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SOLVABILITY FOR IMPULSIVE FRACTIONAL LANGEVIN EQUATION*

Mengrui Xu[1,†], Shurong Sun[2] and Zhenlai Han[2]

Abstract We investigate impulsive fractional Langevin equation involving two fractional Caputo derivatives with boundary value conditions. By Banach contraction mapping principle and Krasnoselskii's fixed point theorem, some results on the existence and uniqueness of solution are obtained.

Keywords Fractional Langevin equation, impulsive, fractional differential equations, boundary value problems.

MSC(2010) 34A37, 34A08, 34B15.

1. Introduction

The states of many processes are often subject to instantaneous perturbations and experience abrupt changes at certain moments of time. The model of impulsive differential equations is better than pure continuous-time or discrete-time model for describing those processes [3,16,17]. Although fractional differential equation is developing rapidly owing to its wide applications of science and engineering in recent decades [1, 8, 9, 13, 15, 18, 20–23, 25], the study of fractional impulsive differential equations has been started quite recently ([17,25]).

There are some ways to consider the concept of a solution to fractional differential equations with impulses. In 2012, Fečkan et al. [5] gave a new concept which is to keep the lower limit t_0 of the fractional derivative for all $t \geq t_0$ but consider different initial conditions on each interval (t_k, t_{k+1}). Fractional derivative provides an excellent instrument for the description of memory and hereditary properties of processes. This is the main advantage of fractional derivatives in comparison with classical integer derivatives [15]. This concept can reflect that fractional derivatives have global property and the memory accumulated by the long time effects in the whole process including impulsive moments. This approach is used in some papers (for example, [2,5–7,14,19,24]).

In this paper, also in this way, we consider the boundary value problem of two-term Caputo fractional impulsive Langevin equation

$$D^{\delta}u(t) + \lambda D^{\delta-1}u(t) + f(t, u(t)) = 0, \quad t \in J' := J\backslash\{t_1, \cdots, t_m\}, \tag{1.1}$$

$$\Delta u|_{t=t_k} = a_k, \quad \Delta u'|_{t=t_k} = b_k, \; k = 1, \cdots, m, \tag{1.2}$$

†the corresponding author. Email address:xumengrui01@163.com(M. Xu)
1Department of Mathematics, Shandong University, South Shanda Road, Jinan, Shandong 250100, China
2School of Mathematical Sciences, University of Jinan, West Nanxinzhuang Road, Jinan, Shandong 250100, China
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$$\lambda u(0) = u(1),\ u'(0) = d, \tag{1.3}$$

where D^{δ} and $D^{\delta-1}$ are the standard Caputo fractional derivatives with the lower limit zero and $1 < \delta \leq 2$, $J = [0, 1]$, $f : J \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, a_k, b_k, λ and $d \in \mathbb{R}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, $\Delta u'|_{t=t_k} = u'(t_k^+) - u'(t_k^-)$.

The Langevin equation was introduced by Langevin in 1908 to give an elaborate description of Brownian motion. It has been widely used to describe the evolution of physical phenomena in fluctuating environments [4]. The nonlinear fractional Langevin equation involving two fractional derivatives as a kind of generalization of Langevin equation has been studied by many researchers [10–12]. In addition, A. Kilbas et al. [8] considered the fractional differential equation with two fractional derivatives of the type: $D^{\alpha}x(t) - \lambda D^{\beta}x(t) = f(t)$, where $\lambda \in \mathbb{R}$, D^{α} and D^{β} denote the Caputo fractional derivatives with the lower limit zero. However, there are less results about multi-term fractional impulsive differential equations and no paper considered the solution for two-term Caputo fractional impulsive Langevin equation with boundary conditions (1.1)-(1.3). What's more, the equation we studied can reduce to single-term fractional differential equations by letting parameter $\lambda = 0$ and reduce to classical Langevin equation by letting order $\delta = 2$. In this article, we will study the existence and uniqueness of solution for BVP (1.1)-(1.3), using Banach contraction mapping principle and Krasnoselskii's fixed point theorem.

The paper is organized as follows. In Section 2, we recall some necessary concepts and results and present preliminary results. In Section 3, some results on the existence and uniqueness of solution are obtained. Two examples are given in Section 4.

2. Preliminaries

In this section, we give some definitions and lemmas which are required for building our theorems.

Definition 2.1 ([15]). The fractional integral of order $\alpha > 0$ of a function $f : [0, +\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,\ t > 0,$$

where $\Gamma(\alpha)$ is the Gamma function, provided the right side is pointwise defined on $[0, +\infty)$.

Definition 2.2 ([15]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $f : [0, +\infty) \to \mathbb{R}$ is given by

$${}^{RL}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha+1} f(s) ds,\ t > 0,$$

where n is the smallest integer greater than or equal to α, provided that the right side is pointwise defined on $[0, +\infty)$.

Definition 2.3 ([15]). The Caputo fractional derivative of order $\alpha > 0$ of a function $f:[0,+\infty)\to\mathbb{R}$ is given by

$$D_{0+}^{\alpha}f(t)={}^{RL}D^{\alpha}\left[f(t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right],\ t>0,$$

where n is the smallest integer greater than or equal to α, provided that the right side is pointwise defined on $[0,+\infty)$.

Lemma 2.1 ([3]). *(Compactness criterion) The set $F\in PC([0,T],\mathbb{R}^n)$ is relatively compact if and only if:*

(1) F is bounded, that is, $||x||\le c$ for each $x\in F$ and some $c>0$;

(2) F is quasiequicontinuous in $[0,T]$.

Lemma 2.2 ([15]). *(Krasnoselskii's fixed point theorem) Let M be a closed, bounded, convex and nonempty subset of a Banach space X. Let A,B be the operators such that*

(i) $Ax+By\in M$ whenever $x,y\in M$;

(ii) A is compact and continuous;

(iii) B is a contraction mapping.

Then there exists $z\in M$ such that $z=Az+Bz$.

Consider the piecewise continuous functions space

$$PC(J,\mathbb{R})=\{u:[0,1]\to\mathbb{R}:u\in C((t_k,t_{k+1}],\mathbb{R})\text{ and there exist }u(t_k^+),\ u(t_k^-)\text{ with }u(t_k^-)=u(t_k),\ k=0,1\cdots,m\}.$$

with the norm $||u||=\sup_{0\le t\le 1}|u(t)|$. Denote $PC^1(J,\mathbb{R})=\{u,u'\in PC(J,\mathbb{R})\}$ with the norm $||u||_1=||u||+||u'||$. Obviously, $PC^1(J,\mathbb{R})$ is Banach space.

Definition 2.4. A function $u\in PC^2(J,\mathbb{R})$ is said to be a solution of (1.1)-(1.3) if u satisfies the equation $D^{\delta}u(t)+\lambda D^{\delta-1}u(t)+f(t,u(t))=0,\ t\in J'$, and the conditions $\Delta u|_{t=t_k}=a_k$, $\Delta u'|_{t=t_k}=b_k$, $k=1\cdots,m$, $\lambda u(0)=u(1)$ and $u'(0)=d$.

Lemma 2.3. *Let $h:J\to\mathbb{R}$ be continuous. A function u is a solution of the boundary value problem*

$$D^{\delta}u(t)+\lambda D^{\delta-1}u(t)+h(t)=0,\quad t\in J',\tag{2.1}$$

$$\lambda u(0)=u(1),\ u'(0)=d,\quad \Delta u|_{t=t_k}=a_k,\quad \Delta u'|_{t=t_k}=b_k,\ k=1,2,\cdots,m,\tag{2.2}$$

if and only if $u\in PC(J,\mathbb{R})$ is a solution of the integral equation

$$u(t)=\begin{cases}p(t)+\lambda\int_0^1H(t,s)u(s)ds+\int_0^1G(t,s)h(s)ds,\ t\in[0,t_1],\\ p(t)+q(t,k)+\lambda\int_0^1H(t,s)u(s)ds+\int_0^1G(t,s)h(s)ds,\ t\in(t_k,t_{k+1}],\\ \qquad k=1,2,\cdots,m,\end{cases}\tag{2.3}$$

where

$$p(t) = -(\lambda t + 1)\left(\sum_{i=1}^{m}\big(a_i + (b_i + \lambda a_i)(1 - t_i)\big)\right) + (-\lambda t + t - 1)d,$$

$$q(t,k) = \sum_{i=1}^{k}\big(a_i + (b_i + \lambda a_i)(t - t_i)\big),$$

$$H(t,s) = \begin{cases} \lambda t, \ 0 \le s < t \le 1, \\ \lambda t + 1, \ 0 \le t < s \le 1, \end{cases}$$

and

$$G(t,s) = \begin{cases} (1+\lambda t)(1-s)^{\delta-1} - (t-s)^{\delta-1}, \ 0 \le s < t \le 1, \\ (1+\lambda t)(1-s)^{\delta-1}, \ 0 \le t < s \le 1. \end{cases}$$

Proof. Assume u satisfies (2.1)-(2.2). If $t \in [0, t_1]$, applying I^δ on both sides of (2.1), one has

$$u(t) = A_0 + B_0 t - \lambda \int_0^t u(s)ds - \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}h(s)ds. \tag{2.4}$$

Note that $u(0) = A_0$. By boundary condition $u'(0) = d$, one has $B_0 = d + \lambda A_0$.

Furthermore, in general, if $t \in (t_k, t_k + 1]$, $k = 1, 2, \cdots, m$, then

$$u(t) = A_k + B_k t - \lambda \int_0^t u(s)ds - \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}h(s)ds, \tag{2.5}$$

$$u'(t) = B_k - \lambda u(t) - \frac{1}{\Gamma(\delta-1)}\int_0^t (t-s)^{\delta-2}h(s)ds. \tag{2.6}$$

By the impulsive conditions $u(t_k^+) = a_k + u(t_k^-)$ and $u'(t_k^+) = b_k + u'(t_k^-)$, we deduce that

$$\begin{aligned} A_k + B_k t_k &= a_k + A_{k-1} + B_{k-1}t_k, \\ B_k - \lambda u(t_k^+) &= b_k + B_{k-1} - \lambda u(t_k^-). \end{aligned} \tag{2.7}$$

Thus,

$$B_k = b_k + \lambda a_k + B_{k-1} = \sum_{i=1}^{k}(b_i + \lambda a_i) + d + \lambda A_0. \tag{2.8}$$

Combining (2.8) with (2.7), we get

$$\begin{aligned} A_k =& a_k + A_{k-1} + B_{k-1}t_k - B_k t_k = a_k - (b_k + \lambda a_k)t_k + A_{k-1} \\ =& \sum_{i=1}^{k}\big(a_i - (b_i + \lambda a_i)t_i\big) + A_0. \end{aligned}$$

Therefore, substituting A_k and B_k into (2.5), one has, for $t \in (t_k, t_{k+1}]$, $k = 1, 2, \cdots, m$,

$$\begin{aligned} u(t) =& A_0 + (d + \lambda A_0)t + \sum_{i=1}^{k}\big(a_i + (b_i + \lambda a_i)(t - t_i)\big) - \lambda \int_0^t u(s)ds \\ &- \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}h(s)ds. \end{aligned} \tag{2.9}$$

In particular, $u(1) = u(t_{m+1}) = A_0 + d + \gamma A_0 + \sum_{i=1}^{m} \big(a_i + (b_i + \gamma a_i)(1 - t_i)\big) - \gamma \int_0^1 u(s)ds - \frac{1}{\Gamma(\delta)} \int_0^1 (1-s)^{\delta-1} h(s)ds$. From boundary condition $\lambda u(0) = u(1)$, we have

$$-A_0 = \sum_{i=1}^{m} \big(a_i + (b_i + \lambda a_i)(1 - t_i)\big) + d - \lambda \int_0^1 u(s)ds - \frac{1}{\Gamma(\delta)} \int_0^1 (1-s)^{\delta-1} h(s)ds.$$

For $t \in [0, t_1]$, substituting A_0 into (2.4), one has $u(t) = p(t) + \lambda \int_0^1 H(t,s)u(s)ds + \int_0^1 G(t,s)h(s)ds$. For $t \in (t_k, t_{k+1}]$, $k = 1, 2, \cdots, m$, according to (2.9), we can get

$$u(t) = p(t) + q(t,k) + \lambda \int_0^1 H(t,s)u(s)ds + \frac{1}{\Gamma(\delta)} \int_0^1 G(t,s)h(s)ds.$$

Conversely, assume that $u(t)$ is a solution of (2.3), we can easily show that $u(t)$ is the solution of (2.1)-(2.2). The proof is complete. □

For convenience, we denote $H = \sup_{t\in[0,1]} \int_0^1 |H(t,s)|ds$, $P = \sup_{t\in[0,1]} |p(t)|$, $G = \sup_{t\in[0,1]} \int_0^1 |G(t,s)|ds$, $Q = \max_k \sup_{t\in(t_k,t_{k+1}]} |q(t,k)|,\ k = 1, 2, \cdots, m$.

For the forthcoming analysis, we need the following hypotheses

(H1) There exists a constant $L > 0$ such that

$$|f(t,u) - f(t,v)| \le L|u - v|, \text{ for each } t \in [0,1], \text{ and all } u, v \in \mathbb{R};$$

(H2) There exists a integrable function $\mu : [0,1] \to \mathbb{R}^+$ such that

$$|f(t,u)| \le \mu(t),\ (t,u) \in [0,1] \times \mathbb{R}.$$

3. Existence and uniqueness of solution

In this section, we will show the existence and uniqueness of solution for boundary value problems (1.1)-(1.3) by Banach contraction mapping principle and Krasnoselskii's fixed point theorem.

Theorem 3.1. *Assume that (H1) holds. If* $|\lambda|H + \frac{LG}{\Gamma(\delta)} < 1$, *then BVP* (1.1)-(1.3) *has a unique solution.*

Proof. Define operator $T : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ by

$$Tu(t) = \begin{cases} p(t) + \lambda \int_0^1 H(t,s)u(s)ds + \frac{1}{\Gamma(\delta)} \int_0^1 G(t,s)f(s,u(s))ds,\ t \in [0,t_1], \\ p(t) + q(t,k) + \lambda \int_0^1 H(t,s)u(s)ds + \frac{1}{\Gamma(\delta)} \int_0^1 G(t,s)f(s,u(s))ds, \\ \qquad t \in (t_k, t_{k+1}], k = 1, \cdots, m. \end{cases}$$

Then T is well-defined and $u \in PC(J,\mathbb{R})$ is a solution to the BVP (1.1)-(1.3), if and only if u is a fixed point of T. It is easy to verify that $Tu \in PC(J,\mathbb{R})$ by Lebesgue's dominated convergence theorem.

For all $u, v \in PC(J,\mathbb{R})$, $t \in [0,1]$, by (H1), we have

$$\begin{aligned} &|Tu(t) - Tv(t)| \\ =&\left|\lambda \int_0^1 H(t,s)\big(u(s) - v(s)\big)ds + \frac{1}{\Gamma(\delta)} \int_0^1 G(t,s)\big(f(s,u(s)) - f(s,v(s))\big)ds\right| \\ \le& |\lambda|||u - v|| \int_0^1 |H(t,s)|ds + \frac{L||u - v||}{\Gamma(\delta)} \int_0^1 |G(t,s)|ds \le \left(|\lambda|H + \frac{LG}{\Gamma(\delta)}\right) ||u - v||. \end{aligned}$$

Hence, T is a contraction mapping and there exists a unique fixed point according to Banach contraction mapping principle. Therefore, (1.1)-(1.3) has a unique solution. □

Theorem 3.2. *Assume that (H2) holds. If* $|\lambda|H < 1$, *then BVP* (1.1)-(1.3) *has at least one solution.*

Proof. Define operators E and F from $PC(J,\mathbb{R})$ into itself by

$$Eu(t) = \begin{cases} p(t) + \lambda \int_0^1 H(t,s)u(s)ds,\ t \in [0,t_1], \\ p(t) + q(t,k) + \lambda \int_0^1 H(t,s)u(s)ds, t \in (t_k,t_{k+1}], k = 1,\cdots,m, \end{cases} \tag{3.1}$$

$$Fu(t) = \frac{1}{\Gamma(\delta)} \int_0^1 G(t,s)f(s,u(s))ds,\ t \in [0,1], \tag{3.2}$$

for $u \in PC(J,\mathbb{R})$. It is easy to verify that E and F are continuous on $PC(J,\mathbb{R})$ by Lebesgue's dominated convergence theorem. Since $|\lambda|H < 1$, we can take $r > 0$ large enough such that $|\lambda|H + \frac{P+Q+\frac{G}{\Gamma(\delta)}||\mu||}{r} < 1$. Set $B_r = \{u \in PC(J,\mathbb{R}) : ||u|| \leq r\}$. Then B_r is a nonempty bounded closed convex subset in $PC(J,\mathbb{R})$. For any $u, v \in B_r$, $k = 1,\cdots,m$, we have

$$|Eu(t)| = \left|p(t) + q(t,k) + \lambda \int_0^1 H(t,s)u(s)ds\right| \leq P + Q + |\lambda|Hr,\ t \in (t_k,t_{k+1}],$$

and $|Eu(t)| \leq P + |\lambda|Hr,\ t \in [0,t_1]$. For all $v \in B_r$, $t \in [0,1]$,

$$|Fv(t)| = \left|\frac{1}{\Gamma(\delta)} \int_0^1 G(t,s)f(s,v(s))ds\right| \leq \frac{G}{\Gamma(\delta)}||\mu||. \tag{3.3}$$

Hence, $|Eu(t) + Fv(t)| \leq |Eu(t)| + |Fv(t)| \leq P + Q + |\lambda|Hr + \frac{G}{\Gamma(\delta)}||\mu|| < r$, which implies that $Eu + Fv \in B_r$, for $t \in [0,1]$. On the other hand, E is a contraction mapping since for all $t \in [0,1]$,

$$\begin{aligned} |Eu(t) - Ev(t)| =& |\lambda| \left|\int_0^1 H(t,s)u(s)ds - \int_0^1 H(t,s)v(s)ds\right| \leq |\lambda|H||u - v|| \\ <& ||u - v||. \end{aligned}$$

Next we prove F is compact. Take any bounded subset $B_\eta = \{u \in PC(J,\mathbb{R}) : ||u|| \leq \eta\}$. According to (3.3), $F(B_\eta)$ is bounded. Taking $\tau_1, \tau_2 \in [0,t_1]$ with $\tau_1 < \tau_2$, for any $u \in B_\eta$, we have

$$\begin{aligned} |Fu(\tau_2) - Fu(\tau_1)| =& \left|\frac{1}{\Gamma(\delta)} \int_0^1 G(\tau_2,s)f(s,u(s)) - \frac{1}{\Gamma(\delta)} \int_0^1 G(\tau_1,s)f(s,u(s))ds\right| \\ \leq& \frac{||\mu||}{\Gamma(\delta)} \left|\int_0^1 (1+\lambda\tau_2)(1-s)^{\delta-1}ds - \int_0^{\tau_2} (\tau_2 - s)^{\delta-1}ds \right. \\ & \left. - \int_0^1 (1+\lambda\tau_1)(1-s)^{\delta-1}ds + \int_0^{\tau_1} (\tau_1 - s)^{\delta-1}ds\right| \\ =& \frac{||\mu||}{\Gamma(\delta+1)} \left(|\lambda(\tau_2 - \tau_1)| + \tau_2^\delta - \tau_1^\delta\right). \end{aligned}$$

As $\tau_1 \to \tau_2$, the right side of the above inequality tends to zero. In general, using the same way, for $\tau_1, \tau_2 \in (t_k, t_{k+1}]$, we can get $|Fu(\tau_2) - Fu(\tau_1)| \to 0$ as $\tau_1 \to \tau_2$. Then $F(B_\eta)$ is quasiequicontinuous on $[0, 1]$. According to Lemma 2.1, F is compact. Now we apply Krasnoselskii's fixed point theorem to the operators E and F to get that there exists at least a $u \in B_r$ such that $u = Eu + Fu$, which is a solution to the BVP (1.1)-(1.3) and the proof is complete. □

4. Examples

Example 4.1. Consider the following BVP for two-term fractional impulsive Langevin equation

$$\begin{cases} D^{\frac{3}{2}}u(t) + \frac{1}{2}D^{\frac{1}{2}}u(t) = \frac{u(t)}{8+u(t)}, \quad t \in [0,1]\backslash\frac{1}{2}, \\ \Delta u|_{t=\frac{1}{2}} = 1, \quad \Delta u'|_{t=\frac{1}{2}} = 1, \quad \frac{1}{2}u(0) = u(1), \ u'(0) = 1. \end{cases} \tag{4.1}$$

We can find $|\frac{u(t)}{8+u(t)} - \frac{v(t)}{8+v(t)}| \le \frac{||u-v||}{8}$, $L = \frac{1}{8}$, $\lambda = \frac{1}{2}$, $H \le |\lambda| + 1 = \frac{3}{2}$, $\delta = \frac{3}{2}$, $G \le \frac{|\lambda|+1}{\delta} = 1$. Therefore, $|\lambda|H + \frac{LG}{\Gamma(\delta)} < 1$ and (H1) are satisfied. According to Theorem 3.1, (4.1) has a unique solution.

Example 4.2. Consider the following BVP for two-term fractional impulsive Langevin equation

$$\begin{cases} D^{\frac{3}{2}}u(t) + \frac{1}{2}D^{\frac{1}{2}}u(t) = sinx(t) + t, \quad t \in [0,1]\backslash\frac{1}{2}, \\ \Delta u|_{t=\frac{1}{2}} = 1, \quad \Delta u'|_{t=\frac{1}{2}} = 1, \quad \frac{1}{2}u(0) = u(1), \ u'(0) = 1. \end{cases} \tag{4.2}$$

Let $\mu(t) = 1 + t$. Obviously, (H2) are satisfied and $|\lambda|H \le \frac{3}{4}$, where $\lambda = \frac{1}{2}$. Furthermore, $||\mu|| = 2$, $P = (|\lambda| + 1)[a_1 + (b_1 + \lambda a_1)(1 - \frac{1}{2})] + (|\lambda| + 2)d = \frac{41}{8}$, $Q = a_1 + (b_1 + \lambda a_1)(1 - \frac{1}{2}) = \frac{7}{4}$, where $a_1 = b_1 = d = 1$. Because $\frac{P+Q+\frac{G}{\Gamma(\delta)}||\mu||}{1-|\lambda|H} \le \frac{55}{2} + \frac{16}{\sqrt{\pi}}$, we can chose $r > \frac{55}{2} + \frac{16}{\sqrt{\pi}}$. Then by Theorem 3.2, (4.2) has at least one solution.

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