Journal of Applied Analysis and Computation Website:http://jaac.ijournal.cn
Volume 9, Number 2, April 2019, 747–764 DOI:10.11948/2156-907X.20180165

DYNAMICS OF A MODIFIED LESLIE-GOWER MODEL WITH GESTATION EFFECT AND NONLINEAR HARVESTING

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Abstract This study focuses on the dynamics of a modified Leslie-Gower predator-prey model where the intake rate of prey is by per capita predator according to Crowley-Martin functional response and prey is harvested through nonlinear harvesting strategy. Further the time-delay (τ) is imposed to utilize gestation period of predations. We investigate the permanence analysis of proposed system. The local stability of non-delayed model at all possible equilibrium points is studied. It is shown that the given model undergoes Hopf bifurcation around positive equilibrium point with respect to delay parameter τ. Subsequently the stability of Hopf bifurcation and its direction are explored through normal and center manifold theories. The derived theoretical results are justified with the help of numerical simulations.

Keywords Predator-Prey Model, prey harvesting, Crowley-Martin functional response, stability Analysis, Hopf bifurcation.

MSC(2010) 34C23, 34C20, 92D25.

1. Introduction

Since the pioneering work of Lotka and Volterra in 1920's, the dynamics of interactions between two species models has been gaining special attention among researchers in both mathematics and biology disciplines. Many remarkable results on dynamics of predator-prey model have been found in [4–6,8,10,21] and references therein. Suppose that predators can switch to alternative food when their favorite food is not sufficiency in abundance. To tackle this issue, authors in [3] have developed the modified edition of general Leslie-Gower model [19] and the model given by:

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right) - yg(x,y), \\ \frac{dy}{dt} &= y\left(c-\frac{dy}{x+l}\right), \end{aligned} \tag{1.1}$$

where $x(t)$ and $y(t)$ are the densities of prey and predator populations respectively at time t, r and k stand for the per capita growth rate of prey and carrying capacity respectively. The predator can consume prey according to the functional

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response $g(\cdot,\cdot)$. The parameters c and d indicate the growth rate and maximum reduction rate of predator respectively and l represents the measure of extent to which environment provides protection to the predator.

The most commonly used functional response in the existing literature is a function of prey's density only (Holling I-III) in which the interference among predators is not utilized whereas this will be common when predators compete for food. To cover this key factor, functional responses (ratio-dependent [2], Beddington-DeAngelis [4,10] and Crowley-Martin [9]) have been introduced which do not depend upon the prey's density alone rather than on both prey's and predator's densities. The demerits of Beddington-DeAngelis function is handling prey and finding prey are taken as mutually exclusive events. The Crowley-Martin function assumes that the interference among predators raises even if individual predator is looking for prey or handling prey and it is given by $g(x,y)=\frac{c_1 x}{1+ax+by+abxy}$, where c_1, a and b are positive parameters that are used for effects of capture rate, handling time and magnitude of interference among predators respectively on the feeding rate. Therefore it is interesting to introduce Crowley-Martin function in modified Leslie-Gower model and corresponding results are found in [1, 26, 31, 34]. This model takes the form

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{k}-\frac{c_1 y}{1+ax+by+abxy}\right), \\ \frac{dy}{dt} &= y\left(c-\frac{dy}{x+l}\right). \end{aligned} \tag{1.2}$$

Ali and Jazar in [1] considered the system (1.2) in which they derived the sufficient conditions for global stability, Hopf bifurcation and the existence and nonexistence of periodic solutions. In [26], the authors have considered the discrete-time version of the model (1.2) and they showed that the system exhibits the flip bifurcation and Neimark-Sacker bifurcation.

By reason of financial income, harvesting of species commonly exists in fishery, forestry and wildlife management. Enthused by this fact, it is meaningful to establish the harvesting of species in prey-predator models and different types of harvesting strategies have been used in the literatures. To mention few, constant-yield harvesting [15, 16, 18], constant-effort harvesting [7], age-selective harvesting [17], nonlinear harvesting [11,12,14,30,32] and so on. Among them, nonlinear harvesting is more applicable from the financial as well as biological point of view rather than the other strategies [14]. The formulation of nonlinear harvesting is

$$H(z)=\frac{qEz}{m_1E+m_2z},$$

where z is the density of prey (or predator), q stands for the catchability coefficient, E represents the external effort applied for harvesting and $m_1,\ m_2$ are positive constants. The problem of modified Leslie-Gower predator-prey model with Holling II function and nonlinear prey harvesting is considered in [11], while predator harvesting on Holling-Tanner type model is dealt with in [14]. In these works, authors show that the system exhibits complex dynamical behaviors by varying harvesting parameters.

Time delays in biological systems are inevitable due to maturation period, gestation period, handling and digesting time, etc. The existence of delays in dynamical systems may enhance the complexity of its dynamics. Some important results on dynamics of delayed predator-prey model are given in [20, 22, 24, 25, 28–30, 33] and references therein. Specifically, gestation delay means that the consumption of prey by predator will be a source of predator growth which is not instantaneous rather lagged by some time delay τ. In [32], the modified Leslie-Gower model with gestation delay and nonlinear harvesting is investigated and it is showed that the system experiences saddle-node-Hopf bifurcation. Recently the dynamics of Holling-Tanner model with Beddington-DeAngelis functional response and linear prey harvesting rate is described in [27]. To our knowledge, there is no literature available on dynamics of system (1.2) with harvesting and gestation effect on predator equation. This facts has inspired our present work. Thus we focus on the modified Leslie-Gower predator-prey model with Crowley-Martin functional response and nonlinear prey harvesting in the form:

$$\begin{aligned} \frac{dX}{dT} &= RX\left(1-\frac{X}{K}\right) - \frac{mXY}{1+AX+BY+ABXY} - \frac{qEX}{m_1E+m_2X}, \\ \frac{dY}{dT} &= Y\left(C - \frac{DY(t-\tau)}{X(t-\tau)+L}\right), \end{aligned} \tag{1.3}$$

subject to the initial conditions $N_0(\theta) = \phi(\theta) > 0,\ \ P_0(\theta) = \psi(\theta) > 0,\ \forall\theta \in [-\tau, 0]$, where $\phi(\theta),\ \psi(\theta)$ are continuous and bounded functions in $[-\tau, 0]$.

2. Mathematical Model

To reduce the complexity of the considered system, we make the following non-dimensional scheme $X \to Kx,\ Y \to Ry,\ T \to \frac{1}{R}t$ and let $\alpha = AK,\ \beta = \frac{BR}{m}$, $g = \frac{qE}{m_2KR},\ h = \frac{m_1E}{m_2K},\ \gamma = \frac{C}{R},\ \delta = \frac{DR}{CmK},\ \rho = \frac{L}{K}$. Then the system (1.3) becomes

$$\begin{aligned} \frac{dx}{dt} &= x\left(1 - x - \frac{y}{(1+\alpha x)(1+\beta y)} - \frac{g}{h+x}\right), \\ \frac{dy}{dt} &= \gamma y\left(1 - \frac{\delta y(t-\tau)}{x(t-\tau)+\rho}\right), \end{aligned} \tag{2.1}$$

subjected to the initial conditions

$$x_0(\theta) = \phi(\theta) > 0,\ \ y_0(\theta) = \psi(\theta) > 0,\ \forall\theta \in [-\tau, 0], \tag{2.2}$$

where $\phi(\theta),\ \psi(\theta)$ are continuous and bounded functions in $[-\tau, 0]$.

2.1. Boundedness and permanence analysis

In this subsection, we present the permanence analysis of solutions of system (2.1). Generally permanence analysis confirms that the prey and predator will always coexist at any time and any location of the inhabited domain. These results will be used in the following sections.

Definition 2.1. System (2.1) is said to be permanent if there exist positive constants $\underline{c}$ and $\bar{c}$ such that $(x(t), y(t))$ of (2.1) with $x_0 \geq 0$ and $y_0 \geq 0$ satisfying

$$\begin{aligned} 0 < \underline{c} \leq \lim_{t\to\infty} \inf x(t) \leq \lim_{t\to\infty} \sup x(t) \leq \bar{c}, \\ 0 < \underline{c} \leq \lim_{t\to\infty} \inf y(t) \leq \lim_{t\to\infty} \sup y(t) \leq \bar{c}. \end{aligned}$$

Lemma 2.1. *If* $a > 0,\ b > 0$ *and* $\frac{dx(t)}{dt} \leq (\geq) x(t)(b - ax(t)),\ x(t_0) > 0,$ *then* $\limsup_{t\to\infty} x(t) \leq \frac{b}{a}$ $(\limsup_{t\to\infty} x(t) \geq \frac{b}{a})$.

Theorem 2.1. *Let* $(x(t), y(t))$ *be the solution of (2.1) with* $x_0 \geq 0,\ y_0 \geq 0.$ *The following hold:*

1. $\lim_{t\to\infty} \sup x(t) \leq 1$ *and*
2. $\lim_{t\to\infty} \sup y(t) \leq \dfrac{(1+\rho)e^{\gamma\tau}}{\delta}.$

Proof. From the prey equation of (2.1), we have that for all $t \in [0, \infty)$

$$\frac{dx(t)}{dt} \leq x(t)(1 - x(t)).$$

Then, by Lemma 2.1, we obtain

$$\lim_{t\to\infty} \sup x(t) \leq 1.$$

Similarly, from the predator equation of (2.1), we have

$$\frac{dy(t)}{dt} < \gamma y(t).$$

Thus for $t > \tau$, integrating above relation from $t - \tau$ to t, we get

$$y(t) \leq y(t-\tau)e^{\gamma\tau},$$

which is equivalent to

$$y(t-\tau) \geq y(t)e^{-\gamma\tau}.$$

Note that there exists a positive integer T for $t > T$ and $x(t) < \bar{x}$. Thus, for $t > T + \tau$,

$$\frac{dy(t)}{dt} \leq \gamma y(t)\left(1 - \frac{\delta e^{-\gamma\tau}}{1+\rho} y(t)\right).$$

Now, by using Lemma 2.1, we have

$$\lim_{t\to\infty} \sup y(t) \leq \frac{(1+\rho)e^{\gamma\tau}}{\delta} \equiv \bar{y}.$$

Therefore, the proof is complete. □

Theorem 2.2. *Let* $(x(t), y(t))$ *be the solution of (2.1) with* $x_0 \geq 0,\ y_0 \geq 0.$ *The following hold:*

1. $\lim_{t\to\infty} \inf x(t) \geq \dfrac{(h-g)(1+\beta\bar{y}) - h\bar{y}}{h(1+\beta\bar{y})}$ *and*
2. $\lim_{t\to\infty} \inf y(t) \geq \dfrac{(\rho + \underline{x})}{\delta} exp\left(-\dfrac{\gamma\delta\bar{y}}{\rho+\underline{x}}\tau\right).$

Proof. It is easy to see that

$$\frac{dx}{dt} \geq x\left(1 - x - \frac{\bar{y}}{1+\beta\bar{y}} - \frac{g}{h}\right).$$

Hence, by Lemma 2.1, we obtain

$$\lim_{t\to\infty}\inf x(t) \geq \frac{(h-g)(1+\beta\bar{y})-h\bar{y}}{h(1+\beta\bar{y})} \equiv \underline{x}.$$

For any $\kappa > 1$, there exists a $T_\kappa > 0$ such that $t > T_\kappa$, $x(t) > \underline{x}/\kappa$ and $y(t) < \kappa\bar{y}$. Then, for $t > T_\kappa + \tau$, we have

$$\frac{dy(t)}{dt} \geq \gamma y(t)\left(1 - \frac{\delta\kappa}{\kappa\rho+\underline{x}}y(t-\tau)\right).$$

For $t > T_\kappa + \tau$, the above inequality gives

$$\frac{dy(t)}{dt} \geq -\frac{\kappa^2\gamma\delta\bar{y}}{\kappa\rho+\underline{x}}y(t),$$

which leads to

$$y(t-\tau) < y(t)exp\left(\frac{\kappa^2\gamma\delta\bar{y}}{\kappa\rho+\underline{x}}\tau\right).$$

Then, for $t > T_\kappa + \tau$,

$$\frac{dy(t)}{dt} \geq \gamma y(t)\left(1 - \frac{\delta\kappa}{\kappa\rho+\underline{x}}exp\left(\frac{\kappa^2\gamma\delta\bar{y}}{\kappa\rho+\underline{x}}\tau\right)y(t)\right)$$

which implies

$$\lim_{t\to\infty}\inf y(t) \geq \frac{(\kappa\rho+\underline{x})}{\delta\kappa}exp\left(-\frac{\kappa^2\gamma\delta\bar{y}}{\kappa\rho+\underline{x}}\tau\right).$$

Putting $\kappa \to 1$, we get

$$\lim_{t\to\infty}\inf y(t) \geq \frac{(\rho+\underline{x})}{\delta}exp\left(-\frac{\gamma\delta\bar{y}}{\rho+\underline{x}}\tau\right) \equiv \underline{y}.$$

□

Let $\underline{c} = \min\{\underline{x}, \underline{y}\}$ and $\bar{c} = \max\{1, \bar{y}\}$. Then, by Definition 1 and Theorems 2.1 and 2.2, we arrive at the following result.

Theorem 2.3. *If*

$$h(1+\beta\bar{y}) > h\bar{y} + g(1+\beta\bar{y}) \tag{2.3}$$

holds, then the system (2.1) is permanent.

2.2. Existence of Equilibria

The equilibria of system (2.1) are given by

$$x\left(1 - x - \frac{y}{(1+\alpha x)(1+\beta y)} - \frac{g}{h+x}\right) = 0,$$
$$y\left(1 - \frac{\delta y}{x+\rho}\right) = 0.$$

Solving the above equations, we get the following equilibrium points:

i. the trivial equilibrium point $E_0 = (0, 0)$.

ii. The predator free axial equilibrium points $E_1^- = (x^-, 0)$ and $E_1^+ = (x^+, 0)$ where x^- and x^+ are the roots of equation $x^2 - (1-h)x + g - h = 0$, that is,

$$x^- = \frac{1-h}{2} - \frac{1}{2}\sqrt{(1-h)^2 - 4(g-h)} \quad \text{and} \quad x^+ = \frac{1-h}{2} + \frac{1}{2}\sqrt{(1-h)^2 - 4(g-h)}.$$

iii. The prey extinction equilibrium point $E_2 = (0, \frac{\rho}{\delta})$.

iv The interior equilibrium point $E_* = (x^*, y^*)$, where $y^* = \frac{x^*+\rho}{\delta}$ and x^* is a root of the following quintic equation in z,

$$az^4 + bz^3 + cz^2 + dz + e = 0, \tag{2.4}$$

where

$$\begin{aligned}
a &= -\alpha\beta,\\
b &= (1-h)\alpha\beta - (\beta + \alpha\delta + \alpha\beta\rho),\\
c &= -(\delta + \beta\rho) + (1-h)(\beta + \alpha\delta + \alpha\beta\rho) + (h-g)\alpha\beta - 1,\\
d &= (h-g)(\delta + \beta\rho) - (h-g)(\beta + \alpha\delta + \alpha\beta\rho) - (\rho + h),\\
e &= (h-g)(\delta + \beta\rho) - \rho h.
\end{aligned}$$

Remark 2.1. The equilibriums E_0 and E_2 always exist. If $g > h$, both equilibriums $E_1^- = (x^-, 0)$ and $E_1^+ = (x^+, 0)$ exist when $1 > h$ and $(1-h)^2 > 4(g-h)$ while if $g < h$ then $E_1^+ = (x^+, 0)$ only exists. It is easy to observe from (2.4) that the leading coefficient a is always negative and e is positive if

$$(h-g)(\delta + \beta\rho) > \rho h \tag{2.5}$$

holds. Hence, if (2.5) is satisfied, the Descartes rule of sign assures that the equation (2.4) possesses at least one positive root. Further equation (2.4) has a unique positive root, say x_*, if (2.5) holds along with any one the following conditions:

H1 $b < 0,\ c < 0$ and $d < 0$,

H2 $b < 0,\ c < 0$ and $d > 0$,

H3 $b < 0,\ c > 0$ and $d > 0$,

H4 $b > 0,\ c > 0$ and $d > 0$.

Hereafter we always assume that the system (2.1) satisfies any one of the above conditions.

3. Local stability analysis and Hopf bifurcation

In this section, we deal with the local stability of the system (2.1) about the possible equilibrium points. We consider two cases.

Case 1: The non-delayed model

Theorem 3.1. *For system (2.1) without delay (i.e. $\tau = 0$),*

i. $E_0 = (0,0)$ *is saddle if* $h < g$ *and unstable if* $h > g$.

ii. $E_1^- = (x^-, 0)$ *is always unstable.*

iii. $E_1^+ = (x^+, 0)$ *is always saddle.*

iv. $E_2 = (0, \frac{\rho}{\delta})$ *is stable if* $1 < \frac{\rho}{\delta+\beta\rho} + \frac{g}{h}$ *and saddle if* $1 > \frac{\rho}{\delta+\beta\rho} + \frac{g}{h}$.

Proof. i. The Jacobian matrix of (2.1) calculated at $E_0 = (0,0)$ is given by

$$J|_{E_0} = \begin{pmatrix} 1 - \frac{g}{h} & 0 \\ 0 & \gamma \end{pmatrix}.$$

The eigenvalues of $J|_{E_0}$ are $\lambda_1 = \frac{h-g}{h}$ and $\lambda_2 = \gamma > 0$. Hence the result.

ii. Evaluation J at E_1^- is given by

$$J|_{E_1^-} = \begin{pmatrix} x\left(-1 + \frac{g}{(h+x)^2}\right) & -\frac{x}{1+\alpha x} \\ 0 & \gamma \end{pmatrix}.$$

The corresponding eigenvalues are $\lambda_1 = x^- \sqrt{(1-h)^2 - 4(g-h)} > 0$ and $\lambda_2 = \gamma > 0$.

iii. Similarly the eigenvalues of J at E_1^+ are $\lambda_1 = -x^+\sqrt{(1-h)^2 - 4(g-h)} < 0$ and $\lambda_2 = \gamma > 0$.

iv. The eigenvalues of J at E_2 are $\lambda_1 = 1 - \frac{\rho}{\delta+\beta\rho} - \frac{g}{h}$ and $\lambda_2 = -\gamma < 0$. Hence the proof is complete. □

Now we look at the local asymptotic stability and Hopf bifurcation of the positive equilibrium E_*.

Theorem 3.2. *Assume that (2.5) holds and*

$$\begin{aligned} &\gamma^* < \gamma, \\ &\gamma^* < \frac{n^*}{\delta(1+\alpha n^*)(1+\beta p^*)^2}. \end{aligned} \tag{3.1}$$

Then the positive equilibrium E_* is locally asymptotically stable.

Proof. The Jacobian matrix of (2.1) at E_* is

$$J|_{E_*} = \begin{pmatrix} j_{11}^* & j_{12}^* \\ j_{21}^* & j_{22}^* \end{pmatrix},$$

where

$$\begin{aligned} j_{11}^* &= x_*\left(\frac{\alpha y_*}{(1+\alpha x_*)^2(1+\beta y_*)} + \frac{g}{(h+x_*)^2} - 1\right) = \gamma^*, \\ j_{12}^* &= -\frac{x_*}{(1+\alpha x_*)(1+\beta y_*)^2} < 0, \quad j_{21}^* = \frac{\gamma}{\delta} > 0, \quad j_{22}^* = -\gamma < 0. \end{aligned}$$

The characteristic equation for $J|_{E_*}$ is

$$\lambda^2 - tr(J|_{E_*})\lambda + det(J|_{E_*}) = 0, \tag{3.2}$$

where $tr(J|_{E_*}) = \gamma^* - \gamma$ and $det(J|_{E_*}) = -\gamma^*\gamma + \dfrac{\gamma x_*}{\delta(1+\alpha x_*)(1+\beta y_*)^2}$. According to Routh-Hurwitz criterion, characteristic equation will have negative real roots if $tr(J|_{E_*}) < 0$ and $det(J|_{E_*}) > 0$. Here, if (3.1) holds (i.e. $\gamma^* < 0$), then we get $tr(J|_{E_*}) < 0$ and $det(J|_{E_*}) > 0$. □

Theorem 3.3. *Assume that $\gamma = \gamma^*$ and*

$$\frac{x_*}{\delta(1+\alpha x_*)(1+\beta y_*)^2} > \gamma. \tag{3.3}$$

Then the system (2.1) exhibits Hopf bifurcation near E_* with respect to γ.

Proof. It is well known that if $tr(J|_{E_*}) = 0$, then the characteristic equation has purely imaginary roots whenever $det(J|_{E_*}) > 0$. Let γ be a bifurcation parameter, that is, when g crosses critical value it loses its stability and a Hopf bifurcation occurs. If $\gamma^{hb} = \gamma^* = x_*\left(\dfrac{\alpha y^*}{(1+\alpha x_*)^2(1+\beta y_*)} + \dfrac{g}{(h+x_*)^2} - 1\right)$, it is evident that (i) $tr(J|_{E_*}) = 0$ and (ii) $det(J|_{E_*}) > 0$ provided (3.3) holds. Moreover transitivity condition is given by (iii) $\frac{d}{d\gamma}(tr(J|_{E_*})) = -1 \neq 0$. These conditions assure the existence of a Hopf bifurcation around E_*. □

Case 2: The delayed model

Now we consider the system (2.1) (with delay) and we seek the local stability around $E_* = (x^*, y^*)$. For simplicity, let us take $u = x - x^*$ and $v = y - y^*$. By using Taylor series expansion of (2.1) at (x^*, y^*), we obtain

$$\begin{aligned}
\dot{u}(t) &= p_{10}u(t) + p_{01}v(t) + \sum_{i+j\geq 2} p_{ij}u^i(t)v^j(t),\\
\dot{v}(t) &= q_{100}u(t-\tau) + q_{001}v(t-\tau) + \sum_{i+j+k\geq 2} g_{ijk}u^i(t-\tau)v^j(t)v^k(t-\tau),
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
p_{10} &= x^*\left(\frac{\alpha y^*}{(1+\alpha x^*)^2(1+\beta y^*)} + \frac{g}{(h+x^*)^2} - 1\right),\\
p_{01} &= -\frac{x_*}{(1+\alpha x_*)(1+\beta y_*)^2}, \quad q_{100} = \frac{\gamma\delta y_*^2}{(x_*+\rho)^2}, \quad q_{001} = -\frac{\gamma\delta y_*}{x_*+\rho},\\
f_{ij} &= \frac{1}{i!j!}\frac{\partial^{i+j} f}{\partial x^i(t)\partial y^j(t)}\bigg|_{(x_*,y_*)},\\
g_{ijk} &= \frac{1}{i!j!k!}\frac{\partial^{i+j+k} g}{\partial x^i(t-\tau)\partial y^j(t)\partial y^k(t-\tau)}\bigg|_{(x_*,y_*)},\\
f &= x\left(1 - x - \frac{y}{(1+\alpha x)(1+\beta y)} - \frac{g}{h+x}\right),\\
g &= \gamma y\left(1 - \frac{\delta y(t-\tau)}{x(t-\tau)+\rho}\right).
\end{aligned}$$

Therefore, linearize the system (3.4) as follows

$$\begin{aligned}\dot{u}(t) &= p_{10}u(t) + p_{01}v(t),\\ \dot{v}(t) &= q_{100}u(t-\tau) + q_{001}v(t-\tau),\end{aligned} \tag{3.5}$$

whose characteristic equation can be given by

$$\begin{vmatrix} p_{10}-\lambda & p_{01} \\ q_{100}e^{-\lambda\tau} & b_{001}e^{-\lambda\tau}-\lambda \end{vmatrix} = 0,$$

$$\text{that is}\quad \lambda^2 - A\lambda - B\lambda e^{-\lambda\tau} + Ce^{-\lambda\tau} = 0, \tag{3.6}$$

where $A = p_{10}$, $B = q_{001}$ and $C = p_{10}q_{001} - p_{01}q_{100}$. Assume that there exists a purely imaginary solution $\lambda(\tau) = \pm i\eta(\tau)$ of (3.6). Substituting this into (3.6) and separating the real and imaginary parts, we get

$$\begin{aligned}B\eta\sin(\eta\tau) - C\cos(\eta\tau) &= -\eta^2,\\ C\sin(\eta\tau) + B\eta\cos(\eta\tau) &= -A\eta.\end{aligned} \tag{3.7}$$

Solving the above equations, we obtain

$$\begin{aligned}\cos(\eta\tau) &= \frac{\eta^2(C-AB)}{B^2\eta^2+C^2},\\ \sin(\eta\tau) &= -\frac{\eta(B\eta^2+AC)}{B^2\eta^2+C^2}.\end{aligned} \tag{3.8}$$

Since $\sin^2(\eta\tau) + \cos^2(\eta\tau) = 1$, we have

$$\eta^4 - (B^2 - A^2)\eta^2 - C^2 = 0. \tag{3.9}$$

Hence the equation (3.9) has unique positive root (say η_0) given by

$$\eta_0 = \sqrt{\frac{(B^2-A^2) + \sqrt{(B^2-A^2)^2+4C^2}}{2}}. \tag{3.10}$$

Since η_0 is a root of equation (3.9), from (3.7), we get the critical delay τ_n as

$$\tau_n = \frac{1}{\eta_0}\arcsin\left(-\frac{\eta_0(B\eta_0^2+AC)}{B^2\eta_0^2+C^2}\right) + \frac{2n\pi}{\eta_0},\ n = 0, 1, 2, \cdots. \tag{3.11}$$

Let $\lambda(\tau) = \zeta(\tau)+i\eta(\tau)$ be the root of the characteristic equation (3.6) with $\zeta(\tau_n) = 0$ and $\eta(\tau_n) = \eta_0$.

Theorem 3.4. *The following transversality condition is satisfied:*

$$\left[\frac{d(R(\lambda(\tau)))}{d\tau}\right]_{\tau=\tau_n} > 0.$$

Proof. Taking $\lambda(\tau) = \zeta(\tau) + i\eta(\tau)$ in (3.6) and taking the derivative with respect to τ, we have

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{(2\lambda - A)e^{\lambda\tau}}{\lambda(C-B\lambda)} + \frac{B}{\lambda(C-B\lambda)} - \frac{\tau}{\lambda}.$$

Then

$$\begin{aligned} R\left(\left[\frac{d\lambda}{d\tau}\right]^{-1}\right)_{\lambda=i\eta_0} &= \left[\frac{(2\lambda-A)e^{\lambda\tau}}{\lambda(C-B\lambda)}+\frac{B}{\lambda(C-B\lambda)}\right]_{\lambda=i\eta_0} \\ &= \left[\frac{-B^2\eta_0+(-AB+2C)\eta_0\cos(\eta_0\tau)+(-2B\eta_0^2-AC)\sin(\eta_0\tau)}{\eta_0(B^2\eta_0^2+C^2)}\right] \\ &= \left[\frac{\sqrt{(B^2-A^2)^2+4C^2}}{\eta_0(B^2\eta_0^2+C^2)}\right] > 0. \end{aligned}$$

□

From the above results, it is easy to derive the following theorem:

Theorem 3.5. *Suppose that* $\frac{\alpha y_*}{(1+\alpha x_*)^2(1+\beta y_*)}+\frac{g}{(h+x_*)^2}<1$ *holds. Then the following results hold:*

i. the interior equilibrium point E_* *of the model (2.1) is locally asymptotically stable for* $\tau\in[0,\tau_0)$.

ii. The interior equilibrium point E_* *of the model (2.1) undergoes Hopf bifurcation around* E_* *at* $\tau=\tau_n,\ (n=0,1,2,\cdots)$.

Remark 3.1. For all $\tau=\tau_n,\ (n=0,1,2,\cdots)$, the transversality condition and Hopf-bifurcation hold. When $\tau=\tau_n$, the equation (3.9) has only one positive root. Because of this, there is no interval for τ exists for which stability of equilibrium point E_* switches from stability to instability to stability.

4. Stability and direction of the Hopf bifurcation

In Section 3, we have shown that the system (2.1) undergoes Hopf bifurcation at critical delay $\tau=\tau_0$. Now we are in a position to investigate the direction of Hopf bifurcation, stability and period of the bifurcating periodic solution from E_*. The method used in this section is based on center manifold theory and normal form theory explored by Hassard et al. [13]. At $\tau=\tau_0$, $\pm i\eta_0$ are the purely imaginary roots of the equation (3.6) at E_*. Assume $\tau=\tau_0+\sigma,\ \sigma\in\mathbb{R}$, which yields Hopf bifurcation at $\sigma=0$ of (2.1). Let $z_1=x-x_*$ and $z_2=y-y_*$. Normalizing the delay with scaling $t\to t/\tau$, equation (2.1) becomes

$$\dot{z}(t)=L_\sigma(z_t)+F(z_t,\sigma), \tag{4.1}$$

where $z(t)=(z_1(t),z_2(t))^T\in\mathbb{R}^2$, $L_\sigma:\mathbb{C}\to\mathbb{R}$ and $F:\mathbb{R}\times\mathbb{C}\to\mathbb{R}$ are respectively given by

$$L_\sigma(\phi)=(\tau_0+\sigma)D_1\begin{pmatrix}\phi_1(0)\\ \phi_2(0)\end{pmatrix}+(\tau_0+\sigma)D_2\begin{pmatrix}\phi_1(-1)\\ \phi_2(-1)\end{pmatrix}, \tag{4.2}$$

where $D_1 = \begin{pmatrix} p_{10} & p_{01} \\ 0 & 0 \end{pmatrix}$, $D_2 = \begin{pmatrix} 0 & 0 \\ q_{100} & q_{001} \end{pmatrix}$ and

$$F(\phi,\sigma) = \begin{pmatrix} \sum\limits_{i+j\geq 2} p_{ij}\phi_1^i(0)\phi_2^j(0) \\ \sum\limits_{i+j+k\geq 2} q_{ijk}\phi_1^i(-1)\phi_2^j(0)\phi_2^k(-1) \end{pmatrix}. \tag{4.3}$$

The values of $p_{10},\ p_{01}, q_{100}$ and q_{001} are same as given in (3.4) and p_{ij} and q_{ijk} are given by:

$$p_{20} = -1 + \frac{gh}{(h+x^*)^3} + \frac{\alpha y^*}{(1+\alpha x^*)^3(1+\beta y^*)},\quad p_{11} = -\frac{1}{(1+\alpha x^*)^2(1+\beta y^*)^2},$$
$$p_{02} = \frac{\beta x^*}{(1+\alpha x^*)(1+\beta y^*)^3},\quad q_{200} = -\frac{\gamma\delta y_*^2}{(x_*+\rho)^3},\quad q_{110} = \frac{\gamma\delta y_*}{(x_*+\rho)^2},$$
$$q_{101} = \frac{\gamma\delta y_*}{(x_*+\rho)^2},\quad q_{011} = -\frac{\gamma\delta}{x_*+\rho},\quad q_{300} = \frac{\gamma\delta y_*^2}{(x_*+\rho)^4},\quad q_{210} = -\frac{\gamma\delta y_*}{(x_*+\rho)^3}.$$

Using Riesz representation theorem, there exists a function $\omega(\theta,\sigma)$ of bounded variation for $\theta\in[-1,0]$, such that

$$L_\sigma(\phi) = \int_{-1}^{0} d\omega(\theta,\sigma)\phi(\theta) \quad \text{for} \quad \phi\in\mathbb{C}. \tag{4.4}$$

In fact, we choose

$$\omega(\theta,\sigma) = (\tau_0+\sigma)D_1\xi(\theta) + (\tau_0+\sigma)D_2\xi(\theta+1), \tag{4.5}$$

where $\xi(\theta)$ is a Dirac delta function. For $\phi\in\mathbb{C}^1([-1,0],\mathbb{R}^2)$, we define

$$A(\sigma)\phi(\theta) = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & \theta\in[-1,0), \\ \displaystyle\int_{-1}^{0} d\omega(m,\sigma)\phi(m), & \theta = 0, \end{cases}$$

and

$$R(\sigma)\phi(\theta) = \begin{cases} 0, & \theta\in[-1,0), \\ F(\sigma,\phi), & \theta = 0. \end{cases}$$

Then the system (4.1) can be rewritten as follows

$$\dot z(t) = A(\sigma)z_t + R(\sigma)z_t, \tag{4.6}$$

where $z_t(\theta) = z(t+\theta)$ for $\theta\in[-1,0]$. Similarly, for $\psi\in\mathbb{C}^1([-1,0],\mathbb{R}^2)$, we assume

$$A^*(\sigma)\psi(m) = \begin{cases} \dfrac{-d\psi(m)}{dm}, & m\in(0,1], \\ \displaystyle\int_{-1}^{0} d\omega^T(0,\sigma)\psi(-t), & m = 0, \end{cases}$$

and a bilinear form:

$$< \psi(s), \phi(\theta) >= \bar{\psi}^T(0)\phi(0) - \int_{\theta=-1}^{0}\int_{\zeta=0}^{\theta} \bar{\psi}^T(\zeta-\theta)d\omega(\theta)\phi(\zeta)d\zeta, \tag{4.7}$$

where $\omega(\theta) = \omega(\theta, 0)$. Then $A(0)$ and A^* are disjoint operators. We discussed from previous section that $\pm i\eta_0$ are the eigenvalues of $A(0)$ as well as eigenvalues of A^*. Let $s(\theta) = (1, \mu)^T e^{i\eta_0\tau_0\theta}$ be the eigenvector of $A(0)$ with respect to eigenvalue $i\eta_0\tau_0$ which yields $A(0)s(\theta) = i\eta_0\tau_0 s(\theta)$. Then it is easy to obtain following:

$$\tau_0 \begin{pmatrix} i\eta_0 - p_{10} & -p_{01} \\ q_{100}e^{-i\eta_0\tau_0} & i\eta_0 - b_{001}e^{-i\eta_0\tau_0} \end{pmatrix} \begin{pmatrix} 1 \\ \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which gives

$$s(0) = (1, \mu)^T = \left(1, \frac{q_{100}e^{-i\eta_0\tau_0}}{i\eta_0 - q_{001}e^{-i\eta_0\tau_0}}\right).$$

On the other hand $s^*(m) = E(1, \mu^*)^T e^{i\eta_0\tau_0 m}$ is the eigenvector of $A^*(0)$ with respect to eigenvalue $-i\eta_0\tau_0$. Therefore

$$E\tau_0 \begin{pmatrix} -i\eta_0 - p_{10} & q_{100}e^{i\eta_0\tau_0} \\ -p_{01} & -i\eta_0 - b_{001}e^{i\eta_0\tau_0} \end{pmatrix} \begin{pmatrix} 1 \\ \mu^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain

$$s^*(0) = E(1, \mu^*)^T = E\left(1, -\frac{i\eta_0 + p_{10}}{q_{100}e^{i\eta_0\tau_0}}\right).$$

We compute E such that $< s^*(m), s(\theta) >= 1$, that is

$$\begin{aligned} < s^*(m), s(\theta) > &= \bar{E}(1, \bar{\mu}^*)(1, \mu)^T - \int_{-1}^{0}\int_{0}^{\theta} \bar{E}(1, \bar{\mu}^*)e^{-i\eta_0\tau_0(\zeta-\theta)}d\eta(\theta)(1, \rho)^T e^{i\omega_0\tau_0\zeta}d\zeta \\ &= \bar{E}\{1 + \bar{\mu}^* q_{100}\tau_0 e^{-i\eta_0\tau_0} + \mu\bar{\mu}^*(1 + q_{001}\tau_0 e^{-i\eta_0\tau_0})\}. \end{aligned}$$

Hence

$$E = \frac{1}{1 + \mu^* q_{100}\tau_0 e^{i\eta_0\tau_0} + \bar{\mu}\mu^*(1 + q_{001}\tau_0 e^{i\eta_0\tau_0})}.$$

The remaining part of the derivation is given in Appendix of [23]. Now we define

$$\begin{aligned} \upsilon_1(0) &= \frac{1}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\ \mu_2 &= \frac{Re\{\upsilon_1(0)\}}{Re\{\lambda'(\tau_0)\}}, \\ \beta_2 &= 2Re\{\upsilon_1(0)\}, \\ T_2 &= -\frac{Im\{\upsilon_1(0)\} + \mu_2 Im\{\lambda'(\tau_0)\}}{\omega_0}, \end{aligned} \tag{4.8}$$

where g_{ij} are given in equations (A.5a)-(A.5d) of [23]. Now we summarize the above results into following theorem:

Theorem 4.1. *From (4.8), we conclude that*

1. *The sign of* μ_2 *determines the direction of the Hopf bifurcation: if* $\mu_2 > 0$ $(\mu_2 < 0)$*, then the Hopf bifurcation is supercritical (sub-critical).*
2. *The sign of* β_2 *determines the stability of the bifurcating periodic solution: the bifurcation periodic solutions are stable (unstable) if* $\beta_2 < 0(\beta_2 > 0)$.
3. *The sign of* T_2 *determines the periodic solution: the period increases (decreases) if* $T_2 > 0(T_2 < 0)$.

5. Numerical examples

In this subsection, we develop numerical simulations to illustrate our proposed theoretical results in the previous sections. We consider two cases.

Case 1: The non-delayed model: Consider the system (4.1) without time delay ($\tau = 0$) and with the following fixed parameters $\alpha = 2.0$, $\beta = 0.2$, $\delta = 0.32$, $\rho = 0.2$, $g = 0.09$, $h = 0.3$ and γ regarded as bifurcation parameter; that is

$$\begin{aligned} \frac{dx}{dt} &= x\left(1 - x - \frac{y}{(1+2.0x)(1+0.2y)} - \frac{0.09}{0.3+x}\right), \\ \frac{dy}{dt} &= \gamma y\left(1 - \frac{0.32y}{x+0.2}\right). \end{aligned} \tag{5.1}$$

The above model has four equilibrium points, namely, $E_0 = (0,0)$, $E_1 = (0.9266, 0)$, $E_2 = (0, 0.625)$ and positive interior equilibrium $(x_*, y_*) = (0.114545, 0.982954)$. The graphical visualization of equilibrium points (red bullets) through nullclines is shown Figure 1. At $\gamma = 0.1$, it can be concluded from simple calculations that E_0 is unstable while E_1 and E_2 are saddle. With the same parameters and $\gamma = 0.1$, the conditions given in Theorem 3.2 are well satisfied as $\gamma^* - \gamma = -0.0299818 < 0$ and $\gamma^* - \frac{x_*}{\delta(1+\alpha x_*)(1+\beta y_*)^2} = -0.133382 < 0$. The equilibrium point E_* is locally asymptotically stable and which also can be confirmed from Figure 2. In addition, we choose γ as a bifurcation parameter and keep other parameters fixed as in the above. We find that $\gamma^{hb} = \gamma^* = 0.07$ and when γ crosses its critical value γ^*, system loses its stability and Hopf bifurcation occurs around E_*, see Figure 3.

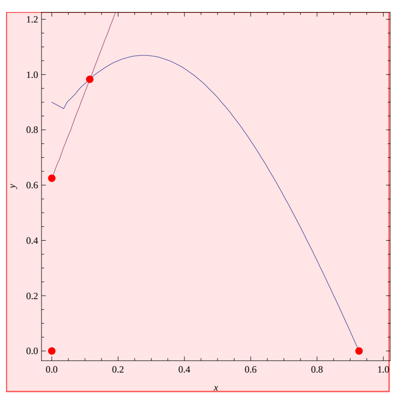

Figure 1. The existence of all possible equilibrium points (red bullets).

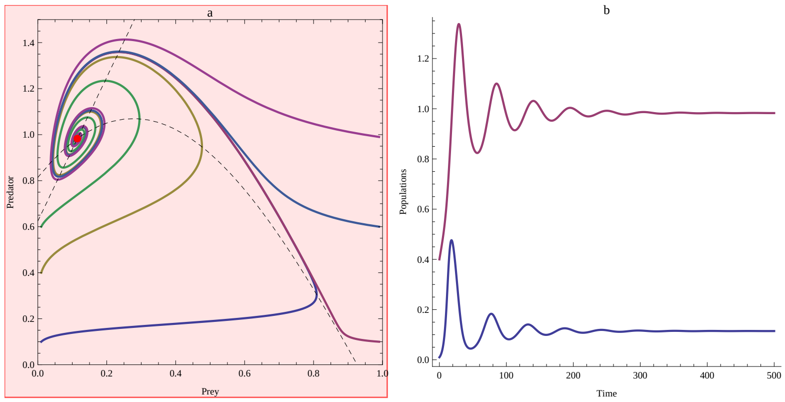

Figure 2. (a) The phase space diagram of (5.1) for $\gamma = 0.1$. (b) The time trajectories of system (5.1) for $\gamma = 0.1$. Thus $E_* = (0.114545, 0.982954)$ is locally asymptotically stable.

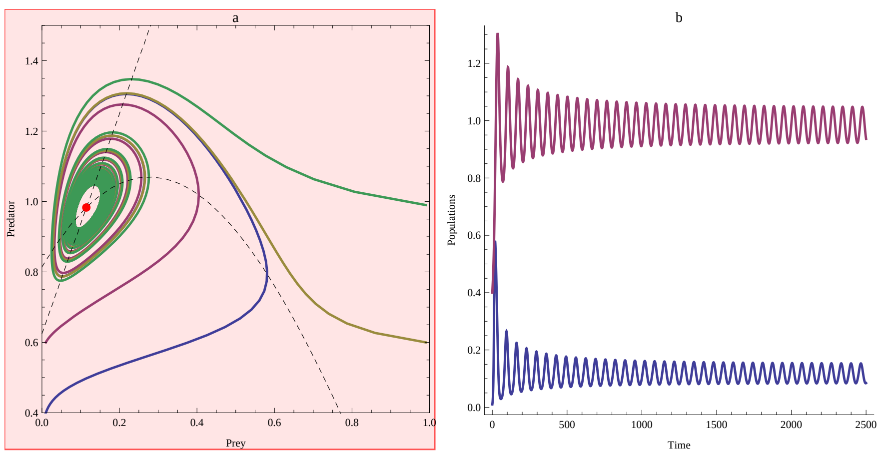

Figure 3. (a) The phase space diagram of (5.1) for $\gamma = 0.07$. (b) The time trajectories of system (5.1) for $\gamma = 0.07$. Thus the model (5.1) undergoes Hopf bifurcation around E_*.

Case 2: The delayed model: We consider the following delayed system

$$\begin{aligned} \frac{dx}{dt} &= x\left(1 - x - \frac{y}{(1+2.0x)(1+0.2y)} - \frac{0.09}{0.3+x}\right), \\ \frac{dy}{dt} &= \gamma y\left(1 - \frac{0.32y(t-\tau)}{x(t-\tau)+0.2}\right). \end{aligned} \tag{5.2}$$

Using (3.10) and (3.11), one can easily obtain $\eta_0 = 0.22244$ and critical delay $\tau_0 = 3.26242$. As stated in Theorem 3.5, when $0 < \tau < \tau_0$, the equilibrium point E_* of system (5.2) is asymptotically stable, which is easily verified from Figure 4. Whenever τ crosses its critical value τ_0, system (5.2) loses its stability and family of periodic solutions (that is, Hopf bifurcation) exists. For this value, the transversality condition holds by Theorem 3.4 since $([\frac{d(R(\lambda(\tau)))}{d\tau}]_{\tau=\tau^+_{1,n}}) > 0$. Figure 5 displays the periodic solution of (5.2) for $\tau = 3.26242$. From the results derived in Section 4, we get $s(0) = (1, 1.07574 - 3.41753i)^T$, $s^*(0) = \bar{E}(1, -0.32 - 0.19188i)^T$ $g_{20} = 5.47613 + 3.15866i$, $g_{11} = 0.093921 + 0.20673i$, $g_{02} = -6.65778 + 0.11011i$ and

$g_{21} = 18.0231 + 31.0934i$ where $\bar{E} = 0.14936 + 0.61924i$. Also we get $\mu_2 < 0$, $\beta > 0$ and $T_2 > 0$. Thus we conclude that Hopf bifurcation at $\tau_0 = 3.26242$ is subcritical.

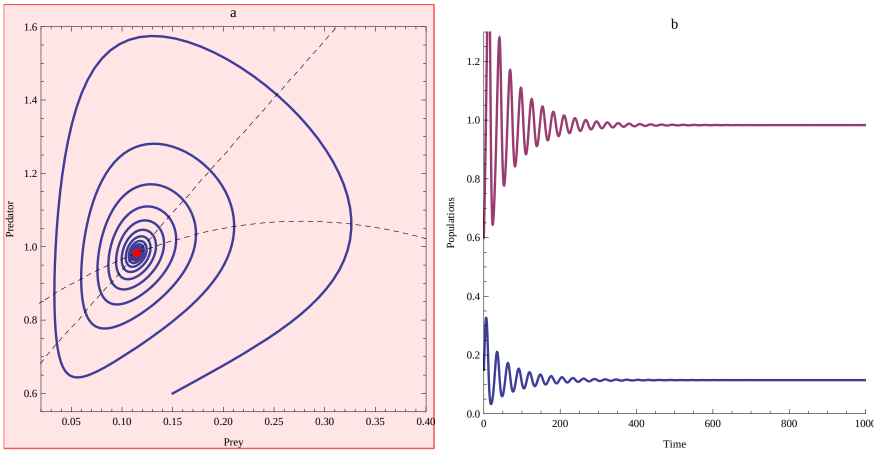

Figure 4. (a) The phase space diagram of (5.2) for $\tau = 3.0$. (b) The time trajectories of system (5.2) for $\tau = 3.0$. Thus E_* is locally asymptotically stable.

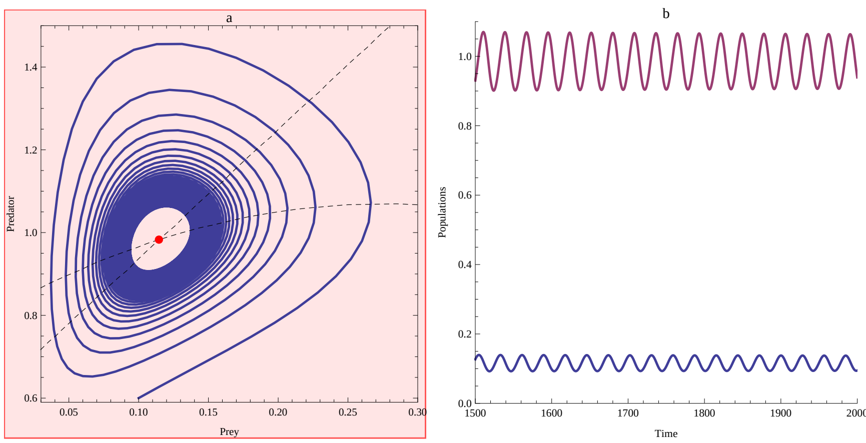

Figure 5. (a) The phase space diagram of (5.2) for $\tau = 3.27$. (b) The periodic solutions of system (5.2) for $\tau = 3.27$. Thus the model (5.2) undergoes Hopf bifurcation around E_*.

6. Conclusion

This work discusses the dynamical properties of modified Leslie-Gower predator-prey model (2.1) with Crowley-Martin functional response. The nonlinear harvesting and gestation delay are respectively utilized into the prey and predator equations to construct the more suitable model. Long term coexistence of both predator and prey is ensured by persistence analysis of model (2.1). We have established the existence of possible equilibrium points and their local stability without delay. Non-delayed version of model (2.1) experiences Hopf bifurcation near E_* when the parameter γ crosses its critical value $\gamma = \gamma^{hb}$. It is clear that from the proposed results that time-delay causes the existence of Hopf bifurcation around positive equilibrium point. With the help of the center manifold theorem, the stability of Hopf bifurcation and its direction are performed. The developed theoretical results are verified by numerical examples.

Acknowledgements. The authors are grateful to the anonymous referees for their useful suggestions which improve the contents of this article.

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