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THE DIRICHLET PROBLEM FOR NONLINEAR ELLIPTIC EQUATIONS WITH VARIABLE EXPONENT

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Abstract In this paper we study the Dirichlet problem for nonlinear elliptic equations with variable exponents in Sobolev spaces with variable exponent. We show that for every continuous function g on the boundary there exists a unique continuous extension of g.

Keywords $p(.)$-Laplacian operator, potential theory, Sobolev spaces with variable exponent.

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1. Introduction

The aim of this work is to consider the problem

$$\begin{cases} -\Delta_{p(.)}u + \mathcal{B}(x,u) = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded regular domain, g is a continuous function on the boundary of Ω and $\mathcal{B} : \Omega \times \mathbb{R} \to \mathbb{R}$ is a given Carathéodory functions. Recall that the operator $\Delta_{p(.)}$ is defined by

$$\Delta_{p(.)}u := div\left(|\nabla u|^{p(.)-2}\nabla u\right)$$

where $p : \Omega \to [1, \infty)$ is a measurable function in Ω called *the variable exponent*. Hence our goal is to show the existence of a function $u \in \mathcal{W}^{1,p(.)}(\Omega)$, the corresponding Sobolev space, such that $u - g \in \mathcal{W}_0^{1,p(.)}(\Omega)$ and $-\Delta_{p(.)}u + \mathcal{B}(x,u) = 0$ in Ω.

It is clear that $\Delta_{p(.)}$ can be seen as a natural extension of the classical $p-$Laplacian operator in p is a constant, however equations with variable exponent are also used to study electrorheological fluids as described in [25].

In the same way and when dealing with study of variational integrals with non-standard growth, the next perturbed equation appears in a natural way,

$$-\Delta_{p(.)}u + \mathcal{B}(x,u) = 0. \tag{1.1}$$

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We refer to [1,20,21,29] for more details in this case. Notice that under suitable hypotheses on $\mathcal{B}$, the authors in [15] were able to show the existence of a weak solution.

Taking into consideration the variational structure of the equation, the $p(.)$-Dirichlet energy minimizing problem was studied by Fan and Zhang in [16]. More precisely, setting

$$J(u) = \int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} dx - \int_{\Omega} F(x,u)\, dx. \tag{1.2}$$

then the authors in [16] where able to show that the regularity of minimizer of J (under suitable hypotheses on F) and then they extend the classical result by Brezis and Nirenberg in [9] to the variable exponent case, that is, any local minimizers in the $\mathcal{C}^1$ topology to J is also a local minimizers of J in the $\mathcal{W}^{1,p(.)}$ topology. For nonvariational case, see [2]. Also, the nonlinear elliptic equations associated with variable exponent has recently attracted attention. For a survey of recent results in the field we refer to [3, 4, 17, 18, 22, 24, 26, 28]. See also [23, 27] for anisotropic equations.

It is clear that one of the main difficulty when dealing with the $\Delta_{p(.)}$ operator in the fact that when u satisfies $-\Delta_{p(.)}u = 0$, then λu, for $\lambda \in \mathbb{R}^*$, does not satisfy the same equation.

For simplicity of typing we set

$$\mathcal{L}u := -\Delta_{p(.)}u + \mathcal{B}(x,u), \tag{1.3}$$

then the main result of this paper is to show that, under suitable hypotheses on $\mathcal{B}$, we get the existence of a solution to the problem $\mathcal{L}u = 0$ in Ω with a continuous boundary data.

The main result of this paper is the following theorem.

Theorem 1.1. *Let Ω be a bounded regular domain and $g \in \mathcal{C}(\partial\Omega)$. Suppose that the next assumptions hold*

(H_1) $|\mathcal{B}(x,\zeta)| \leq a(x) + C|\zeta|^{p(x)-1}$ *a.e. x in Ω and for all $\zeta \in \mathbb{R}^d$, where $a : \Omega \to \mathbb{R}^+$ is a measurable function lying in $L^{p'(.)}(\Omega)$ and $C > 0$.*

(H_2) $\zeta \to \mathcal{B}(x,\zeta)$ *is increasing function for every $x \in \Omega$.*

Then there exists a unique continuous extension $w^g_{p(.)}$ of g in $\overline{\Omega}$.

A typical example when the assumptions (H_1) and (H_2) hold is $\mathcal{B}(x,u) = |u|^{p(x)-2}u$.

To prove the main result we will follows by approximation as in [**?**]. It is clear that our approach is different than the one used in [**?**].

In fact, using uniform approximations, it is possible to define solutions in the Sobolev space including for continuous boundary values, and these turn out to coincide with the corresponding Brelot-Perron-Wiener solutions.

This paper is organized as follows. In Section 2 we introduce some preliminary results, including the variable Lebesgue and Sobolev spaces and some of their properties.

The $p(.)$-Poisson problem is discussed in Section 3. Duality arguments are used in order to show the existence and the uniqueness of the weak solution. In Subsection 3.1 we give the definition of weak sub and super-solutions, the comparison principle and some properties of the solution to the $p(.)$-Poisson problem as the Hölder continuity of the solution.

In Section 4 we treat the case of the operator $\mathcal{L}$ defined above. We begin by considering datum in the Sobolev space $\mathcal{W}^{1,p(\cdot)}(\Omega)$ and small domain Ω. Then using approximating arguments we get the extension result without any smallness condition on the domain. The main extension result is proved in Section 5.

2. Preliminaries and functional setting

We begin by defining the variable exponent Lebesgue and Sobolev spaces, we refer to [?, ?, 12, 24] for more properties of these spaces.

Let $p : \Omega \to [1, \infty)$ be a measurable function (called *the variable exponent* on Ω), we set $p^+ = \operatorname{ess\,sup}_{x\in\Omega} p(x)$ and $p^- = \operatorname{ess\,inf}_{x\in\Omega} p(x)$.

The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) = \left\{ \begin{array}{c} u : \Omega \to \mathbb{R} \text{ measurable, } \rho_{p(.)}(\lambda u) = \int_\Omega |\lambda u|^{p(x)}\, dx < \infty \\ \text{for some } \lambda > 0. \end{array} \right\}$$

The function $\rho_{p(.)} : L^{p(\cdot)}(\Omega) \to [0, \infty)$ is called *the modular* of the space $L^{p(\cdot)}(\Omega)$. We define a norm, the so-called *Luxembourg norm*, in this space by

$$\|u\|_{p(.)} = \inf\left\{ \lambda > 0 : \rho_{p(.)}\left(\frac{u}{\lambda}\right) \le 1 \right\}.$$

As in the classical case, the *dual variable exponent* function p' of p is given by $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ and dual space for $L^{p(\cdot)}(\Omega)$ is $L^{p'(\cdot)}(\Omega)$.

In a natural way we define the *variable exponent Sobolev space* $\mathcal{W}^{1,p(\cdot)}(\Omega)$ by

$$\mathcal{W}^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega),\ \ |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}.$$

The space $\mathcal{W}^{1,p(\cdot)}(\Omega)$ is endowed with the norm

$$\|u\|_{1,p(.)} = \|u\|_{p(.)} + \|\nabla u\|_{p(.)}.$$

It is not difficult to show that $\mathcal{W}^{1,p(\cdot)}(\Omega)$ is a Banach space.

The Sobolev exponent p^* of p is $p^*(x) = \frac{dp(x)}{d-p(x)}$ if $p(x) < d$ and $p^*(x) = \infty$ otherwise.

Now, the space $\mathcal{W}_0^{1,p(\cdot)}(\Omega)$ (the variable exponent Sobolev space with zero boundary values) is defined as the closure of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm of $\mathcal{W}^{1,p(\cdot)}(\Omega)$. The dual space of $\mathcal{W}_0^{1,p(\cdot)}(\Omega)$ will be denote by $\mathcal{W}^{-1,p'(\cdot)}(\Omega)$.

As in the classical case, we have the next Sobolev inequality.

Theorem 2.1. *Assume that $u \in \mathcal{W}_0^{1,p(\cdot)}(\Omega)$, then there exists a positive constant C that depends only on the variable exponent p such that*

$$||u||_{p^*(.)} \le C||u||_{\mathcal{W}_0^{1,p(.)}(\Omega)}.$$

To develop a regularity theory for the equation $\mathcal{L}u = 0$, we will assume that the variable exponent p satisfies the logarithmic Hölder continuity condition introduced by Zhikov in [**?**], namely we suppose that

$$(H_3) : |p(x) - p(y)| \leq \frac{C_1}{-\log|x-y|} \text{ for } (x,y) \in \Omega \times \Omega \text{ with } |x-y| < \frac{1}{2},$$

where $C_1 > 0$ is independent of (x, y). Under this assumption and as it was observed in [**?**], the space of smooth functions is dense in the variable exponent Sobolev space. We refer also to the monograph [12] for the details in this direction.

For technical reasons, we will also assume that $1 < p^- \leq p^+ < d$.

3. Existence and regularity results for the $p(.)$-Poisson problem

The Poisson problem is one of the most classical problems treated in the theory of partial differential equations. Beside its importance in itself it is also very used as an auxiliary problem in the treatment of nonlinear problems. In this section, we are concerned with the existence, uniqueness of the weak solution to $p(.)$-Poisson problem.

More precisely for $f \in L^{p^{*'}(.)}(\Omega)$, we will consider the following problems

$$\begin{cases} \Delta_{p(.)} u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

To define weak solution of (3.1), we set

$$a(u, \varphi) = \int_\Omega |\nabla u|^{p(x)-2} \nabla u . \nabla \varphi \, dx$$

where $\varphi \in \mathcal{W}_0^{1,p(.)}(\Omega)$.

Definition 3.1. Let $f \in \mathcal{W}^{-1,p'(.)}(\Omega)$. We say that u is a weak solution of (3.1) if $u \in \mathcal{W}_0^{1,p(.)}(\Omega)$ and $a(u, \varphi) = \langle f, \varphi \rangle$ for all $\varphi \in \mathcal{W}_0^{1,p(.)}(\Omega)$. Where $\langle ., . \rangle$ denotes duality product on $\mathcal{W}_0^{1,p(.)}(\Omega)$.

Note that the definition of a weak solution makes sense only if $f \in \mathcal{W}^{-1,p'(.)}(\Omega)$. By Sobolev inequality, we have $\mathcal{W}_0^{1,p(.)}(\Omega) \hookrightarrow L^{p^*(.)}(\Omega)$ and by duality,
$L^{p^{*'}(.)}(\Omega) \hookrightarrow \mathcal{W}^{-1,p'(.)}(\Omega)$.

On the other hand, since $|\Omega| < \infty$, then $L^{r(.)}(\Omega) \hookrightarrow L^{p^{*'}(.)}(\Omega)$ if $r(x) \geq p^{*'}(x)$ *a.e.* in Ω.

Hence we consider (3.1) for $f \in L^{r(.)}(\Omega)$ with r as above. Let us begin by the next auxiliary results.

Theorem 3.1. *Assume that $p^{*'}(x) \leq r(x) < \infty$ a.e. in Ω. Then for all $f \in L^{r(.)}(\Omega)$, there exists a unique weak solution $u \in \mathcal{W}_0^{1,p(.)}(\Omega)$ to the Poisson problem (3.1) such that*

$$\|u\|_{p^*(.)}^{p(x)-1} \leq C \|f\|_{r(.)}, \tag{3.2}$$

where $C > 0$ is independent of f.

Proof. The existence and the uniqueness of a weak solution $u \in \mathcal{W}_0^{1,p(.)}(\Omega)$ is a direct consequence of [**?**]. Let prove now the estimate (3.2). In what follows, we denote by $C_1, C_2, ..$ any generic constant which may vary from line to line and that is independent of f. By the definition of u we have

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla u \, dx = \int_\Omega f u \, dx.$$

Hence

$$\rho_{p(.)}(\nabla u) = \int_\Omega f u \, dx.$$

Since u is a weak solution and $f \in L^{r(\cdot)}(\Omega)$, then by Lemma 3.2.20 in [12], it holds

$$\rho_{p(.)}(\nabla u) \le 2 \|f\|_{r(.)} \|u\|_{r'(.)}.$$

Thus, as in Lemma 3.2.4 in [12], we reach that

$$\|\nabla u\|_{p(.)}^{p^+} \le 2 \|f\|_{r(.)} \|u\|_{r'(.)}.$$

By Sobolev inequality, we get the existence of a positive constant C_1 such that

$$\|u\|_{p^*(.)} \le C_1 \|\nabla u\|_{p(.)}.$$

Taking into consideration that $p^+ > 1$,

$$\|u\|_{p^*(.)}^{p^+} \le (C_1)^{p^+} \|\nabla u\|_{p(.)}^{p^+} \le 2 (C_1)^{p^+} \|f\|_{r(.)} \|u\|_{r'(.)}.$$

Since $p^*(x) \ge r'(x)$ a.e. in Ω, then

$$\|u\|_{r'(.)} \le C_2 \|u\|_{p^*(.)}.$$

Hence

$$\|u\|_{p^*(.)}^{p^+-1} \le 2 (C_1)^{p^+} C_2 \|f\|_{r(.)}, \tag{3.3}$$

whenever $\|\nabla u\|_{p(.)} \le 1$. On the other hand, if $\|\nabla u\|_{p(.)} \ge 1$, then

$$\|u\|_{p^*(.)}^{p^--1} \le 2 (C_1)^{p^-} C_2 \|f\|_{r(.)}. \tag{3.4}$$

Combining inequalities (3.3) and (3.4), it holds,

$$\begin{aligned} \|u\|_{p^*(.)}^{p(x)-1} &\le \max\left\{ \|u\|_{p^*(.)}^{p^+-1}, \|u\|_{p^*(.)}^{p^--1} \right\} \\ &\le 2C_2 \max\left\{ (C_1)^{p^+}, (C_1)^{p^-} \right\} \|f\|_{r(.)}. \end{aligned}$$

Hence

$$\|u\|_{p^*(.)}^{p(x)-1} \le C \|f\|_{r(.)}$$

where $C := 2C_2 \max\left\{ (C_1)^{p^+}, (C_1)^{p^-} \right\}$. $\square$

As a consequence we get the next result.

Lemma 3.1. *Assume that $f \in L^{r(.)}$ and define the operator*

$$\mathcal{T} : L^{r(.)}(\Omega) \to L^{p(.)}(\Omega)$$

where $\mathcal{T}(f) = u_f$ is the solution of the problem (3.1), then $\mathcal{T}$ is completely continuous.

Proof. It is not difficult to show that $\mathcal{T}$ is continuous. Now taking into consideration that the inclusion $\mathcal{W}_0^{1,p(.)}(\Omega) \hookrightarrow L^{p(.)}(\Omega)$ is compact, we get easily the desired result. □

3.1. $\mathcal{L}$-solutions and $\mathcal{L}$-sub-super-solution

In this subsection we will be discuss the solution and sub-super-solution of nonlinear partial differential equation

$$\mathcal{L}u := -\Delta_{p(.)}u + \mathcal{B}(x,u) = 0 \text{ in } \Omega. \tag{3.5}$$

and we prove some useful properties of the solution to the above equation as locally Hölder continuity uniform boundlessness of the gradient of a such solutions in the space $\mathcal{W}^{1,p(.)}(\Omega)$.

Let us begin by the next definitions.

Definition 3.2. Assume that r is as in Theorem 3.1. We say that $u \in \mathcal{W}^{1,p(.)}(\Omega)$ is a weak $\mathcal{L}$-solution of (3.5), if $\mathcal{B}(x,u) \in L^{r(.)}(\Omega)$ and

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u.\nabla\varphi dx + \int_\Omega \mathcal{B}(x,u)\varphi dx = 0$$

for all $\varphi \in \mathcal{W}_0^{1,p(.)}(\Omega)$.

We say that a function $u \in \mathcal{W}^{1,p(.)}(\Omega)$ is a weak $\mathcal{L}$-super-solution (resp. $\mathcal{L}$-sub-solution) of (3.5), if $\mathcal{B}(x,u) \in L^{r(.)}(\Omega)$ and

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u.\nabla\varphi dx + \int_\Omega \mathcal{B}(x,u)\varphi dx \geq 0 \text{ (resp. } \leq 0)$$

for every nonnegative function $\varphi \in \mathcal{W}_0^{1,p(.)}(\Omega)$.

Proposition 3.1. *Let u and v be two $\mathcal{L}$-sub-solutions of (3.5) in Ω, then* $\max(u,v)$ *is also a $\mathcal{L}$-sub-solution. A similar statement holds for the minimum of two $\mathcal{L}$-super-solution.*

Proof. Let u, v be two $\mathcal{L}$-sub-solutions of (3.5) in Ω. Fix $\varphi \in \mathcal{C}_0^\infty(\Omega)$, $\varphi \geq 0$ and define $\Omega_1 = \{x \in \Omega, u > v\}$, $\Omega_2 = \{x \in \Omega, u \leq v\}$.

We have

$$\nabla(\max(u,v)) = \begin{cases} \nabla u(x), \text{ for a.e. } x \in \Omega_1, \\ \nabla v(x), \text{ for a.e. } x \in \Omega_2. \end{cases}$$

Hence

$$\begin{aligned} &\int_\Omega |\nabla \max(u,v)|^{p(x)-2} \nabla \max(u,v).\nabla\varphi dx \\ &= \int_{\Omega_1} |\nabla u|^{p(x)-2} \nabla u.\nabla\varphi dx + \int_{\Omega_2} |\nabla v|^{p(x)-2} \nabla v.\nabla\varphi dx. \end{aligned}$$

It is clear that

$$\begin{aligned}
I &= \int_{\Omega} |\nabla \max(u,v)|^{p(x)-2} \nabla \max(u,v) . \nabla\varphi dx \\
&= \int_{\Omega_1} |\nabla u|^{p(x)-2} \nabla u . \nabla\varphi dx + \int_{\Omega_2} |\nabla v|^{p(x)-2} \nabla v . \nabla\varphi dx \\
&= I_1 + I_2 .
\end{aligned}$$

Let $\phi_n \in \mathcal{C}^1(\mathbb{R})$ be a cutoff function defined by

$$\phi_n(t) = \begin{cases} 1 \text{ if } \ t \geq \frac{1}{n}; \\ 0 \text{ if } \ t \leq 0; \end{cases}$$

with $\phi_n^{'} > 0$ on $\left]0, \frac{1}{n}\right[$.

Consider $q_n(x) = \phi_n((u-v)(x))$, then $q_n \in \mathcal{W}^{1,p(.)}(\Omega)$, and

$$\nabla q_n = \begin{cases} \phi_n^{'}(u-v) \nabla (u-v) \ \text{ if } \ u - v \geq \frac{1}{n}, \\ \qquad 0 \text{ if } \ u-v \leq 0. \end{cases}$$

It is clear that $q_n \to \chi_{\Omega_1}$, $\|q_n\|_\infty \leq 1$ a.e. in Ω_1 and $1 - q_n \to \chi_{\Omega_2}$, $\|1-q_n\|_\infty \leq 2$ a.e. in Ω_2.

By the dominated convergence theorem, it follows that

$$I_1 = \lim_{n\to\infty} \int_{\Omega_1} q_n |\nabla u|^{p(x)-2} \nabla u . \nabla\varphi \, dx$$

and

$$I_2 = \lim_{n\to\infty} \int_{\Omega_2} (1-q_n) |\nabla v|^{p(x)-2} \nabla v . \nabla\varphi \, dx .$$

Hence

$$\begin{aligned}
&\int_{\Omega} q_n |\nabla u|^{p(x)-2} \nabla u . \nabla\varphi dx \\
&= \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u . \nabla(q_n\varphi) \, dx - \int_{\Omega} \varphi |\nabla u|^{p(x)-2} \nabla u . \nabla(q_n) \, dx \\
&\leq - \int_{\Omega} \mathcal{B}(x,u)(q_n\varphi) \, dx - \int_{\Omega_n} \varphi |\nabla u|^{p(x)-2} \nabla u . \nabla(q_n) \, dx
\end{aligned}$$

where $\Omega_n = \left\{x \in \Omega, v < u < v + \frac{1}{n}\right\}$.

Setting

$$I_n = \int_{\Omega} q_n |\nabla u|^{p(x)-2} \nabla u . \nabla\varphi dx$$

and

$$J_n = \int_{\Omega} (1-q_n) |\nabla v|^{p(x)-2} \nabla v . \nabla\varphi dx,$$

in a similar way

$$\begin{aligned}
&\int_{\Omega} (1-q_n) |\nabla v|^{p(x)-2} \nabla v . \nabla\varphi dx \\
&\leq - \int_{\Omega} (1-q_n) \mathcal{B}(x,v) \varphi dx + \int_{\Omega_n} \left(|\nabla v|^{p(x)-2} \nabla v\right) \varphi . \nabla(q_n) \, dx .
\end{aligned}$$

Hence, we get

$$\begin{aligned}
I_n + J_n \leq & -\int_{\Omega} \mathcal{B}(x,u)(q_n\varphi)\,dx - \int_{\Omega}(1-q_n)\,\mathcal{B}(x,v)\,\varphi dx \\
& + \int_{\Omega_n}\left(|\nabla v|^{p(x)-2}\nabla v - |\nabla u|^{p(x)-2}\nabla u\right)\varphi.\nabla(q_n)\,dx \\
\leq & -\int_{\Omega} \mathcal{B}(x,u)(q_n\varphi)\,dx - \int_{\Omega}(1-q_n)\,\mathcal{B}(x,v)\,\varphi dx \\
& + \int_{\Omega_n}\varphi\phi_n^{'}(u-v)\left(|\nabla v|^{p(x)-2}\nabla v - |\nabla u|^{p(x)-2}\nabla u\right).(\nabla(u-v))\,dx.
\end{aligned}$$

From Proposition 17.3 in [**?**], we deduce that

$$\left(|\nabla v|^{p(x)-2}\nabla v - |\nabla u|^{p(x)-2}\nabla u\right).(\nabla(u-v)) \leq 0.$$

Using the fact that $\varphi \geq 0$ and $\phi_n^{'} > 0$, it holds

$$I_n + J_n \leq -\int_{\Omega} \mathcal{B}(x,u)(q_n\varphi)\,dx - \int_{\Omega}(1-q_n)\,\mathcal{B}(x,v)\,\varphi dx.$$

Hence

$$\int_{\Omega}|\nabla \max(u,v)|^{p(x)-2}\nabla \max(u,v).\nabla\varphi dx + \int_{\Omega}\mathcal{B}(x,\max(u,v))\,\varphi dx \leq 0$$

for all $\varphi \in \mathcal{C}_0^{\infty}(\Omega)$, $\varphi \geq 0$. Since the space of smooth functions is dense in the Sobolev space, we conclude that

$$\int_{\Omega}|\nabla \max(u,v)|^{p(x)-2}\nabla \max(u,v).\nabla\varphi dx + \int_{\Omega}\mathcal{B}(x,\max(u,v))\,\varphi dx \leq 0$$

for all $\varphi \in \mathcal{W}_0^{1,p(.)}(\Omega)$, $\varphi \geq 0$, which completes the proof. □

Remark 3.1. From (H_2), we obtain that for all $k > 0$, if u is a $\mathcal{L}$-super-solution (resp. $\mathcal{L}$-sub-solution) to (3.5), then $u + k$ (resp. $u - k$) is also a $\mathcal{L}$-super-solution (resp. $\mathcal{L}$-sub-solution) to (3.5). If $C \in \mathbb{R}^d$, then $C - u$ and $u - C$ solve (3.5).

One of the main properties of the solutions of the equation (3.5) is the fact that any non-negative $\mathcal{L}$-solution in an open set Ω satisfies the Harnack's inequality

$$\operatorname{ess\,sup}_{x\in B_R} u(x) \leq C\left(\operatorname{ess\,inf}_{x\in B_R} u(x) + R\right)$$

with $C > 0$ independent of u and B_R is an open ball such that $B_{4R} \subset \Omega$, see [**?**] for more details.

It is clear that the Harnack's inequality holds for any compact K.

Harnack's inequality can be iterated to obtain the local Hölder continuity of $\mathcal{L}-$solutions. The proof of Harnack's inequality is based on two weak Harnack estimates. One (the infimum estimate) holds for $\mathcal{L}$-super-solutions and the other (the supremum estimate) for $\mathcal{L}$-sub-solutions.

For a measurable set $E \subset \Omega$, we denote $\operatorname{osc}_{x\in E} u = \operatorname{ess\,sup}_{x\in E} u(x) - \operatorname{ess\,inf}_{x\in E} u(x)$.

As a consequence we get the next result.

Proposition 3.2. *Let u be a weak $\mathcal{L}$-solution of (3.5). Assume that $B(x,R) \subset \Omega$ and fix $r < R$, then*

$$\operatorname{osc}_{B(x,r)} u \leq 2^k \left(\frac{r}{R}\right)^k \operatorname{osc}_{B(x,R)} u + K(r)$$

where $k \in [0,1]$.

Proof. We set

$$m(r) := \operatorname{ess\,inf}_{B(r)} u \text{ and } M(r) := \operatorname{ess\,sup}_{B(r)} u.$$

By applying the Harnack inequality respectively to the nonnegative functions $u - m(r)$ and $M(r) - u$, we obtain

$$M\left(\frac{r}{2}\right) - m(r) \leq C\left(m\left(\frac{r}{2}\right) - m(r) + \frac{r}{2}\right),$$

and

$$M(r) - M\left(\frac{r}{2}\right) \leq C\left(M(r) - M\left(\frac{r}{2}\right) + \frac{r}{2}\right).$$

Adding this inequalities, we get

$$M(r) - m(r) \leq C\left(m\left(\frac{r}{2}\right) - m(r) + \frac{r}{2} + M(r) - M\left(\frac{r}{2}\right) + \frac{r}{2}\right).$$

Thus

$$\operatorname{osc}_{B\left(x,\frac{r}{2}\right)} u \leq \frac{C-1}{C} \operatorname{osc}_{B(x,r)} u + Cr.$$

To complete the proof, we iterate this inequality.

Let $m \in \mathbb{N}^*$ be such that $2^{m-1} \leq \frac{R}{r} < 2^m$, then in a continuous way, we get

$$\operatorname{osc}_{B(x,2^{m-2}r)} u \leq \frac{C-1}{C} \operatorname{osc}_{B(x,2^{m-1}r)} u + 2^{m-1}Cr \leq \frac{C-1}{C} \operatorname{osc}_{B(x,R)} u + CR.$$

Hence

$$\operatorname{osc}_{B(x,r)} u \leq \left(\frac{C-1}{C}\right)^{m-1} \operatorname{osc}_{B(x,R)} u + r\sum_{j=2}^{m-2}\left(2^j (C-1)^{j-1}\right) + (3C-1)R.$$

We set $k = -\frac{\ln\left(\frac{C-1}{C}\right)}{\ln(2)} \leq 1$ where $C \geq 1$, then

$$\sum_{j=2}^{m-2}\left(2^j (C-1)^{j-1}\right) = \begin{cases} -\frac{4}{2C-3}(C-1) + \frac{(2C-2)^{m-1}}{(2C-3)(C-1)} & \text{if } C \neq \frac{3}{2}, \\ 2m-6 & \text{if } C = \frac{3}{2}. \end{cases}$$

Therefore, if $C \neq \frac{3}{2}$, it holds

$$\begin{aligned} \operatorname{osc}_{B(x,r)} u \leq{}& \left(\frac{C-1}{C}\right)^{m-1} \operatorname{osc}_{B(x,R)} u + r\left(-\frac{4}{2C-3}(C-1) + \frac{(2C-2)^{m-1}}{(2C-3)(C-1)}\right) \\ &+ (3C-1)R \\ \leq{}& 2^k\left(\frac{r}{R}\right)^k \operatorname{osc}_{B(x,R)} u + r\left(-\frac{4}{2C-3}(C-1) + \frac{(2C-2)^{m-1}}{(2C-3)(C-1)}\right) \\ &+ (3C-1)R. \end{aligned}$$

If $C=\frac{3}{2}$, we have

$$\begin{aligned}\operatorname{osc}_{B(x,r)} u &\leq \left(\frac{C-1}{C}\right)^{m-1} \operatorname{osc}_{B(x,R)} u + r\,(2m-6) + (3C-1)\,R \\ &\leq 2^k \left(\frac{r}{R}\right)^k \operatorname{osc}_{B(x,R)} u + r\,(2m-6) + (3C-1)\,R.\end{aligned}$$

Let $K\,(r)$ be a constant such that

$$K\,(r) = \begin{cases} r\,(2m-6) + (3C-1)\,R & \text{if } C=\frac{3}{2}, \\ r\left(-\frac{4}{2C-3}\,(C-1) + \frac{(2C-2)^{m-1}}{(2C-3)(C-1)}\right) + (3C-1)\,R & \text{if } C\neq\frac{3}{2}. \end{cases}$$

Hence

$$\operatorname{osc}_{B(x,r)} u \leq 2^k \left(\frac{r}{R}\right)^k \operatorname{osc}_{B(x,R)} u + K\,(r)\,.$$

□

Lemma 3.2. *Let u be a $\mathcal{L}$-solution of (3.5). Then u is locally uniformly bounded in $\mathcal{W}^{1,p(.)}\,(\Omega)$.*

Proof. Let $\eta \in \mathcal{C}_0^\infty\,(\Omega)\,,\ 0\leq\eta\leq 1$ and $\eta=1$ on $\omega\subset\overline{\omega}\subset\Omega$. Using $\varphi=\eta u\in\mathcal{W}_0^{1,p(.)}\,(\Omega)$ as a test function in (3.5), it follows that

$$\begin{aligned}0 &= \int_\Omega |\nabla u|^{p(x)-2}\,\nabla u\,(u\nabla\eta+\eta\nabla u)\,dx + \int_\Omega \mathcal{B}\,(x,u)\,\eta u dx \\ &= \int_\Omega |\nabla u|^{p(x)-2}\,\nabla u\,(u\nabla\eta)\,dx + \int_\Omega \eta\,|\nabla u|^{p(x)}\,dx + \int_\Omega \mathcal{B}\,(x,u)\,\eta u dx.\end{aligned}$$

Hence

$$\int_\Omega \eta\,|\nabla u|^{p(x)}\,dx \leq \int_\Omega |\nabla u|^{p(x)-1}\,|u|\,|\nabla\eta|\,dx + \int_\Omega |\mathcal{B}\,(x,u)|\,|\eta|\,|u|\,dx,$$

and therefore

$$\begin{aligned}\int_\omega |\nabla u|^{p(x)}\,dx &\leq \int_\omega |\mathcal{B}\,(x,u)|\,|u|\,dx \\ &\leq \int_\omega \left(a\,(x) + c\,|u|^{p(x)-1}\right)|u|\,dx \\ &\leq \int_\omega a\,(x)\,|u|\,dx + c\int_\omega |u|^{p(x)}\,dx \\ &\leq \int_\omega a\,(x)\,|u|\,dx + c\int_\omega \left(|u|^{p^+} + |u|^{p^-}\right)dx.\end{aligned}$$

By Proposition 3.2, we have $|u|\leq M$ on ω, where M is a positive real. From Hölder's inequality, we get

$$\int_\omega |\nabla u|^{p(x)}\,dx \leq C\left(M, |\omega|\,, \|a\|_{p'(.)}\right)$$

and this complete the proof. □

Remark 3.2. When $|\omega|$ is small enough, we can choose the constant C independent of ω.

We close this subsection by recalling the following fundamental comparison principle. Hence we can solve the Dirichlet problem in our case once proving the existence of sub and super-solutions.

Theorem 3.2 ([?]). *Let u be a $\mathcal{L}-$super-solution and v is a $\mathcal{L}-$sub-solution of (3.5), on Ω, such that*

$$\limsup_{x\to y} v(x) \leq \liminf_{x\to y} u(x)$$

for all $y\in\partial\Omega$ and both sides of the inequality are not simultaneously $+\infty$ or $-\infty$, then $v\leq u$ in Ω.

4. $\mathcal{L}$-variational Dirichlet problem

In this section we look for the existence and uniqueness of variational Dirichlet problem

$$\begin{cases} \mathcal{L}u := -\Delta_{p(.)}u + \mathcal{B}(x,u) = 0 \text{ in } \Omega, \\ u - g \in \mathcal{W}_0^{1,p(.)}(\Omega), \end{cases} \tag{4.1}$$

under the Assumption (H_1), (H_2) and (H_3).

From Theorem 3.1, we know that for arbitrary all $f\in L^{r(.)}(\Omega)$, there exists a unique weak solution $u_f\in\mathcal{W}_0^{1,p(.)}(\Omega)$ of the Poisson problem (3.1) which satisfies the estimate

$$\|u_f\|_{p^*(.)}^{p(x)-1} \leq C\,\|f\|_{r(.)},$$

where the constant C is independent of f.

Setting $\widetilde{u} = u - g$ and $\widetilde{\mathcal{B}}(x,u) = \mathcal{B}(x,u+g)$, then u is a solution of the problem (4.1) if and only if $\widetilde{u}$ is a solution of

$$\begin{cases} \widetilde{\mathcal{L}}u \equiv \Delta_{p(x)}(\widetilde{u}\ + g) + \widetilde{\mathcal{B}}(x,u) = 0 \text{ in } \Omega, \\ \widetilde{u}\in\mathcal{W}_0^{1,p(.)}(\Omega). \end{cases} \tag{4.2}$$

Using (H_1), we get

$$\left|\widetilde{\mathcal{B}}(x,u)\right| \leq b(x) + C'\,|u|^{p(x)-1}$$

where $b(x) = a(x) + \left(1+\frac{1}{\epsilon}\right)^{p^+-1}|g|^{p(x)-1}$ and $C' = C(1+\epsilon)^{p^+-1} > 1$. Hence $\widetilde{\mathcal{B}}$ is a carathéodory function which satisfies

$$\begin{cases} \left|\widetilde{\mathcal{B}}(x,\sigma)\right| \leq b(x) + C'\,|\sigma|^{p(x)-1} \\ \text{a.e. in } \Omega \text{ and for all } \sigma\in\mathbb{R}. \end{cases} \tag{4.3}$$

We define the superposition (Nemytskii) operator associated to $\widetilde{\mathcal{B}}$, acting on the measurable function $u:\Omega\to\mathbb{R}$ by

$$\mathcal{N}_{\widetilde{\mathcal{B}}}(u)(x) := \widetilde{\mathcal{B}}(x,u(x))$$

for all $x \in \Omega$. Then we call $u \in \mathcal{W}^{1,p(.)}(\Omega)$ is a weak $\mathcal{L}$-solution of (3.5), if

$$\begin{cases} \mathcal{N}_{\widetilde{\mathcal{B}}}(u) \in L^{r(.)}(\Omega) \\ a(u,\varphi) = \left\langle \mathcal{N}_{\widetilde{\mathcal{B}}}(u), \varphi \right\rangle \end{cases}$$

for all $\varphi \in \mathcal{W}_0^{1,p(.)}(\Omega)$. The above problem make sense if $\mathcal{N}_{\widetilde{\mathcal{B}}}(u) \in L^{r(.)}(\Omega)$.

To show that we need the next condition on r. Assuming that

$$p^{*'}(x) \leq r(x) \leq p'(x) \ a.e \text{ in } \Omega. \tag{4.4}$$

Let now prove that $\mathcal{N}_{\widetilde{\mathcal{B}}}(u) \in L^{r(.)}(\Omega)$.

Lemma 4.1. *Suppose that (H_1) and (4.4) hold. Then $\mathcal{N}_{\widetilde{\mathcal{B}}}(u) \in L^{r(.)}(\Omega)$ for all $u \in \mathcal{W}^{1,p(.)}(\Omega)$. Moreover there exist a positive constant k such that*

$$\left\|\mathcal{N}_{\widetilde{\mathcal{B}}}(u)\right\|_{r(.)} \leq 2k \begin{pmatrix} \|b\|_{p'(.)} + c' 2^{p^+-1} |\Omega|^{\frac{p^+-1}{d}} \|u\|_{p^*(.)}^{p^+-1} \\ +c' 2^{p^--1} |\Omega|^{\frac{p^--1}{d}} \|u\|_{p^*(.)}^{p^--1} \end{pmatrix}.$$

Proof. By the Riesz representation Theorem,

$$\begin{aligned} \left\|\mathcal{N}_{\widetilde{\mathcal{B}}}(u)\right\|_{p'(.)} &= \sup_{\|v\|_{p(.)} \leq 1} \left| \int_\omega \mathcal{N}_{\widetilde{\mathcal{B}}}(u)\, v dx \right| \\ &= \sup_{\|v\|_{p(.)} \leq 1} \left| \left\langle \mathcal{N}_{\widetilde{\mathcal{B}}}(u), v \right\rangle \right|. \end{aligned}$$

Given $v \in \mathcal{W}_0^{1,p(.)}(\Omega)$, we have

$$\begin{aligned} \left| \left\langle \mathcal{N}_{\widetilde{\mathcal{B}}}(u), v \right\rangle \right| &\leq \left| \left\langle \left|\mathcal{N}_{\widetilde{\mathcal{B}}}(u)\right|, |v| \right\rangle \right| \\ &\leq \langle |b|, |v| \rangle + c' \left\langle |u|^{p(x)-1}, |v| \right\rangle. \end{aligned}$$

Using Hölder's inequality, we get

$$\left| \left\langle \mathcal{N}_{\widetilde{\mathcal{B}}}(u), v \right\rangle \right| \leq 2 \|b\|_{p'(.)} \|v\|_{p(.)} + 2c' \left\| |u|^{p(x)-1} \right\|_{p'(.)} \|v\|_{p(.)}.$$

Thus

$$\begin{aligned} &\sup_{\|v\|_{p(.)} \leq 1} \left| \left\langle \mathcal{N}_{\widetilde{\mathcal{B}}}(u), v \right\rangle \right| \\ &\leq 2 \|b\|_{p'(.)} \sup_{\|v\|_{p(.)} \leq 1} \|v\|_{p(.)} + 2c' \sup_{\|v\|_{p(.)} \leq 1} \left\| |u|^{p(x)-1} \right\|_{p'(.)} \|v\|_{p(.)} \\ &\leq 2 \|b\|_{p'(.)} \sup_{\|v\|_{p(.)} \leq 1} \|v\|_{p(.)} + 2c' \sup_{\|v\|_{p(.)} \leq 1} \left\| |u|^{p(x)-1} \right\|_{p'(.)} \|v\|_{p(.)} \\ &\leq 2 \|b\|_{p'(.)} + 2c' \left\| |u|^{p(x)-1} \right\|_{p'(.)}. \end{aligned}$$

By Lemma 2.1 in [13], we deduce that

$$\left\| |u|^{p(x)-1} \right\|_{p'(.)} \leq \|u\|_{p(.)}^{p^+-1} + \|u\|_{p(.)}^{p^--1}.$$

Hence

$$\left\|\mathcal{N}_{\widetilde{\mathcal{B}}}(u)\right\|_{p'(.)} \leq 2\|b\|_{p'(.)} + 2c' \left(\|u\|_{p(.)}^{p^+-1} + \|u\|_{p(.)}^{p^--1}\right).$$

Since $r(x) \leq p'(x)$ a.e. in Ω, then there exist a constant $k > 0$ such that

$$\begin{aligned}\left\|\mathcal{N}_{\widetilde{\mathcal{B}}}(u)\right\|_{r(.)} &\leq 2k\|b\|_{p'(.)} + 2kc' \left(\|u\|_{p(.)}^{p^+-1} + \|u\|_{p(.)}^{p^--1}\right) \\ &\leq 2k\left(\|b\|_{p'(.)} + c'\|u\|_{p(.)}^{p^+-1} + c'\|u\|_{p(.)}^{p^--1}\right).\end{aligned}$$

Hence $\mathcal{N}_{\widetilde{\mathcal{B}}}(u) \in L^{r(.)}(\Omega)$, as claimed. Since $\frac{1}{p^*(x)} + \frac{1}{d} = \frac{1}{p(x)}$, we get from Hölder's inequality

$$\|u\|_{p(.)}^{p^+-1} \leq 2^{p^+-1}|\Omega|^{\frac{p^+-1}{d}}\|u\|_{p^*(.)}^{p^+-1} \text{ and } \|u\|_{p(.)}^{p^--1} \leq 2^{p^--1}|\Omega|^{\frac{p^--1}{d}}\|u\|_{p^*(.)}^{p^--1}.$$

Hence

$$\left\|\mathcal{N}_{\widetilde{\mathcal{B}}}(u)\right\|_{r(.)} \leq 2k\begin{pmatrix}\|b\|_{p'(.)} + c'2^{p^+-1}|\Omega|^{\frac{p^+-1}{d}}\|u\|_{p^*(.)}^{p^+-1} \\ +c'2^{p^--1}|\Omega|^{\frac{p^--1}{d}}\|u\|_{p^*(.)}^{p^--1}\end{pmatrix}$$

and this complete the proof. □

Lemma 4.2. *Suppose that (H_1) and (4.4) hold, then the operator*

$$\mathcal{N}_{\widetilde{\mathcal{B}}} : L^{p(.)}(\Omega) \to L^{r(.)}(\Omega)$$

is continuous and bounded.

Proof. We first note that

$$\frac{r(x)}{p(x)(p(x)-1)} + \frac{p(x)(p(x)-1) - r(x)}{p(x)(p(x)-1)} = 1.$$

Hence by Young's inequality, we have

$$\begin{aligned}\left|\widetilde{\mathcal{B}}(x,u(x))\right| &\leq \frac{p(x)(p(x)-1) - r}{p(x)(p(x)-1)}\left(C'\right)^{\frac{p(x)(p(x)-1)}{p(x)(p(x)-1)-r(x)}} \\ &\quad + \frac{r(x)}{p(x)(p(x)-1)}|u|^{\frac{p(x)}{r(x)}} + b(x) \\ &\leq \left(1 - \frac{r^-}{p^+(p^+-1)}\right)\left(C'\right)^{\frac{p(x)(p(x)-1)}{p(x)(p(x)-1)-r(x)}} \\ &\quad + b(x) + \frac{r^+}{p^-(p^--1)}|u|^{\frac{p(x)}{r(x)}} \\ &\leq \left(1 - \frac{r^-}{p^+(p^+-1)}\right)\left(C'\right)^{\frac{p^+\left(p^+-1\right)}{p^-(p^--1)-r^+}} \\ &\quad + b(x) + \frac{r^+}{p^-(p^--1)}|u|^{\frac{p(x)}{r(x)}} \\ &\leq \left(1 - \frac{r^-}{p^+(p^+-1)}\right)\left(C'\right)^{\frac{p^+\left(p^+-1\right)}{p^-(p^--1)-r^+}} + a(x) \\ &\quad + \left(1 + \frac{1}{\epsilon}\right)^{p^+-1}|g|^{p(x)-1} + \frac{r^+}{p^-(p^--1)}|u|^{\frac{p(x)}{r(x)}}.\end{aligned}$$

Hence

$$\begin{aligned}&\left\|\left(1-\frac{r^{-}}{p^{+}\left(p^{+}-1\right)}\right)\left(C^{'}\right)^{\frac{p^{+}\left(p^{+}-1\right)}{p^{-}\left(p^{-}-1\right)-r^{+}}}+b\left(x\right)\right\|_{r(.)}\\ &\leq\left(1-\frac{r^{-}}{p^{+}\left(p^{+}-1\right)}\right)\left(C^{'}\right)^{\frac{p^{+}\left(p^{+}-1\right)}{p^{-}\left(p^{-}-1\right)-r^{+}}}\left\|1\right\|_{r(.)}\\ &\quad+\left\|a\left(x\right)\right\|_{r(.)}+\left(1+\frac{1}{\epsilon}\right)^{p^{+}-1}\left\|\left|g\right|^{p(x)-1}\right\|_{r(.)}.\end{aligned}$$

Now recall that $a\in L^{p^{'}(.)}\left(\Omega\right)$ and $r\left(x\right)\leq p^{'}\left(x\right)$ a.e. in Ω. Hence $a\in L^{r(.)}\left(\Omega\right)$ and there exist a constant $C>0$ such that

$$\begin{aligned}\left\|\left|g\right|^{p(x)-1}\right\|_{r(.)}&\leq C\left\|\left|g\right|^{p(x)-1}\right\|_{p^{'}(.)}\\ &\leq C\max\left\{\left\|g\right\|_{p(.)}^{p^{-}-1},\left\|g\right\|_{p(.)}^{\left(p^{+}-1\right)}\right\}<\infty.\end{aligned}$$

Since $\left\|1\right\|_{r(.)}\leq 2\max\left\{\left|\Omega\right|^{\frac{1}{r^{+}}},\left|\Omega\right|^{\frac{1}{r^{-}}}\right\}<\infty$, then

$$\begin{aligned}\left|\widetilde{\mathcal{B}}\left(x,u\left(x\right)\right)\right| \leq &\left(1-\tfrac{r^{-}}{p^{+}\left(p^{+}-1\right)}\right)\left(C^{'}\right)^{\frac{p^{+}\left(p^{+}-1\right)}{p^{-}\left(p^{-}-1\right)-r^{+}}}\\ &+b\left(x\right)+\tfrac{r^{+}}{p^{-}\left(p^{-}-1\right)}\left|u\right|^{\frac{p(x)}{r(x)}}\end{aligned}\tag{4.5}$$

with

$$\left(\left(1-\frac{r^{-}}{p^{+}\left(p^{+}-1\right)}\right)\left(C^{'}\right)^{\frac{p^{+}\left(p^{+}-1\right)}{p^{-}\left(p^{-}-1\right)-r^{+}}}+b\left(x\right)\right)\in L^{r(.)}\left(\Omega\right).$$

Now combining (4.5) and Theorem 1.16 in [?], it holds that the operator $\mathcal{N}_{\widetilde{\mathcal{B}}}$ is continuous and bounded from $L^{p(.)}\left(\Omega\right)$ to $L^{r(.)}\left(\Omega\right)$. $\square$

In the case of small domains, we have

Theorem 4.1. *Suppose that the assumptions* (H_1), (H_2) *hold with* $C\geq 1$. *Assume that* r *verifies (4.4). If* $|\Omega|$ *is small enough, then for every* $g\in\mathcal{W}^{1,p(.)}\left(\Omega\right)$, *there exists a unique* $\mathcal{L}-$*solution* $u\in\mathcal{W}^{1,p(.)}\left(\Omega\right)$ *of (4.1) such that* $u-g\in\mathcal{W}_0^{1,p(.)}\left(\Omega\right)$.

Proof. Let $M>0$. We set

$$\mathcal{K}^{M}:=\left\{f\in L^{r(.)}\left(\Omega\right);\left\|f\right\|_{r(.)}\leq M\right\}.$$

Consider the operator $\mathcal{F}$ defined by

$$\mathcal{F}f:=\left(\mathcal{N}_{\widetilde{\mathcal{B}}}\circ\mathcal{T}\right)\left(f\right),$$

with $f\in\mathcal{K}^{M}$.

By Lemma 4.1,

$$\left\|\mathcal{F}f\right\|_{r(.)}=\left\|\mathcal{N}_{\widetilde{\mathcal{B}}}\left(\widetilde{u}_f\right)\right\|_{r(.)}\leq 2k\begin{pmatrix}\left\|b\right\|_{p^{'}(.)}+c^{'}2^{p^{+}-1}\left|\Omega\right|^{\frac{p^{+}-1}{d}}\left\|\widetilde{u}_f\right\|_{p^{*}(.)}^{p^{+}-1}\\ +c^{'}2^{p^{-}-1}\left|\Omega\right|^{\frac{p^{-}-1}{d}}\left\|\widetilde{u}_f\right\|_{p^{*}(.)}^{p^{-}-1}.\end{pmatrix}$$

Using Theorem 3.1, we get

$$\|\widetilde{u}_f\|_{p^*(.)}^{p^+-1} \leq C\,\|f\|_{r(.)} \text{ and } \|\widetilde{u}_f\|_{p^*(.)}^{p^--1} \leq C\,\|f\|_{r(.)}\ ,$$

where $C > 0$ is independent of $f \in L^{r(.)}(\Omega)$. Hence

$$\begin{aligned}\|\mathcal{F}f\|_{r(.)} &\leq 2k\begin{pmatrix}\|b\|_{p'(.)} + c' 2^{p^+-1}\,|\Omega|^{\frac{p^+-1}{d}}\,C\,\|f\|_{r(.)} \\ +c' 2^{p^--1}\,|\Omega|^{\frac{p^--1}{d}}\,C\,\|f\|_{r(.)}\end{pmatrix} \\ &\leq 2k\begin{pmatrix}\|b\|_{p'(.)} \\ +Cc' \max\left\{2^{p^+-1}\,|\Omega|^{\frac{p^+-1}{d}}, 2^{p^--1}\,|\Omega|^{\frac{p^--1}{d}}\right\}\|f\|_{r(.)}\end{pmatrix} \\ &\leq 2k\left(\|b\|_{p'(.)} + MCc' \max\left\{2^{p^+-1}\,|\Omega|^{\frac{p^+-1}{d}}, 2^{p^--1}\,|\Omega|^{\frac{p^--1}{d}}\right\}\right).\end{aligned}$$

Since $|\Omega|$ is small enough, we can choose

$$M_\Omega := \frac{2k\,\|b\|_{p'(.)}}{1 - kc_pCc' \max\left\{2^{p^+-1}\,|\Omega|^{\frac{p^+-1}{d}}, 2^{p^--1}\,|\Omega|^{\frac{p^--1}{d}}\right\}}.$$

Consequently

$$\mathcal{F}\left(\mathcal{K}^M\right) \subset \mathcal{K}^M, \forall M \geq M_\Omega.$$

Notice that $\mathcal{K}^M$ is a no empty closed convex subset of $L^{r(.)}(\Omega)$, by Lemmas 4.2 and 3.1, we have $\mathcal{F} : L^{r(.)}(\Omega) \to L^{r(.)}(\Omega)$ is completely continuous, hence by the Shauder fixed point theorem, $\mathcal{F}$ admits a fixed point in $\mathcal{K}^M$. Hence there exists a $\mathcal{L}-$solution $u \in \mathcal{W}^{1,p(.)}(\Omega)$ such that $u - g \in \mathcal{W}_0^{1,p(.)}(\Omega)$. By the comparison principle, we have the uniqueness of the solution. □

Finally, we give a result with no restriction on the measure of Ω.

Theorem 4.2. *Suppose that $\mathcal{B}$ verify the assumptions (H_1) and (H_2) with $C \geq 1$. Assume that the condition (4.4) holds. Then for all $g \in \mathcal{W}^{1,p(.)}(\Omega)$, there exists a unique $\mathcal{L}-$solution $u \in \mathcal{W}^{1,p(.)}(\Omega)$ of (4.1) such that $u - g \in \mathcal{W}_0^{1,p(.)}(\Omega)$.*

Proof. By Theorem 4.1, there exists a unique $\mathcal{L}-$solution denotes by $u_g^\Omega \in \mathcal{W}^{1,p(.)}(\Omega)$ of (4.1) such that $u_g^\Omega - g \in \mathcal{W}_0^{1,p(.)}(\Omega)$ when $|\Omega|$ is small enough.

Let $(g_i)_i$ be a sequence of functions belong to $\mathcal{W}^{1,p(.)}(\Omega)$ and $(\omega_i)_i$ be increasing sequence of open subset such that $\Omega = \bigcup_i \omega_i$ and $\Omega\backslash\omega_i$ is a bounded regular domain for all $i \in \mathbb{N}^*$. We choose $i_0 \in \mathbb{N}^*$ such that $\Omega\backslash\omega_{i_0}$ is a bounded regular domain with $|\Omega\backslash\omega_{i_0}|$ is small enough.

We define the sequence of functions $(f_{\omega_i})_i$ by $f_0 = u_g^\Omega$ in Ω and

$$f_{i+1} = \begin{cases} u_{g_i}^{\omega_{i+1}} & \text{in } \omega_{i+1}, \\ f_i & \text{in } \Omega\backslash\omega_{i+1}, \end{cases}$$

for all $i \in \mathbb{N}$. Since any function in the space $\mathcal{W}_0^{1,p(.)}(\omega_{i+1})$ can be extended by 0 in $\Omega\backslash\omega_{i+1}$, we get that $f_{\omega_i} \in \mathcal{W}^{1,p(.)}(\Omega)$.

By proposition 3.2, $(f_{\omega_i})_i$ is locally uniformly bounded in $\mathcal{W}^{1,p(.)}(\Omega)$, then we can extract a subsequence $f_{\omega_{\varphi(n)}}$ converges weakly in $\mathcal{W}^{1,p(.)}(\Omega)$ to a function w^{Ω}. To complete the proof we show that w_g^{Ω} is a $\mathcal{L}-$solution. By Lemmas 3.2 and 3.3 in [?], we have

$$\mathcal{B}\left(x, f_{\omega_{\varphi(n)}}\right) \to \mathcal{B}\left(x, w_g^{\Omega}\right)$$

and

$$\left|\nabla f_{\omega_{\varphi(n)}}\right|^{p(x)-2} \nabla f_{\omega_{\varphi(n)}} \to \left|\nabla w_g^{\Omega}\right|^{p(x)-2} \nabla w_g^{\Omega}$$

a.e. in Ω. In addition the sequence $\left(\left|\nabla f_{\omega_{\varphi(n)}}\right|^{p(x)-2} \nabla f_{\omega_{\varphi(n)}}\right)$ is bounded, hence by Vitali's theorem we deduce that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \int_{\Omega} \left|\nabla f_{\omega_{\varphi(n)}}\right|^{p(x)-2} \nabla f_{\omega_{\varphi(n)}} \varphi dx + \int_{\Omega} \mathcal{B}\left(x, f_{\omega_{\varphi(n)}}\right) \varphi dx \\ &= \int_{\Omega} \left|\nabla w_g^{\Omega}\right|^{p(x)-2} \nabla w_g^{\Omega} \varphi dx + \int_{\Omega} \mathcal{B}\left(x, w_g^{\Omega}\right) \varphi dx \end{aligned}$$

for all $\varphi \in \mathcal{W}_0^{1,p(.)}(\Omega)$. So w_g^{Ω} it is a $\mathcal{L}-$solution in Ω and this complete the proof. $\square$

5. $\mathcal{L}$-harmonic functions

We have proved that if Ω is bounded regular domain and $g \in \mathcal{W}^{1,p(.)}(\Omega)$, then there exist a unique function $u \in \mathcal{W}^{1,p(.)}(\Omega)$ solution of (4.1) such that $u-g \in \mathcal{W}_0^{1,p(.)}(\Omega)$. We deal now with the case $g \in \mathcal{C}(\partial\Omega)$.

The main question in this direction is the following: for $g \in \mathcal{C}(\partial\Omega)$, is there a unique continuous extension of g in $\overline{\Omega}$, that is u_g which is a continuous weak solution of (3.5)?

Before answering the above question, we need some tools.

Definition 5.1. A function $h : \Omega \to \mathbb{R}$ is said to be $\mathcal{L}$-harmonic in Ω if it is a continuous weak solution to (3.5). We set

$$\mathcal{H}_{\mathcal{L}}(\Omega) = \{h : h \text{ is } \mathcal{L} - \text{harmonic in } \Omega\}.$$

Definition 5.2. Let g be a continuous function on $\partial\Omega$. We say that w_g solves the Dirichlet problem with boundary value g if $w_g \in \mathcal{H}_{\mathcal{L}}(\Omega)$ and

$$\lim_{x\to y} w_g(x) = g(y),$$

for all $y \in \partial\Omega$.

We denote by $w_{\mathcal{L}}^g$ the solution of (4.1), by Theorem 4.2 and proposition 3.2, the unique solution $w_{\mathcal{L}}^g \in \mathcal{H}_{\mathcal{L}}(\Omega)$. Moreover if $w_{\mathcal{L}}^g$ solves the Dirichlet problem with boundary value $g \in \mathcal{C}(\partial\Omega)$, then $w_{\mathcal{L}}^g$ is a continuous extension of g in $\overline{\Omega}$.

Notice that the comparison principle given in Theorem 3.2 can be extended to functions in $\mathcal{C}(\partial\Omega)$ in the following way.

Lemma 5.1. *If $g_1, g_2 \in \mathcal{C}(\partial\Omega)$ and $g_1 \leq g_2$ on $\partial\Omega$, then $w_{\mathcal{L}}^{g_1} \leq w_{\mathcal{L}}^{g_2}$ in Ω.*

Hence we have

Theorem 5.1. *Let $g \in \mathcal{C}(\partial\Omega)$. Then there exists a unique continuous $\mathcal{L}$-harmonic extension $w_{\mathcal{L}}^{g}$ of g in $\overline{\Omega}$.*

Proof. It is clear that if g is the trace on $\partial\Omega$ of a function $\hat{g} \in \mathcal{C}^{\infty}\left(\overline{\Omega}\right)$, then the result holds trivially. To prove the general case $g \in \mathcal{C}(\partial\Omega)$, we follow by approximation.

Let $g_i \in \mathcal{C}^{\infty}\left(\overline{\Omega}\right)$ be such that

$$\sup_{\partial\Omega} |g - g_i| \leq \frac{1}{2^i}, i = 1, 2, \ldots.$$

Then the comparison principle, it holds that

$$\left|w_{\mathcal{L}}^{g_i} - w_{\mathcal{L}}^{g_j}\right| \leq \frac{1}{2^i} + \frac{1}{2^j}.$$

Hence, the sequence $\left(w_{\mathcal{L}}^{g_i}\right)_i$ converges uniformly on $\overline{\Omega}$ to a continuous function $w_{\mathcal{L}}^{g}$. By Lemmas 3.2, 3.3 in [**?**], we deduce that the limit $w_{\mathcal{L}}^{g}$ it is a solution of (3.5), hence it is a $\mathcal{L}$-harmonic function in Ω. $\square$

Notice that by the uniqueness of the solution, we deduce that the $\mathcal{L}$-harmonic function $H_{\mathcal{L}}^{g}$ and the solution given by Brelot-Perron-Wiener Method coincide. In this case the space of continuous functions on $\partial\Omega$ is resolutive.

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