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UNIQUE EXISTENCE RESULTS AND NUMERICAL SOLUTIONS FOR FOURTH-ORDER IMPULSIVE DIFFERENTIAL EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract The work is concerned with three kinds of fourth-order impulsive differential equations with nonlinear boundary conditions. We at first focused on studying the existence and uniqueness of positive solutions for these kinds of problems. By converting the problem to an equivalent integral equation, then applying the new class of fixed point theorems for the sum operator on cone, we obtain the sufficient conditions which not only guarantee the existence of a unique positive solution, but also be applied to construct two iterative sequences for approximating it. Further, we present the numerical methods for solving the fourth-order differential equations. At last, some examples are given with numerical verifications to illustrate the main results.

Keywords Existence and uniqueness, positive solution, impulsive differential equations, fixed point theorem for sum operator, numerical solution.

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1. Introduction

During the last decades, nonlinear boundary value problem arises in a variety of different areas of applied mathematics, physics, chemistry and biology, which can be found in the elastics stability, in chemical or biological problems, and in thermal ignition of gases and so on(see [1–11, 14–20, 22–28, 30–32, 34, 35, 37]). Thereinto, fourth-order boundary value problems are extensively applied to mechanics, engineering and physics (see [2, 11, 15]). These applications are intended to motivate authors investigation of the solutions for the fourth-order nonlinear boundary value problems. On the one hand, some authors have studied the existence and multiplicity of positive solutions by using the shooting method, the lower and upper solution method, the Leray-Schauder continuation method, the topological degree theory and fixed point theorems and so on (see [3–6, 8, 22, 23, 25, 32, 35] and the references therein). Moreover, there are some papers concerned with the uniqueness of posi-

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tive solutions (see [9,18,19,31,34,37]). For example, in [19], Li and Zhang utilized a fixed point theorem of generalized concave increasing operators to investigate the existence and uniqueness of the positive solutions for an elastic beam equation with nonlinear boundary conditions. In [34], by using a fixed point theorems for a class of general mixed monotone operators, Zhai and Jiang obtain some sufficient conditions which guarantee the existence of unique monotone positive solutions for an elastic beam equation. In [31], the authors get the existence, nonexistence, and uniqueness of convex monotone positive solutions of an elastic beam equation with a parameter via a fixed point theorem of cone expansion and a fixed point theorem of generalized concave operators.

Besides, It is well recognised that the theorem of impulsive differential equations is a nature framework for a mathematical modelling of many nature phenomena, and many authors have already paid much attention to the solutions of differential equations with impulsive effects. (see [1, 7, 16, 17, 20, 27, 30] and the references therein). To the best of our knowledge, among these existence literatures, most studies related to boundary value problem for first-order or second-order impulsive differential equation, and there are few papers focused on the study of fourth-order impulsive differential equations with nonlinear boundary conditions.

Motivated by the above work, in this paper, the authors consider the following boundary value problem for fourth-order impulsive differential equations:

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t)) + g(t, u(t)), & t \in J', \\ \Delta u(t_k) = I_k(u(t_k), u'(t_k)), & t = t_k, \quad k = 1, 2, ..., m, \\ u(0) = a, \quad u'(0) = u''(1) = 0, \\ u'''(1) = -q(u(1)), \end{cases} \tag{1.1}$$

where impulsive points $\{t_k\}_{k \in N^+}$ satisfy $0 < t_1 < ... < t_k < ... < t_m < 1$, $J = [0, 1]$, $J' = J \setminus \{t_1, t_2, ..., t_m\}$, $f \in C(J \times \mathbf{R} \times \mathbf{R})$, $g \in C(J \times \mathbf{R})$, $q \in C(\mathbf{R})$, and $I_k \in C(\mathbf{R} \times \mathbf{R})(k = 1, 2, ...m)$. $\Delta u(t_k)$ denotes the jump of $u(t)$ at $t = t_k$, that is, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, with $u(t_k^+) = \lim_{t \to t_k^+} u(t)$, $u(t_k^-) = \lim_{t \to t_k^-} u(t)$. The constant $a > 0$.

Furthermore, when the variable $x \equiv 0$ or $y \equiv 0$ in impulsive term $I_k(x, y)$, that is to say, the impulsive term $I_k = I_k(y)$, and $I_k = I_k(x)$, the problem (1.1) reduces to the following boundary value problems:

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t)) + g(t, u(t)), & t \in J', \\ \Delta u(t_k) = I_k(u'(t_k)), & t = t_k, \quad k = 1, 2, ..., m, \\ u(0) = a, \quad u'(0) = u''(1) = 0, \\ u'''(1) = -q(u(1)), \end{cases} \tag{1.2}$$

and

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t)) + g(t, u(t)), & t \in J', \\ \Delta u(t_k) = I_k(u(t_k)), & t = t_k, \quad k = 1, 2, ..., m, \\ u(0) = a, \quad u'(0) = u''(1) = 0, \\ u'''(1) = -q(u(1)), \end{cases} \tag{1.3}$$

where $I_k \in C(\mathbf{R})(k = 1, 2, ..., m)$. Under some hypotheses, by using different fixed theorems of sum operator on cone, we research the existence and uniqueness of positive solutions for problems (1.1)-(1.3), and also construct two iterative sequences for approximating the positive solution. Also, we introduce numerical methods to deal with the fourth-order impulsive differential equations, and obtain the numerical solution for some concrete examples.

Our work presented in this paper has the following new features. First, we not only prove the existence and uniqueness of positive solution for fourth-order impulsive differential equations, but also present the numerical methods for solving the equations. So, for the concrete examples, we at first prove its unique existence of positive solution, then give its the numerical solution, which makes the property of unique positive solution more clearly. Second, the sum operator is an efficient way to solve the existence and uniqueness of positive solution for differential equations. Especially, the method used in this paper is about the sum of three operators with different property, which is new to the existing literature. Third, the nonlinear term and impulsive term does not depend solely on the unknown solution but on its first order derivative, and the equations in this paper is the generalization of the equations in [18], the equations (1.1)-(1.3) reduces to equations in [18] if $a = 0$, $I_k \equiv 0$, $g(t, u(t)) \equiv 0$ and the authors provide some alternative approaches to study these kinds of equations. Fourth, our conclusions not only obtain the existence of unique positive solution, but also be applied to construct two iterative sequences for approximating it. Moreover, for the problem (1.1)-(1.3), the estimate of unique positive solution is derived with $\mu(t^2 + \frac{1}{2}) \preceq u^* \preceq \lambda(t^2 + \frac{1}{2})$ for some $\lambda > \mu > 0$. Consequently, the results obtained in this paper are more general and complement many previous known conclusions.

With this content in mind, the outline of this paper is organized as follows. Section 2 review some of the standard facts on definitions, fixed point theorems and derive the integral equation for problem (1.1). In Section 3, by using the fixed point theorems of sum operator, our main results are stated and proved. In section 4, the numerical methods will be discussed and applied to some concrete examples to illustrate the effectiveness of our main results in practise.

2. Preliminaries

As for prerequisites, we briefly present some necessary definitions in ordered Banach spaces, some lemmas that will be used in the proofs of our theorems. For more details, we refer the reader to [12, 13, 21, 36].

Recall that a non-empty closed convex set $P \subset E$ is a cone if it satisfies (i) $x \in P, \lambda \geq 0 \Rightarrow \lambda x \in P$; (ii) $x \in P, -x \in P \Rightarrow x = \theta$, in which $(E, \|\cdot\|)$ is a real Banach space with partially ordered by a cone $P \subset E$, i.e., $x \leq y$ if and only if $y - x \in P$, and θ is the zero element of E. P is called normal if there exists a constant $N > 0$ such that, for all $x, y \in E, \theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$, and N is called the normality constant of P. For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$ (i.e., $h \geq \theta$ and $h \neq \theta$), we denote by P_h the set $P_h = \{x \in E \mid x \sim h\}$. It is easy to see that $P_h \subset P$.

We say that an operator $A : E \to E$ is increasing (decreasing) if $x \leq y$ implies $Ax \leq Ay (Ax \geq Ay)$.

Definition 2.1 (see [12]). $A : P \times P \to P$ is said to be a mixed monotone operator if $A(x, y)$ is increasing in x and decreasing in y, i.e., $u_i,\ v_i (i = 1, 2) \in P,\ u_1 \leq u_2,\ v_1 \geq v_2$ imply $A(u_1, v_1) \leq A(u_2, v_2)$. Element $x \in P$ is called a fixed point of

A if $A(x, x) = x$.

Definition 2.2 (see [12]). An operator $A : P \to P$ is said to be sub-homogeneous if it is satisfies

$$A(tx) \geq tA(x), \ \forall t \in (0, 1), \ x \in P.$$

An operator A is said to be $\alpha-$concave if it satisfies

$$A(tx) \geq t^{\alpha} A(x), \ \forall t \in (0, 1), \ x \in P,$$

where α be a real number with $0 \leq \alpha < 1$.

Lemma 2.1 (Theorem 3.1, [29]). *Let P be a normal cone in E. Let $\alpha \in (0, 1)$. Suppose that $A : P \to P$ is an increasing sub-homogeneous operator, $B : P \to P$ is a decreasing operator, $C : P \times P \to P$ is a mixed monotone operator, and satisfy the following conditions:*

$$B(t^{-1}y) \geq tBy, \quad C(tx, t^{-1}y) \geq t^{\alpha} C(x, y), \quad \forall t \in (0, 1), \quad x, y \in P. \tag{2.1}$$

Assume that:

(H_1) there is $h \in P_h$ such that $Ah \in P_h$, $Bh \in P_h$, $C(h, h) \in P_h$;

(H_2) there exists a constant $\delta_0 > 0$ such that $C(x, y) \geq \delta_0(Ax + By)$, $\forall x, y \in P$.

Then:

(i) the operator equation $Ax + Bx + C(x, x) = x$ has a unique solution x^* in P_h;

(ii) for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences

$$x_n = Ax_{n-1} + By_{n-1} + C(x_{n-1}, y_{n-1}), \ y_n = Ay_{n-1} + Bx_{n-1} + C(y_{n-1}, x_{n-1}),$$

$n = 1, 2, \ldots$, we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.

Lemma 2.2 (Theorem 3.7, [29]). *Let P be a normal cone, $\alpha \in (0, 1)$. Suppose that $A : P \to P$ is an increasing sub-homogeneous operator, $B : P \to P$ is a decreasing operator, $C : P \times P \to P$ is a mixed monotone operator, and satisfy:*

$$B(t^{-1}y) \geq t^{\alpha} By, \quad C(tx, t^{-1}y) \geq tC(x, y), \quad \forall t \in (0, 1), \quad x, y \in P. \tag{2.2}$$

Assume (H_1) holds and

(H_2') there exists a constant $\delta_0 > 0$ such that $Ax + C(x, y) \leq \delta_0 By$, $\forall x, y \in P$.

Then, the conclusions $(i) - (ii)$ of Lemma 2.1 hold.

Lemma 2.3 (Theorem 3.8, [29]). *Let P be a normal cone, $\alpha \in (0, 1)$. Suppose that $A : P \to P$ is an increasing $\alpha-$ concave operator, $B : P \to P$ is a decreasing operator, $C : P \times P \to P$ is a mixed monotone operator, and satisfy:*

$$B(t^{-1}y) \geq tBy, \quad C(tx, t^{-1}y) \geq tC(x, y), \quad \forall t \in (0, 1), \quad x, y \in P. \tag{2.3}$$

Assume (H_1) holds and

(H_2'') there exists a constant $\delta_0 > 0$ such that $By + C(x, y) \leq \delta_0 Ax$, $\forall x, y \in P$.

Then, the conclusions $(i) - (ii)$ of Lemma 2.1 hold.

Remark 2.1. It is shown that the Lemmas 2.1-2.3 can be treated as a special case of Corollary 3.7 in [21]. Besides, If we take $B \equiv \theta$ in Lemma 2.1 and Lemma 2.3, then the corresponding conclusions are still true(see Theorem 2.1 and Theorem 2.4 in [33]).

In what follows, for convenience, we give some notations:

Let $PC[J,R] = \{x | x : J \to R,\ x(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$, and the right limit $x(t_k^+)$ exists for $k = 1, 2, 3...., m\}$. Then $PC[J,R]$ is a Banach space with the norm $\|x\|_{PC} = \sup_{t\in J} |x(t)|$.

$X = PC^1[J,R] = \{x | x \in PC[J,R]$, such that $x'(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$, and the right limit $x'(t_k^+)$ exists for $t = 1, 2, 3...., m\}$. Then $PC^1[J,R]$ is a Banach space with the norm $\|x\|_{PC^1} = \max\{\|x\|_{PC}, \|x'\|_{PC}\}$.

Lemma 2.4. *If $f : J \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, $g : J \times \mathbf{R} \to \mathbf{R}$, $q : \mathbf{R} \to \mathbf{R}$ are continuous, then $u \in PC^1[J,R] \cap C^4[J',R]$ is the solution of the problem (1.1) if and only if $u \in PC^1[J,R]$ is the solution of the integral equation*

$$\begin{aligned} u(t) =& a + \int_0^1 G(t,s)[f(s,u(s),u'(s)) + g(s,u(s))]ds + q(u(1))\psi(t) \\ &+ \sum_{0<t_k<t} I_k(u(t_k),u'(t_k)), \quad \forall t \in J, \end{aligned} \tag{2.4}$$

where

$$G(t,s) = \frac{1}{6}\begin{cases} s^2(3t-s), & 0 \le s \le t \le 1; \\ t^2(3s-t), & 0 \le t \le s \le 1, \end{cases} \tag{2.5}$$

and

$$\psi(t) = \frac{t^2}{2} - \frac{t^3}{6}, \qquad \forall t \in J. \tag{2.6}$$

Proof. At first, we prove the sufficiency.

Assuming that $u \in PC^1[J,R] \cap C^4[J',R]$ is the solution of problem (1.1), for $u^{(4)}(t) = f(t,u(t),u'(t)) + g(t,u(t))$, combined with the boundary conditions $u'''(1) = -q(u(1))$, we integrate it from t to 1:

$$u'''(t) = -q(u(1)) - \int_t^1 [f(s,u(s),u'(s)) + g(s,u(s))]ds, \qquad \forall t \in J.$$

Next, we continue to integrate $u'''(t)$ from t to 1:

$$u''(t) = q(u(1))(1-t) + \int_t^1 [f(s,u(s),u'(s)) + g(s,u(s))](s-t)ds, \quad \forall t \in J.$$

Then, combined with $u'(0) = 0$, we integrate the above formula from 0 to t:

$$\begin{aligned} u'(t) =& q(u(1))\left(t - \frac{t^2}{2}\right) + \int_0^t \frac{s^2}{2}[f(s,u(s),u'(s)) + g(s,u(s))]ds \\ &+ \int_t^1 \left(ts - \frac{t^2}{2}\right)[f(s,u(s),u'(s)) + g(s,u(s))]ds, \qquad \forall t \in J. \end{aligned}$$

At last, integrating $u'(t)$ from 0 to t, and using $u(0)=a$, $\Delta u|_{t=t_k} = I_k((u(t_k),u'(t_k))$, we have

$$u(t) = a + q(u(1))\left(\frac{t^2}{2} - \frac{t^3}{6}\right) + \int_0^t \left(\frac{s^2 t}{2} - \frac{s^3}{6}\right)[f(s,u(s),u'(s)) + g(s,u(s))]ds$$

$$+\int_t^1 \left(\frac{st^2}{2}-\frac{t^3}{6}\right)[f(s,u(s),u'(s))+g(s,u(s))]ds+\sum_{0<t_k<t} I_k(u(t_k),u'(t_k))$$

$$=a+\int_0^1 G(t,s)[f(s,u(s),u'(s))+g(s,u(s))]ds+q(u(1))\psi(t)+\sum_{0<t_k<t} I_k(u(t_k),u'(t_k)),$$

where $G(t,s)$ and $\psi(t)$ are defined by (2.5) and (2.6), respectively.

Next, we prove the necessity. Suppose that $u(t)$ is the solution of the integral equation (2.4), obviously, $\Delta u|_{t=t_k} = I_k((u(t_k), u'(t_k))$. Then direct differentiation of (2.4) implies, for $t \neq t_k$, $u^{(4)}(t) = f(s,u(s),u'(s)) + g(s,u(s))$. Further, it is easy to verify that $u(0) = a,\ u'(0) = 0,\ u''(1) = 0,\ \ u'''(1) = -q(u(1))$. The Lemma is proved. □

Lemma 2.5 (Lemma 2.1, [19]). *For any $t,s \in [0,1]$, we have*

$$\frac{1}{3}s^2t^2 \leq G(t,s) \leq \frac{1}{2}st^2, \qquad \frac{1}{3}t^2 \leq \psi(t) \leq \frac{1}{2}t^2.$$

$$\frac{1}{2}s^2t \leq \frac{\partial G(t,s)}{\partial t} \leq st, \qquad \frac{1}{2}t \leq \psi'(t) \leq 2t.$$

3. Main results

In this section, we shall apply Lemma 2.1-2.3 to study the equations (1.1)-(1.3) and obtain new results on the existence and uniqueness of positive solutions. We will work in the Banach apace $X = PC^1[J,R]$ equipped with the norm $\|x\| = \max\{\sup_{t\in J}|x(t)|,\ \sup_{t\in J}|x'(t)|\}$ and a partial order given by:

$$x,y \in PC^1[J,R],\ \ x \preceq y \ \Leftrightarrow\ x(t) \leq y(t),\ \ x'(t) \leq y'(t) \text{ for } t \in J.$$

Define a cone P in X as follows:

$P = \{x \in PC^1[J,R] : x(t) \geq 0, x'(t) \geq 0,\ t \in J\}$, the standard cone. It is clear that P is a normal cone in $PC^1[J,R]$ and the normality constant is 1.

Theorem 3.1. *The basic assumptions are the following:*

(L_1) $f : [0,1]\times[0,+\infty)\times[0,+\infty) \to [0,+\infty)$ *is continuous,* $f(t,x,y)$ *is increasing in* $x \in [0,+\infty)$ *for fixed* $t \in J$, $y \in [0,+\infty)$ *and decreasing in* $y \in [0,+\infty)$ *for fixed* $t \in J$, $x \in [0,+\infty)$*;*

(L_2) $g : [0,1]\times[0,+\infty) \to [0,+\infty)$ *is continuous, and* $g(t,x)$ *is increasing in* $x \in [0,+\infty)$ *for fixed* $t \in J$*;* $q : [0,+\infty) \to [0,+\infty)$ *is continuous, and* $q(y)$ *is decreasing in* $y \in [0,+\infty)$*;*

(L_3) *For any* $k = 1,2,...,m$, $I_k : [0,+\infty)\times[0,+\infty) \to [0,+\infty)$ *are continuous.* $I_k(x,y)$ *is nondecreasing in* $x \in [0,+\infty)$ *for fixed* $y \in [0,+\infty)$*, and nonincreasing in* $y \in [0,+\infty)$ *for fixed* $x \in [0,+\infty)$*.*

(L_4) $g(t,\lambda x) \geq \lambda g(t,x)$ *for* $\lambda \in (0,1),\ t \in J,\ x \in [0,+\infty)$*, and* $q(\lambda^{-1}y) \geq \lambda q(y)$ *for* $\lambda \in (0,1),\ y \in [0,+\infty)$*, and there exist constants* $\alpha_1,\alpha_2 \in (0,1)$, $\forall t \in J,\ \lambda \in (0,1),\ x,y \in [0,+\infty)$*, such that*

$$f(t,\lambda x,\lambda^{-1}y) \geq \lambda^{\alpha_1} f(t,x,y), \qquad I_k(\lambda x,\lambda^{-1}y) \geq \lambda^{\alpha_2} I_k(x,y), \quad k = 1,2,...,m.$$

(L_5) *There exist constants* $\sigma > 0$, $\delta > 0$, *such that* $f(t,x,y),\ g(t,x) \geq \sigma \geq q(y) > 0$ *and* $f(t,x,y) \geq \delta g(t,x)$, $\forall t \in J,\ x,y \in [0,+\infty)$.

Then:

(1) *problem (1.1) has a unique positive solution* u^* *in* P_h, *where* $h(t)=t^2+\frac{1}{2}, \forall t \in J$;

(2) *for any* $x_0, y_0 \in P_h$, $t \in J$, *constructing successively the sequences*

$$\begin{aligned} x_n(t) =& a+\int_0^1 G(t,s)[f(s,x_{n-1}(s),y'_{n-1}(s))+g(s,x_{n-1}(s))]ds+q(y_{n-1}(1))\psi(t) \\ &+\sum_{0<t<t_k} I_k(x_{n-1}(t_k),y'_{n-1}(t_k)), \quad n=1,2,\ldots, \end{aligned} \tag{3.1}$$

$$\begin{aligned} y_n(t) =& a+\int_0^1 G(t,s)[f(s,y_{n-1}(s),x'_{n-1}(s))+g(s,y_{n-1}(s))]ds+q(x_{n-1}(1))\psi(t) \\ &+\sum_{0<t<t_k} I_k(y_{n-1}(t_k),x'_{n-1}(t_k)), \quad n=1,2,\ldots, \end{aligned} \tag{3.2}$$

we have $x_n(t) \to u^*(t)$ *and* $y_n(t) \to u^*(t)$ *in* $PC^1[J,R]$ *as* $n \to \infty$.

Proof. For any $u, v \in P$, we define three operators $A : P \to X$, $B : P \to X$, $C : P \times P \to X$ by

$$Au(t) = \frac{a}{4} + \int_0^1 G(t,s)g(s,u(s))ds, \qquad Bv(t) = \frac{a}{4} + q(v(1))\psi(t),$$
$$C(u,v)(t) = \frac{a}{2} + \int_0^1 G(t,s)f(s,u(s),v'(s))ds + \sum_{0<t_k<t} I_k(u(t_k),v'(t_k)).$$

Then

$$(Au)'(t) = \int_0^1 G_t(t,s)g(s,u(s))ds, \qquad (Bv)'(t) = q(v(1))\psi'(t),$$
$$(C(u,v))'(t) = \int_0^1 G_t(t,s)f(s,u(s),v'(s))ds.$$

Evidently, by Lemma 2.4, $u \in PC^1[J,R] \cap C^4[J',R]$ is the solution of problem (1.1) if and only if $u \in PC^1[J,R]$ solves the operator equation $u = Au+Bu+C(u,u)$. From (L_1)-(L_3) and Lemma 2.5, we know that $A : P \to P,\ B : P \to P,\ C : P \times P \to P$.

Next, the proof will be divided into four steps to check the operators A, B, C satisfy all the conditions of Lemma 2.1.

Step 1: We prove that C is a mixed monotone operator, A is an increasing operator, B is a decreasing operator.

In fact, for $u_i, v_i \in P,\ \ i = 1,2$ with $u_1 \succeq u_2,\ v_1 \preceq v_2$, we know that $u_1(t) \geq u_2(t),\ v_1(t) \leq v_2(t),\ u'_1(t) \geq u'_2(t),\ v'_1(t) \leq v'_2(t),\ t \in J$. It follows from ($L_1$), ($L_3$) and Lemma 2.5 that

$$\begin{aligned} C(u_1,v_1)(t) &= \frac{a}{2} + \int_0^1 G(t,s)f(s,u_1(s),v'_1(s))ds + \sum_{0<t_k<t} I_k(u_1(t_k),v'_1(t_k)) \\ &\geq \frac{a}{2} + \int_0^1 G(t,s)f(s,u_2(s),v'_2(s))ds + \sum_{0<t_k<t} I_k(u_2(t_k),v'_2(t_k)) \\ &= C(u_2,v_2)(t). \end{aligned} \tag{3.3}$$

$$
\begin{aligned}
(C(u_1,v_1))'(t) &= \int_0^1 G_t(t,s)f(s,u_1(s),v_1'(s))ds \\
&\geq \int_0^1 G_t(t,s)f(s,u_2(s),v_2'(s))ds = (C(u_2,v_2))'(t).
\end{aligned} \tag{3.4}
$$

Hence, $C(u_1,v_1) \succeq C(u_2,v_2)$, C is a mixed monotone operator. For any $u,v \in P$ with $u \preceq v$, so we have $u(t) \leq v(t),\ u'(t) \leq v'(t),\ t \in J$, Similar to the argument of (3.3) and (3.4), by (L_2) and Lemma 2.5, we get $Au \preceq Av$, A is an increasing operator. $Bu \succeq Bv$, B is a decreasing operator.

Step 2: We prove that A is a sub-homogeneous operator, and operator B, C satisfy (2.1) in Lemma 2.1.

Firstly, for any $\lambda \in (0,1)$, according to the condition (L_4), we have

$$
A(\lambda u)(t) = \frac{a}{4} + \int_0^1 G(t,s)g(s,\lambda u(s))ds \geq \lambda\left(\frac{a}{4} + \int_0^1 G(t,s)g(s,u(s))ds\right) = \lambda Au(t),
$$
$$
(A(\lambda u))'(t) = \int_0^1 G_t(t,s)g(s,\lambda u(s))ds \geq \lambda \int_0^1 G_t(t,s)g(s,u(s))ds = (\lambda Au)'(t).
$$

So $A(\lambda u) \succeq \lambda Au$ for $\lambda \in (0,1),\ u \in P$. That is A is a sub-homogeneous operator.

Further, for any $\lambda \in (0,1),\ v \in P$, from (L_4) we know that

$$
\begin{aligned}
B(\lambda^{-1}v)(t) &= \frac{a}{4} + q(\lambda^{-1}v(1))\psi(t) \geq \frac{a}{4}\lambda + \lambda q(v(1))\psi(t) = \lambda Bv(t), \\
(B(\lambda^{-1}v))'(t) &= q(\lambda^{-1}v(1))\psi'(t) \geq \lambda q(v(1))\psi'(t) = (\lambda Bv)'(t).
\end{aligned}
$$

This means $B(\lambda^{-1}v) \succeq \lambda Bv$ holds for $\lambda \in (0,1),\ v \in P$. Also for any $\lambda \in (0,1)$, $u,v \in P$, if we set $\alpha = \max\{\alpha_1,\alpha_2\}$, together with (L_4), we have

$$
\begin{aligned}
C(\lambda u,\lambda^{-1}v)(t) &= \frac{a}{2} + \int_0^1 G(t,s)f(s,\lambda u(s),\lambda^{-1}v'(s))ds + \sum_{0<t_k<t} I_k(\lambda u(t_k),\lambda^{-1}v'(t_k)) \\
&\geq \frac{a}{2} + \lambda^{\alpha_1}\int_0^1 G(t,s)f(s,u(s),v'(s))ds + \lambda^{\alpha_2}\sum_{0<t_k<t} I_k(u(t_k),v'(t_k)) \\
&\geq \lambda^{\alpha}\left(\frac{a}{2} + \int_0^1 G(t,s)f(s,u(s),v'(s))ds + \sum_{0<t_k<t} I_k(u(t_k),v'(t_k))\right) \\
&= \lambda^{\alpha}C(u,v)(t),
\end{aligned}
$$

$$
\begin{aligned}
(C(\lambda u,\lambda^{-1}v))'(t) &= \int_0^1 G_t(t,s)f(s,\lambda u(s),\lambda^{-1}v'(s))ds \\
&\geq \lambda^{\alpha_1}\int_0^1 G_t(t,s)f(s,u(s),v'(s))ds \\
&\geq \lambda^{\alpha}\int_0^1 G_t(t,s)f(s,u(s),v'(s))ds = (\lambda^{\alpha}C(u,v))'(t).
\end{aligned}
$$

That is, $C(\lambda u,\lambda^{-1}v) \succeq \lambda^{\alpha}C(u,v)$ for $\lambda \in (0,1),\ u,v \in P$. Hence the operator B, C satisfy (2.1) in Lemma 2.1.

Step 3: We need to prove $Ah \in P_h,\ Bh \in P_h,\ C(h,h) \in P_h$. Let $h(t) = t^2 + \frac{1}{2}$, according to (L_2) and Lemma 2.5, for any $t \in J$, we have

$$\begin{aligned}
Ah(t) &= \frac{a}{4} + \int_0^1 G(t,s)g(s,s^2+\frac{1}{2})ds \le \frac{a}{4} + \int_0^1 \frac{1}{2}st^2 g(s,\frac{3}{2})ds \\
&\le \left(\frac{a}{2} + \frac{1}{2}\int_0^1 sg(s,\frac{3}{2})ds\right)\cdot h(t), \\
Ah(t) &\ge \frac{a}{4} + \int_0^1 \frac{1}{3}s^2t^2 g(s,\frac{1}{2})ds \ge \min\left\{\frac{a}{2}, \frac{1}{3}\int_0^1 s^2 g(s,\frac{1}{2})ds\right\}\cdot h(t), \\
(Ah)'(t) &= \int_0^1 G_t(t,s)g(s,s^2+\frac{1}{2})ds \le \int_0^1 stg(s,\frac{3}{2})ds = \frac{1}{2}\int_0^1 sg(s,\frac{3}{2})ds\cdot h'(t), \\
(Ah)'(t) &\ge \int_0^1 \frac{1}{2}s^2 tg(s,\frac{1}{2})ds = \frac{1}{4}\int_0^1 s^2 g(s,\frac{1}{2})ds\cdot h'(t).
\end{aligned}$$

Let $c_1 = \min\left\{\frac{a}{2}, \frac{1}{4}\int_0^1 s^2 g(s,\frac{1}{2})ds\right\}, \qquad c_2 = \frac{a}{2} + \frac{1}{2}\int_0^1 sg(s,\frac{3}{2})ds.$

From $(L_2),\ (L_5)$, we have $c_2 \ge c_1 > 0$. As a result,

$$c_1h(t) \le Ah(t) \le c_2h(t),\ (c_1h)'(t) \le (Ah)'(t) \le (c_2h)'(t),\ t \in J.$$

Consequently, $c_1h \preceq Ah \preceq c_2h$. Namely, $Ah \in P_h$.

Similarly, from (L_2) and Lemma 2.5, for any $t \in J$, we have

$$\begin{aligned}
Bh(t) &= \frac{a}{4} + q(h(1))\psi(t) \le \frac{a}{4} + q(\frac{3}{2})\cdot\frac{1}{2}t^2 \le \max\left\{\frac{a}{2}, \frac{1}{2}q(\frac{3}{2})\right\}\cdot h(t), \\
Bh(t) &\ge \frac{a}{4} + q(\frac{3}{2})\cdot\frac{1}{3}t^2 \ge \min\left\{\frac{a}{2}, \frac{1}{3}q(\frac{3}{2})\right\}\cdot h(t), \\
\frac{1}{4}q(\frac{3}{2})\cdot h'(t) &= q(\frac{3}{2})\cdot\frac{1}{2}t \le (Bh)'(t) = q(h(1))\psi'(t) \le q(\frac{3}{2})\cdot 2t = q(\frac{3}{2})\cdot h'(t).
\end{aligned}$$

Let $c_3 = \min\left\{\frac{a}{2}, \frac{1}{4}q(\frac{3}{2})\right\}, \quad c_4 = \max\left\{\frac{a}{2}, q(\frac{3}{2})\right\}$. From $(L_2), (L_5)$, we have $c_4 \ge c_3 > 0$, and thus $c_3h(t) \le Bh(t) \le c_4h(t),\ (c_3h)'(t) \le (Bh)'(t) \le (c_4h)'(t),\ t \in J$. Therefore $c_3h \preceq Bh \preceq c_4h$, that is $Bh \in P_h$.

Also, according to (L_1), (L_3) and Lemma 2.5, for any $t \in J$, we have

$$\begin{aligned}
C(h,h)(t) &= \frac{a}{2} + \int_0^1 G(t,s)f(s,h(s),h'(s))ds + \sum_{0<t_k<t} I_k(h(t_k),h'(t_k)) \\
&= \frac{a}{2} + \int_0^1 G(t,s)f(s,s^2+\frac{1}{2},2s)ds + \sum_{0<t_k<t} I_k(t_k^2+\frac{1}{2},2t_k) \\
&\le \frac{a}{2} + \int_0^1 \frac{1}{2}t^2 sf(s,\frac{3}{2},0)ds + \sum_{k=1}^m I_k(\frac{3}{2},0) \\
&\le \left(a + \frac{1}{2}\int_0^1 sf(s,\frac{3}{2},0)ds + 2\sum_{k=1}^m I_k(\frac{3}{2},0)\right)\cdot h(t), \\
C(h,h)(t) &\ge \frac{a}{2} + \int_0^1 \frac{1}{3}t^2s^2 f(s,\frac{1}{2},2)ds
\end{aligned}$$

$$\geq \min\left(a, \frac{1}{3}\int_0^1 s^2 f(s, \frac{1}{2}, 2)ds\right) \cdot h(t).$$

$$(C(h,h))'(t) = \int_0^1 G_t(t,s) f(s, s^2 + \frac{1}{2}, 2s)ds \leq \int_0^1 stf(s, \frac{3}{2}, 0)ds$$

$$= \frac{1}{2}\int_0^1 sf(s, \frac{3}{2}, 0)ds \cdot h'(t),$$

$$(C(h,h))'(t) \geq \int_0^1 \frac{1}{2}s^2 tf(s, \frac{1}{2}, 2)ds = \frac{1}{4}\int_0^1 s^2 f(s, \frac{1}{2}, 2)ds \cdot h'(t).$$

Let $c_5 = \min\left\{a,\ \frac{1}{4}\int_0^1 s^2 f(s, \frac{1}{2}, 2)ds\right\}$, $c_6 = a + \frac{1}{2}\int_0^1 sf(s, \frac{3}{2}, 0)ds + 2\sum_{k=1}^m I_k(\frac{3}{2}, 0)$. Applying the conditions (L_1), (L_3), (L_5), we get $c_6 \geq c_5 > 0$. In consequence,

$$c_5 h(t) \leq C(h,h)(t) \leq c_6 h(t),\ (c_5 h)'(t) \leq (C(h,h))'(t) \leq (c_6 h)'(t), \quad \forall t \in J.$$

Thus, $c_5 h \preceq C(h,h) \preceq c_6 h$. That is $C(h,h) \in P_h$. Hence, the condition (H_1) in Lemma 2.1 is proved.

Step 4: We verify that the operators A, B, C satisfy the condition (H_2).

For $u, v \in P$, and any $t \in J$, taking (L_1), (L_3), (L_5) and Lemma 2.5 into consideration, we get

$$C(u,v)(t) = \frac{a}{2} + \int_0^1 G(t,s) f(s, u(s), v'(s))ds + \sum_{0<t_k<t} I_k(u(t_k), v'(t_k))$$

$$\geq \frac{a}{2} + \frac{1}{2}\int_0^1 G(t,s)\delta g(s, u(s))ds + \frac{1}{2}\int_0^1 \frac{1}{3}t^2 s^2 f(s, u(s), v'(s))ds$$

$$\geq \min\left\{1, \frac{\delta}{2}\right\}\left(\frac{a}{4} + \int_0^1 G(t,s) g(s, u(s))ds\right) + \frac{1}{6}t^2\int_0^1 s^2\sigma ds + \frac{a}{4}$$

$$= \min\left\{1, \frac{\delta}{2}\right\} Au(t) + \frac{1}{18}\sigma t^2 + \frac{a}{4} \geq \min\left\{1, \frac{\delta}{2}\right\} Au(t) + \frac{t^2}{18}q(v(1)) + \frac{a}{4}$$

$$\geq \min\left\{1, \frac{\delta}{2}\right\} Au(t) + \frac{1}{9}\left(q(v(1))\psi(t) + \frac{a}{4}\right)$$

$$= \min\left\{1, \frac{\delta}{2}\right\} Au(t) + \frac{1}{9}Bv(t).$$

$$(C(u,v))'(t) = \int_0^1 G_t(t,s) f(s, u(s), v'(s))ds$$

$$\geq \frac{1}{2}\int_0^1 G_t(t,s)\delta g(s, u(s))ds + \frac{1}{2}\int_0^1 \frac{1}{2}ts^2 f(s, u(s), v'(s))ds$$

$$\geq \frac{\delta}{2}(Au)'(t) + \frac{1}{4}t\int_0^1 s^2\sigma ds = \frac{\delta}{2}(Au)'(t) + \frac{1}{12}\sigma t$$

$$\geq \frac{\delta}{2}(Au)'(t) + \frac{t}{12}q(v(1)) \geq \frac{\delta}{2}(Au)'(t) + \frac{1}{24}q(v(1))\psi'(t)$$

$$= \frac{\delta}{2}(Au)'(t) + \frac{1}{24}(Bv)'(t).$$

Let $\delta_0 = \min\{\frac{\delta}{2}, \frac{1}{24}\}$. Then

$$C(u,v)(t) \geq \delta_0[Au(t) + Bv(t)],\ (C(u,v))'(t) \geq \delta_0[(Au)'(t) + (Bv)'(t)],\ \forall t \in J.$$

In other words, $C(u,v) \succeq \delta_0(Au+Bv)$ for $u,v \in P$.

Therefore, the operators A, B, C satisfy all the conditions of Lemma 2.1, by application of Lemma 2.1, we have the operator equation $Au + Bu + C(u,u) = u$ has a unique positive solution u^* in P_h. Consequently, problem (1.1) has a unique positive solution $u^*(t) \in P_h$, where $h(t) = t^2 + \frac{1}{2}$. And for any $x_0, y_0 \in P_h$, $t \in J$, constructing successively the sequences (3.1) and (3.2), we have $x_n(t) \to u^*(t)$ and $y_n(t) \to u^*(t)$ in $PC^1[J,R]$ as $n \to \infty$. □

When the variable $x \equiv 0$ or $y \equiv 0$ in impulsive term $I_k(x,y)$, that is to say, $I_k(x,y) = I_k(y)$, or $I_k(x,y) = I_k(x)$, by using the Lemma 2.2 and Lemma 2.3, we can obtain the following results.

Theorem 3.2. *Let f, g, q satisfy the assumptions (L_1)-(L_2) and*

(L_6) *For any $k = 1,2,...,m$, $I_k \in C([0,+\infty),[0,+\infty))$, and $I_k(y)$ is nonincreasing in $y \in [0,+\infty)$.*

(L_7) *$f(t,\lambda x,\lambda^{-1}y) \geq \lambda f(t,x,y)$, $\forall t \in J$, $\lambda \in (0,1)$, $x,y \in [0,+\infty)$; $g(t,\lambda x) \geq \lambda g(t,x)$ for $\lambda \in (0,1)$, $t \in J$, $x \in [0,+\infty)$. Besides, there exist constants $\alpha_1, \alpha_2 \in (0,1)$, for $\lambda \in (0,1)$, $y \in [0,+\infty)$, such that*

$$q(\lambda^{-1}y) \geq \lambda^{\alpha_1} q(y), \qquad I_k(\lambda^{-1}y) \geq \lambda^{\alpha_2} I_k(y), \qquad k = 1,2,...,m.$$

(L_8) *There exist constant $\delta' > 0$, such that $f(t,x,y) + g(t,x) \leq \delta' \leq q(y)$, and $f(t,\frac{1}{2},2) \not\equiv 0$, $g(t,\frac{1}{2}) \not\equiv 0$.*

Then:

(1) the fourth-order impulsive differential equations (1.2) has a unique positive solution u^* in P_h, where $h(t) = t^2 + \frac{1}{2}, t \in J$;

(2) for any initial values $x_0, y_0 \in P_h$, $t \in J$, the sequences $\{x_n\}$, $\{y_n\}$ of successive approximations defined

$$\begin{aligned} x_n(t) =& a + \int_0^1 G(t,s)[f(s,x_{n-1}(s),y'_{n-1}(s)) + g(s,x_{n-1}(s))]ds \\ &+ q(y_{n-1}(1))\psi(t) + \sum_{0<t<t_k} I_k(y'_{n-1}(t_k)), \quad n = 1,2,\ldots, \end{aligned} \tag{3.5}$$

$$\begin{aligned} y_n(t) =& a + \int_0^1 G(t,s)[f(s,y_{n-1}(s),x'_{n-1}(s)) + g(s,y_{n-1}(s))]ds \\ &+ q(x_{n-1}(1))\psi(t) + \sum_{0<t<t_k} I_k(x'_{n-1}(t_k)), \quad n = 1,2,\ldots, \end{aligned} \tag{3.6}$$

both converge uniformly to $u^*(t)$ in $PC^1[J,R]$ as $n \to \infty$.

Proof. At first, we define three operators $A : P \to X$, $B : P \to X$, $C : P \times P \to X$ by

$$Au(t) = \frac{a}{4} + \int_0^1 G(t,s)g(s,u(s))ds, \quad Bv(t) = \frac{a}{2} + q(v(1))\psi(t) + \sum_{0<t_k<t} I_k(v'(t_k)),$$

$$C(u,v)(t) = \frac{a}{4} + \int_0^1 G(t,s)f(s,u(s),v'(s))ds, \quad \forall u,v \in P.$$

Then

$$(Au)'(t) = \int_0^1 G_t(t,s)g(s,u(s))ds, \qquad (Bv)'(t) = q(v(1))\psi'(t),$$

$$(C(u,v))'(t) = \int_0^1 G_t(t,s)f(s,u(s),v'(s))ds, \quad \forall u,v \in P.$$

Definitely, by Lemma 2.4, $u \in PC^1[J,R] \cap C^4[J',R]$ is the solution of problem (1.2) if and only if $u \in PC^1[J,R]$ solves the operator equation $u = Au + Bu + C(u,u)$. Similar to the proof of Theorem 3.1, from (L_1)-(L_2), (L_6)-(L_7) and Lemma 2.5, we obtain that $A : P \to P$ is an increasing sub-homogeneous operator, $B : P \to P$ is a decreasing operator, $C : P \times P \to P$ is a mixed monotone operator, and operators B, C satisfy (2.2).

Next, we will prove $Ah \in P_h,\ Bh \in P_h,\ C(h,h) \in P_h$. Let $h(t) = t^2 + \frac{1}{2}$, by (L_1), (L_2), (L_6) and Lemma 2.5, for any $t \in J$, analysis similar to that in the proof of Theorem 3.1 shows that

$$\begin{aligned}
\min\left\{\frac{a}{2}, \frac{1}{3}\int_0^1 s^2 g(s,\frac{1}{2})ds\right\} \cdot h(t) \leq & Ah(t) \leq \left(\frac{a}{2} + \frac{1}{2}\int_0^1 sg(s,\frac{3}{2})ds\right) \cdot h(t), \\
\frac{1}{4}\int_0^1 s^2 g(s,\frac{1}{2})ds \cdot h'(t) \leq & (Ah)'(t) \leq \frac{1}{2}\int_0^1 sg(s,\frac{3}{2})ds \cdot h'(t), \\
\min\left\{a, \frac{1}{3}q(\frac{3}{2})\right\} \cdot h(t) \leq & Bh(t) \leq \left(a + \frac{1}{2}q(\frac{3}{2}) + 2\sum_{k=1}^m I_k(0)\right) \cdot h(t), \\
\frac{1}{4}q(\frac{3}{2}) \cdot h'(t) \leq & (Bh)'(t) \leq q(\frac{3}{2}) \cdot h'(t), \\
\min\left\{\frac{a}{2}, \frac{1}{3}\int_0^1 s^2 f(s,\frac{1}{2},2)ds\right\} \cdot h(t) \leq & C(h,h)(t) \leq \left(\frac{a}{2} + \frac{1}{2}\int_0^1 sf(s,\frac{3}{2},0)ds\right) \cdot h(t), \\
\frac{1}{4}\int_0^1 s^2 f(s,\frac{1}{2},2)ds \cdot h'(t) \leq & (C(h,h))'(t) \leq \frac{1}{2}\int_0^1 sf(s,\frac{3}{2},0)ds \cdot h'(t),
\end{aligned}$$

where we have set

$$c_1 = \min\left\{\frac{a}{2}, \frac{1}{4}\int_0^1 s^2 g(s,\frac{1}{2})ds\right\}, \qquad c_2 = \frac{a}{2} + \frac{1}{2}\int_0^1 sg(s,\frac{3}{2})ds.$$

Furthermore, we set

$$c_7 = \min\left\{a, \frac{1}{4}q(\frac{3}{2})\right\}, \qquad c_8 = a + q(\frac{3}{2}) + 2\sum_{k=1}^m I_k(0),$$

$$c_9 = \min\left\{\frac{a}{2},\ \frac{1}{4}\int_0^1 s^2 f(s,\frac{1}{2},2)ds\right\}, \qquad c_{10} = \frac{a}{2} + \frac{1}{2}\int_0^1 sf(s,\frac{3}{2},0)ds.$$

From (L_1), (L_2), (L_6) and (L_8), we have $c_2 \geq c_1 > 0$, $c_8 \geq c_7 > 0$, $c_{10} \geq c_9 > 0$. Therefore, we can easily deduce that $c_1 h \preceq Ah \preceq c_2 h$, $c_7 h \preceq Bh \preceq c_8 h$, $c_9 h \preceq C(h,h) \preceq c_{10} h$, that is $Ah \in P_h,\ Bh \in P_h,\ C(h,h) \in P_h$.

At last, we show that the operators A, B, C satisfy the condition (H_2') in Lemma 2.2. For $u, v \in P$, and any $t \in J$, from (L_8) and Lemma 2.5, we have that

$$\begin{aligned} C(u,v)(t) + Au(t) &= \frac{a}{2} + \int_0^1 G(t,s)f(s,u(s),v'(s))ds + \int_0^1 G(t,s)g(s,u(s))ds \\ &\le \frac{a}{2} + \int_0^1 \frac{1}{2}st^2\delta' ds = \frac{a}{2} + \frac{1}{4}t^2\delta' \le \frac{a}{2} + \frac{1}{4}t^2 q(v(1)) \\ &\le \frac{a}{2} + q(v(1))\psi(t) + \sum_{0<t_k<t} I_k(v'(t_k)) = Bv(t). \end{aligned}$$

$$\begin{aligned} (C(u,v))'(t) + (Au)'(t) &= \int_0^1 G_t(t,s)f(s,u(s),v'(s))ds + \int_0^1 G_t(t,s)g(s,u(s))ds \\ &\le \int_0^1 G_t(t,s)\delta' ds \le \int_0^1 st\delta' ds = \frac{1}{2}t\delta' \\ &\le \frac{1}{2}tq(v(1)) \le \psi'(t)q(v(1)) = (Bv)'(t). \end{aligned}$$

Let $\delta_0 = 1$. Then

$$C(u,v)(t) + Au(t) \le \delta_0 Bv(t), \quad (C(u,v))'(t) + (Au)'(t) \le \delta_0 (Bv)'(t), \quad t \in J.$$

As a result, $C(u,v) + Au \preceq \delta_0 Bv$ for $u, v \in P$. Finally, an application of Lemma 2.2 implies u^* is a positive solution of problem (3.4), Besides, we also construct the convergence of the sequences $\{x_n\}$ and $\{y_n\}$ as show in (3.5), (3.6) to u^* in $PC^1[J,R]$. This makes end to the proof. □

Theorem 3.3. *Suppose f, g, q satisfy the assumptions (L_1)-(L_2) and*

- (L_9) *For any $k = 1,2,...,m$, $I_k \in C([0,+\infty),[0,+\infty))$, and $I_k(x)$ is nondecreasing in $x \in [0,+\infty)$.*
- (L_{10}) *$f(t,\lambda x,\lambda^{-1}y) \ge \lambda f(t,x,y)$, $\forall t \in J$, $\lambda \in (0,1)$, $x,y \in [0,+\infty)$; $q(\lambda^{-1}y) \ge \lambda q(y)$ for $\lambda \in (0,1)$, $t \in J$, $y \in [0,+\infty)$. There also exist a constant $\alpha_1, \alpha_2 \in (0,1)$, for $\lambda \in (0,1)$, $x \in [0,+\infty)$, such that*

$$g(t,\lambda x) \ge \lambda^{\alpha_1} g(t,x), \qquad I_k(\lambda x) \ge \lambda^{\alpha_2} I_k(x), \qquad k = 1,2,...,m.$$

- (L_{11}) *There exist two constants $\delta_1, \delta_2 > 0$, such that $f(t,x,y) \le \delta_1 g(t,x)$, $q(y) \le \delta_2 \le g(t,x)$ and $f(t,\frac{1}{2},2) \not\equiv 0$, $g(t,\frac{1}{2}) \not\equiv 0$, $q(\frac{3}{2}) \not\equiv 0$.*

Then:

(1) the problem (1.3) has a unique positive solution u^* in P_h, where $h(t) = t^2 + \frac{1}{2}, t \in J$;

(2) for any $x_0, y_0 \in P_h$, $t \in J$, constructing successively the sequences

$$\begin{aligned} x_n(t) = a &+ \int_0^1 G(t,s)[f(s,x_{n-1}(s),y'_{n-1}(s)) + g(s,x_{n-1}(s))]ds \\ &+ q(y_{n-1}(1))\psi(t) + \sum_{0<t<t_k} I_k(x_{n-1}(t_k)), \quad n = 1,2,\ldots, \end{aligned} \tag{3.7}$$

$$y_n(t) = a + \int_0^1 G(t,s)[f(s,y_{n-1}(s),x'_{n-1}(s)) + g(s,y_{n-1}(s))]ds$$
$$+ q(x_{n-1}(1))\psi(t) + \sum_{0<t<t_k} I_k(y_{n-1}(t_k)), \quad n = 1,2,\ldots, \tag{3.8}$$

we have $x_n(t) \to u^*(t)$ *and* $y_n(t) \to u^*(t)$ *in* $PC^1[J,R]$ *as* $n \to \infty$.

Proof. We first define three operators $A : P \to X$, $B : P \to X$, $C : P \times P \to X$ by

$$Au(t) = \frac{a}{2} + \int_0^1 G(t,s)g(s,u(s))ds + \sum_{0<t_k<t} I_k(u(t_k)), \quad Bv(t) = \frac{a}{4} + q(v(1))\psi(t),$$
$$C(u,v)(t) = \frac{a}{4} + \int_0^1 G(t,s)f(s,u(s),v'(s))ds, \quad \forall u,v \in P.$$

Then

$$(Au)'(t) = \int_0^1 G_t(t,s)g(s,u(s))ds, \quad (Bv)'(t) = q(v(1))\psi'(t),$$
$$(C(u,v))'(t) = \int_0^1 G_t(t,s)f(s,u(s),v'(s))ds, \quad \forall u,v \in P.$$

Clearly, by Lemma 2.4, $u \in PC^1[J,R] \cap C^4[J',R]$ is the solution of problem (1.3) if and only if $u \in PC^1[J,R]$ solves the operator equation $u = Au + Bu + C(u,u)$. we can now proceed analogously to the proof of Theorem 3.1 and obtain that $A : P \to P$ is an increasing $\alpha-$ concave operator, $B : P \to P$ is a decreasing operator, $C : P \times P \to P$ is a mixed monotone operator, and operators B, C satisfy (2.3).

Next, we will prove $Ah \in P_h$, $Bh \in P_h$, $C(h,h) \in P_h$. This follows by the same method as in Theorem 3.1. Let $h(t) = t^2 + \frac{1}{2}$, by (L_1), (L_2), (L_9) and Lemma 2.5, for any $t \in J$, we have

$$\min\left\{a, \frac{1}{3}\int_0^1 s^2 g(s,\frac{1}{2})ds\right\} \cdot h(t) \le Ah(t) \le \left(a + \frac{1}{2}\int_0^1 sg(s,\frac{3}{2})ds + 2\sum_{k=1}^m I_k(\frac{3}{2})\right) \cdot h(t),$$
$$\frac{1}{4}\int_0^1 s^2 g(s,\frac{1}{2})ds \cdot h'(t) \le (Ah)'(t) \le \frac{1}{2}\int_0^1 sg(s,\frac{3}{2})ds \cdot h'(t),$$
$$\min\left\{\frac{a}{2}, \frac{1}{3}q(\frac{3}{2})\right\} \cdot h(t) \le Bh(t) \le \max\left\{\frac{a}{2}, \frac{1}{2}q(\frac{3}{2})\right\} \cdot h(t),$$
$$\frac{1}{4}q(\frac{3}{2}) \cdot h'(t) \le (Bh)'(t) \le q(\frac{3}{2}) \cdot h'(t),$$
$$\min\left\{\frac{a}{2}, \frac{1}{3}\int_0^1 s^2 f(s,\frac{1}{2},2)ds\right\} \cdot h(t) \le C(h,h)(t) \le \left(\frac{a}{2} + \frac{1}{2}\int_0^1 sf(s,\frac{3}{2},0)ds\right) \cdot h(t),$$
$$\frac{1}{4}\int_0^1 s^2 f(s,\frac{1}{2},2)ds \cdot h'(t) \le (C(h,h))'(t) \le \frac{1}{2}\int_0^1 sf(s,\frac{3}{2},0)ds \cdot h'(t).$$

If we set $c_{11} = \min\left\{a, \frac{1}{4}\int_0^1 s^2 g(s,\frac{1}{2})ds\right\}$, $c_{12} = a + \frac{1}{2}\int_0^1 sg(s,\frac{3}{2})ds + 2\sum_{k=1}^m I_k(\frac{3}{2})$, combining c_3, c_4, c_9, c_{10} and (L_1), (L_2), (L_9), (L_{11}), we get $c_4 \ge c_3 > 0$, $c_{10} \ge$

$c_9 > 0$, $c_{12} \geq c_{11} > 0$. Therefore, we can easily deduce that $c_{11}h \preceq Ah \preceq c_{12}h$, $c_3h \preceq Bh \preceq c_4h$, $c_9h \preceq C(h,h) \preceq c_{10}h$, that is $Ah \in P_h,\ Bh \in P_h,\ C(h,h) \in P_h$.

At last, we show that the operators A, B, C satisfy the condition (H_2'') in Lemma 2.3. For $u, v \in P$, and any $t \in J$, from (L_{11}) and Lemma 2.5, we have that

$$\begin{aligned}
C(u,v)(t) + Bv(t) &= \frac{a}{2} + q(v(1))\psi(t) + \int_0^1 G(t,s)f(s,u(s),v'(s))ds \\
&\leq \frac{a}{2} + \frac{1}{2}t^2\delta_2 + \delta_1 \int_0^1 G(t,s)g(t,u(s))ds \\
&= \frac{a}{2} + \frac{9}{2}\int_0^1 \frac{1}{3}t^2 s^2 \delta_2 ds + \delta_1 \int_0^1 G(t,s)g(t,u(s))ds \\
&\leq \frac{a}{2} + \frac{9}{2}\int_0^1 G(t,s)g(t,u(s))ds + \delta_1 \int_0^1 G(t,s)g(t,u(s))ds \\
&\leq \frac{a}{2} + \left(\frac{9}{2} + \delta_1\right)\int_0^1 G(t,s)g(t,u(s))ds + \sum_{0<t_k<t} I_k(u(t_k)) \\
&\leq \left(\frac{9}{2}+\delta_1\right)\left(\frac{a}{2}+\int_0^1 G(t,s)g(t,u(s))ds + \sum_{0<t_k<t} I_k(u(t_k))\right) \\
&= \left(\frac{9}{2} + \delta_1\right) Au(t), \\
(C(u,v))'(t) + (Bv)'(t) &= q(v(1))\psi'(t) + \int_0^1 G_t(t,s)f(s,u(s),v'(s))ds \\
&\leq 2t\delta_2 + \delta_1 \int_0^1 G_t(t,s)g(s,u(s))ds \\
&= 12\int_0^1 \frac{1}{2}s^2 t\delta_2 ds + \delta_1 \int_0^1 G_t(t,s)g(s,u(s))ds \\
&\leq 12\int_0^1 G_t(t,s)g(s,u(s))ds + \delta_1 \int_0^1 G_t(t,s)g(s,u(s))ds \\
&\leq (12+\delta_1)\int_0^1 G_t(t,s)g(s,u(s))ds = (12+\delta_1)(Au)'(t).
\end{aligned}$$

Let $\delta_0 = 12 + \delta_1$. Then

$$C(u,v)(t) + Bv(t) \leq \delta_0 Au(t), \quad (C(u,v))'(t) + (Bv)'(t) \leq \delta_0 (Au)'(t), \quad t \in J.$$

As a result, $C(u,v) + Bv \preceq \delta_0 Au$ for $u, v \in P$. Finally, an application of Lemma 2.3 implies u^* is a positive solution of problem (1.3), Besides, we also construct the convergence of the sequences $\{x_n\}$ and $\{y_n\}$ to u^* in $PC^1[J,R]$. This makes end to the proof. □

If we take $g(t,u) \equiv 0$ in problem (1.1) and problem (1.3), from Remark 2.1, we obtain the following corollaries.

Corollary 3.1. *Let assumptions (L_1), (L_3) hold, and*

(L_2') $q : [0,+\infty) \to [0,+\infty)$ *is continuous, $q(x)$ is increasing in $x \in [0,+\infty)$;*

(L_4') *$q(\lambda x) \geq \lambda q(x)$ for $\lambda \in (0,1),\ x \in [0,+\infty)$, and there exist constants $\alpha_1, \alpha_2 \in (0,1)$, $\forall t \in [0,1],\ \lambda \in (0,1),\ x,y \in [0,+\infty)$, such that*

$$f(t,\lambda x,\lambda^{-1}y) \geq \lambda^{\alpha_1} f(t,x,y), \qquad I_k(\lambda x,\lambda^{-1}y) \geq \lambda^{\alpha_2} I_k(x,y), \quad k=1,2,...,m.$$

(L_5') *There exist constant $\sigma > 0$, such that $f(t,x,y) \geq \sigma \geq q(x) > 0, \forall t \in J,\ x,y \in [0,+\infty)$.*

Then the problem

$$\begin{cases} u^{(4)}(t) = f(t,u(t),u'(t)), \quad t \in J'; \\ \Delta u|_{t=t_k} = I_k((u(t_k),u'(t_k)),\ k=1,2,...m, \\ u(0)=a, \quad u'(0)=u''(1)=0, \quad u'''(1) = -q(u(1)), \end{cases} \tag{3.9}$$

has a unique positive solution $u^* \in P_h$, where $h(t) = t^2 + \frac{1}{2}$. Moreover, for any $x_0, y_0 \in P_h$, $t \in J$, constructing successively the sequences

$$\begin{aligned} x_n(t) &= a + \int_0^1 G(t,s) f(s,x_{n-1}(s),y'_{n-1}(s))ds + q(x_{n-1}(1))\psi(t) \\ &\quad + \sum_{0<t<t_k} I_k(x_{n-1}(t_k),y'_{n-1}(t_k)), \\ y_n(t) &= a + \int_0^1 G(t,s) f(s,y_{n-1}(s),x'_{n-1}(s))ds + q(y_{n-1}(1))\psi(t) \\ &\quad + \sum_{0<t<t_k} I_k(y_{n-1}(t_k),x'_{n-1}(t_k)), \end{aligned}$$

we have $||x_n - u^*|| \to 0$ and $||y_n - u^*|| \to 0$ in $PC^1[J,R]$ as $n \to \infty$.

Proof. *$\forall u,v \in P$, we define two operators $A : P \to X$, $C : P \times P \to X$ by*

$$\begin{aligned} Au(t) &= \frac{a}{2} + q(u(1))\psi(t), \\ C(u,v)(t) &= \frac{a}{2} + \int_0^1 G(t,s) f(s,u(s),v'(s))ds + \sum_{0<t_k<t} I_k(u(t_k),v'(t_k)). \end{aligned}$$

Then $(Au)'(t) = q(u(1))\psi'(t)$, $\quad (C(u,v))'(t) = \int_0^1 G_t(t,s) f(s,u(s),v'(s))ds$. Clearly, $u \in PC^1[J,R] \cap C^4[J',R]$ is the solution of problem (3.9) if and only if $u \in PC^1[J,R]$ solves the operator equation $u = Au + C(u,u)$. Next, according to Remark 2.1, similar to the proof of Theorem 3.1-Theorem 3.3, we can easily obtain the results. Here we omit this proof. □

Corollary 3.2. *Let assumptions (L_1), (L_9), (L_2') hold, and*

(L_4'') *$f(t,\lambda x,\lambda^{-1}y) \geq \lambda f(t,x,y)$ for $\lambda \in (0,1),\ x,y \in [0,+\infty)$, and there exist constants $\alpha_1, \alpha_2 \in (0,1)$, $\forall t \in [0,1],\ \lambda \in (0,1),\ x \in [0,+\infty)$, such that*

$$q(\lambda x) \geq \lambda^{\alpha_1} q(x), \qquad I_k(\lambda x) \geq \lambda^{\alpha_2} I_k(x), \quad k=1,2,...,m.$$

(L_5'') *There exist constant $\sigma > 0$, such that $f(t,x,y) \leq \sigma \leq q(x), \forall t \in J,\ x,y \in [0,+\infty)$, and $f(t,\frac{1}{2},2) \not\equiv 0$.*

Then the problem

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t)), & t \in J'; \\ \Delta u|_{t=t_k} = I_k((u(t_k)), & k = 1, 2, ...m, \\ u(0) = a, \quad u'(0) = u''(1) = 0, \quad u'''(1) = -q(u(1)), \end{cases} \tag{3.10}$$

has a unique positive solution $u^* \in P_h$*, where* $h(t) = t^2 + \frac{1}{2}$*. Moreover, for any* $x_0, y_0 \in P_h$*,* $t \in J$*, constructing successively the sequences*

$$x_n(t) = a + \int_0^1 G(t,s) f(s, x_{n-1}(s), y'_{n-1}(s)) ds + q(x_{n-1}(1))\psi(t) + \sum_{0<t<t_k} I_k(x_{n-1}(t_k)),$$

$$y_n(t) = a + \int_0^1 G(t,s) f(s, y_{n-1}(s), x'_{n-1}(s)) ds + q(y_{n-1}(1))\psi(t) + \sum_{0<t<t_k} I_k(y_{n-1}(t_k)),$$

we have $||x_n - u^*|| \to 0$ *and* $||y_n - u^*|| \to 0$ *in* $PC^1[J, R]$ *as* $n \to \infty$*.*

Proof. $\forall u, v \in P$, we define two operators $A : P \to X$, $C : P \times P \to X$ by

$$Au(t) = \frac{a}{2} + q(u(1))\psi(t) + \sum_{0<t_k<t} I_k(u(t_k)),$$

$$C(u,v)(t) = \frac{a}{2} + \int_0^1 G(t,s) f(s, u(s), v'(s)) ds.$$

Then $(Au)'(t) = q(u(1))\psi'(t)$, $\quad (C(u,v))'(t) = \int_0^1 G_t(t,s) f(s, u(s), v'(s)) ds$. Obviously, $u \in PC^1[J, R] \cap C^4[J', R]$ is the solution of problem (3.10) if and only if $u \in PC^1[J, R]$ solves the operator equation $u = Au + C(u, u)$. Similarly, according to Remark 2.1, and the proof of Theorem 3.1-Theorem 3.3, we can easily obtain the results. Here we omit the proof. □

Remark 3.1. If $I_k \equiv 0$, $a = 0$ in Corollaries 3.1 and 3.2. From Remark 2.1, the same result has been obtained independently by Li and Zhai in [18]. In this sense, our results extends and supplements that of [18].

4. Numerical methods and examples

In this section, we will present the numerical methods for solving the fourth-order impulsive differential equations and apply the methods to some examples.

Numerical methods. Theorem 3.1 gives an iterative methods to calculate the solution u^*. Starting from any $x_0, y_0 \in P_h$, by the iterative process given by equation (3.1) and (3.2), we have x_n, $y_n \to u^*$. Note that initially, we can set $x_0 = y_0$, so the process can be simplified as following

$$\begin{aligned} x_n(t) =& a + \int_0^1 G(t,s)[f(s, x_{n-1}(s), x'_{n-1}(s)) + g(s, x_{n-1}(s))] ds + q(x_{n-1}(1))\psi(t) \\ &+ \sum_{0<t<t_k} I_k(x_{n-1}(t_k), x'_{n-1}(t_k)), \quad n = 1, 2, \cdots. \end{aligned}$$

Numerically, we can divide the intervals $[0, t_1]$, ..., $[t_k, t_{k+1}]$, ... each into some equal spaced subintervals. For example, use $a_0 = t_k, a_1 = a_0 + h, ..., a_N = t_{k+1}$ for interval $[t_k, t_{k+1}]$, where $h = (t_{k+1} - t_k)/N$. We can use the Trapozoidal rule for the numerical integration on each intervals. As for the only first order derivative $x'(s)$, we can use the central difference. $x'(a_i) = (x(a_{i+1}) - x(a_{i-1}))/2h$, for $i = 1, ..., N-1$. As for the ending points, if unknown, we can simply set $x'(a_N) = x'(a_{N-1})$ and $x'(a_0) = x'(a_1)$. In the following, we will first use an example to verify the convergence of our numerical methods. Then we will apply it to some practical problems.

Example 4.1. Consider the following boundary value problem of fourth-order impulsive differential equation:

$$\begin{cases} u^{(4)}(t) = \frac{\pi^4}{16}\sqrt{1 - \frac{4}{\pi^2}(u'(t))^2}, \\ \Delta u|_{t=\frac{1}{2}} = 1, \\ u(0) = 1, \quad u'(0) = u''(1) = 0, \\ u'''(1) = -\frac{\pi^3}{8}u(1). \end{cases} \tag{4.1}$$

Where $J = [0, 1]$, $t_1 = \frac{1}{2}$, constant $a = 1$. The exact solution of this equation is

$$u(t) = \begin{cases} \cos(\frac{\pi}{2}t) & \text{for} \quad 0 \le t \le \frac{1}{2}, \\ 1 + \cos(\frac{\pi}{2}t) & \text{for} \quad \frac{1}{2} < t \le 1. \end{cases}$$

Let

$$f(t, x, y) = \frac{\pi^4}{16}\sqrt{1 - \frac{4}{\pi^2}y^2}, \qquad I_1(x, y) = 1, \qquad g(t, x) = 0, \qquad q(y) = -\frac{\pi^3}{8}y.$$

Note that if $1 - \frac{4}{\pi^2}y^2 < 0$, set $f(t, x, y) = 0$. It is easy to see that $f(t, x, y)$ is decreasing in y, $q(y)$ is decreasing in y.

Numerical experiment shows that the iteration (4.1) converges fast. Figure 1 shows the numerical solution compared with exact solution. In this example, we used 40 sample points. And 20 iterations are used. As for the initial $x_0(t)$, we simply set $x_0(t) = a$ as a constant function. This example shows that our numerical method is correct. And in the following we will apply our methods to a series problems.

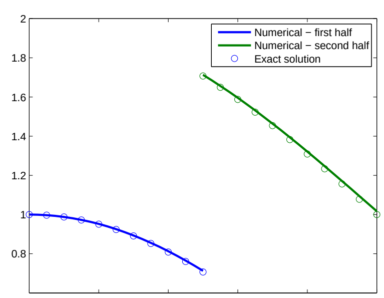

Figure 1. Numerical solution compared with exact solution.

Example 4.2. Consider the following boundary value problem of fourth-order impulsive differential equation:

$$\begin{cases} u^{(4)}(t) = u^{\frac{1}{3}}(t) + (u'(t)+1)^{-\frac{1}{3}} + t^2 + 3 + \frac{u(t)}{1+u(t)}e^t, \quad t \in J, \ t \neq \frac{1}{2}, \\ \Delta u|_{t=\frac{1}{2}} = u^{\frac{1}{2}}(\frac{1}{2}) + (u'(\frac{1}{2})+1)^{-\frac{1}{3}}, \\ u(0) = 2, \quad u'(0) = u''(1) = 0, \\ u'''(1) = -\frac{1}{u(1)+1}, \end{cases} \tag{4.2}$$

where $J = [0,1]$, $t_1 = \frac{1}{2}$, constant $a = 2$.

Obviously, problem (4.2) fits the framework of problem (1.1). Let

$$f(t,x,y) = x^{\frac{1}{3}} + (y+1)^{-\frac{1}{3}} + t^2 + 2, \qquad I_1(x,y) = x^{\frac{1}{2}} + (y+1)^{-\frac{1}{3}},$$
$$g(t,x) = \frac{x}{1+x}e^t + 1, \qquad q(y) = \frac{1}{y+1}.$$

Then, $f : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$, $I : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$, $g : [0,1] \times [0,+\infty) \to [0,+\infty)$ and $q : [0,+\infty) \to [0,+\infty)$ are continuous. It is easy to see that $f(t,x,y)$ is increasing in $x \in [0,+\infty)$ for fixed $t \in [0,1]$, $y \in [0,+\infty)$ and decreasing in $y \in [0,+\infty)$ for fixed $t \in [0,1]$, $x \in [0,+\infty)$; $I(x,y)$ is nondecreasing in $x \in [0,+\infty)$ for fixed $y \in [0,+\infty)$ and nonincreasing in $y \in [0,+\infty)$ for fixed $x \in [0,+\infty)$; $g(t,x)$ is increasing in $x \in [0,+\infty)$ for fixed $t \in [0,1]$, and $q(y)$ is decreasing in $y \in [0,+\infty)$. Moreover, for $t \in J,\ x,y \in [0,+\infty)$, $\lambda \in (0,1)$, $\alpha_1 = \frac{1}{3}$, $\alpha_2 = \frac{1}{2}$ we can obtain

$$\begin{aligned} f(t,\lambda x,\lambda^{-1}y) &= (\lambda x)^{\frac{1}{3}} + (\lambda^{-1}y+1)^{-\frac{1}{3}} + t^2 + 2 \geq \lambda^{\frac{1}{3}}(x^{\frac{1}{3}} + (y+\lambda)^{-\frac{1}{3}} + t^2 + 2) \\ &\geq \lambda^{\frac{1}{3}}(x^{\frac{1}{3}} + (y+1)^{-\frac{1}{3}} + t^2 + 2) = \lambda^{\alpha_1} f(t,x,y), \\ I_1(\lambda x,\lambda^{-1}y) &= \lambda^{\frac{1}{2}}x^{\frac{1}{2}} + \lambda^{\frac{1}{3}}(y+\lambda)^{-\frac{1}{3}} \geq \lambda^{\frac{1}{2}}(x^{\frac{1}{2}} + (y+1)^{-\frac{1}{3}}) = \lambda^{\alpha_2} I_1(x,y), \\ g(t,\lambda x) &= \frac{\lambda x}{1+\lambda x}e^t + 1 \geq \frac{\lambda x}{1+x}e^t + \lambda = \lambda g(t,x), \\ q(\lambda^{-1}y) &= (\lambda^{-1}y+1)^{-1} = \lambda(y+\lambda)^{-1} \geq \lambda(y+1)^{-1} = \lambda q(y). \end{aligned}$$

Besides, let $\sigma = 1, \quad \delta \in \left(0, \frac{1}{e}\right]$,

$$\begin{aligned} f(t,x,y) &= x^{\frac{1}{3}} + (y+1)^{-\frac{1}{3}} + t^2 + 2 \geq \sigma = 1 \geq \frac{1}{y+1} = q(y) > 0, \\ g(t,x) &= \frac{x}{1+x}e^t + 1 \geq \sigma \geq q(y) > 0, \\ f(t,x,y) &= x^{\frac{1}{3}} + (y+1)^{-\frac{1}{3}} + t^2 + 2 \geq 2 = \frac{1}{e}\cdot e + 1 \\ &\geq \delta\left(\frac{x}{1+x}\right)e^t + \delta = \delta g(t,x). \end{aligned}$$

Therefore, all the conditions of Theorem 3.1 are satisfied. By the application of Theorem 3.1, we deduce that the problem (4.2) has a unique positive solution $u^* \in P_h$. The numerical result is given in Figure 2.

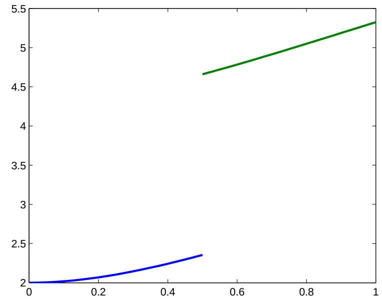

Figure 2. Numerical solution of Example 4.2.

Example 4.3. Consider the following fourth-order impulsive differential equations:

$$\begin{cases} u^{(4)}(t) = \frac{u(t)}{1+u(t)} + \frac{1}{1+u'(t)} + \cos^2 t + \frac{t(u(t)+1)}{1+tu(t)} \quad t \in J,\ t \neq \frac{1}{2}, \\ \Delta u|_{t=\frac{1}{3}} = (u'(\frac{1}{3}) + 1)^{-\frac{1}{2}}, \\ u(0) = 4, \quad u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = -\frac{1}{\sqrt[3]{u(1)+1}} - 4. \end{cases} \tag{4.3}$$

Let

$$f(t,x,y) = \cos^2 t + \frac{x}{1+x} + \frac{1}{1+y}, \qquad I_1(y) = (y+1)^{-\frac{1}{2}},$$
$$g(t,x) = \frac{t(x+1)}{1+tx}, \qquad q(y) = (y+1)^{-\frac{1}{3}} + 4.$$

It is easy to see that $f : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$, $g : [0,1] \times [0,+\infty) \to [0,+\infty)$, $q : [0,+\infty) \to [0,+\infty)$ and $I : [0,+\infty) \to [0,+\infty)$ are continuous with $f(t,\frac{1}{2},2) \not\equiv 0$, $g(t,\frac{1}{2}) \not\equiv 0$, and $f(t,x,y)$ is increasing in $x \in [0,+\infty)$ for fixed $t \in [0,1]$, $y \in [0,+\infty)$ and decreasing in $y \in [0,+\infty)$ for fixed $t \in [0,1]$,$x \in [0,+\infty)$; $I(y)$ is nonincreasing in $y \in [0,+\infty)$; $g(t,x)$ is increasing in $x \in [0,+\infty)$ for fixed $t \in [0,1]$, $q(y)$ is decreasing in $y \in [0,+\infty)$. Besides, for $t \in [0,1]$, $x,y \in [0,+\infty)$, $\alpha_1 = \frac{1}{3}$, $\alpha_2 = \frac{1}{2}$, we have

$$f(t,\lambda x,\lambda^{-1}y) = \cos^2 t + \frac{\lambda x}{1+\lambda x} + \frac{1}{1+\lambda^{-1}y} \geq \cos^2 t + \frac{\lambda x}{1+\lambda x} + \frac{\lambda}{1+y} \geq \lambda f(t,x,y),$$
$$g(t,\lambda x) = \frac{t(\lambda x+1)}{1+\lambda tx} \geq \frac{\lambda t(x+1)}{1+tx} = \lambda g(t,x),$$
$$q(\lambda^{-1}y) = \lambda^{\frac{1}{3}}(\lambda+y)^{-\frac{1}{3}} + 4 \geq \lambda^{\frac{1}{3}}[(\lambda+y)^{-\frac{1}{3}} + 4] \geq \lambda^{\frac{1}{3}}[(1+y)^{-\frac{1}{3}} + 4] = \lambda^{\frac{1}{3}} q(y),$$
$$I_1(\lambda^{-1}y) = \lambda^{\frac{1}{2}}(\lambda+y)^{-\frac{1}{2}} \geq \lambda^{\frac{1}{2}}(1+y)^{-\frac{1}{2}} = \lambda^{\frac{1}{2}} I_1(y).$$

Moreover, Let $\delta' = 4$

$$f(t,x,y) + g(t,x) = \cos^2 t + \frac{x}{1+x} + \frac{1}{1+y} + \frac{t(x+1)}{1+tx} \leq 4 = \delta' \leq (y+1)^{-\frac{1}{3}} + 4 = q(y).$$

As a result, problem (4.3) fits the framework of problem (1.2), and we have proved all the conditions of Theorem 3.2. By the application of Theorem 3.2, we can obtain

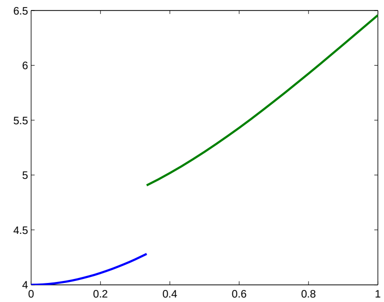

Figure 3. Numerical solution of Example 4.3.

the problem (4.3) has a unique positive solution $u^* \in P_h$. The numerical result is given in Figure 3.

Example 4.4. Consider the following fourth-order impulsive boundary value problem:

$$\begin{cases} u^{(4)}(t) = 2tu^{\frac{1}{4}}(t) + \frac{1}{1+u'(t)} + t^3 + 2, \quad t \in J, \ t \neq \frac{1}{2}, \\ \Delta u|_{t=\frac{1}{4}} = \frac{\sqrt{u(\frac{1}{4})}}{1+\sqrt{u(\frac{1}{4})}}, \\ u(0) = 3, \quad u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = -\frac{1}{1+\sqrt{u(1)}}. \end{cases} \tag{4.4}$$

Let

$$f(t,x,y) = t^3 + tx^{\frac{1}{4}} + \frac{1}{1+y}, \quad I_1(x) = \frac{\sqrt{x}}{1+\sqrt{x}}, \quad g(t,x) = 2 + tx^{\frac{1}{4}}, \quad q(y) = \frac{1}{1+\sqrt{y}}.$$

It is easy to see that $f : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$, $g : [0,1] \times [0,+\infty) \to [0,+\infty)$, $q : [0,+\infty) \to [0,+\infty)$ and $I : [0,+\infty) \to [0,+\infty)$ are continuous with $f(t, \frac{1}{2}, 2) \not\equiv 0$, $g(t, \frac{1}{2}) \not\equiv 0$, $q(\frac{3}{2}) \not\equiv 0$ and $f(t,x,y)$ is increasing in $x \in [0,+\infty)$ for fixed $t \in [0,1]$, $y \in [0,+\infty)$ and decreasing in $y \in [0,+\infty)$ for fixed $t \in [0,1]$, $x \in [0,+\infty)$; $I(x)$ is nondecreasing in $y \in [0,+\infty)$; $g(t,x)$ is increasing in $x \in [0,+\infty)$ for fixed $t \in [0,1]$; $q(y)$ is decreasing in $y \in [0,+\infty)$. Besides, for $t \in [0,1]$, $x, y \in [0,+\infty)$, $\alpha_1 = \frac{1}{3}$, $\alpha_2 = \frac{1}{2}$, we have

$$\begin{aligned} f(t, \lambda x, \lambda^{-1}y) &= t^3 + t(\lambda x)^{\frac{1}{4}} + \frac{1}{1+\lambda^{-1}y} \geq \lambda(t^3 + tx^{\frac{1}{4}} + \frac{1}{1+y}) = \lambda f(t,x,y), \\ q(\lambda^{-1}y) &= \frac{1}{1+\sqrt{\lambda^{-1}y}} = \frac{\sqrt{\lambda}}{\sqrt{\lambda}+\sqrt{y}} \geq \frac{\lambda}{1+\sqrt{y}} = \lambda q(y), \\ g(t, \lambda x) &= 2 + t(\lambda x)^{\frac{1}{4}} \geq \lambda^{\frac{1}{4}} 2 + \lambda^{\frac{1}{4}} tx^{\frac{1}{4}} = \lambda^{\frac{1}{4}} g(t,x), \\ I_1(\lambda x) &= \frac{\sqrt{\lambda x}}{1+\sqrt{\lambda x}} \geq \frac{\lambda^{\frac{1}{2}}\sqrt{x}}{1+\sqrt{x}} = \lambda^{\frac{1}{2}} I_1(x). \end{aligned}$$

Furthermore, Let $\delta_1 = 1$, $\delta_2 = 2$,

$$f(t,x,y) = t^3 + tx^{\frac{1}{4}} + \frac{1}{1+y} \leq 2 + tx^{\frac{1}{4}} = \delta_1 g(t,x), \ \ q(y) = \frac{1}{1+\sqrt{y}} \leq \delta_2 \leq 2 + tx^{\frac{1}{4}} = g(t,x).$$

In consequence, an application of Theorem 3.3 means that the problem (4.4) has a unique positive solution $u^* \in P_h$. The numerical result is given in Figure 4.

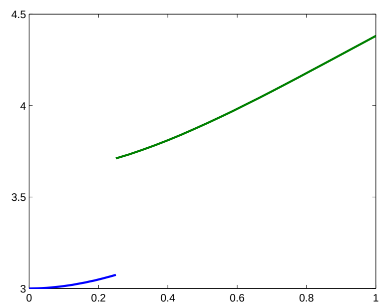

Figure 4. Numerical solution of Example 4.4.

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