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FIXED POINT THEOREMS FOR MULTIVALUED NON-LINEAR *F*-CONTRACTIONS ON QUASI METRIC SPACES WITH AN APPLICATION

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Abstract In this paper, we introduce (α, β)-type $F-\tau$ contraction and utilize the same to prove some fixed point results for multivalued mappings in quasi metric spaces. Furthermore, we furnish with some examples to exhibit the utility of our results. As an application, we establish the existence of a solution for a non-linear integral equation.

Keywords Quasi metric space, left $\mathcal{K}$-completeness, left $\mathcal{M}$-completeness, (α, β)-type $F-\tau$ contraction.

MSC(2010) 54H25, 47H10.

1. Introduction

In 1922, Banach formulated his celebrated contraction principle which has been extended and generalized in many directions with several applications in varied domains. The enormous applications of fixed point theory had always inspired the research activity of this domain. In 2012, Wardowski [22] obtained a novel generalization of Banach contraction when he coined the idea of F-contraction and proved that every F-contraction mapping on a complete metric space possesses a unique fixed point. Thereafter, several authors enriched this concept in several respects (*e.g.* [1,6,10,12,13,15,18,21,23]).

Recently, Acar et al. [2] introduced the concept of generalized multivalued F-contraction (mappings). This idea attracted the attention of several researchers (*e.g.* [4,8,9,11,17]). In 2017, Dag et al. [8] established fixed point results for multivalued F-contraction on quasi metric spaces under suitable types of completeness.

Inspired by the work of Dag et al. [8], in this paper, we introduce a generalized F-contraction namely "(α, β)-type $F-\tau$ contraction" and utilize the same to prove some fixed point results for multi-valued F-contraction mappings in quasi metric spaces using certain types of completeness assumptions. An example is also furnished to exhibit that our results are proper generalizations of the corresponding results contained in Dag et al. [8]. Additionally, we point out some fallacies in the results due to Iqbal et al. [14]. We also deduce a fixed point result for single valued mappings in the setting of Hausdorff T_1-quasi metric space and likewise adopt

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an example to exhibit that Hausdorffness condition can not be dropped. Finally, as an application, we establish the existence of a solution for a non-linear integral equation.

2. Preliminaries

In this section, we present some definitions, related notions and basic results needed in our subsequent discussions.

Definition 2.1 ([24]). Let M be a non-empty set. A mapping $q : M \times M \to [0, \infty)$ is said to be a quasi-pseudo metric if it satisfies the following conditions ($\forall x, y, z \in M$):

(i) $q(x, x) = 0$;

(ii) $q(x, y) \leq q(x, z) + q(z, y)$.

If, in addition, q satisfies the following:

(iii) $q(x, y) = q(y, x) = 0 \implies x = y$,

then q is said to be a quasi metric. Further, q is called T_1-quasi metric if, instead of (iii), the following condition is satisfied:

(iii)* $q(x, y) = 0 \implies x = y$.

The pair (M, q) is said to be quasi-pseudo, quasi and T_1-quasi metric space respectively.

Remark 2.1.

(i) Every metric is T_1-quasi metric, every T_1-quasi metric is quasi metric and every quasi metric is quasi-pseudo metric.

(ii) Each quasi-pseudo metric q on M generates a topology τ_q on M which has a base the family of open balls $\{B_q(z, \epsilon) : z \in M, \epsilon > 0\}$, where $B_q(z, \epsilon) = \{y \in M : q(z, y) < \epsilon\}$.

(iii) If q is a quasi-pseudo metric on M, then q^{-1} defined by $q^{-1}(z, y) = q(y, z)$ is also a quasi-pseudo metric.

(iv) The closure of a subset $A \subset M$ with respect to τ_q and τ_q^{-1} are denoted by $Cl_q(A)$ and $Cl_{q^{-1}}(A)$, respectively. If (M, q) is a quasi metric space, A a non-empty subset of M and $z \in M$, then

$$z \in Cl_q(A) \iff q(z, A) = \inf\{q(z, a) : a \in A\} = 0$$

and

$$z \in Cl_{q^{-1}}(A) \iff q(A, z) = \inf\{q(a, z) : a \in A\} = 0.$$

(v) If (M, q) is a metric space and A is a compact subset of M, then for each $z \in M$, there is $a \in A$ such that $q(z, a) = q(z, A)$. But in the case of quasi metric space, this property is not satisfied.

Let (M, q) be a quasi-pseudo metric space. Then the upper (resp. lower) Hausdorff quasi-pseudo metric H_q^+ (resp. H_q^-) on $\mathcal{P}(M)$ is defined by:

$$H_q^+(A_1, A_2) = \sup_{a_2 \in A_2} q(A_1, a_2) \text{ (resp. } H_q^-(A_1, A_2) = \sup_{a_1 \in A_1} q(a_1, A_2)), \ \forall A_1, A_2 \in \mathcal{P}(M),$$

where $\mathcal{P}(M)$ denotes the power set of M. Moreover,

$$H(A_1, A_2) = \max\{H_q^-(A_1, A_2), H_q^+(A_1, A_2)\}$$

is called Hausdorff quasi-pseudo metric on $CB_q(M)$, the family of all non-empty τ_q-closed and bounded subsets of M. For further details, one can consult [5, 16, 19].

Definition 2.2 ([7]). Let (M, q) be a quasi metric space. A sequence $\{z_n\}$ in M is q-convergent (resp. q^{-1}-convergent) if $\{z_n\}$ converges to z with respect to τ_q (resp. $\tau_{q^{-1}}$) and is denoted by $z_n \overset{q}{\to} z$ (resp. $z_n \overset{q^{-1}}{\to} z$), i.e.

$$z_n \overset{q}{\to} z \implies \lim_{n\to\infty} q(z, z_n) = 0,$$

$$\text{(resp. } z_n \overset{q^{-1}}{\to} z \implies \lim_{n\to\infty} q(z_n, z) = 0).$$

Definition 2.3 ([7]). Let (M, q) be a quasi metric space. Then (M, q) is said to be

(i) left $\mathcal{K}$-Cauchy (/forward Cauchy) if for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\forall n, k,\ n \geq k \geq n_0$, $q(z_k, z_n) < \epsilon$ or we can say $\sum_{n=1}^{\infty} q(z_n, z_{n+1}) < \infty$.

(ii) right $\mathcal{K}$-Cauchy (/backward Cauchy) if for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\forall n, k,\ n \geq k \geq n_0$, $q(z_n, z_k) < \epsilon$ or we can say $\sum_{n=1}^{\infty} q(z_{n+1}, z_n) < \infty$.

If a sequence is left $\mathcal{K}$-Cauchy with respect to q, then it is right $\mathcal{K}$-Cauchy with respect to q^{-1}.

Definition 2.4 ([7]). Let (M, q) be a quasi metric space. Then (M, q) is said to be

(i) left (right) $\mathcal{K}$-complete if every left (right) $\mathcal{K}$-Cauchy sequence is q-convergent.

(ii)left (right) $\mathcal{M}$-complete if every left (right) $\mathcal{K}$-Cauchy sequence is q^{-1}-convergent.

For more details about the convergence, Cauchyness and completeness of quasi metric, one can go through [7].

In 2012, Wardowski [22] introduced the notion of F-contraction (which we denote by Ξ) and to accomplish this, he utilized the following:

Definition 2.5. Let $F : (0, \infty) \to \mathbb{R}$ be a function satisfying the following conditions:

(F_1) F is strictly increasing;

(F_2) for each sequence $\{p_n\}_{n=1}^{\infty}$ of positive numbers,

$$\lim_{n\to\infty} p_n = 0 \iff \lim_{n\to\infty} F(p_n) = -\infty;$$

(F_3) there exists $k \in (0, 1)$ such that $\lim_{p\to 0^+} p^k F(p) = 0$;

(F_4) $F(\inf A) = \inf F(A)$, $\forall A \subset (0, \infty)$ with $\inf A > 0$.

We denote the family of all functions F satisfying $(F_1)-(F_3)$ by Ξ, while Ξ_* stands for the family of all functions F in Ξ which satisfy (F_4).

Definition 2.6 ([22]). Let (M, q) be a metric space. A mapping $S : M \to M$ is said to be an F-contraction on (M, q), if there exist $F \in \Xi$ and $\tau > 0$ such that $\forall z, y \in M$,

$$q(Sz, Sy) > 0 \implies \tau + F(q(Sz, Sy)) \leq F(q(z, y)).$$

Wardowski proved the following fixed point result involving F-contraction:

Theorem 2.1 ([22]). *Let (M, q) be a complete metric space and $S : M \to M$ an F-contraction. Then S has a unique fixed point $z^* \in M$ and for every $z \in M$, the sequence $\{S^n z\}_{n\in\mathbb{N}}$ converges to z^*.*

Let (M, q) be a quasi metric space, $C_q(M)$ the family of all non-empty τ_q-closed subsets of M and $K_q(M)$ the family of all non-empty τ_q-compact subsets of M. Dag et al. [8] defined a new class $A_q(M)$ of subsets of M as:

$$A_q(M) = \{A \subseteq M : \text{there exists } y = y(z) \in A, \text{ such that } q(z, A) = q(z, y), \ \forall z \in M\}.$$

Definition 2.7 ([8]). Let (M, q) be a quasi metric space and $S : M \to \mathcal{P}(M)$ a multivalued mapping, $F \in \Xi$ and $\mu \geq 0$. For $z \in M$ with $q(z, Sz) > 0$, define a set $F^z_\mu \subseteq M$ as

$$F^z_\mu = \{y \in Sz : F(q(z, y)) \leq F(q(z, Sz)) + \mu\}.$$

Dag et al. [8] established the following facts:

- If $S : M \to A_q(M)$, then $F^z_\mu \neq \emptyset$, $\forall \mu \geq 0$ and $z \in M$.
- If $S : M \to K_q(M)$, then F^z_μ may be empty for some $z \in M$ and $\mu > 0$.
- If $S : M \to C_q(M)$, then F^z_μ may be empty for some $z \in M$ and $\mu > 0$ but if $F \in \Xi_*$, then $F^z_\mu \neq \emptyset$, $\forall z \in M$ and $\mu > 0$.

Definition 2.8 ([20]). Let $S : M \to M$ be a self-mapping and $\alpha, \beta : M \times M \to [0, \infty)$. Then S is said to be

(i) an α-admissible mapping if

$$\alpha(z, y) \geq 1 \implies \alpha(Sz, Sy) \geq 1, \ \forall z, y \in M.$$

(ii) a β-subadmissible mapping if

$$\beta(z, y) \leq 1 \implies \alpha(Sz, Sy) \leq 1, \ \forall z, y \in M.$$

Definition 2.9 ([3]). Let $S : M \to \mathcal{P}(M)$ be a multivalued mapping and $\alpha, \beta : M \times M \to [0, \infty)$. Then S is said to be

(i) a generalized α_*-admissible mapping if for $z, y \in M$,

$$\alpha(z, y) \geq 1 \implies \alpha(u, v) \geq 1, \ \forall u \in Sz, \ v \in Sy.$$

(ii) a generalized β_*-subadmissible mapping if for $z, y \in M$,

$$0 < \beta(z, y) \leq 1 \implies 0 < \beta(u, v) \leq 1, \ \forall u \in Sz, \ v \in Sy.$$

Iqbal et al. [14] introduced the notion of α-type $F-\tau$ contraction as follows:

Definition 2.10 ([14])**.** Let (M,q) be a metric space, $S: M \to \mathcal{P}(M)$ a multivalued mapping and $F \in \Xi$. Then we say that S is an α-type $F-\tau$ contraction on M, if there exists $\mu > 0$ with $\tau : (0,\infty) \to (\mu,\infty)$ and $\alpha : M \times M \to \{-\infty\} \cup (0,\infty)$ such that $\forall z \in M$, $y \in F^z_\mu$ with $q(z,Sz) > 0$ satisfying

$$\tau(q(z,y)) + \alpha(z,y)F(q(y,Sy)) \leq F(\mathcal{M}_q(z,y)),$$

where

$$\mathcal{M}_q(z,y) = \max\left\{q(z,y), q(z,Sz), q(y,Sy), \frac{q(z,Sy)+q(y,Sz)}{2},\right.$$
$$\left.\frac{q(y,Sy)(1+q(z,Sz))}{1+q(z,y)}, \frac{q(y,Sz)(1+q(z,Sy))}{1+q(z,y)}\right\}. \tag{2.1}$$

In [14], the following result was established:

Theorem 2.2 ([14])**.** *Let (M,q) be a complete metric space and $S: M \to K_q(M)$ an α-type $F-\tau$ contraction on X. Suppose that the following conditions are satisfied:*

(i) S is generalized α_*-admissible mapping;

(ii) the map $z \to q(z,Sz)$ is lower semi-continuous;

(iii) there exists $z_0 \in M$ and $z_1 \in Sz_0$ such that $\alpha(z_0,z_1) \geq 1$;

(iv) τ satisfies

$$\liminf_{s\to\omega^+} \tau(s) > \mu, \ \forall \omega \geq 0.$$

Then S has a fixed point in M.

Remark 2.2. Theorem 2.2 is incorrect in its presented form. See Examples 2.1 and 2.2 in [12].

3. Main Results

Before presenting our main results, we introduce the below definition:

Definition 3.1. Let (M,q) be a quasi metric space, $S: M \to \mathcal{P}(M)$ a multivalued mapping and $F \in \Xi$. Then we say that S is an (α,β)-type $F-\tau$ contraction on M, if $\forall z \in M$ with $q(z,Sz) > 0$, there exists $\mu > 0$ such that $y \in F^z_\mu$ satisfying

$$\tau(q(z,y)) + F(\alpha(z,y)q(y,Sy)) \leq F(\beta(z,y)\mathcal{M}_q(z,y)), \tag{3.1}$$

where $\tau : (0,\infty) \to (\mu,\infty)$, $\alpha, \beta : M \times M \to (0,\infty)$.

Now, we are equipped to state and prove our main result as follows:

Theorem 3.1. *Let (M,q) be a left $\mathcal{K}$-complete T_1-quasi metric space and $S: M \to A_q(M)$ an (α,β)-type $F-\tau$ contraction on M. Suppose that the following conditions are satisfied:*

(i) S is generalized α_*-admissible and β_*-subadmissible mapping;

(ii) $z \to q(z, Sz)$ *is lower semicontinuous with respect to* τ_q*;*

(iii) *there exists* $z_0 \in M$ *and* $z_1 \in F_\mu^{z_0}$ *such that* $0 < \beta(z_0, z_1) \leq 1 \leq \alpha(z_0, z_1)$*;*

(iv) τ *satisfies*

$$\liminf_{s \to \omega^+} \tau(s) > \mu, \ \forall \omega \geq 0.$$

Then S *has a fixed point in* M*.*

Proof. Let on contrary that S has no fixed point. Then $\forall z \in M$, we have $q(z, Sz) > 0$. Since $Sz \in A_q(M)$ for every $z \in M$, so for any $\mu > 0$, $F_\mu^z \neq \emptyset$. Choose $z_0 \in M$ as in the condition *(iii)* and $z_1 \in F_\mu^{z_0}$ such that $0 < \beta(z_0, z_1) \leq 1 \leq \alpha(z_0, z_1)$. Observe that $z_1 \notin Sz_1$, otherwise z_1 is a fixed point of Sz_1, a contradiction. On using (3.1), we have

$$\tau(q(z_0, z_1)) + F(\alpha(z_0, z_1)q(z_1, Sz_1)) \leq F(\beta(z_0, z_1)\mathcal{M}_q(z_0, z_1)), \tag{3.2}$$

where

$$\begin{aligned}
\mathcal{M}_q(z_0, z_1) = & \max\left\{q(z_0, z_1), q(z_0, Sz_0), q(z_1, Sz_1), \frac{q(z_1, Sz_0) + q(z_0, Sz_1)}{2},\right. \\
& \left.\frac{q(z_1, Sz_1)(1 + d(z_0, Sz_0))}{1 + q(z_0, z_1)}, \frac{q(z_1, Sz_0)(1 + q(z_0, Sz_1))}{1 + q(z_0, z_1)}\right\} \\
\leq & \max\left\{q(z_0, z_1), q(z_0, z_1), q(z_1, y), \frac{q(z_1, z_1) + q(z_0, y)}{2},\right. \\
& \left.\frac{q(z_1, y)(1 + q(z_0, z_1))}{1 + q(z_0, z_1)}, \frac{q(z_1, z_1)(1 + q(z_0, y))}{1 + q(z_0, z_1)}\right\} \ \text{(for any } y \in Sz_1) \\
= & \max\left\{q(z_0, z_1), q(z_1, y), \frac{q(z_0, y)}{2}\right\} \\
\leq & \max\left\{q(z_0, z_1), q(z_1, y), \frac{q(z_0, z_1) + q(z_1, y)}{2}\right\} \\
= & \max\{q(z_0, z_1), q(z_1, y)\}.
\end{aligned}$$

Suppose that $q(z_1, y) > q(z_0, z_1)$, then $\mathcal{M}_q(z_0, z_1) \leq q(z_1, y)$.

Now, using (3.2), we have

$$\begin{aligned}
\tau(q(z_0, z_1)) + F(q(z_1, Sz_1)) \leq & \tau(q(z_0, z_1)) + F(\alpha(z_0, z_1)q(z_1, Sz_1)) \\
\leq & F(\beta(z_0, z_1)q(z_1, y)), \ \text{(for any } y \in Sz_1) \\
\leq & F(q(z_1, y))
\end{aligned}$$

so that $\tau(q(z_0, z_1)) \leq 0$, which is a contradiction. Hence, we must infer that $q(z_0, z_1) \geq q(z_1, y)$. Therefore, we have

$$\tau(q(z_0, z_1)) + F(\alpha(z_0, z_1)q(z_1, Sz_1)) \leq F(\beta(z_0, z_1)q(z_0, z_1)).$$

Now, for $z_1 \in M$, there exists $z_2 \in F_\mu^{z_1}$ with $z_2 \notin Sz_2$. As S is generalized α_*-admissible and β_*-subadmissible, we have $0 < \beta(z_1, z_2) \leq 1 \leq \alpha(z_1, z_2)$. Again on the similar lines as above, we obtain

$$\tau(q(z_1, z_2)) + F(\alpha(z_1, z_2)q(z_2, Sz_2)) \leq F(\beta(z_1, z_2)q(z_1, z_2)).$$

Recursively, we get a sequence $\{z_n\}$ in M, where $z_{n+1} \in F_\mu^{z_n}$, $z_{n+1} \notin Sz_{n+1}$, $0 < \beta(z_n, z_{n+1}) \leq 1 \leq \alpha(z_n, z_{n+1})$ and $M(z_n, z_{n+1}) \leq q(z_n, z_{n+1})$ satisfying

$$\tau(q(z_n, z_{n+1})) + F(q(z_{n+1}, Sz_{n+1})) \leq F(q(z_n, z_{n+1})). \tag{3.3}$$

Now, as $z_{n+1} \in F_\mu^{z_n}$, we have

$$F(q(z_n, z_{n+1})) \leq F(q(z_n, Sz_n)) + \mu. \tag{3.4}$$

Using (3.3) and (3.4), we get

$$F(q(z_{n+1}, Sz_{n+1})) \leq F(q(z_n, Sz_n)) + \mu - \tau(q(z_n, z_{n+1})) \tag{3.5}$$

and

$$F(q(z_{n+1}, z_{n+2})) \leq F(q(z_n, z_{n+1})) + \mu - \tau(q(z_n, z_{n+1})). \tag{3.6}$$

Denote $w_n = q(z_n, z_{n+1})$, $\forall n \in \mathbb{N}$. Then $w_n > 0$ and from (3.6), the sequence $\{w_n\}$ is a decreasing sequence of non-negative real numbers. Therefore, there exists $l \geq 0$ such that $\lim_{n\to\infty} w_n = l$.

We assert that $l = 0$. On contrary, let $l > 0$, then

$$\begin{aligned} F(w_{n+1}) \leq & F(w_n) + \mu - \tau(w_n) \\ \leq & F(w_{n-1}) + 2\mu - \tau(w_n) - \tau(w_{n-1}) \\ & \vdots \\ \leq & F(w_0) + (n+1)\mu - \tau(w_n) - \tau(w_{n-1}) - ... - \tau(w_0). \end{aligned} \tag{3.7}$$

Let $\tau(w_{p_n}) = \min\{\tau(w_0), \tau(w_1), ..., \tau(w_n)\}$, $\forall n \in \mathbb{N}$. Hence, from (3.7), we obtain that

$$F(w_n) \leq F(w_0) + n(\mu - \tau(w_{p_n})). \tag{3.8}$$

Similarly, using (3.5), we get

$$F(q(z_n, Sz_n)) \leq F(q(z_0, Sz_0)) + n(\mu - \tau(w_{p_n})). \tag{3.9}$$

For the sequence $\{\tau(w_{p_n})\}$, two cases arise:

Case-I: If for each $n \in \mathbb{N}$, there is $m > n$ such that $\tau(w_{p_n}) > \tau(w_{p_m})$, then we have a subsequence $\{w_{p_{n_k}}\}$ of $\{w_{p_n}\}$ with $\tau(w_{p_{n_k}}) > \tau(w_{p_{n_k+1}})$, $\forall k \in \mathbb{N}$.
Since $w_{p_{n_k}} \to l$ and $l > 0$, so we have

$$\liminf_{k\to\infty} \tau(w_{p_{n_k}}) > \mu.$$

Hence,

$$F(w_{n_k}) \leq F(w_0) + n_k(\mu - \tau(w_{p_{n_k}})),\ \forall k \in \mathbb{N}.$$

Consequently, $\lim_{k\to\infty} F(w_{n_k}) = -\infty$ and by (F_2), we get $\lim_{k\to\infty} w_{p_{n_k}} = 0$, a contradiction.

Case-II: If there exists some $n_0 \in \mathbb{N}$ such that $\tau(w_{p_{n_0}}) = \tau(w_m)$, $\forall m > n_0$, then

$$F(w_m) \leq F(w_0) + m(\mu - \tau(w_{p_{n_0}})),\ \forall m > n_0.$$

This implies $\lim_{m\to\infty} F(w_m) = -\infty$ and hence by (F_2), $\lim_{m\to\infty} w_{p_m} = 0$, a contradiction. Therefore, from both the cases, we conclude that $l = 0$ and hence,

$$\lim_{n\to\infty} q(z_n, z_{n+1}) = 0.$$

Thus, in view of (F_3), there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} w_n^k F(w_n) = 0.$$

Now, from (3.8), we obtain

$$w_n^k F(w_n) - w_n^k F(w_0) \leq w_n^k n(\mu - \tau(w_{p_n})) \leq 0.$$

Letting $n \to \infty$ in the above equation, we obtain that $\lim_{n\to\infty} n w_n^k = 0$. So, there exists $n_0 \in \mathbb{N}$ such that $n w_n^k \leq 1$, $\forall n \geq n_0$, i.e.,

$$w_n \leq \frac{1}{n^{\frac{1}{k}}},\ \forall n \geq n_0.$$

To show that $\{z_n\}$ is left $\mathcal{K}$-Cauchy sequence, consider $m, n \in \mathbb{N}$, with $m > n \geq n_0$ and using triangle inequality, we have

$$\begin{aligned} q(z_n, z_m) \leq& q(z_n, z_{n+1}) + q(z_{n+1}, z_{n+2}) + \ldots + q(z_{m-1}, z_m) \\ =& \sum_{i=n}^{m-1} q(z_i, z_{i+1}) \leq \sum_{i=n}^{\infty} q(z_i, z_{i+1}) \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

Since $\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ is convergent, therefore on taking limit $n \to \infty$, we get

$$\lim_{m,n\to\infty} q(z_n, z_m) = 0.$$

Hence, $\{z_n\}$ is a left $\mathcal{K}$-Cauchy sequence in T_1-quasi metric space (M, q) and since (M, q) is left $\mathcal{K}$-complete, there exists $z \in M$ such that $\{z_n\}$ is q-convergent to z, i.e.

$$\lim_{n\to\infty} q(z, z_n) = 0.$$

Also, from (3.9) and (F_2), we have

$$\lim_{n\to\infty} q(z_n, Sz_n) = 0.$$

As $z \to q(z, Sz)$ is lower semi continuous with respect to τ_q, we have

$$0 < q(z, Sz) \leq \liminf_{n\to\infty} q(z_n, Sz_n) = 0,$$

a contradiction. Therefore, S must have a fixed point. This concludes the proof. □

Remark 3.1. Theorem 3.1 is a corrected as well as a sharpened version of Theorem 2.2 in the setting of T_1-quasi metric space.

Next, we prove a result analogous to Theorem 3.1 in case (M, q) is left $\mathcal{M}$-complete T_1-quasi metric space.

Theorem 3.2. *Let (M,q) be a left $\mathcal{M}$-complete T_1-quasi metric space, $S : M \to A_q(M)$ an (α,β)-type $F-\tau$ contraction on M. If we replace condition (ii) of Theorem 3.1 by the following one (besides retaining the rest of the hypotheses):*

*(ii)** $z \to q(z,Sz)$ *is lower semicontinuous with respect to* $\tau_{q^{-1}}$,

then S has a fixed point.

Proof. Let on contrary that S has no fixed point. On the lines of the proof of Theorem 3.1, we can construct a left $\mathcal{K}$-Cauchy sequence $\{z_n\}$. Now, using the left $\mathcal{M}$-completeness of (M,q), there exists $z \in M$ such that $\{z_n\}$ is q^{-1}-convergent to z, i.e., $q(z_n,z) \to 0$ as $n \to \infty$. Observe that $\lim_{n\to\infty} q(z_n,Sz_n) = 0$ and $z \to q(z,Sz)$ is lower semi-continuous under the topology $\tau_{q^{-1}}$ so that $0 < q(z,Sz) \leq \liminf_{n\to\infty} q(z_n,Sz_n) = 0$, a contradiction. Hence, S must have a fixed point. □

In the following results, we demonstrate that the existence of fixed point of $S : M \to C_q(M)$ (in our above results) can be ascertained, if we take $F \in \Xi_*$.

Theorem 3.3. *Let (M,q) be a $\mathcal{K}$-complete quasi metric space and $S : M \to C_q(M)$ an (α,β)-type $F-\tau$ contraction on M with $F \in \Xi_*$, satisfying all the hypotheses of Theorem 3.1. Then S has a fixed point.*

Proof. Suppose S has no fixed point, then $q(z,Sz) > 0$. Otherwise, if $q(z,Sz) = 0$, then $z \in Cl_q(Sz) = Sz$ and hence, z is a fixed point of S, a contradiction. Since $Sz \in Cl_q(M)$ for every $z \in M$, so for any $\mu > 0$, $F^z_\mu \neq \emptyset$ for $F \in \Xi_*$.

In the same way as in Theorem 3.1, choose $z_0 \in M$ and $z_1 \in F^{z_0}_\mu$ such that $0 < \beta(z_0,z_1) \leq 1 \leq \alpha(z_0,z_1)$ with $z_1 \notin Sz_1$. On using (2.1), we have

$$\tau(q(z_0,z_1)) + F(\alpha(z_0,z_1)q(z_1,Sz_1)) \leq F(\beta(z_0,z_1)\mathcal{M}_q(z_0,z_1)), \tag{3.10}$$

where

$$\begin{aligned}\mathcal{M}_q(z_0,z_1) = \max\Bigg\{&q(z_0,z_1), q(z_0,Sz_0), q(z_1,Sz_1), \frac{q(z_1,Sz_0)+q(z_0,Sz_1)}{2},\\ &\frac{q(z_1,Sz_1)(1+q(z_0,Sz_0))}{1+q(z_0,z_1)}, \frac{q(z_1,Sz_0)(1+q(z_0,Sz_1))}{1+q(z_0,z_1)}\Bigg\}.\end{aligned}$$

On the lines of the proof of Theorem 3.1, we obtain

$$\mathcal{M}_q(z_0,z_1) \leq \max\{q(z_0,z_1), q(z_1,y)\}, \text{ for any } y \in Sz_1.$$

Now, suppose $q(z_1,y) > q(z_0,z_1)$, then $\mathcal{M}_q(z_0,z_1) \leq q(z_1,y)$.

On using (3.10), we have

$$\tau(q(z_0,z_1)) + F(q(z_1,Sz_1)) \leq F(q(z_1,y))$$

and for $y \in F^{z_1}_\mu$, we obtain $\tau(q(z_0,z_1)) \leq \mu$, which is a contradiction to our assertion. Hence, we must have $q(z_0,z_1) \geq q(z_1,y)$, for every $y \in Sz_1$ so that

$$\tau(q(z_0,z_1)) + F(\alpha(z_0,z_1)q(z_1,Sz_1)) \leq F(\beta(z_0,z_1)q(z_0,z_1)).$$

Rest of the proof can be completed on the lines of the proof of Theorem 3.1. □

Similarly, we can also prove the following result:

Theorem 3.4. *Let (M,q) be a left $\mathcal{M}$-complete quasi metric space, $S : M \to C_q(M)$ an (α,β)-type $F-\tau$ contraction on M and $F \in \Xi_*$ satisfying all the hypotheses of Theorem 3.2. Then S has a fixed point.*

The following example is adopted to demonstrate that Theorem 3.4 is a proper generalization of Theorem 4 of [8].

Example 3.1. Let $M = \left\{\frac{1}{2^{n-1}}; n \in \mathbb{N}\right\} \cup \{0\}$ endowed with the metric $q : M \times M \to [0,\infty)$ defined by

$$q(z,y) = \begin{cases} 0, & \text{if } z = y, \\ |y|, & \text{if } z \neq y. \end{cases}$$

Then (M,q) is left $\mathcal{M}$-complete quasi metric space. Define $S : M \to C_q(M)$, $\alpha,\beta : M \times M \to [0,\infty)$ and $F : (0,\infty) \to \mathbb{R}$ by

$$Sz = \begin{cases} \left\{\frac{1}{2^{n+1}}, 1\right\}, & \text{if } z = \frac{1}{2^n}, \\ \{0,1\}, & \text{if } z \in \{0,1\}. \end{cases}$$

$$\alpha(z,y) = \begin{cases} 2, & \text{if } z,y \in \{0,1\}, \\ \frac{1}{2}, & \text{otherwise }. \end{cases}$$

$$\beta(z,y) = 1,\ \forall z,y \in M$$

and

$$F(\alpha) = \ln \alpha,\ \forall \alpha \in (0,\infty),$$

wherein we have

$$q(z,Sz) = \begin{cases} \frac{1}{2^{n+1}}, & \text{if } z = \frac{1}{2^n}, \\ 0, & \text{if } z \in \{0,1\}. \end{cases}$$

Observe that $z \to q(z,Sz)$ is lower semi-continuous with respect to $\tau_{q^{-1}}$.

Let $q(z,Sz) > 0$, then $z = \frac{1}{2^n}$, for $n \in \mathbb{N}$ so that $Sz = \left\{\frac{1}{2^{n+1}}, 1\right\}$ and $y = \frac{1}{2^{n+1}} \in Sz$ such that

$$F(q(z,y)) - F(q(z,Sz)) = F\left(\frac{1}{2^{n+1}}\right) - F\left(\frac{1}{2^{n+1}}\right) = 0.$$

Thus, on taking $\tau(t) = 1,\ \forall t \in (0,\infty)$ for $\mu = \frac{1}{2}$, we get $y \in F_\mu^z$ so that

$$\mathcal{M}_q(z,y) = \max\left\{\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}, \frac{\frac{1}{2^{n+2}} + \frac{1}{2^{n+1}}}{2}, \frac{1}{2^{n+2}}, \frac{\frac{1}{2^{n+1}}(1 + \frac{1}{2^{n+2}})}{1 + \frac{1}{2^{n+1}}}\right\} = \frac{1}{2^{n+1}}.$$

Henceforth,

$$\begin{aligned} \tau(q(z,y)) + F(\alpha(z,y)q(y,Sy)) &= \tau\left(\frac{1}{2^{n+1}}\right) + F\left(\frac{1}{2} \times \frac{1}{2^{n+2}}\right) \\ &= 1 + \ln\left(\frac{1}{2^{n+3}}\right) \\ &= 1 - \ln(2^{n+3}) \\ &< -\ln 2^{n+1} \\ &= F(\beta(z,y)\mathcal{M}_q(z,y)). \end{aligned}$$

Thus, S is (α, β)-type $F - \tau$ contraction. Also S is generalized α_*-admissible and β_*-subadmissible mapping. Hence, all the conditions of Theorem 3.4 are satisfied so that S has a fixed point.

Observe that for $z = \frac{1}{2}$, we have $y = \frac{1}{4} \in F_\mu^{\frac{1}{2}}$ (for $\mu = \frac{1}{2}$) and

$$\begin{aligned}\tau(q(z,y)) + F(q(y,Sy)) &= \tau\left(\frac{1}{2}\right) + F\left(\frac{1}{8}\right) \\ &= 1 + \ln\left(\frac{1}{8}\right) \\ &= -1.07944 \\ &> \ln\left(\frac{1}{4}\right) \\ &= F(q(z,y)),\end{aligned}$$

so that Theorem 4 of [8] is not applicable in the context of the present example which shows that Theorem 3.4 is a genuine extension of Theorem 4 of [8].

Now, we deduce the following result using Theorem 3.1.

Theorem 3.5. *Let (M, q) be a left $\mathcal{K}$-complete T_1-quasi metric space, $S : M \to A_q(M)$ a τ_q-continuous mapping and $F \in \Xi$. Suppose the following conditions are satisfied:*

(i) S is generalized α_*-admissible and β_*-subadmissible mapping;

(ii) there exists $z_0 \in M$ and $z_1 \in F_\mu^{z_0}$ such that $0 < \beta(z_0, z_1) \leq 1 \leq \alpha(z_0, z_1)$;

(iii) there exists $\tau : (0, \infty) \to (\mu, \infty)$ satisfying

$$\liminf_{s \to \omega^+} \tau(s) > \mu, \ \forall \omega \geq 0$$

and $\forall z, y \in M$ with $H(Sz, Sy) > 0$, there exist $\alpha, \beta : M \times M \to (0, \infty)$ satisfying

$$\tau(q(z,y)) + F(\alpha(z,y)H(Sz,Sy)) \leq F(\beta(z,y)\mathcal{M}_q(z,y)).$$

Then S has a fixed point.

Proof. If S is τ_q-continuous mapping, then the mapping $z \to q(z, Sz)$ is lower semi-continuous with respect to τ_q and for $z \in M$ with $q(z, Sz) > 0$ and $y \in F_\mu^z$, we have

$$\begin{aligned}\tau(q(z,y)) + F(\alpha(z,y)q(y,Sy)) &\leq \tau(q(z,y)) + F(\alpha(z,y)H(Sz,Sy)) \\ &\leq F(\beta(z,y)\mathcal{M}_q(z,y)).\end{aligned}$$

Thus, all the conditions of Theorem 3.1 are satisfied. Hence, S has a fixed point. $\square$

Next, we deduce the following natural theorem for single valued mapping from Theorem 3.5.

Theorem 3.6. *Let (M, q) be a Hausdorff left $\mathcal{K}$-complete T_1-quasi metric space, $S : M \to M$ a τ_q-continuous mapping and $F \in \Xi$. Assume that the following conditions are satisfied:*

(i) *S is α-admissible and β-subadmissible mapping;*

(ii) *there exist $z_0, z_1 \in M$ such that $0 < \beta(z_0, z_1) \leq 1 \leq \alpha(z_0, z_1)$;*

(iii) *$\tau : (0, \infty) \to (0, \infty)$ such that*

$$\liminf_{s \to \omega^+} \tau(s) > 0, \ \forall \omega \geq 0$$

and $\forall z, y \in M$ with $q(Sz, Sy) > 0$, there exist functions $\alpha, \beta : M \times M \to (0, \infty)$ satisfying

$$\tau(q(z, y)) + F(\alpha(z, y)q(Sz, Sy)) \leq F(\beta(z, y)\mathcal{M}_q(z, y)). \tag{3.11}$$

Then S has a fixed point.

Now, we present the following example which exhibits that the Hausdorffness condition in Theorem 3.6 is unavoidable.

Example 3.2. Let $M = \{\frac{1}{n} : n \in \mathbb{N}\}$ and define $q : M \times M \to [0, \infty)$ as

$$q(z, y) = \begin{cases} 0, & \text{if } z = y, \\ y, & \text{if } z \neq y. \end{cases}$$

Then (M, q) is a left $\mathcal{K}$-complete T_1-quasi metric space but it is not Hausdorff, as the underlying topology τ_q is cofinite. Define $S : M \to M$ by $Sz = \frac{z}{2}$ and $F : (0, \infty) \to \mathbb{R}$ by $F\alpha = \ln \alpha$, $\forall \alpha > 0$. Also set $\alpha(z, y) = \beta(z, y) = 1$, $\forall z, y \in M$ and $\tau(t) = \ln 2$, $\forall t \in (0, \infty)$. Then S satisfies conditions (i)-(iii) of Theorem 3.6, but S has no fixed point.

The following corollary shows that our earlier results are generalizations of results contained in [8].

Corollary 3.1. *Theorems 1-4 of [8] follow immediately from 3.1-3.4 respectively.*

Proof. Take $\alpha(z, y) = \beta(z, y) = 1$, $\tau(t) = \tau_0$ where τ_0 is a constant $> \mu$. □

4. Application

In this section, as an application of Theorem 3.6, we study the existence of a solution of the non-linear integral equation

$$z(t) = a(t) + \int_a^t K(t, s, z(s))ds \tag{4.1}$$

where $t \in \Omega = [a, b]$, $a : \Omega \to \mathbb{R}$ and $K : \Omega \times \Omega \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Denote $M = C(\Omega, \mathbb{R})$ with usual sup norm, i.e.

$$\|z\| = \max_{t \in \Omega} |z(t)|.$$

Theorem 4.1. *Suppose that the following conditions hold:*

(i) *for any $z, y \in M$ with $z \neq y$, we have*

$$\int_a^t K(t, s, z(s))ds \neq \int_a^t K(t, s, y(s))ds$$

for each $t \in \Omega$,

(ii) for each $t, s \in \Omega$ and $z, y \in M$,

$$|K(t,s,z(s)) - K(t,s,y(s))| \leq \frac{1}{b-a} e^{-\left(\frac{1}{2}+\frac{1}{\|z-y\|}\right)} |z(s) - y(s)|.$$

Then (4.1) *has a solution in M.*

Proof. Consider $q : M \times M \to (0,\infty)$ defined by $q(z,y) = \|z-y\|$, where $\|z-y\| = \sup_{t\in\Omega} |z(t) - y(t)|$. Then (M,q) is a Hausdorff left $\mathcal{K}$-complete T_1-quasi metric space. Define $S : M \to M$ by:

$$Sz(t) = a(t) + \int_a^t K(t,s,z(s))ds, \ \forall z \in M.$$

Observe that $z \in M$ is a fixed point of S iff it is a solution of (4.1).

Now, define $F : (0,\infty) \to \mathbb{R}$ by $F(\alpha) = \ln \alpha$, $\forall \alpha > 0$. Then $\forall z, y \in M$, we have

$$\begin{aligned}
|Sz(t) - Sy(t)| &= \left| \int_a^t (K(t,s,z(s)) - K(t,s,y(s))ds \right| \\
&\leq \int_a^t |K(t,s,z(s)) - K(t,s,y(s))|ds \\
&\leq \int_a^t \frac{1}{b-a} e^{-\left(\frac{1}{2}+\frac{1}{\|z-y\|}\right)} |z(s) - y(s)|ds \\
&\leq \|z-y\| e^{-\left(\frac{1}{2}+\frac{1}{\|z-y\|}\right)} \int_a^t \frac{1}{b-a} ds \\
&\leq \frac{1}{b-a} \|z-y\| e^{-\left(\frac{1}{2}+\frac{1}{\|z-y\|}\right)} \int_a^b ds \\
&= \|z-y\| e^{-\left(\frac{1}{2}+\frac{1}{\|z-y\|}\right)}
\end{aligned}$$

which implies that $\|Sz - Sy\| \leq \|z-y\| e^{-\left(\frac{1}{2}+\frac{1}{\|z-y\|}\right)}$. Thus, for $q(z,y) > 0$ and $\tau(t) = \frac{1}{t} + \frac{1}{2}$, we have

$$\begin{aligned}
q(Sz, Sy) &= \|Sz - Sy\| \\
&\leq \|z-y\| e^{-\left(\frac{1}{2}+\frac{1}{\|z-y\|}\right)} \\
&= q(z,y) e^{-\left(\frac{1}{2}+\frac{1}{d(z,y)}\right)},
\end{aligned}$$

which implies that

$$\ln q(Sz, Sy) \leq -\left(\frac{1}{2} + \frac{1}{q(z,y)}\right) + \ln(q(z,y)),$$

or

$$\frac{1}{2} + \frac{1}{q(z,y)} + \ln q(Sz, Sy) \leq \ln(q(z,y)),$$

yielding thereby

$$\tau(q(z,y)) + F(q(Sz, Sy)) \leq F((q(z,y))).$$

Hence, all the hypotheses of Theorem 3.6 are satisfied (with $\alpha(z,y) = \beta(z,y) = 1$, $\forall z, y \in M$) ensuring the existence of a fixed point of S and hence, (4.1) has a solution in M. $\square$

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