, \infty).$$

If $\hat{t}_1 < \infty$, then we consider two cases that $\hat{t}_1$ is an impulse time and $\hat{t}_1$ is not an impulse time.

When $\hat{t}_1$ is an impulse time, then it follows from (2.3) that

$$V(x(\hat{t}_1)) \leq \exp(-d)V\big(x(\hat{t}_1-\tau_{\sigma(\hat{t}_1)})\big) + \mathcal{X}(\|u\|_{[t_0,\hat{t}_1-\tau_{\sigma(\hat{t}_1)}]}),$$

where $\sigma(\hat{t}_i) := \big\{m \in \mathbb{Z}_+ : \hat{t}_i = t_m\big\}$ and

$$V\big(x(\hat{t}_1-\tau_{\sigma(\hat{t}_1)})\big) \leq \begin{cases} \rho\mathcal{X}(\|u\|_{[t_0,\hat{t}_1-\tau_{\sigma(\hat{t}_1)}]}), & \check{t}_1 \leq \hat{t}_1-\tau_{\sigma(\hat{t}_1)} < \hat{t}_1 \\ \exp\big(\bar{\mu}-\bar{\lambda}(\hat{t}_1-\tau_{\sigma(\hat{t}_1)}-t_0)\big)V_0, & t_0 \leq \hat{t}_1-\tau_{\sigma(\hat{t}_1)} < \check{t}_1 \\ V_0, & \hat{t}_1-\tau_{\sigma(\hat{t}_1)} < t_0 \end{cases}$$

$$\leq \exp\big(\bar{\mu} - \bar{\lambda}(\hat{t}_1 - \tau_{\sigma(\hat{t}_1)} - t_0)\big)V_0 + \rho\mathcal{X}(\|u\|_{[t_0,\hat{t}_1]}).$$

Thus it is easy to see that

$$\begin{aligned} V(x(\hat{t}_1)) &\leq \exp\big(-d + \bar{\mu} - \bar{\lambda}(\hat{t}_1 - \tau_{\sigma(\hat{t}_1)} - t_0)\big)V_0 + (\exp(-d)\rho + 1)\mathcal{X}(\|u\|_{[t_0,\hat{t}_1]}) \\ &\leq \exp\big(-\bar{d} + \bar{\mu} - \bar{\lambda}(\hat{t}_1 - \tau_{\sigma(\hat{t}_1)} - t_0)\big)V_0 + \exp(-\bar{d})\rho\mathcal{X}(\|u\|_{[t_0,\hat{t}_1]}). \end{aligned}$$

When $\hat{t}_1$ is not an impulse time, one has

$$V(x(\hat{t}_1)) = \rho\mathcal{X}(\|u\|_{[t_0,\hat{t}_1]}).$$

Then it can be deduced that

$$\exp\big(-\bar{d} + \bar{\mu} - \bar{\lambda}(\hat{t}_1 - \tau_{\sigma(\hat{t}_1)} - t_0)\big)V_0 + \exp(-\bar{d})\rho\mathcal{X}(\|u\|_{[t_0,\hat{t}_1]}) \geq \rho\mathcal{X}(\|u\|_{[t_0,\hat{t}_1]}).$$

In either case, one obtain

$$V(x(\hat{t}_1)) \leq \exp\big(-\bar{d} + \bar{\mu} - \bar{\lambda}(\hat{t}_1 - \tau_{\sigma(\hat{t}_1)} - t_0)\big)V_0 + \exp(-\bar{d})\rho\mathcal{X}(\|u\|_{[t_0,\hat{t}_1]}),$$

which implies that

$$\begin{aligned} V(x(t)) &\leq \exp\big(\bar{\mu} - \bar{\lambda}(t - \hat{t}_1)\big)V(x(\hat{t}_1)) \\ &\leq \exp\big(2\bar{\mu} - \bar{d} + \bar{\lambda}\tau_{\sigma(\hat{t}_1)} - \bar{\lambda}(t - t_0)\big)V_0 \\ &\quad + \exp\big(\bar{\mu} - \bar{d} - \bar{\lambda}(t - \hat{t}_1)\big)\rho\mathcal{X}(\|u\|_{[t_0,t]}), \forall t \in [\hat{t}_1, \check{t}_2). \end{aligned}$$

It then follows from the fact $\mu - d + \lambda\tau_k < 0$ that

$$V(x(t)) \leq \exp\big(\bar{\mu} - \bar{\lambda}(t - t_0)\big)V_0 + \rho\mathcal{X}(\|u\|_{[t_0,t]}), \forall t \in [\hat{t}_1, \check{t}_2).$$

Let $\hat{t}_2 = \inf\big\{t > \check{t}_2 : V(x(t)) \geq \rho(\mathcal{X}(\|u\|_{[t_0,t]})\big\} \leq \infty$. If $\hat{t}_2 = \infty$, then it's easy to get that

$$V(x(t)) \leq \exp\big(\bar{\mu} - \bar{\lambda}(t - t_0)\big)V_0 + \rho\mathcal{X}(\|u\|_{[t_0,t]}), \forall t \in [t_0, t).$$

If $\hat{t}_2 < \infty$, then we consider two cases that $\hat{t}_2$ is an impulse time and $\hat{t}_2$ is not an impulse time. If $\hat{t}_2$ is an impulse time, then it follows from (2.3) that

$$V(x(\hat{t}_2)) \leq \exp(-d)V\big(x(\hat{t}_2 - \tau_{\sigma(\hat{t}_2)})\big) + \mathcal{X}(\|u\|_{[t_0,\hat{t}_2-\tau_{\sigma(\hat{t}_2)}]}),$$

where

$$\begin{aligned} &V\big(x(\hat{t}_2 - \tau_{\sigma(\hat{t}_2)})\big) \\ &\leq \begin{cases} \rho\mathcal{X}(\|u\|_{[t_0,\hat{t}_2-\tau_{\sigma(\hat{t}_2)}]}), & \check{t}_2 \leq \hat{t}_2 - \tau_{\sigma(\hat{t}_2)} < \hat{t}_2 \\ \exp\big(\bar{\mu} - \bar{\lambda}(\hat{t}_2 - \tau_{\sigma(\hat{t}_2)} - t_0)\big)V_0 & \\ +\rho\mathcal{X}(\|u\|_{[t_0,\hat{t}_2-\tau_{\sigma(\hat{t}_2)}]}), & \hat{t}_1 \leq \hat{t}_2 - \tau_{\sigma(\hat{t}_2)} < \check{t}_2 \\ \rho\mathcal{X}(\|u\|_{[t_0,\hat{t}_2-\tau_{\sigma(\hat{t}_2)}]}), & \check{t}_1 \leq \hat{t}_2 - \tau_{\sigma(\hat{t}_2)} < \hat{t}_1 \\ \exp\big(\bar{\mu} - \bar{\lambda}(\hat{t}_2 - \tau_{\sigma(\hat{t}_2)} - t_0)\big)V_0, & t_0 \leq \hat{t}_2 - \tau_{\sigma(\hat{t}_2)} < \check{t}_1 \\ V_0, & \hat{t}_2 - \tau_{\sigma(\hat{t}_2)} < t_0 \end{cases} \end{aligned}$$

$$\leq \exp\big(\bar{\mu} - \bar{\lambda}(\hat{t}_2 - \tau_{\sigma(\hat{t}_2)} - t_0)\big)V_0 + \rho(\|u\|_{[t_0, \hat{t}_2 - \tau_{\sigma(\hat{t}_2)}]})$$

and therefore

$$\begin{aligned} V(x(\hat{t}_2)) &\leq \exp\big(\bar{\mu} - d - \bar{\lambda}(\hat{t}_2 - \tau_{\sigma(\hat{t}_2)} - t_0)\big)V_0 + \big(\exp(-d)\rho + 1\big)\mathcal{X}(\|u\|_{[t_0,\hat{t}_2]}) \\ &\leq \exp\big(\bar{\mu} - \bar{d} - \bar{\lambda}(\hat{t}_2 - \tau_{\sigma(\hat{t}_2)} - t_0)\big)V_0 + \exp(-\bar{d})\rho\mathcal{X}(\|u\|_{[t_0,\hat{t}_2]}). \end{aligned} \tag{3.11}$$

If $\hat{t}_2$ is not an impulse time, then it is obvious that

$$V(x(\hat{t}_2)) = \rho\mathcal{X}(\|u\|_{[t_0,\hat{t}_2]}). \tag{3.12}$$

Combining (3.11) with (3.12), one obtain

$$V(x(\hat{t}_2)) \leq \exp\big(\bar{\mu} - \bar{d} - \bar{\lambda}(\hat{t}_2 - \tau_{\sigma(\hat{t}_2)} - t_0)\big)V_0 + \exp(-\bar{d})\rho\mathcal{X}(\|u\|_{[t_0,\hat{t}_2]}),$$

then

$$V(x(t)) \leq \exp\big(\bar{\mu} - \bar{\lambda}(t - t_0)\big)V_0 + \rho\mathcal{X}(\|u\|_{[t_0,t]}), \forall t \in [\hat{t}_2, \check{t}_3).$$

In this way, we finally arrive at

$$V(x(t)) \leq \exp\big(\bar{\mu} - \bar{\lambda}(t - t_0)\big)V_0 + \rho\mathcal{X}(\|u\|_{[t_0,t]}), \forall t \in [t_0, t). \tag{3.13}$$

Note that the function V is positive definite and radially unbounded, that is, it satisfies $\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$ for some $\alpha_1, \alpha_2 \in \mathcal{K}$. Therefore, (3.13) implies the uniform ISS with $\beta(s,t) = \alpha_1^{-1}\big(2\exp(\bar{\mu} - \bar{\lambda}(t))\alpha_2(s)\big)$ and $\gamma(s) = \alpha_1^{-1}\big(2\rho\mathcal{X}(s)\big)$. The proof is completed. □

Let $\mathcal{S}_{avg}[\tau^*, N_0]$ denote the class of ADT impulse time sequences which satisfy $t_k - t_{k-1} \geq \tau_k$ and $N(t,s) \leq \frac{t-s}{\tau^*} + N_0, \forall t \geq s \geq t_0$, and let $\mathcal{S}_{r-avg}[\tau^*, N_0]$ denote the class of reverse ADT impulse time sequences which satisfy $t_k - t_{k-1} \geq \tau_k$ and $N(t,s) \geq \frac{t-s}{\tau^*} - N_0, \forall t \geq s \geq t_0$. Assume that $\sum_{k\in\mathbb{Z}_+} \tau_k := P < \infty$, then the following result follows from Theorem 3.2.

Corollary 3.2 (ADT ISS)**.** *Let V be an exponential ISS-Lyapunov function for system (2.1) with rate coefficients $c < 0, d > 0$, then the system (2.1) is uniformly ISS over $\mathcal{S}_{r-avg}[\tau^*, N_0]$ for all $\tau^* < \frac{d}{|c|}$ and $N_0 > 0$, where τ^* denotes the average dwell time for impulse time sequence.*

Proof. To prove above result, pick some $\tau^* < \frac{d}{|c|}$ and take an arbitrary impulse time sequence in $\mathcal{S}_{r-avg}[\tau^*, N_0]$, then we have that $N(t,s) \geq \frac{-(c-\lambda)(t-s)}{d} - N_0, \forall t \geq s \geq t_0$, for $\lambda := c + \frac{d}{\tau^*} > 0$, from which we conclude that (3.8) holds with $\mu := dN_0 + cP$. *Uniformly ISS* then follows from Theorem 3.1. □

Remark 3.2. When $\bar{c} > 0, \bar{d} < 0$, condition (3.9) and (3.10) imply that the continuous dynamics are stabilizing and the discontinuous dynamics are not, see [14]. In this case, the destabilizing impulses should not occur too frequently. Conversely, when $\bar{c} < 0$, condition (3.9) implies that the continuous dynamics are destabilizing. Nevertheless, $\bar{d} > 0$ implies that the discontinuous dynamics play a positive impact to stability. In this case, the reverse ADT constant meets $\tau^* < \frac{d}{\lambda - c}$, which indicates that the average dwell time must not be overly long intervals between impulses. When there is no delay effect, i.e., $\tau(t) = 0$, the $ISS/iISS$ of system (2.1) has been extensively studied in [12] and a set of Lyapunov-based sufficient conditions was provided.

Remark 3.3. In Theorem 3.1, we consider the case that the rates coefficients $c > 0$ and $d_k \in \mathbb{R}$, which implies that the continuous dynamics of the system is stabilizing and the hybrid impulses (i.e., stabilizing impulses and destabilizing impulses) are fully considered. While in Theorem 3.2, we consider the case that the rate coefficients $c < 0, d_k \equiv d > 0$, which means that discrete dynamics of impulses are stabilizing but the continuous dynamics of the system are not. However, the case of destabilizing continuous dynamics with hybrid impulses is not addressed in this paper due to the technical difficulty of handling destabilizing flow. More methods and tools should be developed and explored on this issue.

Corollary 3.3. *Under the same hypotheses in Theorem 3.2, system (2.1) is uniformly iISS over* $S[\mu, \lambda]$.

4. Numerical examples

In this section, examples are given to show the effectiveness of our obtained results.

Example 4.1. Consider the following impulsive system with hybrid delayed impulses

$$\dot{x}(t) = -sat(x(t)) + 0.7sat(u(t)), t \neq t_k,$$
$$x(t) = \begin{cases} x(t^- - \tau_1) + 0.5sat(u(t^- - \tau_1)), & t = t_{2k-1} \\ 0.1x(t^- - \tau_2) + 0.1sat(u(t^- - \tau_2)), & t = t_{2k}, \end{cases}$$

where $k \in \mathbb{N}_+, \tau_k \in \mathbb{R}_+$, $sat(s) = \frac{1}{2}(|s+1| - |s-1|)$. Choose Lyapunov function

$$V(x(t)) = \begin{cases} x^2, & |x| \leq 1, \\ \exp(2|x| - 1), & |x| > 1. \end{cases}$$

When $t \neq t_k, k \in \mathbb{Z}_+$, if $|x| \leq 1$, then it is easy to check that

$$\nabla V(x) \cdot f \leq -(2 - 0.7)V(x) + 0.7sat^2(u).$$

If $|x| > 1$, then $\nabla V(x) \cdot f \leq -0.6V(x)$. Thus (2.2) holds with $\mathcal{X}(s) = s^2$, $c = 0.6$. When $t = t_{2k-1}$, if $|x + 0.5sat(u)| \leq 1$, then we have that

$$V(g(x, u)) = x^2 + 0.25sat^2(u) + x \cdot sat(u) \leq \exp(1 + \ln 1.5)V(x) + 0.75u^2.$$

If $|x + 0.5sat(u)| > 1$, that is, $|x| > 0.5$, then it follows that

$$\begin{aligned} V(g(x, u)) &= V\big(x + 0.5sat(u)\big) \\ &\leq \exp\big(2|x + 0.5sat(u)| - 2\big) \\ &\leq \begin{cases} 1.5eV(x), & 1 < |x|, \\ \exp(2|x| - 2)e, & 0.5 < |x| \leq 1, \end{cases} \\ &\leq \begin{cases} 1.5eV(x), & 1 < |x|, \\ 1.5ex^2, & 0.5 < |x| \leq 1, \end{cases} \end{aligned}$$

$$\le \exp(1 + \ln 1.5)V(x).$$

When $t = t_{2k}$, if $|0.1x + 0.1sat(u)| > 1$, then it is easy to see that

$$V(g(x,u)) \le \exp\big(2|0.1x + 0.1sat(u)| - 2\big) \le \exp(-1.6)V(x).$$

If $|0.1x + 0.1sat(u)| \le 1$, then we have

$$V(g(x,u)) \le \exp(-1.6)V(x) + u^2.$$

Thus in all cases, (2.3) holds with $d_1 = -(\ln 1.5 + 1), d_2 = 1.6, \mathcal{X}(s) = s^2$. Let $\tau_1 = 0.2, \tau_2 = 0.1$, then it is easy to check that condition (3.1) holds with $M = 1.6$, which implies that the system (2.1) is *uniformly ISS* over $\mathcal{F}_\tau$. If the impulsive time sequence $t_{2n-1} = 2n - 1, t_{2n} = 2n$, $n \in \mathbb{Z}_+$ and $x_0 = 1, u = 2x$ are given, then Figure 1 illustrates the state trajectory of the system (2.1).

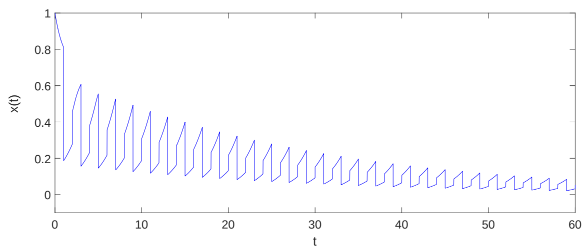

Figure 1. Simulation results of *ISS* for Example 4.1.

Example 4.2. Consider the following 2D impulsive system with delayed impulses

$$\dot{x}(t) = \begin{pmatrix} 0.1x_1 + x_1x_2 \\ 0.1x_2 - x_1^2 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, t \neq t_k,$$
$$x(t) = \begin{pmatrix} 0.2x_1(t^- - \tau_k) \\ 0.2x_2(t^- - \tau_k) \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, t = t_k.$$

We choose Lyapunov function $V(x(t)) = x_1^2 + x_2^2$. When $t \neq t_k$, then it is easy to see that

$$\begin{aligned} \nabla V(x) \cdot f &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2 \\ &= 0.2x_1^2 + 2x_1u_1 + 0.2x_2^2 + 2x_2u_2 \\ &\le 1.2V(x) + (u_1^2 + u_2^2). \end{aligned}$$

When $t = t_k$, then it follows that

$$\begin{aligned} V(g(x)) &= 0.04x_1^2 + 0.04x_2^2 + u_1^2 + u_1^2 + 0.4x_1u_1 + 0.4x_2u_2 \\ &\le 0.24V(x) + 1.2u_1^2 + 1.2u_2^2 \\ &= \exp(\ln 0.24)V(x) + 1.2u_1^2 + 1.2u_2^2. \end{aligned}$$

Thus, it is obvious that (2.2) and (2.3) hold with $\mathcal{X}(s) = 1.2(s_1^2 + s_2^2), c = -1.2$, and $d = -\ln 0.24$. In particular, let $\mu = \lambda = 0.5, \tau_{3n-2} = 0.2, \tau_{3n-1} = 0.3, \tau_{3n} =$

$0.5, n \in \mathbb{Z}_+$ and $x_1(0) = 2, x_2(0) = 1, u_1 = \cos(x_1), u_2 = \sin(x_2)$ and the impulsive time sequence $t_{3n-2} = 2n - 1.5, t_{3n-1} = 2n - 1, t_{3n} = 2n, n \in \mathbb{Z}_+$ are given, then it is easy to check that condition (3.8) is satisfied. According to Theorem 3.2, the system (2.1) is ISS over $\mathcal{S}[\mu, \lambda]$. Then the Figure 2 is given to illustrate the 2-norm of the state trajectory of the system (2.1).

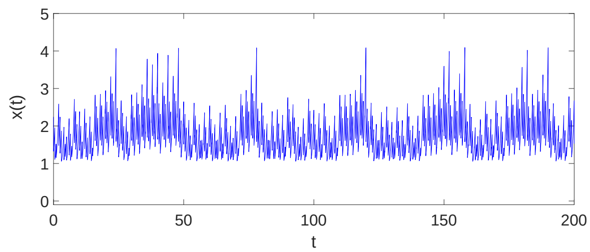

Figure 2. Simulation results of *ISS* for Example 4.2.

5. Conclusion

Utilizing the concept of exponential ISS-Lyapunov function we have presented a series of theorems which provide sufficient conditions for $ISS/iISS$ of impulsive systems with hybrid delayed impulses. When the continuous behaviors is stabilizing, the delayed impulses may destroy the ISS property, thus the rate coefficients should be restrained to ensure the ISS property. We have also shown that even the continuous behaviors is destabilizing, the designed ADT scheme coupled with stabilizing impulses with delay is sufficient to stabilize the system in ISS sense. Those conditions established the relationship between impulsive frequency and the time delay existing in hybrid impulses, and showed the effect of hybrid delayed impulses on $ISS/iISS$. Our results developed a procedure for $ISS/iISS$ analysis of impulsive systems with different impulsive time sequences. Examples have been given to illustrate the efficiency of the obtained results. However, it is worth mentioning that our results only apply to the systems with delayed impulses in which the interval lengths between two consecutive impulsive times are greater than time delays in impulses. In other words, the time delays in impulses only occur between two consecutive impulsive times. Thus investigating other possible cases is a topic for future research. Another topic is the development of ISS conditions for impulsive switched systems with delayed impulses in which state-dependent impulse and switching are fully involved.

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