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POSITIVE SOLUTION FOR NONLINEAR THIRD-ORDER MULTI-POINT BOUNDARY VALUE PROBLEM AT RESONANCE

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Abstract In this paper, positive solutions for a kind of third-order multi-point boundary value problem at resonance are investigated. By using the Leggett-Williams norm-type theorem due to O'Regan and Zima, existence result of at least one positive solution is established. An example is given to demonstrate the main results.

Keywords M-point boundary value problem, resonance, positive solution, Topological degree, fixed point.

MSC(2010) 34B10, 34B15.

1. Introduction

This paper is motivated by the existence of positive solution for the third-order m-point boundary value problem

$$\begin{cases} x'''(t) + f(t, x(t)) = 0, \ t \in [0,1] \\ x''(0) = 0, \ x'(0) = x'(1), \ x(0) = \displaystyle\sum_{i=1}^{m-2} \beta_i x(\xi_i), \end{cases} \tag{1.1}$$

where

$0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1, \ 0 \le \beta_i \le 1, \ i = 1, 2, \cdots, m-2,$
$f \in C([0,1] \times [0, \infty), R),$

with the resonant condition $\sum_{i=1}^{m-2} \beta_i = 1$. It is well known that under this resonant condition the associated linear operator is uninvertible.

Third order differential equations arise in a variety of different areas of applied mathematics and physics, as the deflection of a curved beam having a constant or varying cross section, three layer beam and so on [20]. Recently much attention has been paid to the existence of solutions, especially for the positive solutions, of third-order multi-point boundary value problems at non-resonance (for details see [1, 21 30, 10, 7, 17, 22, 12, 27]).

Anderson [1] established the existence of at least three positive solutions to

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problem

$$\begin{cases} -x'''(t) + f(x(t)) = 0, \ t \in (0,1), \\ x(0) = x'(t_2) = x''(1) = 0, \end{cases}$$

where $f : R \to [0, +\infty)$ is continuous and $1/2 \le t_2 < 1$.

By using the well-known Guo-Krasnoselskiĭ fixed point theorem [8], Palamides and Smyrlis [21] proved that there exist at least one positive solution for third-order three-point problem

$$\begin{cases} x'''(t) = a(t)f(t, x(t)), \ t \in (0,1), \\ x''(\eta) = 0, \ x(0) = x(1) = 0, \ \eta \in (0,1). \end{cases}$$

Yang [30] studied the existence of positive solutions for the third-order m-point boundary value problem

$$\begin{cases} x'''(t) + f(t, \ x(t), \ x'(t), \ x''(t)) = 0, \ t \in [0,1], \\ x''(0) = 0, \ x'(0) = \displaystyle\sum_{i=1}^{m-2} \alpha_i x'(\xi_i), \ x(1) = \displaystyle\sum_{i=1}^{m-2} \beta_i x(\xi_i), \end{cases}$$

where

$$0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1, \ 0 \le \alpha_i < 1, \ 0 \le \beta_i < 1, \ i = 1, 2, \cdots, m-2,$$
$$\sum_{i=1}^{m-2} \alpha_i < 1, \ \sum_{i=1}^{m-2} \beta_i < 1, \text{ and } f \in C([0,1] \times [0, +\infty) \times R^2, \ [0, +\infty)).$$

By using the Avery-Peterson fixed point theorem, the author established the existence of at least three positive solutions of this problem.

For resonant problem of second-order or higher-order differential equations, many existence results of solutions have been established, see [6, 23, 24, 11, 13, 14, 15, 3, 5, 16, 18, 19, 4]. In [4], the authors considered the third-order problem

$$\begin{cases} x'''(t) = f(t, x, x') + e(t), \ t \in (0,1), \\ x'(0) = 0, \ x(1) = \beta x(\eta), \ x(0) = \displaystyle\sum_{i=1}^{m-2} \alpha_i x(\xi_i). \end{cases}$$

By using Mawhin continuation theorem, the existence results of solutions are established under the resonant condition $\beta = 1, \sum_{i=1}^{m-2} \alpha_i = 1, \ \sum_{i=1}^{m-2} \alpha_i \xi_i^2 = 0$ and $\beta = \frac{1}{\eta}, \sum_{i=1}^{m-2} \alpha_i = 1, \ \sum_{i=1}^{m-2} \alpha_i \xi_i^2 = 0$ respectively.

It is well known that the problem of existence for positive solution to nonlinear boundary value problem is very difficult when the resonant case is considered. Only few work gave the approach in this area for first and second-order differential equations [2, 25, 26, 9, 28, 29]. To our best knowledge, few paper deal with the existence result of positive solution for resonant third-order boundary value problems. Motivated by the approach in [25, 26, 9], we study the positive solution for problem (1.1) under the resonant condition. By using the norm-type Leggett-Williams fixed point theorem, we establish the existence results of positive solutions.

2. Preliminaries

Operator $L : domL \subset X \to Y$ is called a Fredholm operator with index zero, that is, ImL is closed and dim Ker L=codim ImL$< \infty$, which implies that there exist continuous projections $P : X \to X$ and $Q : Y \to Y$ such that $ImP = KerL$ and $KerQ = ImL$. Moreover, since dim Im Q=codim Im L, there exists an isomorphism $J : ImQ \to KerL$. Denote by L_P the restriction of L to $KerP \cap domL$ to ImL and its inverse by K_P, so $K_P : ImL \to KerP \cap domL$ and the coincidence equation $Lx = Nx$ is equivalent to

$$x = (P + JQN)x + K_P(I - Q)Nx.$$

Denote $\gamma : X \to C$ be a retraction, that is, a continuous mapping such that $\gamma x = x$ for all $x \in C$ and

$$\Psi := P + JQN + K_P(I - Q)N,$$
$$\Psi_\gamma := \Psi \circ \gamma.$$

Lemma 2.1 (Leggett-Williams norm-type theorem, [25])**.** *Let C be a cone in X, Ω_1, Ω_2 be open bounded subsets of X with $\overline{\Omega}_1 \subset \Omega_2$, $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \neq \emptyset$. Assume that $L : domL \subset X \to Y$ is a Fredholm operator of index zero and*
$(C1)$ *$QN : X \to Y$ is continuous and bounded, $K_P(I - Q)N : X \to X$ is compact on every bounded subset of X,*
$(C2)$ *$Lx \neq \lambda Nx$ for all $x \in C \cap \partial\Omega_2 \cap domL$ and $\lambda \in (0, 1)$,*
$(C3)$ *γ maps subsets of $\overline{\Omega}_2$ into bounded subsets of C,*
$(C4)$ *$d_B([I - (P + JQN)\gamma]|_{kerL}, KerL \cap \Omega_2, 0) \neq 0$, where d_B stands for the Brouwer degree,*
$(C5)$ *There exists $u_0 \in C\backslash\{0\}$ such that $\|x\| \leq \sigma(u_0)\|\Psi x\|$ for $x \in C(u_0) \cap \partial\Omega_1$, where $C(u_0) = \{x \in C : \mu u_0 \leq x\}$ for some $\mu > 0$ and $\sigma(u_0)$ is such that $\|x + u_0\| \geq \sigma(u_0)\|x\|$ for every $x \in C$,*
$(C6)$ *$(P + JQN)\gamma(\partial\Omega_2) \subset C$,*
$(C7)$ *$\Psi_\gamma(\overline{\Omega}_2\backslash\Omega_1) \subset C$,*
then the equation $Lx = Nx$ has a solution in the set $C \cap (\overline{\Omega}_2\backslash\Omega_1)$.

3. Main results

We define the spaces $X = Y = C[0, 1]$ endowed with the maximum norm. It is well known that X and Y are Banach spaces.

Define the linear operator $L : domL \subset X \to Y$,

$$(Lx)(t) = -x'''(t),\ t \in [0, 1],$$

where

$$domL = \{x \in X |\ x''' \in C[0, 1],\ x''(0) = 0,\ x'(0) = x'(1),\ x(0) = \sum_{i=1}^{m-2} \beta_i x(\xi_i)\},$$

and the nonlinear operator $N :\ X \to Y$ with

$$(Nx)(t) = f(t, x(t)),\ t \in [0, 1].$$

It is obvious that $KerL = \{x \in domL : x(t) \equiv c\ , t \in [0,1]\}$. Denote the function $G(s),\ s \in [0,1]$ as follow:

$$G(s) = 1 - s,\ 0 \le s \le 1.$$

Denote $\beta_0 = 0,\ \xi_0 = 0,\ \beta_{m-1} = 0,\ \xi_{m-1} = 1$. Define the function $k(t,s)$ as follow:

$$k(t,s) = \begin{cases} \dfrac{2t-1}{2\sum\limits_{k=0}^{m-1}\beta_i\xi_i} \sum\limits_{i=j+1}^{m-1} \beta_i(\xi_i s - \frac{1}{2}s^2 - \frac{1}{2}\xi_i^2) - \frac{1}{6}(1-s)^3 + \frac{1}{2}(t-s)^2, \\ t \ge s,\ \xi_{j-1} \le s \le \xi_j, \\ \dfrac{2t-1}{2\sum\limits_{k=0}^{m-1}\beta_i\xi_i} \sum\limits_{i=j+1}^{m-1} \beta_i(\xi_i s - \frac{1}{2}s^2 - \frac{1}{2}\xi_i^2) - \frac{1}{6}(1-s)^3, t \le s,\ \xi_{j-1} \le s \le \xi_j, \end{cases}$$

for $j = 1, 2, \cdots, m-1$.

Define the functions $U(t,s)$ and positive number κ as follow:

$$U(t,s) = k(t,s) + \frac{G(s)}{\int_0^1 G(s)ds}(1 - \int_0^1 k(t,s)ds),\ t,\ s \in [0,1],$$

$$\kappa := \min\left\{\frac{1}{2}, \min_{t,s\in[0,1]} \frac{1}{U(t,s)}\right\}.$$

Theorem 3.1. *Suppose that there exist positive constant $R \in (0,\infty)$ such that*

$$f : [0,1] \times [0,R] \to (-\infty,\ +\infty)$$

is continuous and satisfies the following conditions:

$(S1)$ $f(t,x) \ge -\kappa x$, *for* $(t,\ x) \in [0,1] \times [0,\ R]$,

$(S2)$ $f(t,x) < 0$ *for* $[t,\ x] \in [0,\ 1] \times [(1 - \frac{\kappa}{2})R,\ R]$,

$(S3)$ *there exists* $r \in (0,R),\ M \in (0,1),\ t_0 \in [0,1],\ a \in (0,1]$ *and continuous functions*

$$g : [0,1] \to [0,+\infty),\ h : (0,r] \to [0,+\infty),$$

such that

$$f(t,x) \ge g(t)h(x),\ [t,x] \in [0,1] \times (0,r],$$

and $h(x)/x^a$ is non-increasing on $(0,r]$ with

$$\frac{h(r)}{r^a}\int_0^1 U(t_0,s)g(s)ds \ge \frac{1-M}{M^a},$$

then resonant problem (1.1-1.2) has at least one positive solution.

Proof. Firstly we may claim that

$$ImL = \{y \in Y |\ \int_0^1 G(s)y(s)ds = 0\}.$$

Indeed, for each $y \in \{y \in Y| \int_0^1 G(s)y(s)ds = 0\}$, we choose

$$x(t) = -\frac{1}{2}\int_0^t (t-s)^2 y(s)ds + \frac{\sum_{i=1}^{m-2}\beta_i t}{2\sum_{i=1}^{m-2}\beta_i\xi_i}\int_0^{\xi_i}(\xi_i - s)^2 y(s)ds.$$

We can check that

$$-x'''(t) = y(t),\ x''(0) = 0,\ x'(0) = x'(1),\ x(0) = \sum_{i=1}^{m-2}\beta_i x(\xi_i),$$

which means $x(t) \in domL$. Thus

$$\{y \in Y| \int_0^1 G(s)y(s)ds = 0\} \subset ImL.$$

On the other hand, for each $y(t) \in ImL$, there exists $x(t) \in domL$ such that

$$-x'''(t) = y(t),\ x''(0) = 0,\ x'(0) = x'(1),\ x(0) = \sum_{i=1}^{m-2}\beta_i x(\xi_i).$$

Integrating both sides on [0, t], we have

$$x(t) = -\frac{1}{2}\int_0^t (t-s)^2 y(s)ds + \frac{1}{2}x''(0)t^2 + x'(0)t + x(0).$$

Considering condition $x''(0) = 0,\ x'(0) = x'(1),\ x(0) = \sum_{i=1}^{m-2}\beta_i x(\xi_i)$ and resonant condition $\sum_{i=0}^{m-1}\beta_i = 1$, we have

$$\int_0^1 (1-s)y(s)ds = \int_0^1 G(s)y(s)ds = 0.$$

Thus,

$$ImL = \{y \in Y| \int_0^1 G(s)y(s)ds = 0\}.$$

It is obvious that dim KerL=1 and ImL is closed.

Secondly we see $Y = Y_1 \bigoplus ImL$, where

$$Y_1 = \{y_1|\ y_1 = \frac{1}{\int_0^1 G(s)ds}\int_0^1 G(s)y(s)ds, y \in Y\}.$$

In fact, for each $y(t) \in Y$, we have

$$\int_0^1 G(s)[y(s) - y_1]ds = 0.$$

This induces that $y-y_1 \in ImL$. Since $Y_1 \cap ImL = \{0\}$, we have $Y = Y_1 \bigoplus ImL$. Thus L is a Fredholm operator with index zero.

Define two projections $P: X \to X,\ Q: Y \to Y$ by

$$Px = \int_0^1 x(s)ds,$$

$$Qy = \frac{1}{\displaystyle\int_0^1 G(s)ds} \int_0^1 G(s)y(s)ds.$$

Clearly, $ImP = KerL,\ KerQ = ImL$ and $KerP = \{x \in X : \int_0^1 x(s)ds = 0\}$. Note that for $y \in ImL$, the inverse K_P of L_P is given by

$$(K_P)y = \int_0^t k(t,\ s)y(s)ds.$$

In fact, It is easy to check that

$$L(K_P)(y) = (-\int_0^1 k(t,s)y(s)ds)''' = y(t),$$

$$K_P(L)(x) = \int_0^1 k(t,s)(-x'''(s))ds = x(t).$$

Next we will check that every condition of Lemma 2.1 is fulfilled. Remark that f can be extended continuously on $[0,1] \times (-\infty, +\infty)$, condition (C1) of Lemma 2.1 is fulfilled.

Define the set of nonnegative functions C and subsets of X $\Omega_1,\ \Omega_2$ by

$$C = \{x \in X:\ x(t) \geq 0,\ t \in\ [0,\ 1]\},$$

$$\Omega_1 = \{x \in X:\ r > |x(t)| > M\|x\|,\ t \in [0,\ 1]\},$$

$$\Omega_2 = \{x \in X:\ \|x(t)\| < R,\ t \in [0,\ 1]\}.$$

Remark that Ω_1 and Ω_2 are open and bounded sets. Furthermore

$$\overline{\Omega}_1 = \{x \in X:\ r \geq |x(t)| \geq M\|x\|,\ t \in [0,\ 1]\} \subset \Omega_2, C \cap \overline{\Omega}_2 \backslash \Omega_1 \neq .$$

Let the isomorphism $J = I$ and $(\gamma x)(t) = |x(t)|$ for $x \in X$. Then γ is a retraction and maps subsets of $\overline{\Omega}_2$ into bounded subsets of C, which ensures that condition (C3) of Lemma 2.1 is fulfilled.

Then we prove that (C2) of Lemma 2.1 is fulfilled. For this purpose, suppose that there exists $x_0 \in C \cap \partial\Omega_2 \cap domL$ and $\lambda_0 \in (0,1)$ such that $Lx_0 = \lambda_0 N x_0$. Then

$$-x_0'''(t) = \lambda_0 f(t, x_0),$$

for all $t \in [0,1]$. Let $x_0(t_0) = \|x_0\| = R$. The proof is divided into two cases:

(1) We show that $t_0 \neq 0$. Suppose, on the contrary, that $x_0(t)$ achieves maximum value R only at $t_0 = 0$. Then the boundary condition $x(0) = \sum_{i=1}^{m-2} \beta_i x_0(\xi_i)$ in combination with the resonant condition $\sum_{i=1}^{m-2} \beta_i = 1$ yields that $\max_{1\leq i \leq m-2} x_0(\xi_i) \geq R$, which is a contradiction.

(2) Thus there exists $t_0 \in (0,1]$ such that $x_0(t_0) = R = \max_{0\le t\le 1} x_0(t)$. We may choose $\eta < t_0$ nearest to t_0 with $x_0''(\eta) = 0$. From the Mean Value theory, we claim that there exists $\xi \in (\eta, t_0)$ such that

$$x_0(\eta) = x_0(t_0) - x_0'(\xi)(t_0 - \eta).$$

However, from

$$x_0''(t) = -\lambda_0 \int_0^t f(s, x_0(s))ds,$$

and

$$x_0'(t) = -\lambda_0 \int_{t_0}^t (t-s) f(s, x_0(s))ds,$$

we have

$$\begin{aligned} x_0'(\xi) =& -\lambda_0 \int_{t_0}^{\xi} (\xi - s) f(s, x_0)ds \le \lambda_0 \kappa \int_{t_0}^{\xi} (\xi - s) x_0(s)ds \le \lambda_0 \kappa R \int_{t_0}^{\xi} (\xi - s)ds \\ =& \frac{1}{2}(t_0 - \xi)^2 \lambda_0 \kappa R. \end{aligned}$$

Thus

$$x_0(\eta) = x_0(t_0) - x_0'(\xi)(t_0 - \eta) \ge R - \frac{1}{2}(t_0 - \xi)^2 \lambda_0 \kappa (t_0 - \eta) R \ge (1 - \frac{\kappa}{2})R.$$

Then

$$0 \ge x_0''(t_0) - x_0''(\eta) = -\lambda_0 \int_{\eta}^{t_0} f(s, x_0(s))ds,$$

which contradict to condition (S2). Thus (C2) of Lemma 2.1 is fulfilled.

Remark. The sign of third order derivative of a function $y(t)$ at point t_0 can not be confirmed when t_0 is a maximal value of $y(t)$. Thus the method in [29] are not applicable directly to problem (1.1-1.2). In our opinion, it is the key that the conditions $(S2)$ in this paper are stronger than that in [29].

For $x \in KerL \cap \Omega_2$, define

$$H(x, \lambda) = x - \lambda |x| - \frac{\lambda}{\displaystyle\int_0^1 G(s)ds} \int_0^1 G(s) f(s, |x|)ds,$$

where $x \in KerL \cap \Omega_2$ and $\lambda \in [0,1]$.

Suppose $H(x, \lambda) = 0$. In view of $(S1)$ we obtain

$$\begin{aligned} c =& \lambda |c| + \frac{\lambda}{\displaystyle\int_0^1 G(s)ds} \int_0^1 G(s) f(s, |c|)ds \\ \ge& \lambda |c| - \frac{\lambda}{\displaystyle\int_0^1 G(s)ds} \int_0^1 G(s) \kappa |c| ds \\ =& \lambda |c|(1 - \kappa) \ge 0. \end{aligned}$$

Hence $H(x,\lambda)=0$ implies $c\geq 0$. Furthermore, if $H(R,\lambda)=0$, we get

$$0\leq R(1-\lambda)\int_0^1 G(s)ds=\lambda\int_0^1 G(s)f(s,R)ds,$$

contradicting to $(S2)$. Thus $H(x,\lambda)\neq 0$ for $x\in\partial\Omega_2$ and $\lambda\in[0,1]$. Therefore

$$d_B(H(x,0),KerL\cap\Omega_2,0)=d_B(H(x,1),$$

$$KerL\cap\Omega_2,0)=d_B(I,KerL\cap\Omega_2,0)=1.$$

This ensures

$$d_B([I-(P+JQN)\gamma]|_{KerL},KerL\cap\Omega_2,0)=d_B(H(x,1),KerL\cap\Omega_2,0)\neq 0.$$

Let $x\in\overline{\Omega}_2\backslash\Omega_1$ and $t\in[0,1]$. From condition $(S1)$, we see

$$\begin{aligned}(\Psi_\gamma x)(t)&=\int_0^1|x(t)|dt+\frac{1}{\displaystyle\int_0^1 G(s)ds}\int_0^1 G(s)f(s,|x(s)|)ds\\&\quad+\int_0^1 k(t,s)[f(s,|x(s)|)-\frac{1}{\displaystyle\int_0^1 G(s)ds}\int_0^1 G(\tau)f(\tau,|x(\tau)|)d\tau]ds\\&=\int_0^1|x(t)|dt+\int_0^1 U(t,s)f(s,|x(s)|)ds\\&\geq\int_0^1|x(s)|ds-\kappa\int_0^1 U(t,s)|x(s)|ds\\&=\int_0^1(1-\kappa U(t,s))|x(s)|ds\geq 0.\end{aligned}$$

Hence $\Psi_\gamma(\overline{\Omega}_2)\backslash\Omega_1\subset C$. Moreover, since for $x\in\partial\Omega_2$, we have

$$\begin{aligned}(P+JQN)\gamma x&=\int_0^1|x(s)|ds+\frac{1}{\displaystyle\int_0^1 G(s)ds}\int_0^1 G(s)f(s,|x(s)|)ds\\&\geq\int_0^1(1-\frac{\kappa}{\displaystyle\int_0^1 G(s)ds}G(s))|x(s)|ds\geq 0,\end{aligned}$$

which means $(P+JQN)\gamma(\partial\Omega_2)\subset C$. These ensure that (C6), (C7) of Lemma 2.1 hold.

At last, we confirm that (C5) is satisfied. Taking $u_0(t)\equiv 1$ on [0,1], we see

$$u_0\in C\backslash\{0\},C(u_0)=\{x\in C|\ x(t)>0\text{ on }[0,1]\}$$

and we can take $\sigma(u_0)=1$. Let $x\in C(u_0)\cap\partial\Omega_1$, we have

$$x(t)>0,\ t\in[0,1],\ 0<\|x\|\leq r\text{ and }x(t)\geq M\|x\|\text{ on }[0,1].$$

Therefore, in view of (S3), we obtain for all $x \in C(u_0) \cap \partial\Omega_1$,

$$\begin{aligned}(\Psi x)(t_0) &= \int_0^1 x(s)ds + \int_0^1 U(t_0,s)f(s,x(s))ds \\ &\geq M\|x\| + \int_0^1 U(t_0,s)g(s)h(x(s))ds \\ &= M\|x\| + \int_0^1 U(t_0,s)g(s)\frac{h(x(s))}{x^a(s)}x^a(s)ds \\ &\geq M\|x\| + \frac{h(r)}{r^a}\int_0^1 U(t_0,s)g(s)M^a\|x\|^a ds \\ &\geq M\|x\| + (1-M)\|x\| = \|x\|.\end{aligned}$$

So $\|x\| \leq \sigma(u_0)\|\Psi x\|$ for all $x \in C(u_0) \cap \partial\Omega_1$, which means (C5) of Lemma 2.1 holds.

Thus by Lemma 2.1, we confirm that the equation $Lx = Nx$ has a solution $x \in C \cap (\overline{\Omega}_2 \backslash \Omega_1)$, which implies that nonlinear resonant third-order multi-point boundary value problem (1.1) has at least one positive solution. □

4. Example

We investigate the resonant third-order three-point boundary value problem

$$\begin{cases} x'''(t) + (-\frac{1}{2}t^2 + \frac{1}{2}t + \frac{3}{8})(x^2 - 4x + \frac{12}{5})\sqrt{x^2 - 6x + 10} = 0, \ t \in [0,1] \\ x''(0) = 0, \ x'(0) = x'(1), \ x(0) = x(\frac{2}{3}) \end{cases}$$

where $\beta = 1, \ \xi = \frac{2}{3}$ and

$$f(t, \ x) = (-\frac{1}{2}t^2 + \frac{1}{2}t + \frac{3}{8})(x^2 - 4x + \frac{12}{5})\sqrt{x^2 - 6x + 10}.$$

Here

$$k(t,s) = \begin{cases} \frac{3(2t-1)}{4}(\frac{2}{3}s - \frac{1}{2}s^2 - \frac{2}{9}) - \frac{1}{6}(1-s)^3 + \frac{1}{2}(t-s)^2, \ t \geq s, \ 0 \leq s \leq \frac{2}{3}, \\ \frac{3(2t-1)}{4}(\frac{2}{3}s - \frac{1}{2}s^2 - \frac{2}{9}) - \frac{1}{6}(1-s)^3, t \leq s, \ 0 \leq s \leq \frac{2}{3}, \\ -\frac{1}{6}(1-s)^3 + \frac{1}{2}(t-s)^2, \ t \geq s, \ \frac{2}{3} \leq s \leq 1, \\ -\frac{1}{6}(1-s)^3, t \leq s, \ \frac{2}{3} \leq s \leq 1 \end{cases}$$

By a simple computation, we have

$$\int_0^1 G(s)ds = \frac{1}{2}, \ \kappa = \frac{1}{2}, \ \int_0^1 U(0,s)ds = 1.$$

Choose $R = 1, \ r = \frac{1}{4}, \ t_0 = 0, \ a = 1, \ M = \frac{1}{2}$.

We take

$$g(t) = -\frac{1}{2}t^2 + \frac{1}{3}t + \frac{3}{8}, \ t \in [0,1], \ h(x) = \sqrt{x^2 - 6x + 10}, \ x \in [0, \ \frac{1}{4}].$$

Then,

$$\frac{3}{8} \leq g(t) \leq \frac{1}{2},\ t \in [0,1],\ x^2 - 4x + \frac{12}{5} \geq -x,\ x \in [0,1].$$

It is easy to check that

(1) $f(t,x) > -\frac{1}{2}x$, for all $(t,x) \in [0,1] \times [0,\ 1]$,

(2) $f(t,x) < 0$, for all $(t,x) \in [0,1] \times [\frac{3}{4},\ 1]$,

(3) $f(t,x) \geq \frac{117}{80}(-\frac{1}{2}t^2 + \frac{1}{3}t + \frac{3}{8})\sqrt{x^2-6x+10} \geq g(t)h(x), [t,x] \in [0,1] \times (0, \frac{1}{4}]$ and $\frac{h(x)}{x} = \frac{\sqrt{x^2-6x+10}}{x}$ is non-increasing on $(0, \frac{1}{4}]$ with

$$\frac{h(r)}{r^a}\int_0^1 U(0,s)g(s)ds > \frac{411}{32} > 1 = \frac{1-M}{M^a}.$$

Thus all the conditions of Theorem 4.1 are satisfied. This ensures that the resonant problem has at least one solution, positive on [0,1].

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