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APPROXIMATE LIE $*$-DERIVATIONS ON ρ-COMPLETE CONVEX MODULAR ALGEBRAS*

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Abstract In this paper, we obtain generalized Hyers–Ulam stability results of a (m, n)-Cauchy-Jensen functional equation associated with approximate Lie $*$-derivations on ρ-complete convex modular $*$-algebras χ_ρ with Δ_μ-condition on the convex modular ρ.

Keywords Modular $*$-algebra, convex modular, Δ_μ-condition, (m, n)-Cauchy-Jensen mapping, Lie $*$-derivation.

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1. Introduction

In 1940, S.M. Ulam [15] raised the question concerning the stability of group homomorphisms: Let G be a group and let G' be a metric group with the metric $d(\cdot,\ \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $f : G \to G'$ satisfies the inequality

$$d(f(xy), f(x)f(y)) < \delta$$

for all $x, y \in G$, then there exists a homomorphism $F : G \to G'$ with $d(f(x), F(x)) < \varepsilon$ for all $x \in G$? D.H. Hyers [3] has solved the problem of Ulam for the case of additive mappings in 1941. The result was generalized by T. Aoki [1] in 1950, by Th.M. Rassias [12] in 1978, by J.M. Rassias [9] in 1992, and by P. Găvruta [2] in 1994. Over the past few decades, many mathematicians have published the generalized Hyers–Ulam stability results of various functional equations [4, 8, 13].

Now, we recall some basic definitions and remarks of modular spaces with modular functions, which are primitive notions corresponding to norms or metrics, as in the followings [6, 7, 14, 16].

Definition 1.1. Let χ be a real linear space.

(a) A function $\rho : \chi \to [0, \infty]$ is called a modular if for arbitrary $x, y \in \chi$,

(1) $\rho(x) = 0$ if and only if $x = 0$,

(2) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$,

(3) $\rho(\alpha x + \beta y) \le \rho(x) + \rho(y)$ for any scalars α, β, where $\alpha + \beta = 1$ and $\alpha, \beta \ge 0$,

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(b) alternatively, if (3) is replaced by

(3)' $\rho(\alpha x+\beta y) \le \alpha\rho(x)+\beta\rho(y)$ for every scalars α, β, where $\alpha+\beta=1$ and $\alpha,\beta \ge 0$,

then we say that ρ is a convex modular. Now, we extend the properties (3) and (3)' in real fields to complex scalar field acting on the space χ, as follows :

(4) $\rho(\alpha x+\beta y) \le \rho(x)+\rho(y)$ for $\alpha,\beta \in \mathbb{C}$ with $|\alpha|+|\beta|=1$,

(4)' $\rho(\alpha x+\beta y) \le |\alpha|\rho(x)+|\beta|\rho(y)$ for $\alpha,\beta \in \mathbb{C}$ with $|\alpha|+|\beta|=1$.

We remark a modular ρ defines a corresponding modular space, i.e., the linear space χ_ρ given by

$$\chi_\rho = \{x \in \chi : \rho(\lambda x) \to 0 \quad \text{as} \quad \lambda \to 0\}.$$

Let ρ be a convex modular. Then, the modular space χ_ρ can be equipped with a norm called the Luxemburg norm, defined by

$$\|x\|_\rho = \inf\{\lambda > 0 : \rho\Big(\frac{x}{\lambda}\Big) \le 1\}.$$

If ρ is a modular on χ, we note that $\rho(tx)$ is an increasing function in $t \ge 0$ for each fixed $x \in \chi$, that is, $\rho(ax) \le \rho(bx)$ whenever $0 \le a < b$. In addition, if ρ is a convex modular on χ, then $\rho(\alpha x) \le \alpha\rho(x)$ for all $x \in \chi$ and $0 \le \alpha \le 1$. Moreover, we see that $\rho(\alpha x) \le |\alpha|\rho(x)$ for all $x \in \chi$ and $|\alpha| \le 1$.

Remark 1.1. (a) In general, we note that $\rho\big(\sum_{i=1}^n \alpha_i x_i\big) \le \sum_{i=1}^n \alpha_i \rho(x_i)$ for all $x_i \in \chi$ and $\alpha_i \ge 0$ $(i=1,\cdots,n)$ whenever $0 < \sum_{i=1}^n \alpha_i \le 1$ [6].

(b) Consequently, we lead to $\rho\big(\sum_{i=1}^n \alpha_i x_i\big) \le \sum_{i=1}^n |\alpha_i|\rho(x_i)$ for all $x_i \in \chi$ and $0 < \sum_{i=1}^n |\alpha_i| \le 1$, where $\alpha_i \in \mathbb{C}$.

Definition 1.2. Let χ_ρ be a modular space and let $\{x_n\}$ be a sequence in χ_ρ. Then,

(1) $\{x_n\}$ is ρ-convergent to $x \in \chi_\rho$ and write $x_n \xrightarrow{\rho} x$ if $\rho(x_n - x) \to 0$ as $n \to \infty$.

(2) $\{x_n\}$ is called ρ-Cauchy in χ_ρ if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.

(3) A subset K of χ_ρ is called ρ-complete if and only if any ρ-Cauchy sequence is ρ-convergent to an element in K.

They say that the modular functional ρ has the Fatou property if and only if $\rho(x) \le \liminf_{n\to\infty} \rho(x_n)$ whenever the sequence $\{x_n\}$ is ρ-convergent to x. A modular ρ is said to satisfy the Δ_2-condition if there exists $\kappa > 0$ such that $\rho(2x) \le \kappa\rho(x)$ for all $x \in \chi_\rho$.

In 2014, G. Sadeghi [14] has established generalized Hyers–Ulam stability via the fixed point method of a generalized Jensen functional equation $f(rx+sy) = rg(x)+sh(y)$ in convex modular spaces with the Fatou property satisfying the Δ_2-condition with $0 < \kappa \le 2$. In [16], the authors have presented the generalized Hyers–Ulam stability of quadratic functional equations via the extensive studies of fixed point theory in the framework of modular spaces whose modulars are convex, lower semicontinuous but do not satisfy any relatives of Δ_2-conditions (see also [5, 7]). Recently, the authors [6] have investigated stability theorems of functional equations in modular spaces without using the Fatou property and Δ_2-condition.

Now, we introduce the concept of convex modular algebras. We say that χ_ρ is called a convex modular algebra if the fundamental space χ is an algebra with convex modular ρ subject to $\rho(ab) \leq \rho(a)\rho(b)$ for all $a, b \in \chi$. A subset K of a convex modular algebra χ_ρ is called ρ-complete if and only if any ρ-Cauchy sequence in K is ρ-convergent to an element in K. In addition, a convex modular algebra χ_ρ is a convex modular $*$-algebra if the convex modular ρ satisfies $\rho(z^*) = \rho(z)$ for all $z \in \chi_\rho$. We say that a linear mapping f is called a Lie $*$-derivation if $f([x,y]) = [f(x), y] + [x, f(y)]$ and $f(z^*) = f(z)^*$ for all vectors x, y, z, where $[a,b] = ab - ba$.

Now, we consider a mapping $f : X \to Y$ satisfying the following functional equation

$$\sum_{\substack{1\leq i_1<\cdots<i_m\leq n\\ 1\leq k_l(\neq i_j,\forall j\in\{1,\cdots,m\})\leq n}} f\left(\frac{\sum_{j=1}^m x_{i_j}}{m} + \sum_{l=1}^{n-m} x_{k_l}\right) = \frac{n-m+1}{n}\binom{n}{m}\sum_{i=1}^n f(x_i) \quad (1.1)$$

for all $x_1, \cdots, x_n \in X$, where $n, m \in \mathbb{N}$ are fixed integers with $n \geq 2$, $1 \leq m \leq n$, which has been introduced in [11].

In this article, we first investigate generalized Hyers–Ulam stability via direct method of the equation (1.1) using necessarily Δ_μ-condition without using the Fatou property in ρ-complete convex modular algebras, where the modular ρ is said to satisfy Δ_μ-condition if there exists $\kappa > 0$ such that $\rho(\mu x) \leq \kappa\rho(x)$ for all $x \in \chi_\rho$, $\mu := n-m+1$, and then present alternatively generalized Hyers–Ulam stability via direct method of the equation (1.1) in ρ-complete convex modular algebras without using both the Fatou property and Δ_2-condition.

2. Generalized Hyers–Ulam Stability of Eq. (1.1)

Throughout the paper, χ_ρ will be denoted by ρ-conplete convex modular $*$-algebras. In this section, we investigate the stability results of Lie $*$-derivation associated with the equation (1.1). First of all, we introduce the following lemma which has been presented [11].

Lemma 2.1. *Let X and Y be linear spaces. For each m with $1 \leq m \leq n$, a mapping $f : X \to Y$ satisfies the equation (1.1) for all $n \geq 2$ if and only if $f - f(0)$ is Cauchy additive, where $f(0) = 0$ if $m < n$.*

For notational convenience, we let the difference operators $D_\lambda f$ of equation (1.1) and $LDf(x,y)$ of Lie derivation as follows:

$$\begin{aligned} D_\lambda f(x_1, \cdots, x_n) :=& \sum_{\substack{1\leq i_1<\cdots<i_m\leq n\\ 1\leq k_l(\neq i_j,\forall j\in\{1,\cdots,m\})\leq n}} f\left(\frac{\sum_{j=1}^m \lambda x_{i_j}}{m} + \sum_{l=1}^{n-m} \lambda x_{k_l}\right) \\ &- \frac{n-m+1}{n}\binom{n}{m}\lambda\sum_{i=1}^n f(x_i), \\ LDf(x,y) :=& f([x,y]) - [f(x), y] - [x, f(y)] \end{aligned}$$

for all x, y in a linear space X and $\lambda \in \Lambda := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

We observe that if the modular ρ satisfies the Δ_μ-condition, then $\kappa \geq 1$ for nontrivial modular ρ, and $\kappa \geq \mu$ for nontrivial convex modular ρ, where $\mu := n - m + 1 \geq 2$. See references [6, 16].

Now, we present a generalized Hyers–Ulam stability of the equation (1.1) using necessarily Δ_μ-condition without Fatou property, where $\mu := n - m + 1 \geq 2$.

Theorem 2.1. *Let χ_ρ be a ρ-complete convex modular $*$-algebra with Δ_μ-condition. Suppose there exist two functions $\varphi_1 : \chi_\rho^{n+1} \to [0, \infty), \varphi_2 : \chi_\rho^2 \to [0, \infty)$ for which a mapping $f : \chi_\rho \to \chi_\rho$ satisfies the following combined functional inequalities*

$$\rho(D_\lambda f(x_1, \cdots, x_n) + f(z^*) - f(z)^*) \leq \varphi_1(x_1, \cdots, x_n, z), \tag{2.1}$$
$$\rho(LDf(x, y)) \leq \varphi_2(x, y),$$

such that

$$\Psi(x_1, \cdots, x_n, z) := \sum_{j=1}^{\infty} \frac{\kappa^{2j}}{\mu^j} \varphi_1(\frac{x_1}{\mu^j}, \cdots, \frac{x_n}{\mu^j}, \frac{z}{\mu^j}) < \infty, \tag{2.2}$$
$$\lim_{s\to\infty} \kappa^{2s} \varphi_2(\frac{x}{\mu^s}, \frac{y}{\mu^s}) = 0$$

for all $x_1, \cdots, x_n, x, y, z \in \chi_\rho$ and $\lambda \in \Lambda$. Then there exists a unique Lie $*$-derivation $F_1 : \chi_\rho \to \chi_\rho$ satisfying the equation (1.1) and

$$\rho(f(x) - F_1(x)) \leq \frac{\kappa}{\binom{n}{m}\mu^2} \Psi(x, \cdots, x, 0) \tag{2.3}$$

for all $x \in \chi_\rho$.

Proof. First, we remark that since $\sum_{j=1}^{\infty} \frac{\kappa^{2j}}{\mu^j} \varphi_1(0, \cdots, 0) = \Psi(0, \cdots, 0) < \infty$ and $\rho(D_\lambda f(0, \cdots, 0)) \leq \varphi_1(0, \cdots, 0)$, we lead to $\varphi_1(0, \cdots, 0) = 0$, $D_\lambda f(0, \cdots, 0) = 0$ and so $f(0) = 0$. Putting $x_i = x, z = 0$ and $\lambda = 1$ in (2.7), we obtain

$$\rho(D_1 f(x, \cdots, x)) = \rho(\binom{n}{m} f(\mu x) - \binom{n}{m} \mu f(x)) \leq \phi_1(x, \cdots, x, 0), \tag{2.4}$$

which yields

$$\rho(f(\mu x) - \mu f(x)) \leq \frac{1}{\binom{n}{m}} \phi_1(x, \cdots, x, 0), \tag{2.5}$$
$$\rho\Big(\mu f\Big(\frac{x}{\mu}\Big) - f(x)\Big) \leq \frac{1}{\binom{n}{m}} \phi_1\Big(\frac{x}{\mu}, \cdots, \frac{x}{\mu}, 0\Big)$$

for all $x \in \chi_\rho$. Using the convexity of the modular ρ and Δ_μ-condition, one obtains the following inequality

$$\begin{aligned}
\rho\Big(f(x) - \mu^s f\Big(\frac{x}{\mu^s}\Big)\Big) &\leq \rho\Big(\sum_{j=0}^{s-1} \frac{1}{\mu^{j+1}} \Big(\mu^{2j+1} f\Big(\frac{x}{\mu^j}\Big) - \mu^{2j+2} f\Big(\frac{x}{\mu^{j+1}}\Big)\Big)\Big) \\
&\leq \sum_{j=0}^{s-1} \frac{\kappa^{2j+1}}{\binom{n}{m}\mu^{j+1}} \varphi_1\Big(\frac{x}{\mu^{j+1}}, \cdots, \frac{x}{\mu^{j+1}}, 0\Big) \\
&= \sum_{j=1}^{s} \frac{\kappa^{2j-1}}{\binom{n}{m}\mu^j} \varphi_1\Big(\frac{x}{\mu^j}, \cdots, \frac{x}{\mu^j}, 0\Big)
\end{aligned}$$

$$\leq \frac{1}{\binom{n}{m}\kappa}\sum_{j=1}^{s}\frac{\kappa^{2j}}{\mu^{j}}\varphi_1\Big(\frac{x}{\mu^{j}},\cdots,\frac{x}{\mu^{j}},0\Big)$$

for all $x \in \chi_\rho$. Now, replacing x by $\mu^{-t}x$ in above inequality, we have

$$\begin{aligned}\rho\Big(\mu^t f\Big(\frac{x}{\mu^t}\Big)-\mu^{s+t}f\Big(\frac{x}{\mu^{s+t}}\Big)\Big) &\leq \kappa^t\rho\Big(f\Big(\frac{x}{\mu^t}\Big)-\mu^s f\Big(\frac{x}{\mu^{s+t}},0\Big)\Big)\\ &\leq \kappa^t\frac{1}{\binom{n}{m}\kappa}\sum_{j=1}^{s}\frac{\kappa^{2j}}{\mu^{j}}\varphi_1\Big(\frac{x}{\mu^{j+t}},\cdots,\frac{x}{\mu^{j+t}},0\Big)\\ &\leq \frac{\kappa^t\mu^t}{\kappa^{2t}\binom{n}{m}\kappa}\sum_{j=1}^{s}\frac{\kappa^{2(j+t)}}{\mu^{j+t}}\varphi_1\Big(\frac{x}{\mu^{j+t}},\cdots,\frac{x}{\mu^{j+t}},0\Big)\\ &= \frac{\mu^t}{\kappa^t\binom{n}{m}\kappa}\sum_{j=t+1}^{s+t}\frac{\kappa^{2j}}{\mu^{j}}\varphi_1\Big(\frac{x}{\mu^{j}},\cdots,\frac{x}{\mu^{j}},0\Big)\end{aligned}$$

which converges to zero as $t \to \infty$ by the assumption (2.2). Thus, the sequence $\{\mu^s f(\frac{x}{\mu^s})\}$ is ρ-Cauchy for all $x \in \chi_\rho$ and so it is ρ-convergent in χ_ρ since the space χ_ρ is ρ-complete. Thus, we may define a mapping $F_1 : \chi_\rho \to \chi_\rho$ as

$$F_1(x) := \rho-\lim_{s\to\infty}\mu^s f\Big(\frac{x}{\mu^s}\Big) \Longleftrightarrow \lim_{s\to\infty}\rho\Big(\mu^s f\Big(\frac{x}{\mu^s}\Big)-F_1(x)\Big)=0,$$

for all $x \in \chi_\rho$.

Claim 1 : F_1 is an additive mapping with the estimation (2.3) near f. By Δ_μ-condition without using the Fatou property, we can see the following inequality

$$\begin{aligned}\rho(f(x)-F_1(x)) &\leq \frac{1}{\mu}\rho\Big(\mu f(x)-\mu\cdot\mu^s f\Big(\frac{x}{\mu^s}\Big)+\mu\cdot\mu^s f\Big(\frac{x}{\mu^s}\Big)-\mu F_1(x)\Big)\\ &\leq \frac{\kappa}{\mu}\rho\Big(f(x)-\mu^s f\Big(\frac{x}{\mu^s}\Big)\Big)+\frac{\kappa}{\mu}\rho\Big(\mu^s f\Big(\frac{x}{\mu^s}\Big)-F_1(x)\Big)\\ &\leq \frac{\kappa}{\mu}\cdot\frac{1}{\binom{n}{m}\mu}\sum_{j=1}^{s}\frac{\kappa^{2j}}{\mu^{j}}\varphi_1\Big(\frac{x}{\mu^{j}},\cdots,\frac{x}{\mu^{j}},0\Big)+\frac{\kappa}{\mu}\rho\Big(\mu^s f\Big(\frac{x}{\mu^s}\Big)-F_1(x)\Big)\end{aligned}$$

which yields the approximation (2.3) by taking $s\to\infty$. Now, setting $(x_1,\cdots,x_n,z):=(\frac{x_1}{\mu^s},\cdots,\frac{x_n}{\mu^s},0)$ in (2.1) and multiplying the resulting inequality by μ^s, we get

$$\begin{aligned}\rho(\mu^s D_\lambda f(\mu^{-s}x_1,\cdots,\mu^{-s}x_n) &\leq \kappa^s\varphi_1(\mu^{-s}x_1,\cdots,\mu^{-s}x_n,0)\\ &\leq \kappa^s\varphi_1(\mu^{-s}x_1,\cdots,\mu^{-s}x_n,0)\cdot\frac{\kappa^s}{\mu^s}\\ &= \frac{\kappa^{2s}}{\mu^s}\varphi_1(\mu^{-s}x_1,\cdots,\mu^{-s}x_n,0)\end{aligned}$$

which tends to zero as $s \to \infty$ for all $x_1,\cdots,x_n \in \chi_\rho$. Thus, it follows from Remark 1.1 (b) that

$$\begin{aligned}&\rho\Big(\frac{1}{R}D_\lambda F_1(x_1,\cdots,x_n)\Big) \qquad (2.6)\\ &=\rho\Big(\frac{1}{R}D_\lambda F_1(x_1,\cdots,x_n)-\frac{\mu^s}{R}D_\lambda f\Big(\frac{x_1}{\mu^s},\cdots,\frac{\lambda x_n}{\mu^s}\Big)+\frac{\mu^s}{R}D_\lambda f\Big(\frac{x_1}{\mu^s},\cdots,\frac{x_n}{\mu^s}\Big)\Big)\end{aligned}$$

$$\leq \frac{1}{R} \sum_{\substack{1\leq i_1<\cdots<i_m\leq n \\ 1\leq k_l(\neq i_j,\forall j\in\{1,\cdots,m\})\leq n}} \rho\Big(\mu^s f\left(\frac{\sum_{j=1}^m \lambda\frac{x_{i_j}}{\mu^s}}{m} + \sum_{l=1}^{n-m}\lambda\frac{x_{k_l}}{\mu^s}\right)$$

$$-F_1\left(\frac{\sum_{j=1}^m \lambda x_{i_j}}{m} + \sum_{l=1}^{n-m}\lambda x_{k_l}\right)\Big)$$

$$+\frac{n-m+1}{Rn}\binom{n}{m}\sum_{i=1}^n \rho\Big(\mu^s f\Big(\frac{x_i}{\mu^s}\Big) - F_1(x_i)\Big) + \frac{\kappa^{2s}}{\mu^s R}\varphi_1\Big(\frac{x_1}{\mu^s},\cdots,\frac{x_n}{\mu^s},0\Big)$$

for all $x_1,\cdots,x_n\in\chi_\rho$, $\lambda\in\Lambda$ and all positive integers s, where $R := \binom{n}{m}(n-m+2)+2$ is a fixed real number. Taking the limit as $s\to\infty$, one obtains $\rho(\frac{1}{R}D_\lambda F_1(x_1,\cdots,x_n)) = 0$, and thus $D_\lambda F_1(x_1,\cdots,x_n)=0$ for all $x_1,\cdots,x_n\in\chi_\rho$. Hence $F_1:\chi_\rho\to\chi_\rho$ satisfies the equation (1.1), and so it is additive.

Claim 2: F_1 is a linear mapping. By (2.6), we have $D_\lambda F_1(x,\cdots,x)=0$, which yields $F_1(\lambda x)=\lambda F_1(x)$ for all $x\in\chi_\rho$ and $\lambda\in\Lambda$. Next, for any $\lambda=\lambda_1+i\lambda_2\in\mathbb{C}$ where $\lambda_1,\lambda_2\in\mathbb{R}$, let $\gamma_1 := \lambda_1-[\lambda_1]$ and $\gamma_2 := \lambda_2-[\lambda_2]$, where $[\lambda]$ denotes the greatest integer part of λ less than or equal to λ. Then one can find unit complex numbers $\gamma_{i,1},\gamma_{i,2}\in\Lambda$ such that $\gamma_i = \frac{\lambda_{i,1}+\lambda_{i,2}}{2}$ $(i=1,2)$. So, it follows that

$$\begin{aligned}
F_1(\lambda x) &= F_1(\lambda_1 x) + iF_1(\lambda_2 x)\\
&= ([\lambda_1]F_1(x) + F_1(\gamma_1 x)) + i([\lambda_2]F_1(x)+F_1(\gamma_2 x))\\
&= \Big([\lambda_1]F_1(x) + \frac{1}{2}F_1(\gamma_{1,1}x+\gamma_{1,2}x)\Big) + i\Big([\lambda_2]F_1(x)+\frac{1}{2}F_1(\gamma_{2,1}x+\gamma_{2,2}x)\Big)\\
&= \Big([\lambda_1]F_1(x) + \frac{1}{2}F_1(\gamma_{1,1}x) + \frac{1}{2}F_1(\gamma_{1,2}x)\Big)\\
&\quad +i\Big([\lambda_2]F_1(x) + \frac{1}{2}F_1(\gamma_{2,1}x) + \frac{1}{2}F_1(\gamma_{2,2}x)\Big)\\
&= \lambda_1F_1(x) + i\lambda_2F_1(x) = \lambda F_1(x)
\end{aligned}$$

for all $x\in\chi_\rho$. Hence F_1 is a linear mapping.

Claim 3: F_1 is a Lie $*$-derivation. From the last inequality in (2.2) and the last condition in (2.1), one obtains that

$$\begin{aligned}
&\rho\Big(\frac{1}{4}LDF_1(x,y)\Big)\\
&= \rho\Big(\frac{1}{4}LDF_1(x,y) - \mu^{2s}\frac{LDf(\mu^{-s}x,\mu^{-s}y)}{4} + \mu^{2s}\frac{LDf(\mu^{-s}x,\mu^{-s}y)}{4}\Big)\\
&\leq \frac{1}{4}\rho\Big(F_1([x,y]) - \mu^{2s}f(\mu^{-2s}[x,y])\Big) + \frac{1}{4}\rho\Big(\mu^s[x,f(\mu^{-s}y)] - [x,F_1(y)]\Big)\\
&\quad +\frac{1}{4}\rho\Big(\mu^s[f(\mu^{-s}x,y] - [F_1(x),y]\Big) + \frac{1}{4}\rho\Big(\mu^{2s}LDf(\mu^{-s}x,\mu^{-s}y)\Big)\\
&\leq \frac{1}{4}\rho\Big(F_1([x,y]) - \mu^{2s}f(\mu^{-2s}[x,y])\Big) + \frac{1}{4}\rho\Big(\mu^s[x,f(\mu^{-s}y)] - [x,F_1(y)]\Big)\\
&\quad +\frac{1}{4}\rho\Big(\mu^s[f(\mu^{-s}x),y] - [F_1(x),y]\Big) + \frac{\kappa^{2s}}{4}\varphi_2\Big(\mu^{-s}x,\mu^{-s}y\Big)
\end{aligned}$$

for all $x,y\in\chi_\rho$, from which $LDF_1(x,y)=0$ by taking $s\to\infty$ and so F_1 is a Lie derivation. In addition, we get the following inequality

$$\rho\Big(\frac{1}{3}\Big(F_1(z^*) - F_1(z)^*\Big)\Big) \leq \frac{1}{3}\rho\Big(F_1(z^*) - \mu^s f\Big(\frac{z^*}{\mu^s}\Big)\Big)$$

$$
\begin{aligned}
&+\frac{1}{3}\rho\Big(\mu^s f\Big(\frac{z}{\mu^s}\Big)^* - F_1(z)^*\Big) + \frac{1}{3}\rho\Big(\mu^s f\Big(\frac{z^*}{\mu^s}\Big) - \mu^s f\Big(\frac{z}{\mu^s}\Big)^*\Big)\\
\le\ & \frac{1}{3}\rho\Big(F_1(z^*) - \mu^s f\Big(\frac{z^*}{\mu^s}\Big)\Big)\\
&+\frac{1}{3}\rho\Big(\mu^s f\Big(\frac{z}{\mu^s}\Big)^* - F_1(z)^*\Big) + \frac{\kappa^s}{3}\varphi_1\Big(0,\cdots,0,\frac{z}{\mu^s}\Big)\cdot\frac{\kappa^s}{\mu^s}
\end{aligned}
$$

for all vector z. Taking $s\to\infty$, one concludes F_1 is a Lie $*$-derivation.

Claim 4: F_1 is unique. To show the uniqueness of F_1, let's assume that there exists a Lie $*$-derivation $G_1 : \chi_\rho \to \chi_\rho$ which satisfies the approximation (2.3). Since F_1 and G_1 are additive mappings, we see from the equality $\mu^s F_1(\mu^{-s}x) = F_1(x)$ and $\mu^s G_1(\mu^{-s}x) = G_1(x)$ that

$$
\begin{aligned}
\rho(G_1(x) - F_1(x)) &= \rho\Big(\frac{\mu^{s+1}}{\mu}\Big(G_1\Big(\frac{x}{\mu^s}\Big) - f\Big(\frac{x}{\mu^s}\Big)\Big) + \frac{\mu^{s+1}}{\mu}\Big(f\Big(\frac{x}{\mu^s}\Big) - F_1\Big(\frac{x}{\mu^s}\Big)\Big)\Big)\\
&\le \frac{\kappa^{s+1}}{\mu}\rho\Big(G_1\Big(\frac{x}{\mu^s}\Big) - f\Big(\frac{x}{\mu^s}\Big)\Big) + \frac{\kappa^{s+1}}{\mu}\rho\Big(f\Big(\frac{x}{\mu^s}\Big) - F_1\Big(\frac{x}{\mu^s}\Big)\Big)\\
&\le \frac{k^{s+1}}{\mu}\cdot\frac{2\kappa}{\binom{n}{m}\mu^2}\sum_{j=1}^{\infty}\frac{\kappa^{2j}}{\mu^j}\varphi_1\Big(\frac{x}{\mu^{j+s}},\cdots,\frac{x}{\mu^{j+s}},0\Big)\cdot\frac{\kappa^s}{\mu^s}\\
&\le \frac{2\kappa^2}{\binom{n}{m}\mu^3}\sum_{j=s+1}^{\infty}\frac{\kappa^{2j}}{\mu^j}\varphi_1\Big(\frac{x}{\mu^j},\cdots,\frac{x}{\mu^j},0\Big)
\end{aligned}
$$

which tends to zero as $s\to\infty$ for all $x\in\chi_\rho$. Hence the mapping F_1 is a unique Lie $*$-derivation satisfying the estimation (2.3) near f. $\square$

Remark 2.1. In Theorem 2.1 if χ_ρ is a Banach $*$-algebra with norm ρ, and so $\rho(\mu x) = \mu\rho(x)$, $\kappa := \mu$, then we see from (2.1) and (2.2) that there exists a unique Lie $*$-derivation $F_1 : \chi_\rho \to \chi_\rho$, defined as $F_1(x) = \lim_{s\to\infty}\mu^s f(\frac{x}{\mu^s})$, $x\in\chi_\rho$, which satisfies the equation (1.1) and

$$
\rho(f(x) - F_1(x)) \le \frac{1}{\mu\binom{n}{m}}\sum_{j=1}^{\infty}\mu^j\varphi_1\Big(\frac{x}{\mu^j},\cdots,\frac{x}{\mu^j},0\Big)
$$

for all $x\in\chi_\rho$, which is exactly the approximation in Theorem [11].

As a corollary of Theorem 2.1, we obtain the following stability result of the equation (1.1), which generalizes stability result on Banach $*$-algebras.

Corollary 2.1. *Suppose χ_ρ is a Banach $*$-algebra with norm $\|\cdot\|$ and $\kappa = 2$. For given real numbers θ_i, $\vartheta \ge 0$, $r_i > 1 (i = 1,\cdots,n+1)$, $a + b > 2$, if a mapping $f : \chi_\rho \to \chi_\rho$ satisfies*

$$
\begin{aligned}
&\|D_\lambda f(x_1,\cdots,x_n) + f(z^*) - f(z)^*\| \le \sum_{i=1}^{n}\theta_i\|x_i\|^{r_i} + \vartheta\|z\|^{r_{n+1}},\\
&\|LDf(x,y)\| \le \vartheta\|x\|^a\|y\|^b
\end{aligned}
$$

for all $x_1,\cdots,x_n,x,y,z\in\chi_\rho$ and $\lambda\in\Lambda$, then there exists a unique Lie $*$-derivation $F_1 : \chi_\rho \to \chi_\rho$ such that

$$
\|f(x) - F_1(x)\| \le \frac{1}{\binom{n}{m}}\sum_{i=1}^{n}\frac{\theta_i}{\mu^{r_i} - \mu}\|x\|^{r_i}
$$

for all $x \in \chi_\rho$.

In the following, we present a generalized Hyers–Ulam stability via direct method of the equation (1.1) in modular $*$-algebra without using both Fatou property and Δ_μ-condition, where $\mu := n - m + 1 > 2$.

Theorem 2.2. *Suppose that a mapping* $f : \chi_\rho \to \chi_\rho$ *satisfies*

$$\begin{aligned} &\rho(D_\lambda f(x_1, \cdots, x_n) + f(z^*) - f(z)^*) \le \phi_1(x_1, \cdots, x_n, z), \\ &\rho(LDf(x, y)) \le \phi_2(x, y) \end{aligned} \tag{2.7}$$

and $\phi_1 : \chi_\rho^{n+1} \to [0, \infty)$,$\phi_2 : \chi_\rho^2 \to [0, \infty)$ *are mappings such that*

$$\begin{aligned} &\Phi(x_1, \cdots, x_n, z) := \sum_{j=0}^{\infty} \frac{\phi_1(\mu^j x_1, \cdots, \mu^j x_n, \mu^j z)}{\mu^j} < \infty, \\ &\lim_{s\to\infty} \frac{\phi_2(\mu^s x, \mu^s y)}{\mu^{2s}} = 0 \end{aligned} \tag{2.8}$$

for all $x_1, \cdots, x_n, x, y, z \in \chi_\rho, \lambda \in \Lambda$. *Then there exists a unique Lie* $*$*-derivation* $F_2 : \chi_\rho \to \chi_\rho$ *which satisfies the equation (1.1) and*

$$\rho(f(x) - F_2(x)) \le \frac{1}{\binom{n}{m}\mu} \Phi(x, \cdots, x, 0) \tag{2.9}$$

for all $x \in \chi_\rho$.

Proof. It follows from the similar way as in (2.5) that

$$\rho\Big(f(x) - \mu f\Big(\frac{x}{\mu}\Big)\Big) \le \frac{1}{\binom{n}{m}} \varphi_1\Big(\frac{x}{\mu}, \cdots, \frac{x}{\mu}, 0\Big)$$

for all $x \in \chi_\rho$. Since $\sum_{j=0}^{s-1} \frac{1}{\mu^{j+1}} \le 1$, we prove the following functional inequality

$$\begin{aligned} \rho\Big(f(x) - \frac{f(\mu^s x)}{\mu^s}\Big) &= \rho\Big[\sum_{j=0}^{s-1} \big(\frac{f(\mu^j x)}{\mu^j} - \frac{f(\mu^{j+1} x)}{\mu^{j+1}}\big)\Big] \\ &= \rho\Big[\sum_{j=0}^{s-1} \frac{1}{\mu^{j+1}} \big(\mu f(\mu^j x) - f(\mu^{j+1} x)\big)\Big] \\ &\le \sum_{j=0}^{s-1} \frac{1}{\mu^{j+1}} \rho\Big(\mu f(\mu^j x) - f(\mu^{j+1} x)\Big) \\ &\le \frac{1}{\binom{n}{m}\mu} \sum_{j=0}^{s-1} \frac{\phi_1(\mu^j x, \cdots, \mu^j x, 0)}{\mu^j} \end{aligned} \tag{2.10}$$

for all $x \in \chi_\rho$ by Remark (a).

Now, replacing x by $\mu^t x$ in (2.10), we have

$$\rho\Big(\frac{f(\mu^t x)}{\mu^t} - \frac{f(\mu^{s+t} x)}{\mu^{s+t}}\Big) \le \frac{1}{\binom{n}{m}\mu} \sum_{j=t}^{s+t-1} \frac{\phi_1(\mu^j x, \cdots, \mu^j x, 0)}{\mu^j} \tag{2.11}$$

which converges to zero as $t \to \infty$ by the assumption (2.8). Thus the above inequality implies that the sequence $\{\frac{f(\mu^s x)}{\mu^s}\}$ is ρ-Cauchy for all $x \in \chi_\rho$ and so it is ρ-convergent in χ_ρ since the space χ_ρ is ρ-complete. Thus, we may define a mapping $F_2 : \chi_\rho \to \chi_\rho$ as

$$F_2(x) := \rho - \lim_{s\to\infty} \frac{f(\mu^s x)}{\mu^s} \Longleftrightarrow \lim_{s\to\infty} \rho\Big(\frac{f(\mu^s x)}{\mu^s} - F_2(x)\Big) = 0,$$

for all $x \in \chi_\rho$.

Claim 1: F_2 is an additive mapping satisfying the approximation (2.9). In fact, if we put $(x_1, \cdots, x_n, z) := (\mu^s x_1, \cdots, \mu^s x_n, 0)$ in (2.7), and then divide the resulting inequality by μ^s, one obtains

$$\rho\Big(\frac{D_\lambda f(\mu^s x_1, \cdots, \mu^s x_n)}{\mu^s}\Big) \le \frac{\rho(D_\lambda f(\mu^s x_1, \cdots, \mu^s x_n))}{\mu^s} \le \frac{\phi_1(\mu^s x_1, \cdots, \mu^s x_n, 0)}{\mu^s} \to 0$$

which tends to zero as $s \to \infty$, for all $x_1, \cdots, x_n \in \chi_\rho$. Thus, for a fixed positive real $R := \binom{n}{m}(n-m+2)+2$, we figure out by use of Remark 1.1 (b).

$$\begin{aligned}
&\rho\Big(\frac{1}{R} D_\lambda F_2(x_1, \cdots, x_n)\Big) \\
&= \rho\Big(\frac{1}{R} D_\lambda F_2(x_1, \cdots, x_n) - \frac{D_\lambda f(\mu^s x_1, \cdots, \mu^s x_n)}{R\cdot\mu^s} + \frac{D_\lambda f(\mu^s x_1, \cdots, \mu^s x_n)}{R\cdot\mu^s}\Big) \\
&\le \frac{1}{R} \sum_{\substack{1\le i_1<\cdots<i_m\le n \\ 1\le k_l(\ne i_j, \forall j\in\{1,\cdots,m\})\le n}} \rho\Big(\frac{1}{\mu^s} f\Big(\frac{\sum_{j=1}^m \mu^s \lambda x_{i_j}}{m} + \sum_{l=1}^{n-m} \lambda \mu^s x_{k_l}\Big) \\
&\qquad\qquad - F_2\left(\frac{\sum_{j=1}^m \lambda x_{i_j}}{m} + \sum_{l=1}^{n-m} \lambda x_{k_l}\right)\Big) \\
&\quad + \frac{n-m+1}{Rn}\binom{n}{m} \sum_{i=1}^n \rho\Big(\frac{f(\mu^s x_i)}{\mu^s} - F_2(x_i)\Big) + \frac{1}{R}\frac{\phi_1(\mu^s x_1, \cdots, \mu^s x_n, 0)}{\mu^s}
\end{aligned}$$

for all $x_1, \cdots, x_n \in \chi_\rho$, $\lambda \in \Lambda$ and all positive integers s. Taking the limit as $s \to \infty$, one obtains $\rho(\frac{1}{R} D_\lambda F_2(x_1, \cdots, x_n)) = 0$, and so $D_\lambda F_2(x_1, \cdots, x_n) = 0$ for all $x_1, \cdots, x_n \in \chi_\rho$. Hence, taking $\lambda = 1$ in $D_\lambda F_2(x_1, \cdots, x_n) = 0$, we conclude that F_2 satisfies the equation (1.1) and so it is additive by Lemma 2.1.

On the other hand, since $\sum_{i=0}^s \frac{1}{\mu^{i+1}} + \frac{1}{\mu} \le 1$, $(\mu > 2)$ for all $s \in \mathbb{N}$, it follows from (2.5) and Remark 1.1 (a) that

$$\begin{aligned}
&\rho(f(x) - F_2(x)) \\
&= \rho\left(\sum_{i=0}^s \frac{1}{\mu^{i+1}}\Big(\mu f(\mu^i x) - f(\mu^{i+1} x)\Big) + \frac{f(\mu^{s+1} x)}{\mu^{s+1}} - \frac{F_2(\mu x)}{\mu}\right) \\
&\le \frac{1}{\binom{n}{m}\mu} \sum_{i=0}^s \frac{1}{\mu^i} \phi_1(\mu^i x, \cdots, \mu^i x) + \frac{1}{\mu}\rho\Big(\frac{f(\mu^s \cdot \mu x)}{\mu^n} - F_2(\mu x)\Big),
\end{aligned}$$

without applying the Fatou property of the modular ρ for all $x \in \chi_\rho$ and all $s \in \mathbb{N}$, from which we obtain the approximation of f by the additive mapping F_2 as follows

$$\rho(f(x) - F_2(x)) \le \frac{1}{\binom{n}{m}\mu} \sum_{i=0}^\infty \frac{1}{\mu^i} \phi_1(\mu^i x, \cdots, \mu^i x, 0) = \frac{1}{\binom{n}{m}\mu} \Phi(x, \cdots, x, 0)$$

for all $x \in \chi_\rho$ by taking $s \to \infty$ in the last inequality.

Claim 2: F_2 is a Lie $*$-derivation. It follows from the same proof of Theorem 2.1 that the mapping F_2 is a linear mapping. From the second inequality in (2.8) and the second condition in (2.7), we arrive at

$$\begin{aligned}
&\rho\Big(\frac{1}{4}LDF_2(x,y)\Big)\\
&= \rho\Big(\frac{1}{4}LDF_2(x,y) - \frac{LDf(\mu^s x,\mu^s y)}{4\cdot\mu^{2s}} + \frac{LDf(\mu^s x,\mu^s y)}{4\cdot\mu^{2s}}\Big)\\
&\le \frac{1}{4}\rho\Big(F_2\big([x,y]\big) - \frac{f\big(\mu^{2s}[x,y]\big)}{\mu^{2n}}\Big) + \frac{1}{4}\rho\Big(\frac{[x,f(\mu^s y)]}{\mu^s} - [x,F_2(y)]\Big)\\
&\quad + \frac{1}{4}\rho\Big(\frac{[f(\mu^s x),y]}{\mu^s} - [F_2(x),y]\Big) + \frac{1}{4}\rho\Big(\frac{LDf\big(\mu^s x,\mu^s y\big)}{\mu^{2s}}\Big)
\end{aligned}$$

for all $x, y \in \chi_\rho$, which tends to zero as s tends to ∞. Therefore, one obtains $\rho(\frac{1}{4}LDF_2(x,y)) = 0$, and so F_2 is a Lie derivation. On the other hand, we observe from (2.8) that

$$\begin{aligned}
\rho\Big(\frac{1}{4}\Big(F_2(z^*) - F_2(z)^*\Big)\Big) &\le \frac{1}{4}\rho\Big(F_2(z^*) - \frac{f(\mu^s z^*)}{\mu^s}\Big) + \frac{1}{4}\rho\Big(F_2(z)^* - \frac{f(\mu^s z)^*}{\mu^s}\Big)\\
&\quad + \frac{1}{4}\rho\Big(\frac{f(\mu^s z^*) - f(\mu^s z)^*}{\mu^s}\Big) + \frac{1}{4}\rho\Big(\frac{1}{\mu^s}\binom{n}{m}(n-m)f(0)\Big)\\
&\le \frac{1}{4}\rho\Big(F_2(z^*) - \frac{f(\mu^s z^*)}{\mu^s}\Big) + \frac{1}{4}\rho\Big(F_2(z)^* - \frac{f(\mu^s z)^*}{\mu^s}\Big)\\
&\quad + \frac{1}{4}\frac{\phi_1(0,\cdots,0,\mu^s z)}{\mu^s} + \frac{1}{4\mu^s}\rho\Big(\binom{n}{m}(n-m)f(0)\Big)
\end{aligned}$$

which tends to zero as $s \to \infty$ for all vector z. Thus F_2 is a Lie $*$-derivation.

Claim 3: F_2 is unique. To show the uniqueness of F_2, let's assume there exists a Lie $*$-derivation $G_2 : \chi_\rho \to \chi_\rho$ which satisfies the inequality (2.9) for all $x \in \chi_\rho$, but suppose $F_2(x_0) \ne G_2(x_0)$ for some $x_0 \in \chi_\rho$. Then there exists a positive constant $\varepsilon > 0$ such that $\varepsilon < \rho(F_2(x_0) - G_2(x_0))$. For such given $\varepsilon > 0$, it follows from the convergence of series (2.8) that there is a positive integer $n_0 \in \mathbb{N}$ such that $\frac{2}{\binom{n}{m}\mu}\sum_{j=n_0}^{\infty}\frac{\phi_1(\mu^j x_0,\cdots,\mu^j x_0,0)}{\mu^j} < \varepsilon$. Since F_2 and G_2 are additive mappings, we see from the equality $F_2(\mu^{n_0}x_0) = \mu^{n_0}F_2(x_0)$ and $G_2(\mu^{n_0}x_0) = \mu^{n_0}G_2(x_0)$ that

$$\begin{aligned}
\varepsilon &< \rho(F_2(x_0) - G_2(x_0))\\
&= \rho\Big(\frac{F_2(\mu^{n_0}x_0) - f(\mu^{n_0}x_0)}{\mu^{n_0}} + \frac{f(\mu^{n_0}x_0) - G_2(\mu^{n_0}x_0)}{\mu^{n_0}}\Big)\\
&\le \frac{1}{\mu^{n_0}}\rho\Big(F_2(\mu^{n_0}x_0) - f(\mu^{n_0}x_0)\Big) + \frac{1}{\mu^{n_0}}\rho\Big(f(\mu^{n_0}x_0) - G_2(\mu^{n_0}x_0)\Big)\\
&\le \frac{1}{\mu^{n_0}}\frac{2}{\binom{n}{m}\mu}\sum_{j=0}^{\infty}\frac{\phi_1(\mu^{j+n_0}x_0,\cdots,\mu^{j+n_0}x_0,0)}{\mu^j}\\
&= \frac{2}{\binom{n}{m}\mu}\sum_{j=n_0}^{\infty}\frac{\phi_1(\mu^j x_0,\cdots,\mu^j x_0,0)}{\mu^j} < \varepsilon,
\end{aligned}$$

which leads a contradiction. Hence the mapping F_2 is a unique Lie $*$-derivation near

f satisfying the approximation (2.9) on the ρ-complete convex modular $*$-algebra χ_ρ. □

As a corollary of Theorem 2.2, we obtain the following stability result of (m, n)-Cauchy-Jensen functional equation (1.1) associated with Lie $*$-derivation on the Banach $*$-algebra χ_ρ, which may be considered as $\rho = \|\cdot\|$.

Corollary 2.2. *Suppose χ_ρ is a Banach $*$-algebra with norm $\|\cdot\|$. For given positive real numbers θ_i, $\vartheta \geq 0$, $r_i < 1 (i = 1, \cdots, n+1)$, and $a + b < 2$, suppose that a mapping $f : \chi_\rho \to \chi_\rho$ satisfies*

$$\|D_\lambda f(\lambda x_1, \cdots .\lambda x_n) + f(z^*) - f(z)^*\| \leq \sum_{i=1}^{n} \theta_i \|x_i\|^{r_i} + \theta_{n+1}\|z\|^{r_{n+1}},$$
$$\|LDf(x, y)\| \leq \vartheta \|x\|^a \|y\|^b$$

for all $x_1, \cdots, x_n, x, y, z \in \chi_\rho$ and $\lambda \in \Lambda$. Then there exists a unique Lie $*$-derivation $F_2 : \chi_\rho \to \chi_\rho$ such that

$$\|f(x) - F_2(x)\| \leq \frac{1}{\binom{n}{m}} \sum_{i=1}^{n} \frac{\theta_i}{\mu - \mu^{r_i}} \|x\|^{r_i}$$

for all $x \in \chi_\rho$.

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