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AN APPLICATION OF JACK-FUKUI-SAKAGUCHI LEMMA

Mamoru Nunokawa[1], Janusz Sokół[2,†] and Huo Tang[3]

Abstract We present some applications of Jack-Fukui-Sakaguchi Lemma which become sufficient criteria for a function to be in the class of strongly starlike, strongly close-to-convex or in the other classes.

Keywords Bazilevič function, close-to-convex functions, convex functions, starlike functions.

MSC(2010) 30C45, 30C80.

1. Introduction

Let $\mathcal{A}$ be the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all univalent functions in $\mathbb{D}$. If $f \in \mathcal{A}$ satisfies

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0,\ z \in \mathbb{D},$$

then $f(z)$ is said to be starlike with respect to the origin in $\mathbb{D}$ and it is denoted by $f(z) \in \mathcal{S}^*$. It is known that $\mathcal{S}^* \subset \mathcal{S}$. For further properties of starlike functions and other functions having a geometric property we refer to [3, 11]. To prove the main results we apply techniques of differential subordinations widely described in the book [10]. We say that an analytic function $f(z)$ is subordinate to an analytic function $g(z)$, univalent or not, and write $f(z) \prec g(z)$, if and only if there exists a function $\omega(z)$, analytic in $\mathbb{D}$ such that $\omega(0) = 0$, $|\omega(z)| < 1$ for $|z| < 1$ and $f(z) = g(\omega(z))$. If we additionally assume that $g(z)$ is univalent in $\mathbb{D}$, then we have the following equivalence:

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(|z| < 1) \subset g(|z| < 1). \tag{1.2}$$

The differential subordinations were deeply developed in the monograph [10] as well as in many of recent papers, see for example in [2, 8, 9, 16, 17]. The following lemma is a generalization of well known Jack lemma, [5].

†the corresponding author. Email address:jsokol@ur.edu.pl(J. Sokół)
1 Departmentof Mathematics, University of Gunma, Hoshikuki-cho 798-8, Chuou-Ward, Chiba, 260-0808, Japan
2 College of Natural Sciences, University of Rzeszów, Prof. Pigonia's Street 1, 35-310 Rzeszów, Poland
3 School of Mathematics and Statistics, Chifeng University, Chifeng 024000, Inner Mongolia, China

Lemma 1.1 ([4]). *Let $w(z) = a_p z^p + a_{p+1} z^{p+1} + \cdots$, $a_p \neq 0$, $1 \leq p$ be analytic in $\mathbb{D}$. If the maximum of $|w(z)|$ on the circle $|z| = r < 1$ is attained at $z = z_0$, then $z_0 w'(z_0)/w(z_0)$ is a real number and*

$$\frac{z_0 w'(z_0)}{w(z_0)} \geq p.$$

A related boundary behavior of analytic functions is considered also in [14]. In this paper we present some applications of the above Jack-Fukui-Sakaguchi Lemma to obtain several sufficient criteria for a function to be in the class of strongly starlike, strongly close-to-convex or in the other classes. A related Sakaguchi's result was recently considered in [12,13].

2. Main results

Theorem 2.1. *Let $q(z) = 1 + c_n z^n + \cdots$ be analytic in $\mathbb{D}$ with $c_n \neq 0$. Assume that for all $z \in \mathbb{D}$ we have $q(z) \neq -1$, $q(z) \neq 0$ and for all $z \in \mathbb{D} \setminus \{0\}$ we have $q(z) \neq 1$. Furthermore, suppose that*

$$\left| \frac{zq'(z)}{q(z)} \right| < n, \quad (z \in \mathbb{D}), \tag{2.1}$$

for some positive integer n. Then we have

$$q(z) \prec \frac{1+z^n}{1-z^n}. \tag{2.2}$$

Proof. Let us consider the function $w(z)$ such that

$$w^n(z) = \begin{cases} \frac{q(z)-1}{q(z)+1}, & z \neq 0, \\ 0, & z = 0. \end{cases} \tag{2.3}$$

Then we have

$$w(z) = \sqrt[n]{\frac{c_n}{2}} z + \cdots,$$

which gives

$$q(z) = \frac{1+w^n(z)}{1-w^n(z)}, \tag{2.4}$$

then it follows that $w(z)$ is analytic in $\mathbb{D}$ and to prove (2.2) we need to show $|w(z)| < 1$.

From Fukui and Sakaguchi's Lemma 1.1, we have that if there exists a point $z_0 \in \mathbb{D}$ such that

$$|w(z)| < |z_0| \text{ for } |z| < |z_0|$$

and

$$|w(z_0)| = |z_0| \quad w(z_0) = e^{i\theta}, \tag{2.5}$$

where θ is a real number and $0 \leq \theta < 2\pi$, then

$$\frac{z_0 w'(z_0)}{w(z_0)} = k \geq 1. \tag{2.6}$$

From (2.3) and (2.4) it follows that $\theta \neq 0$ and $\theta \neq \pi$ and so $w(z_0) \neq \pm 1$. On the other hand, from (2.4), we have

$$\frac{zq'(z)}{q(z)} = \frac{2nzw'(z)w^{n-1}(z)}{1-w^{2n}(z)}. \tag{2.7}$$

Therefore, by (2.5) and (2.6) the equality (2.7) becomes

$$\begin{aligned}\left|\frac{z_0q'(z_0)}{q(z_0)}\right| &= \left|\frac{2nz_0w'(z_0)w^{n-1}(z_0)}{1-w^{2n}(z_0)}\right| \geq \left|\frac{2nkw^n(z_0)}{1-w^{2n}(z_0)}\right| \\ &= 2nk\left|\frac{e^{in\theta}}{1-e^{2in\theta}}\right| = 2nk\frac{1}{2|\sin n\theta|} \geq n.\end{aligned}$$

This contradicts (2.1) and so, it completes the proof. □

Notice that in the subordination (2.2) the function $(1+z^n)/(1-z^n)$ is not univalent and it makes the calculations more difficult. Usually in $p(z) \prec q(z)$ it is considered univalent function $q(z)$. Several classes of functions connected with subordination under not-univalent function of the type

$$q(z) \prec \frac{1+Az^n}{1+Bz^n},$$

where A, B are some complex numbers, were considered in [6] and [7].

If $q^{\alpha}(z) = zf'(z)/f(z)$, the Theorem 2.1 becomes the following corollary.

Corollary 2.1. *Let $(zf'(z)/f(z))^{1/\alpha} = 1 + c_nz^n + \cdots$, $(zf'(z)/f(z))^{1/\alpha} \neq -1$, $zf'(z)/f(z) \neq 0$ be analytic in $\mathbb{D}$ for some positive real α. Suppose also that*

$$\left|1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right| < \alpha n\ , \quad (z \in \mathbb{D}), \tag{2.8}$$

for some positive integer n. Then we have

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z^n}{1-z^n}\right)^{\alpha}. \tag{2.9}$$

It is easy to see, that (2.9) implies

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha}, \tag{2.10}$$

which means that $f(z)$ is strongly starlike functions of order α. We say that a function $f \in \mathcal{S}^*$ is strongly starlike of order β if and only if

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi}{2}\beta, \quad (z \in \mathbb{D}),$$

for some β $(0 < \beta \leq 1)$, where the function arg is chosen with values in an interval between $-\pi$ and π. Let $\mathcal{SS}^*(\beta)$ denote the class of strongly starlike functions of order β. The class $\mathcal{SS}^*(\beta)$ was introduced independently in [18, 19] and in [1]. Recall also, that if there exists a function $g(z) \in \mathcal{S}^*$ for which the function $f(z) \in \mathcal{A}$ satisfies the condition

$$\left|\arg\left(\frac{zf'(z)}{g(z)}\right)\right| < \frac{\pi}{2}\alpha, \quad (z \in \mathbb{D}),$$

then we say that $f(z)$ is strongly close-to-convex of order α, $0 < \alpha \leq 1$. For some recent results on strongly starlike functions we refer to [15]. Putting $q^{\alpha}(z) = zf'(z)/g(z)$ in Theorem 2.1 we get the following sufficient condition for $f(z)$ to be strongly close-to-convex of order α.

Corollary 2.2. *Assume that $f(z) \in \mathcal{A}$, $g(z) \in \mathcal{S}^*$ and that for some positive real α, $0 < \alpha \leq 1$ the function $(zf'(z)/g(z))^{1/\alpha} = 1 + c_n z^n + \cdots$ is analytic in $\mathbb{D}$ with $(zf'(z)/g(z))^{1/\alpha} \neq -1$, $zf'(z)/g(z) \neq 0$. Furthermore, suppose that*

$$\left| 1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} \right| < \alpha n, \quad (z \in \mathbb{D}), \tag{2.11}$$

for some positive integer n. Then we have

$$\frac{zf'(z)}{g(z)} \prec \left(\frac{1+z^n}{1-z^n} \right)^{\alpha}, \tag{2.12}$$

which follows that $f(z)$ is strongly close-to-convex of order α.

Moreover, if there exists a function $g(z) \in \mathcal{S}^*$ such that $f(z) \in \mathcal{A}$ satisfies the condition

$$\left| \arg \left(\frac{zf'(z)}{f^{1-\beta}(z) g^{\beta}(z)} \right) \right| < \frac{\pi}{2}\alpha, \quad (z \in \mathbb{D}),$$

then we call that $f(z)$ is strongly Bazilevič function of type β, $0 < \beta$ and of order α, $0 < \alpha \leq 1$.

Corollary 2.3. *Assume that $f(z) \in \mathcal{A}$, $g(z) \in \mathcal{S}^*$ and that for some positive real α, β, $0 < \alpha \leq 1$ the function $(zf'(z)/f^{1-\beta}(z)g^{\beta}(z))^{1/\alpha} = 1 + c_n z^n + \cdots$ is analytic in $\mathbb{D}$ with $(zf'(z)/f^{1-\beta}(z)g^{\beta}(z))^{1/\alpha} \neq -1$, $zf'(z)/f^{1-\beta}(z)g^{\beta}(z) \neq 0$. Furthermore, suppose that*

$$\left| 1 + \frac{zf''(z)}{f'(z)} - (1-\beta)\frac{zf'(z)}{f(z)} - \beta \frac{zg'(z)}{g(z)} \right| < \alpha n, \quad (z \in \mathbb{D}), \tag{2.13}$$

for some positive integer n. Then we have

$$\frac{zf'(z)}{f^{1-\beta}(z) g^{\beta}(z)} \prec \left(\frac{1+z^n}{1-z^n} \right)^{\alpha}, \tag{2.14}$$

which follows that $f(z)$ is strongly Bazilevič function of type β, $0 < \beta$ and of order α.

If $q(z)$ is of the form $q(z) = (p(z))^{1/\alpha}$, then Theorem 2.1 becomes the following corollary.

Corollary 2.4. *Let $(p(z))^{1/\alpha} = 1 + c_n z^n + \cdots$, $(p(z))^{1/\alpha} \neq -1$, $p(z) \neq 0$ be analytic in $\mathbb{D}$, and suppose that*

$$\left| \frac{zp'(z)}{p(z)} \right| < \alpha n, \quad (z \in \mathbb{D}), \tag{2.15}$$

for some positive real α and for some positive integer n. Then we have

$$p(z) \prec \left(\frac{1+z^n}{1-z^n} \right)^{\alpha}. \tag{2.16}$$

Theorem 2.2. *Let* $(\log\{p(z)\})^{1/\alpha} = 1 + c_1 z + \cdots$, $\log\{p(z)\} \neq 0$, $(\log\{p(z)\})^{1/\alpha} \neq -1$, *be analytic in* $\mathbb{D}$ *and suppose that*

$$\left|\frac{zp'(z)}{p(z)}\right| < \frac{\alpha}{2\left(\frac{1+\alpha}{2}\right)^{(1+\alpha)/2} + \left(\frac{1-\alpha}{2}\right)^{(1-\alpha)/2}}, \quad (z \in \mathbb{D}), \tag{2.17}$$

for some positive real $\alpha < 1$. *Then we have* $p(z) = e + d_1 z + \cdots$ *and*

$$p(z) \prec e^{\left(\frac{1+z}{1+z}\right)^{\alpha}}, \quad (z \in \mathbb{D}). \tag{2.18}$$

Proof. Let us put

$$w(z) = \frac{(\log\{p(z)\})^{1/\alpha} - 1}{(\log\{p(z)\})^{1/\alpha} + 1}, \qquad w(0) = 0 \tag{2.19}$$

or

$$p(z) = e^{\left(\frac{1+w(z)}{1-w(z)}\right)^{\alpha}}, \tag{2.20}$$

then it follows that $w(z)$ is analytic in $\mathbb{D}$, $w(0) = 0$ and

$$\begin{aligned}\frac{zp'(z)}{p(z)} &= \alpha z \left(\frac{1+w(z)}{1-w(z)}\right)' \left(\frac{1+w(z)}{1-w(z)}\right)^{\alpha-1} \\ &= \frac{2\alpha z w'(z)}{(1-w(z))^2} \left(\frac{1+w(z)}{1-w(z)}\right)^{\alpha-1} \\ &= \frac{2\alpha z w'(z)}{1-w^2(z)} \left(\frac{1+w(z)}{1-w(z)}\right)^{\alpha}.\end{aligned}$$

To prove (2.18) we need $|w(z)| < 1$ in $\mathbb{D}$. If there exists a point $z_0 \in \mathbb{D}$ such that

$$|w(z)| < 1 \quad \text{for} \quad |z| < |z_0|$$

and

$$|w(z_0)| = 1, \quad w(z_0) = e^{i\theta}, \tag{2.21}$$

for some real θ, $\theta \in [0, 2\pi) \setminus \{0, \pi\}$ because from the hypothesis and from (2.19) it follows that $w(z_0) \neq \pm 1$. Then from Jack [5] and Fukui and Sakaguchi's [4] Lemma 1.1, we have that

$$\frac{z_0 w'(z_0)}{w(z_0)} = k \geq 1.$$

Then it follows that

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{2\alpha z_0 w'(z_0)}{1-w^2(z_0)} \left(\frac{1+w(z_0)}{1-w(z_0)}\right)^{\alpha} = 2\alpha k \frac{w(z_0)}{1-w^2(z_0)} \left(\frac{1+w(z_0)}{1-w(z_0)}\right)^{\alpha}$$

and for $\theta \in (0, 2\pi) \setminus \{\pi\}$

$$\begin{aligned}\frac{w(z_0)}{1-w^2(z_0)} &= \frac{e^{i\theta}}{1-e^{2i\theta}} = \frac{i}{2\sin\theta}, \\ \frac{1+w(z_0)}{1-w(z_0)} &= \frac{2i\sin\theta}{2(1-\cos\theta)} = i\frac{\cos(\theta/2)}{\sin(\theta/2)}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\left|\frac{2z_0w'(z_0)}{1-w^2(z_0)}\left(\frac{1+w(z_0)}{1-w(z_0)}\right)^{\alpha}\right| &= \left|\frac{i}{2\sin\theta}\right|\left|i\frac{\cos(\theta/2)}{\sin(\theta/2)}\right|^{\alpha}\\ &= \frac{1}{2}\left|\frac{1}{(\sin(\theta/2))(\cos(\theta/2))}\right|\left|\frac{\cos(\theta/2)}{\sin(\theta/2)}\right|^{\alpha}\\ &= \frac{1}{2}\frac{1}{(|\sin(\theta/2)|^{1+\alpha}|\cos(\theta/2)|^{1-\alpha}}.\end{aligned}$$

Putting

$$\begin{aligned}g(x) &= (\sin x)^{1+\alpha}(\cos x)^{1-\alpha}, \quad 0<x<\pi/2\\ h(x) &= (\sin x)^{1+\alpha}(-\cos x)^{1-\alpha}, \quad \pi/2<x<\pi\end{aligned}$$

shows that

$$\begin{aligned}g'(x) &= (1+\alpha)\left(\frac{\sin x}{\cos x}\right)^{\alpha}\left\{\cos^2 x-\frac{1-\alpha}{1+\alpha}\sin^2 x\right\}\\ &= (1+\alpha)\left(\frac{\sin x}{\cos x}\right)^{\alpha}\left\{1-\frac{2}{1+\alpha}\sin^2 x\right\}.\end{aligned}$$

Therefore, for $0\le x<\pi/2$ we have

$$g'(x)=0 \;\Leftrightarrow\; \left(\sin x=0 \;\vee\; \sin x=\sqrt{\frac{1+\alpha}{2}}\right)$$

which gives

$$g'(x)=0 \;\Leftrightarrow\; \left(x=0 \quad\vee\; x=\sin^{-1}\sqrt{\frac{1+\alpha}{2}}\right).$$

It follows that

$$\max_{0<x<\pi/2}|g(x)| = \left(\frac{1+\alpha}{2}\right)^{(1+\alpha)/2}+\left(\frac{1-\alpha}{2}\right)^{(1-\alpha)/2}.$$

Also

$$\max_{\pi/2<x<\pi}|h(x)| = \left(\frac{1+\alpha}{2}\right)^{(1+\alpha)/2}+\left(\frac{1-\alpha}{2}\right)^{(1-\alpha)/2}.$$

Therefore, we have

$$\left|\frac{z_0p'(z_0)}{p(z_0)}\right| \ge \frac{\alpha k}{2\left(\frac{1+\alpha}{2}\right)^{(1+\alpha)/2}\left(\frac{1-\alpha}{2}\right)^{(1-\alpha)/2}} \ge \frac{\alpha}{2\left(\frac{1+\alpha}{2}\right)^{(1+\alpha)/2}\left(\frac{1-\alpha}{2}\right)^{(1-\alpha)/2}}.$$

This contradicts (2.17) and so, we have (2.18). □

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