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STRUCTURAL BIFURCATION OF DIVERGENCE-FREE VECTOR FIELDS NEAR NON-SIMPLE DEGENERATE POINTS WITH SYMMETRY

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Abstract In this study, topological features of an incompressible two-dimensional flow far from any boundaries is considered. A rigorous theory has been developed for degenerate streamline patterns and their bifurcation. The homotopy invariance of the index is used to simplify the differential equations of fluid flows which are parameter families of divergence-free vector fields. When the degenerate flow pattern is perturbed slightly, a structural bifurcation for flows with symmetry is obtained. We give possible flow structures near a bifurcation point. A flow pattern is found where a degenerate cusp point appears on the x-axis. Moreover, we also show that bifurcation of the flow structure near a non-simple degenerate critical point with double symmetry is generic away from boundaries. Finally, we give an application of the degenerate flow patterns emerging when index 0 and -2 in a double lid driven cavity and in two dimensional peristaltic flow.

Keywords Structural stability, divergence-free vector field, Lagrangian dynamics, structural bifurcation, singularity classification.

MSC(2010) 34D, 35Q35, 58F76, 86A10.

1. Introduction

The main objective of this article is to determine the degenerate flow structures and their associated bifurcations of divergence-free vector field on 2D compact Riemannian manifolds with symmetry conditions. For bifurcation near simple-degenerate critical points away from boundaries, we refer the interested reader to ([8,28]). In this paper we address degenerate critical points near non-simple degenerate critical points.

Streamline topologies near non-simple degenerate critical points in a two-dimensional flow with symmetry about an axis and double symmetry away from boundaries were studied by Deliceoğlu and Gürcan ([14,15]). They found degenerate flow patterns and their bifurcations near a non-simple degenerate critical point via qualitative properties of steady flows based on polynomial expansions of the streamfunction. They used a series of canonical transformation to find the normal form of streamfunction. These techniques were first used by Brøns and Hartnack [8] and by others ([6,9,10,22]). Also, Brøns et al. [11], Dam et al. [13] and Heil et

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al. [23] have studied flow bifurcations by using time as a parameter. Jiménez-Lozano and Sen [25] studied streamline topologies of two-dimensional peristaltic flow and their bifurcations up to codimension two. They found an analytical solution for the stream-function under a long-wavelength and low-Reynolds number approximation. They showed that experimental observations of trapping agree with the theoretical findings. A bifurcation analysis of streamline pattern was also studied by ([1–3, 5, 16]).

The stability and transitions of the structure of incompressible flows with their applications to fluid dynamics were studied by Ma and Wang [29]. They presented a geometric theory for incompressible flows and its application to fluid dynamics. A global structural stability theorem of divergence-free vector fields, providing necessary and sufficient conditions for structural stability was proved by Ma and Wang [30]. Topological ideas have been applied in several other studies in the investigation of structural bifurcation of divergence free-vector fields (see e.g. [19, 24]).

Recently, the structural bifurcation and boundary layer separation for 2-D incompressible fluid have been investigated in many studies ([12, 20, 21, 27, 31–33]). Ghil et al. [20] studied the detailed process of bifurcation in the flows topological structure for a two-dimensional (2-D) incompressible flow subject to no-slip boundary conditions and its connection with boundary-layer separation. Luo et al. [27] considered the solutions of Navier-Stokes equations with Dirichlet boundary conditions governing 2-D incompressible fluid flows. A condition for boundary layer was obtained. The solutions of Navier-Stokes equations governing 2-D incompressible flows with the Dirichlet boundary condition have been analyzed by Wang et al. [33].

Hsia et al. [24] studied the structural stability and bifurcation for 2-D incompressible flows with symmetry. They showed that the symmetric divergence-free velocity field keeps stable if an interior saddle is connected to its symmetric image. The patterns and bifurcations found theoretically was also demonstrated by a numerical example of a Boussinesq flow induced by a temperature jump. They used a fourth order finite difference method proposed by Liu et al. [26] and Wang et al. [34] in the numerical simulation of the physical process.

Ma and Wang [29] showed that a homoclinic orbit, i.e., one center interior separation, is generic in the interior structural bifurcation. Recently in [7], the authors study interior structural bifurcation of two-dimensional symmetric incompressible flows. They obtained the two structural bifurcation scenarios are indeed generic for flows near simple-degenerate critical points with certain symmetries. In this paper, our aim is to obtain new structural bifurcation scenarios and to determine which of them are generic in the flow near the non-simple degenerate critical points.

The scope of the present paper is to make connections between the index of a divergence-free vector field $u(.,t)$ near a non-simple degenerate critical point and its Taylor expansion. The number of free parameters in the Taylor expansion of a divergence-free vector field is reduced by using the homotopy invariance of the index. This method significantly reduces the computational cost of finding a normal form of streamfunction preserving the flow structure under the transformation. This method was also used by the authors ([19, 28, 30]). In this study, the works by Deliceoğlu and Gürcan ([14, 15]) will be revisited through the index theory, as will be seen, structural classification of divergence-free vector fields near the non-simple critical points is simplified significantly.

The main goal of this paper is

(i) to investigate structural bifurcation of $u(.,t)$ at t_0 near a non-simple degenerate point and its evolution in time,

(ii) to obtain some theorems characterizing degenerate singular point with zero Jacobian, and

(iii) to give kinematic conditions which orbit structures appears inside viscous fluid flows near non-simple degenerate critical points,

(iv) to show that the structural bifurcation scenarios near the non-simple degenerate critical point with double symmetry are generic in the flow.

2. Preliminaries

We first introduce some basic definitions, lemmas and theorems which are useful for structural stability and bifurcation of divergence-free vector fields. Let $M \subset \mathbb{R}^2$ be a closed and bounded domain with C^r $(r \geq 1)$ boundary ∂M. Let TM be the tangent bundle of M, and $C^r(TM)$ be the space of all C^r vector fields on M. Denote

$$C_n^r(TM) = \{\mathbf{u} \in C^r(TM) \mid \mathbf{u}_n|_{\partial M} = 0\},$$

$$D^r(TM) = \{\mathbf{u} \in C^r(TM) | \ \mathbf{u_n}|_{\partial M} = 0, \ \nabla.\mathbf{u} = \mathbf{0}\},$$

where $\mathbf{u_n} = \mathbf{u}.\mathbf{n}$, while $\mathbf{n}$ is the normal vector on ∂M.

Definition 2.1 ([28]). Two vector fields $\mathbf{u}, \mathbf{v} \in D^r(TM)$ are called topologically equivalent in $D^r(TM)$ if there exists a homomorphism of $\varphi : M \to M$, which maps orbits of $\mathbf{u}$ to orbits of $\mathbf{v}$ and preserves their orientation.

Definition 2.2 ([28]). A vector field $\mathbf{u} \in D^r(TM)$ is called structurally stable in $D^r(TM)$ if there exists a neighborhood $O \subset D^r(TM)$ of $\mathbf{u}$ such that for any $\mathbf{v} \in O$, $\mathbf{v}$ and $\mathbf{u}$ are topologically equivalent.

Definition 2.3 ([28]). Let $\mathbf{u} \in C^1([0,T], D^r(TM))$. We say that $\mathbf{u}(x,t)$ has a bifurcation in its local structure in a neighborhood $U \subset M$ of x_0 at $t_0 (0 < t_0 < T)$ if, for any $t^- < t_0$ and $t_0 < t^+$ with t^- and t^+ sufficiently close to t_0, the vector fields $\mathbf{u}(\cdot, \mathbf{t}^-)$ and $\mathbf{u}(\cdot, t^+)$ are not topologically equivalent locally in $U \subset M$, and we say that $\mathbf{u}(\cdot, t)$ has a bifurcation at t_0 in its global structure if $U = M$.

A point p is called a singular point of $\mathbf{u} \in D^r(TM)$ if $\mathbf{u}(p) = 0$; a singular point p of $\mathbf{u}$ is called non-degenerate if the Jacobian matrix $D\mathbf{u}(p)$ is invertible; $\mathbf{u}$ is called regular if all singular points are non-degenerate; an interior non-degenerate singular point of $\mathbf{u}$ can be either a center or a saddle, and a non-degenerate boundary singularity must be saddle; $\mathbf{u}$ is structurally stable near each non-degenerate singular point of $\mathbf{u}$. For more details and discussions, see Ref. [28].

Theorem 2.1 ([30]). *A divergence-free vector field $\mathbf{u} \in D^r(TM) (r \geq 1)$ is structurally stable in $D^r(TM)$ if and only if*

(1) $\mathbf{u}$ is regular;

(2) all interior saddle points of $\mathbf{u}$ are self-connected; and

(3) each saddle points of $\mathbf{u}$ on ∂M is connected only to saddle points on the same connected component of ∂M.

Theorem 2.2 ([19]). *Let $p \in M$ be an isolated singular point of $\mathbf{u} \in D^r(TM) (r \geq 1)$. Then p is connected only to a finite number of orbits and the stable and unstable orbits connected to p alternate when tracing a closed curve around p. Furthermore,* *(1) when* $p \in \overset{\circ}{M}$, *p has* $2n(n \geq 0)$ *orbits, n of which are stable, and other n unstable, while the index of p is*

$$ind(\mathbf{u},p) = 1 - n,$$

(2) when $p \in \partial M$, *p has* $n + 2(n \geq 2)$ *orbits, two of which are on the boundary* ∂M, *and the index of p is*

$$ind(\mathbf{u},p) = -\frac{n}{2}.$$

Let the divergence-free vector field $\mathbf{u}$ be anti-symmetric with respect to the origin point $O(0,0)$. The vector field $\mathbf{u}(x,y)$ satisfies

$$\mathbf{u}(-x,-y) = -\mathbf{u}(x,y),$$

and we denote

$$R^r(TM) = \{\mathbf{u} \in D^r(TM) | \ \mathbf{u}(-x,-y) = -\mathbf{u}(x,y)\}.$$

We say that P' is the symmetric image of a point P if $P' = -P$. Similarly, the symmetric image of a set N is represented as $N' = \{-P \mid P \in N\}$. Then the following theorem provides necessary and sufficient conditions for structural stability of a divergence-free vector fields in $R^r(TM)$.

Theorem 2.3 ([24]). *Let* $\mathbf{u} \in R^r(TM)(r \geq 2)$. *Then* $\mathbf{u}$ *is structurally stable in* $R^r(TM)$ *if and only if*

(1) $\mathbf{u}$ *is regular;*

(2) any interior saddle point P of $\mathbf{u}$ *is either self-connected or connected to its symmetric image* P'; *and*

(3) each boundary saddle point P (with symmetric image P')$\in N \subset M$, *N being a connected component of* ∂M *with symmetric image* $N' \subset M$, *is connected to a* $\partial-$*saddle* $Q \in N \cup N' \setminus \left\{P, P'\right\}$.

Lemma 2.1 ([28]). *Let* $\mathbf{u} \in D^r(TM)$ $(r \geq 1)$, *and* $x_0 \in \overset{\circ}{M}$ *be an isolated singular point of* $\mathbf{u}$. *If the index* $ind(\mathbf{u}, x_0) \neq 1, 0, -1$, *then the Jacobian matrix*

$$D\mathbf{u}(x_0) = 0.$$

Lemma 2.2 ([28]). *Let* $\mathbf{u} \in D^r(TM)$ $(r \geq 1)$, *and* $x_0 \in \overset{\circ}{M}$ *be an isolated singular point of* $\mathbf{u}$. *If the index* $ind(\mathbf{u}, x_0) = 0$, *and the angle* θ *between the two orbits connected to* x_0 *is different from 0, then* $D\mathbf{u}(x_0) = 0$ *holds true.*

3. Interior Degenerate Singularities of Divergence-Free Vector Field with Symmetry

3.1. Degenerate singularities of divergence-free vector field with symmetry about an axis

In this section we will derive some relations between the coefficients of the Taylor expansion of $\mathbf{u}(x,y,t_0) = \mathbf{u}^0(x,y)$ near a non-simple-degenerate point x_0 with symmetry about an axis. We suppose that the divergence free vector field $\mathbf{u}^0(x,y)$

is symmetric about y axis, that is, for $\mathbf{u}^0(x,y)=(u_1(x,y),u_2(x,y))$ the vector field $\mathbf{u}^0(x,y)$ satisfies the symmetry condition

$$u_1(x,y)=u_1(-x,y), \quad u_2(x,y)=-u_2(-x,y). \tag{3.1}$$

We define the set of all such vector fields as follows:

$$E^r(TM)=\left\{\mathbf{u^0} \in D^r(TM)|\ u_1(x,y)=u_1(-x,y), \quad u_2(x,y)=-u_2(-x,y)\right\}.$$

Let $x_0(0,0)\in \overset{\circ}{M}$ be an isolated non-simple degenerate singular point of $\mathbf{u^0}$. Then the Jacobian matrix of $\mathbf{u^0}$ can be written as follows:

$$D\mathbf{u^0}(x_0)=D\mathbf{u^0}(0)=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.2}$$

We also assume the following conditions:

$$\frac{\partial^2\left(\mathbf{u^0}(x_0)\cdot e_1\right)}{\partial e_1^2}\neq 0, \tag{3.3}$$

$$\frac{\partial^m\left(\mathbf{u^0}(x_0)\cdot e_1\right)}{\partial e_2^m}\begin{cases} =0,\ 2\leq m<n, \\ \neq 0,\ m=n. \end{cases} \tag{3.4}$$

Here e_1 and e_2 are unit vectors of the Cartesian coordinate system. Under the condition (3.2), (3.3) and (3.4), $\mathbf{u^0}$ is given by

$$\mathbf{u^0}(x,y)=\begin{cases} \alpha x^2+\beta y^n+o\left(x^2,|y|^n\right), \\ -2\alpha xy+o\left(x^2,y^2\right), \end{cases} \tag{3.5}$$

where α,$\beta\neq 0$ and $n\geq 2$.

The following lemma is useful in determining the index of flow structure near the degenerate critical points.

Lemma 3.1. *Let $x_0\in \overset{\circ}{M}$ be an isolated degenerate singular point of $\mathbf{u^0}$ satisfying (3.5). In a small neighborhood of x_0, the index of $\mathbf{u^0}$ is stated as follows:*

$$ind\left(\mathbf{u^0},x_0\right)=\begin{cases} -2,\ as\ n=even,\ n\geq 2\ and\ \alpha\beta<0, \\ \ \ 0,\ as\ n=even,\ n\geq 2\ and\ \alpha\beta>0, \\ -1,\ as\ n=odd\ and\ n\geq 3. \end{cases} \tag{3.6}$$

Proof. Let

$$\mathbf{u}_t^0(x,y)=\begin{cases} \alpha x^2+\beta y^n+t\left(o\left(x^2,|y|^n\right)\right), \\ -2\alpha xy+t\left(o\left(x^2,y^2\right)\right), \end{cases}$$

where $0\leq t\leq 1$ and $n\geq 2$. There exists a neighborhood $G\subset M$ of $x_0(=0)$, such that $\mathbf{u}_t^0(x,y)$ has the unique singular point $x=(0,0)$ in G for all $t\in[0,1]$. By the homotopy invariance of the index, we derive that

$$ind(\mathbf{u}_0^0,x_0)=ind(\mathbf{u}_1^0,x_0)=ind(\mathbf{u^0},x_0).$$

In a small neighborhood of x_0, orbits of $\mathbf{u}_0^0 = \big(\alpha x^2 + \beta y^n, -2\alpha xy\big)$ are given by the following equations:

$$\frac{\beta}{n+1} y^{n+1} + \alpha x^2 y = C, \quad 0 \le |C| < \delta. \tag{3.7}$$

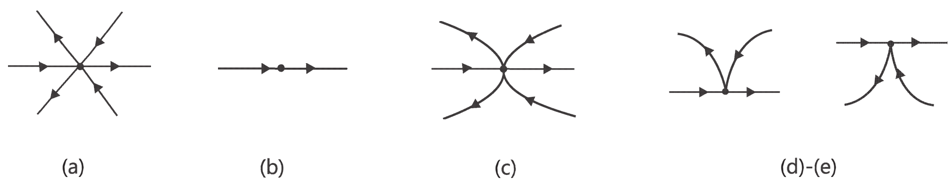

Figure 1. Index of $\mathbf{u}^0$ given by (3.5) about a point x_0 with symmetry about an axis. (a) A degenerate saddle point with $ind(u^0, x_0) = -2$ for the case where $n = 2$ and $\alpha\beta < 0$; (b) A degenerate point on the x-axis with $ind(u^0, x_0) = 0$ for the case where n is even and $n \ge 2$, $\alpha\beta > 0$; (c) A degenerate saddle point with $ind(u^0, x_0) = -2$ for the case where n is even and $n \ge 4$, $\alpha\beta < 0$; (d-e) A degenerate cusp point on the x-axis with $ind(u^0, x_0) = -1$ for the case where n is odd and $n \ge 3$.

(i) When n is even and $n \ge 2$, (3.7) becomes

$$y = 0, \quad \frac{\beta}{n+1} y^n + \alpha x^2 = C. \tag{3.8}$$

For $\alpha\beta < 0$ and $n = 2$, the orbits of $\mathbf{u}_0^0$ in a small neighborhood of x_0 has six separatrices from a single saddle point and this case is denoted a topological saddle with index -2, see Fig. 1(a). When $\alpha\beta > 0$, there are two orbits, one of which is stable, and the other is unstable and the index of x_0 is 0, see Fig. 1(b). When $n \ge 4$ and $\alpha\beta < 0$, the index of the orbit is -2 (see Fig. 1(c)).

(ii) If n is odd and $n \ge 3$. Then equation (3.7) becomes

$$y = 0, \quad \frac{\beta}{n+1} y^n + \alpha x^2 = C. \tag{3.9}$$

By using (3.9), we obtain flow pattern with index -1, where a degenerate cusp point is located on the x-axis, see Fig. 1(d) and Fig. 1(e). This completes the proof. □

3.2. Degenerate Singularities of Divergence-Free Vector Field with Double Symmetry

We now give the Taylor expansion of a divergence-free vector field $\mathbf{u^0}$ near x_0 with double symmetry together with its connection to the index of the underlying flow patterns. For this purpose we consider the degenerate structures of divergence free vector field $\mathbf{u^0} \in D^r(TM)$ with double-symmetry. Let the vector field $\mathbf{u^0} \in D^r(TM)$ be double-symmetric with respect to both the x and the y-axis, that is, $\mathbf{u^0}$ satisfies,

$$u_1(x,y) = u_1(-x,y) \;\text{ and }\; u_1(x,y) = u_1(x,-y), \tag{3.10}$$

$$u_2(x,y) = -u_2(-x,y) \;\text{ and }\; u_2(x,y) = -u_2(x,-y). \tag{3.11}$$

We denote the set of these vector fields as follows:

$$F^r(TM) = \left\{ \mathbf{u^0} \in D^r(TM) | \ \mathbf{u^0} \ \text{ satisfies (3.10) and (3.11)} \right\}.$$

Let $x_0(0,0) \in \overset{\circ}{M}$ be an isolated degenerate singular point of $\mathbf{u^0}$. Then the Jacobian matrix of $\mathbf{u^0}$ can be written as follows:

$$D\mathbf{u^0}(x_0) = D\mathbf{u}^0(0) = \begin{pmatrix} 0 \ 0 \\ 0 \ 0 \end{pmatrix}. \tag{3.12}$$

Assume further that

$$\frac{\partial^2 \left(\mathbf{u^0}(x_0) \cdot e_1 \right)}{\partial e_2^2} \neq 0, \tag{3.13}$$

$$\frac{\partial^m \left(\mathbf{u^0}(x_0) \cdot e_1 \right)}{\partial e_1^m} \begin{cases} = 0, \ 2 \leq m < 2k, \\ \neq 0, \ m = 2k. \end{cases} \tag{3.14}$$

Under the conditions (3.12), (3.13) and (3.14), the vector field $\mathbf{u^0}$ is written as

$$\mathbf{u^0}(x,y) = \begin{cases} \alpha y^2 + \beta x^{2k} + o(y^2, x^{2k}), \\ -2k\beta x^{2k-1} y - o(y, x^{2k-1}) \end{cases} \tag{3.15}$$

where α,$\beta \neq 0$ and $k = (1,2,...)$.

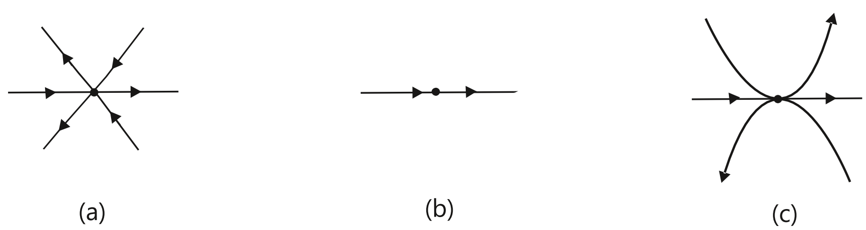

Figure 2. Index of non-simple degenerate singular points of $\mathbf{u}^0$ given by (3.15) with double symmetry. (a) A degenerate saddle point with $ind(u^0, x_0) = -2$ for the case $k = 1$ and $\alpha\beta < 0$; (b) A degenerate point on the x-axis with $ind(u^0, x_0) = 0$ for the case $\alpha\beta > 0$; (c) A degenerate saddle point with $ind(u^0, x_0) = -2$ for the case $k > 1$ and $\alpha\beta < 0$.

Lemma 3.2. *Let $x_0 \in \overset{\circ}{M}$ be an isolated degenerate singular point of $\mathbf{u^0}$ satisfying (3.15). In a small neighborhood of x_0, the index of $\mathbf{u^0}$ is stated as follows:*

$$ind\left(\mathbf{u^0}, x_0\right) = \begin{cases} -2, \ as \ \ \alpha\beta < 0, \\ \ \ 0, \ as \ \ \alpha\beta > 0. \end{cases} \tag{3.16}$$

Proof. Consider the vector field

$$\mathbf{u}_t^0(x,y) = \begin{cases} \alpha y^2 + \beta x^{2k} + t\left(o(y^2, x^{2k})\right), \\ -2k\beta x^{2k-1} y + t\left(o(y, x^{2k-1})\right), \end{cases}$$

where $0 \leq t \leq 1$. As in section 3.1, we may find flow topology in the neighborhood of non-simple degenerate critical points. There exist a neighborhood $G \subset M$ of $x_0(=0)$, such that $\mathbf{u}_t^0(x,y)$ has a unique singular point $x=(0,0)$ in G for all $t \in [0,1]$. By the homotopy invariance of the index, we derive that

$$ind(\mathbf{u}_0^0, x_0) = ind(\mathbf{u}_1^0, x_0) = ind(\mathbf{u^0}, x_0).$$

In a small neighborhood of x_0, we obtain from $\mathbf{u}_0^0 = \left(\alpha y^2 + \beta x^{2k}, -2k\beta x^{2k-1}y\right)$,

$$\frac{\alpha}{3}y^3 + \beta x^{2k}y = C, \quad 0 \leq |C| < \delta. \tag{3.17}$$

From (3.17), it easy to see that

$$y = 0, \quad \frac{\alpha}{3}y^2 + \beta x^2 = C \ \text{ for } \ k = 1. \tag{3.18}$$

This means that if $\alpha\beta < 0$ (respectively, $\alpha\beta > 0$), the non-simple degenerate point has a index -2, see Fig. 2(a) (respectively, as shown in Fig. 2(b)). If $k > 1$ in (3.17), then

$$y = 0, \quad \frac{\alpha}{3}y^2 + \beta x^{2k} = C. \tag{3.19}$$

In this case, we can see apparently from (3.19) if $\alpha\beta < 0$, the orbits of $\mathbf{u}_0$ in a small neighborhood of $x_0 = (0,0)$ is as illustrated in *Fig. 2(c).* □

Now we assume that $x_0(0,0) \in \overset{\circ}{M}$ is an isolated non-simple degenerate singular point of $\mathbf{u^0} \in F^r(TM)$ satisfying the following conditions:

$$\frac{\partial^2\left(\mathbf{u^0}(x_0)\cdot e_1\right)}{\partial e_1^2} \neq 0, \tag{3.20}$$

$$\frac{\partial^m\left(\mathbf{u^0}(x_0)\cdot e_1\right)}{\partial e_2^m} \begin{cases} = 0,\ 2 \leq m < 2n, \\ \neq 0,\ m = 2n. \end{cases} \tag{3.21}$$

Under the conditions (3.20) and (3.21), the vector field $\mathbf{u^0}$ is given by

$$\mathbf{u^0}(x,y) = \begin{cases} \beta x^2 + \alpha y^{2n} + o(x^2, |y|^{2n}), \\ -2\beta xy + o(x,y), \end{cases} \tag{3.22}$$

where $\alpha,\beta \neq 0$ and $n \geq 1$.

Lemma 3.3. *Let $x_0 \in \overset{\circ}{M}$ be an isolated degenerate singular point of $\mathbf{u^0}$ satisfying (3.22). In a small neighborhood of x_0, the index of $\mathbf{u^0}$ is stated as follows:*

$$ind\left(\mathbf{u^0}, x_0\right) = \begin{cases} -2,\ as\ n = 1 \ \ and\ \alpha\beta < 0, \\ \ \ 0,\ as\ n = 1\ and\ \alpha\beta > 0, \\ -2,\ as\ n > 1, and\ \alpha\beta < 0. \end{cases} \tag{3.23}$$

Proof. The process of proof is similar to the proof of Lemma 3.2. Here we omit the details. □

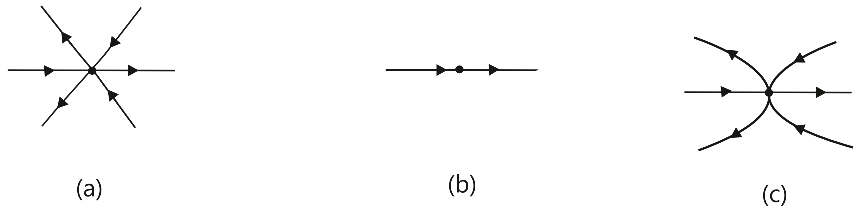

Figure 3. Index of non-simple degenerate singular points of $\mathbf{u}^0$ given by (3.22) with double symmetry. (a) A degenerate saddle point with $ind(u^0, x_0) = -2$ for the case $n = 1$ and $\alpha\beta < 0$; (b) A degenerate point on the x-axis with $ind(u^0, x_0) = 0$ for the case $\alpha\beta > 0$; (c) A degenerate saddle point with $ind(u^0, x_0) = -2$ for the case $n > 1$ and $\alpha\beta < 0$.

4. Structural Bifurcation Near Non-Simple Degenerate Singular Point

In this section we consider the transitions in the topological structure of a divergence free vector field near a non-simple degenerate interior point with symmetry about an axis. Let $\mathbf{u} \in \mathbf{C}^1\left([0,T], E^r(TM)\right)$ $(r \geq 1)$ be a one parameter family of divergence-free vector fields. We consider the Taylor expansion of $\mathbf{u}(x,t)$ at t_0 $(0 < t_0 < T)$,

$$\begin{aligned} \mathbf{u}(x,t) &= \mathbf{u^0}(x) + (t-t_0)\,\mathbf{u}^1(x) + o\left(|\ t-t_0|\right), \\ \mathbf{u}^0(x) &= \mathbf{u}(x,t_0), \\ \mathbf{u}^1(x) &= \tfrac{\partial}{\partial t}\mathbf{u}(x,t_0). \end{aligned} \tag{4.1}$$

We start with the following assumptions for the structural bifurcations.

Assumption (1): Let $x_0 \in \overset{\circ}{M}$ be an isolated degenerate singular point of $\mathbf{u}^0(x)$ satisfying the condition of *Lemma 3.1.* Suppose that

$$ind(\mathbf{u}^0, x_0) = 0, \tag{4.2}$$

$$D\mathbf{u}^0(x_0) = 0, \tag{4.3}$$

$$\mathbf{u}^1(x_0) \cdot e_1 \neq 0, \tag{4.4}$$

where e_1 is the unit vector and we also assume that $\mathbf{u}^0 \in C^n$ near $x_0 \in \overset{\circ}{M}$ for some $m \geq 1$, and

$$\frac{\partial^m\left(\mathbf{u}^0(x_0)\cdot e_1\right)}{\partial e_2^m}\begin{cases} = 0,\ 1 \leq m < 2, \\ \neq 0,\ m = 2n. \end{cases} \tag{4.5}$$

Assumption (2): Let $x_0 \in \overset{\circ}{M}$ be an isolated degenerate singular point of $\mathbf{u}^0(x)$ satisfying the condition of *Lemma 3.1.* Suppose that

$$ind(\mathbf{u}^0, x_0) = -2, \tag{4.6}$$

$$D\mathbf{u}^0(x_0) = 0, \tag{4.7}$$

$$\mathbf{u}^1(x_0) \cdot e_1 \neq 0, \tag{4.8}$$

where e_1 is the unit vector and we also assume that $\mathbf{u}^0 \in C^n$ near $x_0 \in \overset{\circ}{M}$ for some $m \geq 1$, and

$$\frac{\partial^m \left(\mathbf{u}^0(x_0) \cdot e_1\right)}{\partial e_2^m} \begin{cases} = 0, \, 1 \leq m < 2, \\ \neq 0, \, m = 2. \end{cases} \tag{4.9}$$

Assumption (3): Let $x_0 \in \overset{\circ}{M}$ be an isolated degenerate singular point of $\mathbf{u}^0(x)$ satisfying the condition of *Lemma 3.1.* Suppose that

$$ind(\mathbf{u}^0, x_0) = -2, \tag{4.10}$$

$$D\mathbf{u}^0(x_0) = 0, \tag{4.11}$$

$$\mathbf{u}^1(x_0) \cdot e_1 \neq 0, \tag{4.12}$$

where e_1 is the unit vector and we also assume that $\mathbf{u}^0 \in C^n$ near $x_0 \in \overset{\circ}{M}$ for some $m \geq 1$, and

$$\frac{\partial^m \left(\mathbf{u}^0(x_0) \cdot e_1\right)}{\partial e_2^m} \begin{cases} = 0, \, 1 \leq m < 4, \\ \neq 0, \, m \geq 4. \end{cases} \tag{4.13}$$

Assumption (4): Let $x_0 \in \overset{\circ}{M}$ be an isolated degenerate singular point of $\mathbf{u}^0(x)$ satisfying the condition of *Lemma 3.1.* Suppose that

$$ind(\mathbf{u}^0, x_0) = -1, \tag{4.14}$$

$$D\mathbf{u}^0(x_0) = 0, \tag{4.15}$$

$$\mathbf{u}^1(x_0) \cdot e_1 \neq 0, \tag{4.16}$$

where e_1 is the unit vector and we also assume that $\mathbf{u}^0 \in C^n$ near $x_0 \in \overset{\circ}{M}$ for some odd $m \geq 1$, and

$$\frac{\partial^m \left(\mathbf{u}^0(x_0) \cdot e_1\right)}{\partial e_2^m} \begin{cases} = 0, \, 1 \leq m < 3, \\ \neq 0, \, m \geq 3. \end{cases} \tag{4.17}$$

Theorem 4.1. *Let $u \in C^1\left([0,T], E^r(TM)\right)$ $(r \geq 1)$ satisfy all conditions of Assumption (1). Then, the unfolding of codimension-one of $\mathbf{u}(x,t)$ bifurcates from (x_0, t_0) exactly four non-degenerate singular points in a small neighborhood of x_0 for any $t < t_0$ (or $t > t_0$), two of which are saddle points and the rest are center points. $\mathbf{u}(x,t)$ has no singular point in a small neighborhood of x_0 for any $t > t_0$ (or $t < t_0$).*

Proof. From *Assumption (1)* and *Lemma 3.1* the vector field $\mathbf{u}^0(x,y)$ has the Taylor expansion at $x_0 = 0$ as follows:

$$\mathbf{u}^0(x,y) = \begin{cases} \alpha x^2 + \beta y^{2n} + o\left(x^2, y^{2n}\right), \\ -2\alpha xy + o(x^2, y^2), \end{cases} \tag{4.18}$$

where $\alpha \neq 0$, $\beta \neq 0$, $\alpha \cdot \beta > 0$ and $n \geq 1$.

By the condition (4.4) and since the orbit of $\mathbf{u}(x,y)$ is symmetric with respect to the y axis, we have

$$\mathbf{u}^1(x,y) = \begin{cases} \lambda + O\left(\left|x^2\right|, |y|\right), \\ xO\left(1\right), \end{cases} \tag{4.19}$$

where $\lambda \neq 0$. We assume that $\alpha > 0$ and $\lambda > 0$ (or other combinations of α and λ), so by taking $\alpha\beta > 0$, we get $\beta > 0$. Now we consider the singular points of the vector field $\mathbf{u}^0 - \epsilon\mathbf{u}^1$ for all $\epsilon > 0$ sufficiently small.

$$\alpha x^2 + \beta y^{2n} + o\left(x^2 + y^{2n}\right) - \epsilon\lambda - \epsilon O\left(|x| + |y|\right) = 0, \tag{4.20}$$

$$-2\alpha xy + o(x^2 + y^2) - x\epsilon O\left(1\right) = 0. \tag{4.21}$$

By the implicit function theorem, in a small neighborhood of $(0,0)$, we can solve (4.20) in terms of ϵ uniquely as

$$\epsilon = \lambda^{-1}(\alpha x^2 + \beta y^{2n}) + o\left(x^2 + y^{2n}\right). \tag{4.22}$$

(4.21) is satisfied for $x = 0$ and from (4.22) we get the solitions

$$P_{1,2} = \pm\left(0, \left(\frac{\epsilon\lambda}{\beta}\right)^{1/2n}\right) + o\left(\epsilon^{1/2n}\right) \tag{4.23}$$

of $\mathbf{u}^0 - \epsilon\mathbf{u}^1 = 0$.

For $x \neq 0$, (4.21) becomes

$$-2\alpha y + \lambda^{-1}\alpha x^2 O(1) + o(x^2 + y^2) = 0, \tag{4.24}$$

which gives the solution

$$y = O(x^2). \tag{4.25}$$

Plugging (4.25) into (4.22), we obtain two more solutions

$$P_{3,4} = \pm\left(\left(\frac{\epsilon\lambda}{\alpha}\right)^{1/2}, 0\right) + o\left(\sqrt{\epsilon}\right) \tag{4.26}$$

of $\mathbf{u}^0 - \epsilon\mathbf{u}^1 = 0$. Finally, we shall show

$$\begin{aligned} \det D(\mathbf{u}^0 - \epsilon\mathbf{u}^1) &= -\left(\frac{\partial}{\partial x}\left(\mathbf{u}_1^0 - \epsilon\mathbf{u}_1^1\right)\right)^2 - \frac{\partial}{\partial y}\left(\mathbf{u}_1^0 - \epsilon\mathbf{u}_1^1\right) \cdot \frac{\partial}{\partial x}\left(\mathbf{u}_2^0 - \epsilon\mathbf{u}_2^1\right) \\ &= -4\alpha^2 x^2 + 4\alpha\beta n y^{2n} + o\left(x^2 + y^{2n}\right), \end{aligned}$$

which yields that

$$\det D\left(\mathbf{u}^0 - \epsilon\mathbf{u}^1\right)|_{P_{1,2}} = 4n\alpha\epsilon\lambda > 0,$$
$$\det D\left(\mathbf{u}^0 - \epsilon\mathbf{u}^1\right)|_{P_{3,4}} = -4\alpha\epsilon\lambda < 0.$$

Thus it is proven that $\mathbf{u}^0 - \epsilon\mathbf{u}^1$ has four singular points for any sufficiently small $\epsilon > 0$. P_1 and P_2 are center points and the rest (P_3 and P_4) are saddle points, as shown in Fig. 4(c).

Similarly, for all (x,y) in a small neighborhood of $(0,0)$ when $\epsilon > 0$, $\alpha > 0$, $\beta > 0$ and $\lambda > 0$. Hence there no bifurcated solutions of $\mathbf{u}^0 + \epsilon\mathbf{u}^1 = 0$ for any $\epsilon > 0$ sufficiently small. See Fig. 4(a). The complete bifurcation diagram is shown in Fig. 4. That finishes the proof. □

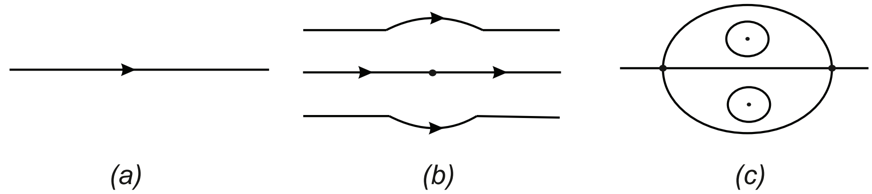

Figure 4. A description of the unfolding of codimension-one singularities for the case index 0 with symmetry about an axis. A similar diagram was also found by Gürcan and Deliceoğlu [15].

Theorem 4.2. *Let $u \in C^1([0,T], E^r(TM))$ $(r \geq 1)$ satisfy all conditions of Assumption (2). Then, the unfolding of codimension-one of $\mathbf{u}(x,t)$ bifurcates from (x_0, t_0) exactly two non-degenerate singular points in a small neighborhood of x_0 for any $t < t_0$ (or $t > t_0$), which are saddle points and $\mathbf{u}(x,t)$ bifurcates from (x_0, t_0) exactly two non-degenerate singular points in a small neighborhood of x_0 for any $t > t_0$ (or $t < t_0$), which are saddle points.*

Proof. From *Assumption (2)* and *Lemma 3.1* the vector field $\mathbf{u}^0(x,y)$ has the Taylor expansion at $x_0 (x = 0)$ as follows:

$$\mathbf{u}(x,y) = \begin{cases} \alpha x^2 + \beta y^2 + f(x) + y g_1(x,y) + o\left(|y|^2\right), \\ -2\alpha xy - y f'(x) + y^2 g_2(x,y), \end{cases} \tag{4.27}$$

where $\alpha \neq 0$, $\beta \neq 0$, $\alpha \cdot \beta < 0$.

By (4.8) and because of the orbit of $\mathbf{u}(x,y)$ is symmetric with respect to the y axis, we have

$$\mathbf{u}^1(x,y) = \begin{cases} \lambda + O\left(|xy|^2\right), \\ O\left(|xy|\right), \end{cases} \tag{4.28}$$

where $\lambda \neq 0$. We assume that $\alpha > 0$ and $\lambda > 0$ (or other combinations of α and λ), so by $\alpha\beta < 0$, then $\beta < 0$. Now we consider the singular points of the vector field $\mathbf{u}^0 - \epsilon \mathbf{u}^1$ for all $\epsilon > 0$ sufficiently small.

$$\alpha x^2 + \beta y^2 + f(x) + y g_1(x,y) + o\left(|y|^2\right) - \epsilon\lambda - \epsilon O\left(|xy|^2\right) = 0, \tag{4.29}$$

$$-2\alpha xy - y f'(x) + y^2 g_2(x,y) - \epsilon O\left(|x|, |y|\right) = 0, \tag{4.30}$$

If $y = 0$ in (4.29), then

$$x = \pm\sqrt{\alpha^{-1}\epsilon\lambda}. \tag{4.31}$$

If $x = 0$ in (4.29), then

$$y = \pm\sqrt{\beta^{-1}\epsilon\lambda}. \tag{4.32}$$

In this case, (4.32) has no solution, then the singular points of $\mathbf{u}^0 - \epsilon\mathbf{u}^1$ is taken as follows:

$$P_1 = \left(\sqrt{\alpha^{-1}\epsilon\lambda}, 0\right), \quad P_2 = \left(-\sqrt{\alpha^{-1}\epsilon\lambda}, 0\right).$$

Finally, we shall show that the singular points of $\mathbf{u}^0 - \epsilon\mathbf{u}^1$ are non-degenerate for all $\epsilon > 0$ sufficiently small. By $\nabla.\mathbf{u} = 0$, we have

$$\det D(\mathbf{u}^0 - \epsilon\mathbf{u}^1) = -\left(\frac{\partial}{\partial x}\left(\mathbf{u}_1^0 - \epsilon\mathbf{u}_1^1\right)\right)^2 - \frac{\partial}{\partial y}\left(\mathbf{u}_1^0 - \epsilon\mathbf{u}_1^1\right) \cdot \frac{\partial}{\partial x}\left(\mathbf{u}_2^0 - \epsilon\mathbf{u}_2^1\right)$$

$$= -4\alpha^2 x^2 + 4\alpha\beta y^2,$$

which yields that

$$\det D\left(\mathbf{u}^0 - \epsilon \mathbf{u}^1\right)|_{P_1} = -4\alpha\epsilon\lambda < 0,$$
$$\det D\left(\mathbf{u}^0 - \epsilon \mathbf{u}^1\right)|_{P_2} = -4\alpha\epsilon\lambda < 0.$$

This implies that $\mathbf{u}^0 - \epsilon\mathbf{u}^1$ has two singular points for any sufficiently small $\epsilon > 0$ and these points (P_1 and P_2) are saddle points on the x-axis, as shown in Fig. 5c.

Similarly, we consider the singular points of $\mathbf{u}^0 + \epsilon\mathbf{u}^1$ for all $\epsilon > 0$ sufficiently small.

$$\alpha x^2 + \beta y^2 + f(x) + y g_1(x,y) + o\left(|xy|^2\right) + \epsilon\lambda + \epsilon O\left(|y|^2\right) = 0, \tag{4.33}$$

$$-2\alpha xy - y f^{'}(x) + y^2 g_2(x,y) + \epsilon O\left(|x|, |y|\right) = 0. \tag{4.34}$$

When we consider (4.33) together with (4.34), if $y = 0$, then

$$x = \pm\sqrt{-\alpha^{-1}\epsilon\lambda}. \tag{4.35}$$

If $x = 0$, then

$$y = \pm\sqrt{-\beta^{-1}\epsilon\lambda}. \tag{4.36}$$

We can clearly see that (4.35) has no solution and the singular points of $\mathbf{u}^0 + \epsilon\mathbf{u}^1$ are taken as follows:

$$P_1 = \left(0, \sqrt{-\beta^{-1}\epsilon\lambda}\right), \quad P_2 = \left(0, -\sqrt{-\beta^{-1}\epsilon\lambda}\right).$$

Now, we shall show that the singular points of $\mathbf{u}^0 + \epsilon\mathbf{u}^1$ are non-degenerate for all $\varepsilon > 0$ sufficiently small. By $\nabla.\mathbf{u} = 0$, we have

$$\det D(\mathbf{u}^0 + \epsilon\mathbf{u}^1) = -\left(\frac{\partial}{\partial x}\left(\mathbf{u}_1^0 + \epsilon\mathbf{u}_1^1\right)\right)^2 - \frac{\partial}{\partial y}\left(\mathbf{u}_1^0 + \epsilon\mathbf{u}_1^1\right)\cdot\frac{\partial}{\partial x}\left(\mathbf{u}_2^0 + \epsilon\mathbf{u}_2^1\right)$$
$$= -4\alpha^2 x^2 + 4\alpha\beta y^2,$$

which yields that

$$\det D\left(\mathbf{u}^0 + \epsilon \mathbf{u}^1\right)|_{P_1} = -4\alpha\varepsilon\lambda < 0,$$
$$\det D\left(\mathbf{u}^0 + \epsilon \mathbf{u}^1\right)|_{P_2} = -4\alpha\epsilon\lambda < 0.$$

In this case we proved that $\mathbf{u}^0 + \epsilon\mathbf{u}^1$ has two singular points for any sufficiently small $\epsilon > 0$. P_1 and P_2 are saddle points away from boundaries, as shown in Fig. 5a. The bifurcation process is shown in Fig. 5. □

Theorem 4.3. *Let $u \in C^1\left([0,T], E^r(TM)\right)$ $(r \geq 1)$ satisfy all conditions of Assumption (3). Then, the unfolding of codimension-one of $\mathbf{u}(x,t)$ bifurcates from (x_0, t_0) exactly two non-degenerate singular points in a small neighborhood of x_0 for any $t < t_0$ (or $t > t_0$), which are saddle points and $\mathbf{u}(x,t)$ bifurcates from (x_0, t_0) exactly two non-degenerate singular points in a small neighborhood of x_0 for any $t > t_0$ (or $t < t_0$), which are saddle points.*

Proof. The proof is similar to the proof of Theorem 4.2. We omit the computational details here. Bifurcation diagram for the Theorem 4.3 is shown in Fig. 6. □

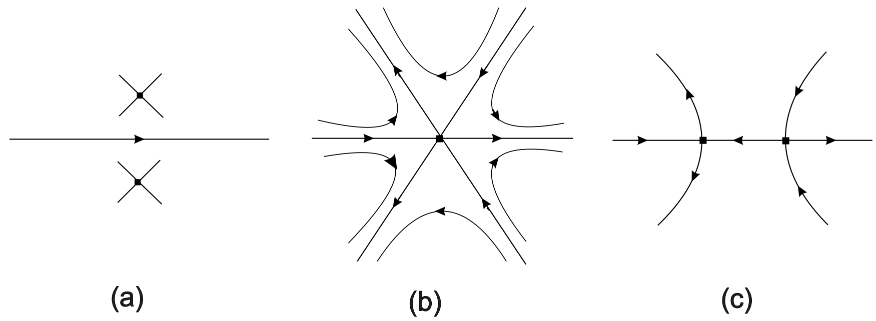

Figure 5. A description of the unfolding of codimension-one singularities for the case index -2 with symmetry about an axis. A similar diagram was also found by Gürcan and Deliceoğlu [15].

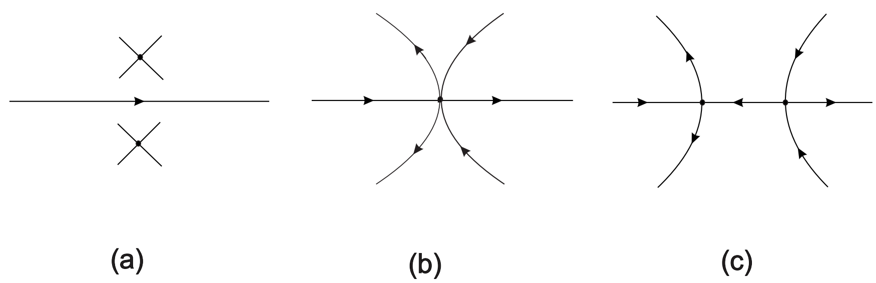

Figure 6. Schematic illustration of the unfolding of codimension-one singularities for Theorem 4.3.

Theorem 4.4. *Let $u \in C^1([0,T], E^r(TM))$ $(r \geq 1)$ satisfy all conditions of Assumption (4). Then, the unfolding of codimension-one of $\mathbf{u}(x,t)$ bifurcates from (x_0, t_0) exactly two non-degenerate singular points in a small neighborhood of x_0 for any $t < t_0$ (or $t > t_0$), which are saddle points and $\mathbf{u}(x,t)$ bifurcates from (x_0, t_0) exactly three non-degenerate singular points in a small neighborhood of x_0 for any $t > t_0$ (or $t < t_0$), two of which are saddle points and another one is a center.*

Proof. The proof is similar to the proof of Theorem 4.2. We omit the computational details here. Bifurcation diagram for the Theorem 4.4 is shown in Fig. 7. □

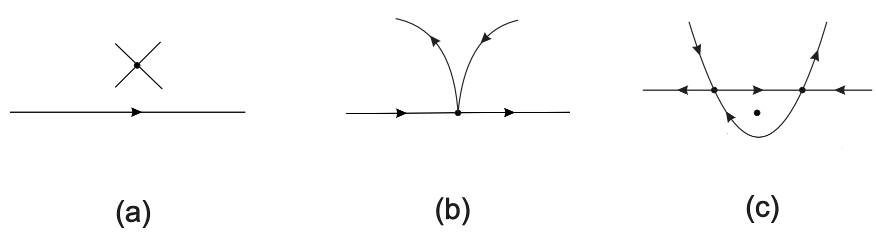

Figure 7. Schematic illustration of the unfolding of codimension-one singularities for the case index -1 with symmetry about an axis.

4.1. Genericity of Structural Bifurcation in Double-Symmetric Flow Near Non-Simple Degenerate Singular Point

Let $x_0 \in \overset{\circ}{M}$ and $0 < t_0 < T$ be given. we consider the space of double-symmetric vector fields

$$S_d = \left\{ \mathbf{u} \in C^1\left([0,T],\ F^r(TM)\right),\ \mathbf{u}^0(\mathbf{x}_0) = 0, \det D\mathbf{u}^0(\mathbf{x}_0) = 0,\ \mathbf{u}^0 = \mathbf{u}(.,t_0)\ \right\}$$

which contains all double-symmetric 2D divergence free vector fields in $C^1([0,T], F^r(TM))$, which have a local bifurcation in their local structure at $(\mathbf{x}_0, t_0)$. İt is easy to see that the subset

$$\widetilde{S_d} = \{\mathbf{u} \in S_d \mid D\mathbf{u}^0(\mathbf{x}_0) = 0,\ \frac{\partial^2(\mathbf{u}(x_0)\cdot e_1)}{\partial e_1^2} \neq 0, \quad (4.37)$$
$$\frac{\partial^2(\mathbf{u}(x_0)\cdot e_1)}{\partial e_2^2} \neq 0,\ \mathbf{u}^1(\mathbf{x}_0)\cdot e_1 \neq 0)\}$$

is open and dense in S_d, where e_1 and e_2 are unit vectors and $\mathbf{u}^1(\mathbf{x}) = \frac{\partial}{\partial t}\mathbf{u}(\mathbf{x}, t_0)$. From *Lemma 3.2,* it follows the genericity theorems of structural bifurcation.

Theorem 4.5. *For any $u \in \widetilde{S_d}$ and $ind\big(\mathbf{u}^0, \mathbf{x}_0\big) = 0$, u has a bifurcation in its local structure at $(\mathbf{x}_0, t_0)$. More precisely the unfolding of codimension-one of $\mathbf{u}(x,t)$ bifurcates from (x_0, t_0) exactly four non-degenerate singular points in a small neighborhood of x_0 for any $t < t_0$ (or $t > t_0$), two of which are saddle points and another ones are center points. $\mathbf{u}(x,t)$ has no singular point in a small neighborhood of x_0 for any $t > t_0$ (or $t < t_0$), as shown in Fig. 4.*

Theorem 4.6. *For any $u \in \widetilde{S_d}$ and ind $\left(\mathbf{u}^0, \mathbf{x}_0\right) = -2$, u has a bifurcation in its local structure at $(\mathbf{x}_0, t_0)$. More precisely the unfolding of codimension-one of $u(x,t)$ bifurcates from (x_0, t_0) exactly two non-degenerate singular points in a small neighborhood of x_0 for any $t < t_0$ (or $t > t_0$), which are saddle points and $u(x,t)$ bifurcates from (x_0, t_0) exactly two non-degenerate singular points in a small neighborhood of x_0 for any $t > t_0$ (or $t < t_0$), which are saddle points, as shown in Fig. 5.*

Proof. If $u \in \widetilde{S_d}$, then $\mathbf{u}^0 \in F^2(TM)$ and $\mathbf{u}(x,y)$ has the Taylor expansion (3.15) for $k = 1$. By using *Lemma 3.2*, we have

$$ind(\mathbf{u}^0, \mathbf{x}_0) = 0,$$

$$ind(\mathbf{u}^0, \mathbf{x}_0) = -2.$$

Then these two Genericity Theorems follow from Theorem 4.1. and Theorem 4.2, respectively. The proofs are complete. □

5. Application

As an application of the bifurcation found in the previous section, we investigate the two dimensional Stokes flow in a double lid driven cavity with two rigid walls using an analytical solution for the stream function. A detailed exploration of streamline patterns and their bifurcation in the double lid driven cavity can be found in Gürcan and Deliceoğlu ([15]) and Gürcan ([17]). They explored the mechanism of eddy generation for the case of lids moving in the same direction and the flow satisfying the symmetry condition (3.10) and (3.11) (reproduced in Figure 8). The parameter space for these cavity flows is two dimensional, described by variations in the aspect ratio A and the speed ratio which we fix $S = 1$. In Figure 9 a series of flow patterns is illustrated for different values of aspect ratio A. By variation of the aspect ratio, the flow structures (a)-(c) in Figure 9 were obtained by Gürcan and Deliceoğlu [15]. In Figure 9(a), there exists a heteroclinic connection between side eddies. By increasing the aspect ratio(A) to around 3.225, these side eddies approach each other and coalesce to produce a non-simple degenerate critical point in the middle cavity; Figure 9(b). As A is further increased, this degenerate saddle point evolves into two separatrix with a saddle point and two centers (see Figure 9(c)). This bifurcation is exactly the same as those shown in Figure 5.

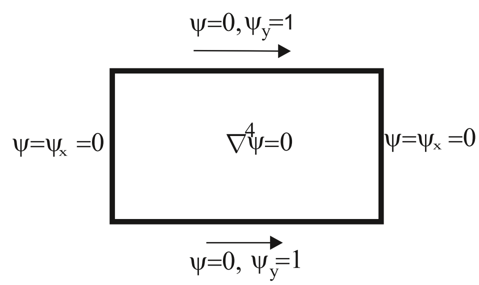

Figure 8. The boundary value problem for the double lid-driven cavity with two solid walls.

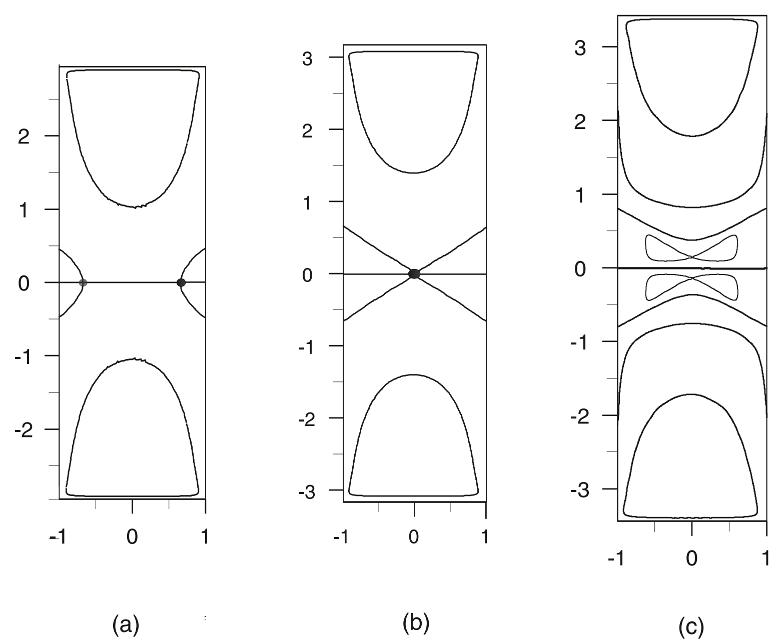

Figure 9. Flow structure development found by Gürcan and Deliceoğlu ([15]) in the double lid driven cavity as A increases from $A = 2.95$ to 3.45 for $S = 1$; (a) $A = 2.95$, (b) $A = 3.225$, (c) $A = 3.45$.

The streamline topologies of two-dimensional planar and axisymmetric peristaltic flow have been investigated by Jiménez-Lozano and Sen [25]. They found flow structures (a)-(e) shown in Figure 10. They obtained two non-simple degenerate critical points as the flow rate q and the amplitude ratio ϕ are varied. The flow patterns (b) and (d) correspond to the non-simple degenerate critical points with index 0 and -1, respectively. The sequence (a)-(c) corresponds to the bifurcation in Figure 4 when ϵ is changed from negative to positive. Jiménez-Lozano and Sen [25] also stated that a degenerate point with six heteroclinic connections can be seen at $q = -2/15$ (Figure 10(d)) which corresponds to the non-simple degenerate point with index -2 in Figure (5).

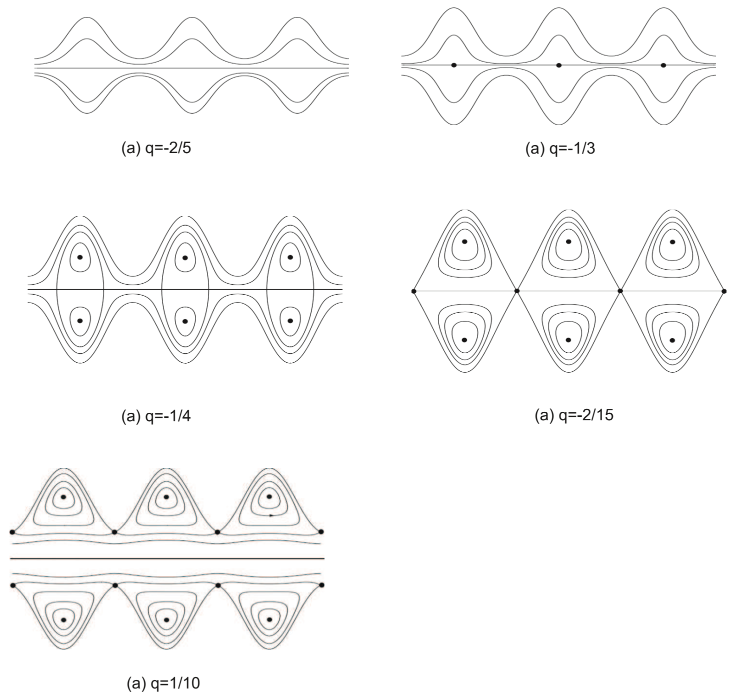

Figure 10. Streamline patters of peristaltic flow in moving frame for different values of the flow rate q with fixed amplitude ratio $\phi = 0.6$, reproduced from Ref. [25].

Bifurcations and eddy genesis of Stokes flow within a sectorial cavity consisting of two stationary side walls and both lids moving studied by Gürcan et al. [18]. Lids moving in the same radial direction with equal speed, (i.e. $S = 1$), the flow structure is symmetric about $\theta = 0$ for all values of A. The corresponding flow structures at the three aspect ratio $A = 1.65$ (before the bifurcation), $A = 1.641$ (the degenerate critical point) and $A = 1.62$ (after the bifurcation) are given in Figure 11. The flow pattern at the aspect ratio $A = 1.641$ is structurally unstable. This numerical result agrees with the result outlined in Figure 5. The two-dimensional flow induced by an oscillatory magnetic obstacle has been analyzed by Beltrn et al. [4]. The problem is characterized by three parameters: the oscillation Reynolds number, Re_ω, the Hartmann number, Ha, and the dimensionless amplitude of the magnetic obstacle oscillation, D. The motion is periodic and can be described as a function of the phase $\pi < \phi < \pi$ in the cycle. They show instantaneous streamlines for $D = 0.01$, $Ha = 100$, and $Re_\omega = 1$ in Figure 12. A more detailed description of this application

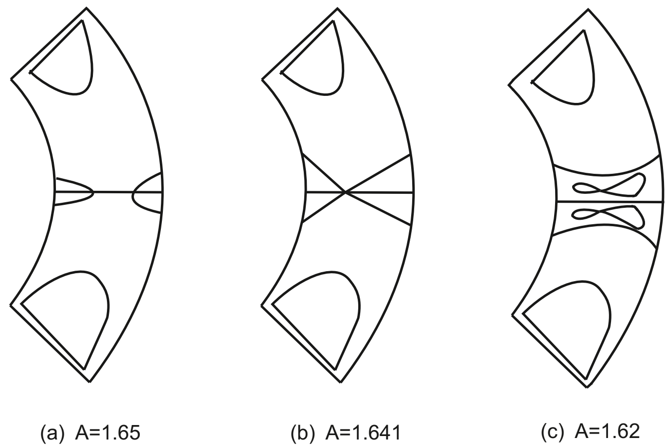

Figure 11. Illustration of flow structure in a sectorial cavity with lids moving in the same direction. (a) $(A, S) = (1.65, 1)$, (b) $(A, S) = (1.641, 1)$, (c) $(A, S) = (1.62, 1)$.

can be found in Beltrn et al. [4].

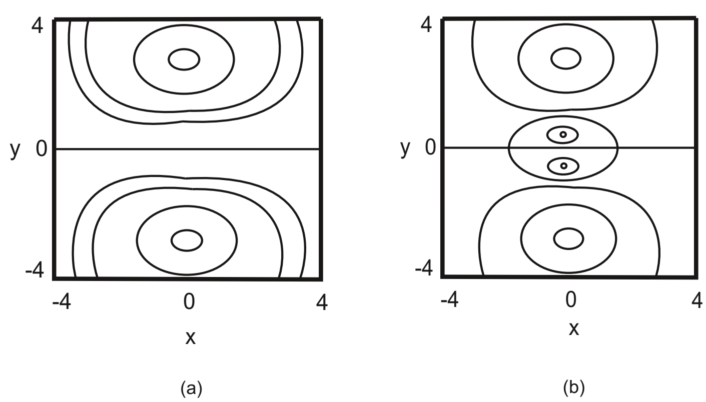

Figure 12. Instantaneous streamlines for $D = 0.01$, $Ha = 100$ and $Re_\omega = 1$: (a) $\phi = -927\pi/1000$ and (b) $\phi = -752\ pi/1000$, reproduced from Ref. [4]

6. Conclusion

In this paper, flow topology near the non-simple-degenerate critical points with symmetry and their bifurcation are investigated from a topological point of view. Using a homotopy invariance of the index we develop a theory for the sufficient and necessary conditions for structural bifurcation of a divergence free-vector fields near non-simple degenerate critical points. We obtain new flow patters (the degenerate cusp point appears on the x-axis) that appears only near the non-simple degenerate critical point. Also, both kinematic and structural bifurcation theories for incompressible flows near non-simple degenerate singular points with double symmetry have been derived. We show that the two structural bifurcation scenarios near the non-simple degenerate points we obtain are indeed generic for flows with double symmetry. The theory was applied to the pattern found numerically in the studies of Stokes flow in a double lid driven cavity and in two dimensional peristaltic flow.

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