Journal of Applied Analysis and Computation
Volume 9, Number 1, February 2019, 271–294
Website:http://jaac-online.com/
DOI:10.11948/2019.271

HOMOCLINIC SOLUTIONS OF DISCRETE NONLINEAR SYSTEMS VIA VARIATIONAL METHOD*

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Abstract Homoclinic solutions arise in various discrete models with variational structure, from discrete nonlinear Schrödinger equations to discrete Hamiltonian systems. In recent years, a lot of interesting results on the homoclinic solutions of difference equations have been obtained. In this paper, we review some recent progress by using critical point theory to study the existence and multiplicity results of homoclinic solutions in some discrete nonlinear systems with variational structure.

Keywords Discrete nonlinear system, homoclinic solutions, variational method.

MSC(2010) 39A12, 39A23.

1. Introduction

In the theory of differential equations, a trajectory which is asymptotic to a constant state as $|t| \to \infty$ (t denotes the time variable) is called a homoclinic solution. It has been found in various models of continuous dynamical systems, such as continuous Hamiltonian system, and which often have tremendous effects on the dynamics of such nonlinear systems [3, 27, 67, 86, 98]. It is well-known that homoclinic solutions play an important role in analyzing the chaos of dynamical systems. If a system has transversely intersected homoclinic solutions, then it must be chaotic. If it has smoothly connected homoclinic solutions, then it cannot withstand the perturbation, in the sense that its perturbed system probably produces chaos. Hence, it is of interest to find homoclinic solutions. Discrete Hamiltonian system, a discretization of the continuous one, can be easily shown that it preserves the symplectic structure. So its solutions can give some desirable numerical features for solutions of the continuous Hamiltonian system [1, 2]. It has been found that the trajectory which is asymptotic to a constant state as $|t| \to \infty$ also exists in discrete Hamiltonian systems. We still call it a homoclinic solution. First we give the definition of homoclinic solutions for discrete nonlinear systems below: If $\overline{x} = \{\overline{x}_n : n \in \mathbb{Z}\}$ is

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*The authors were supported by National Natural Science Foundation of China (11471085, 11271380) and Guangdong Province Key Laboratory of Computational Science.

a solution of a discrete system, another solution $x = \{x_n : n \in \mathbb{Z}\}$ will be called a homoclinic solution emanating from $\overline{x}$ if $|x_n - \overline{x}_n| \to 0$ as $|n| \to \infty$. In this paper, it is of interest to consider a homoclinic solution emanating from 0.

In the past years, the discrete nonlinear Schrödinger (DNLS) equation, which is a nonlinear lattice system that appears in many areas of physics, received great attention. Discrete solitons or standing waves which exist in the DNLS systems also yield a great deal of interest, from photorefractive media [46], biomolecular chains [45], to Bose-Einstein condensates [58]. The experimental observations of discrete solitons in nonlinear lattice systems have been reported in [29]. Many methods, such as the principle of anticontinuity [4] and centre manifold reduction [41], are used to study the existence of discrete solitons of DNLS equations. It is interesting that homoclinic solutions also appear when we look for the discrete solitons of DNLS equations. For example, consider the following DNLS equation:

$$i\dot{\psi_n} = -\Delta\psi_n - c_n|\psi_n|^2\psi_n, \ n \in \mathbb{Z}, \quad \lim_{|n|\to\infty} \psi_n = 0, \tag{1.1}$$

where $\Delta\psi_n = \psi_{n+1} + \psi_{n-1} - 2\psi_n$ is the discrete Laplacian in one spatial dimension, the given sequence $\{c_n\}$ is real-valued. Making use of $\psi_n = u_n e^{-i\omega t}$, where $\{u_n\}$ is a real valued sequence and $\omega \in \mathbb{R}$ is the temporal frequency, equation (1.1) becomes

$$-\Delta u_n - \omega u_n = c_n|u_n|^2 u_n, \ n \in \mathbb{Z}, \quad \lim_{|n|\to\infty} u_n = 0. \tag{1.2}$$

Clearly, discrete solitons of (1.1) correspond to homoclinic solutions emanating from 0 of (1.2). Moreover, it has been seen that the variational structure in (1.2) leads to find the homoclinic solutions of (1.2). Generally, the problem of finding homoclinic solutions of discrete nonlinear systems with variational structure, can be reformulated as a problem of finding critical point of the corresponding functional. Critical point theory has been used for a long time to study the existence of homoclinic solutions to continous Newtonian and Hamilton systems, semi-linear elliptic differential equations, nonlinear differential Schrödinger equations and so on; see monographs [75,95] and the reference therein for more information. Only in the past decade has critical point theory been widely used to study the existence of homoclinic solutions for discrete nonlinear systems with variational structure. The purpose of this paper is to review some progress on the existence and multiple results of homoclinic solutions of discrete nonlinear systems with variational structure via critical point theory.

Critical point theory was first introduced in 2003 by Guo-Yu [31–33] for studying the existence of periodic and subharmonic solutions in second order difference equations and discrete Hamiltonian systems. It has been shown to be a powerful tool in the study of the existence of homoclinic solutions of discrete nonlinear systems. The essential part of the usage of critical point theory is to make a transformation of a discrete problem into a continuously differentiable one, we may refer to a recent survey article [6] on the development of periodic solutions to discrete systems. In 2006-2007, Ma-Guo [60,61] first used critical point theory to study homoclinic solutions of a class of discrete Hamiltonian systems. It was found by Ma-Guo [60,61] that the existence of homoclinic solutions for discrete systems can be changed to the existence of critical points for the corresponding variational functional on suitable vector space, for example, the space l^2 of two-sided infinite sequences, and then it is made possible to use critical point theory. Pankov [68, 69] almost simultaneously found that the standing waves of a periodic DNLS equation can be changed to critical points of some variational functional. These pioneering works have produced great attentions in the field and many more novel and interesting results on the existence of homoclinic solutions for discrete systems are expanding based on this approach. The most difficult obstacle on the existence of homoclinic solutions for discrete systems by using critical point theory is to overcome the loss of compactness due to the fact that the problem is set on the domain $\mathbb{Z}$. To do this, effective methods like periodic approximation and compact embedding are developed [51, 60, 61, 109], which were used for a long time in studying the differential systems.

The remainder of this paper proceeds as follows. In Section 2, we focus on the existence of homoclinic solutions for nonlinear difference equations, which is associated with the standing waves of DNLS equations. In Section 3, we review some progress on the existence of nontrivial homoclinic solutions of discrete Hamiltonian systems. Finally, in Section 4, we present results in the direction of discrete p-Laplacian equations as well as difference equations containing both advanced and retarded arguments.

2. Homoclinic solutions of DNLS equations

2.1. Periodic case

In the past decade, many scholars have focused on the existence of homoclinic solutions for the following nonlinear difference equation

$$Lu_n - \omega u_n = \sigma f_n(u_n), \quad n \in \mathbb{Z}, \tag{2.1}$$

where $f_n(0) = 0$, $f_n(u)$ is continuous in u, $f_{n+T}(u) = f_n(u)$ for each $n \in \mathbb{Z}$, $\sigma = \pm 1$, and L is a Jacobi operator [93] given by

$$Lu_n = a_n u_{n+1} + a_{n-1} u_{n-1} + b_n u_n,$$

where $\{a_n\}$, $\{b_n\}$ are real valued T-periodic sequences.

This problem appears when we look for the standing waves of the periodic discrete nonlinear Schrödinger (DNLS) equation

$$i\dot{\psi}_n = -\Delta\psi_n + v_n\psi_n - \sigma f_n(\psi_n), \quad n \in \mathbb{Z}, \tag{2.2}$$

where $\sigma = \pm 1$, $\Delta\psi_n = \psi_{n+1} + \psi_{n-1} - 2\psi_n$ is the discrete Laplacian in one spatial dimension, the given sequence $\{v_n\}$ is assumed to be T-periodic, and $f_n(u)$ is a T-periodic function in n of nonlinearities. Typical representatives of saturable nonlinearities are

$$f_n(u) = l_n \frac{|u|^p u}{1 + |u|^p}, \quad l_n \neq 0, \ p > 0,$$

and

$$f_n(u) = \chi_n(1 - e^{-h_n|u|^2})u, \quad \chi_n \neq 0, \ h_n > 0,$$

where $\{l_n\}$, $\{\chi_n\}$, and $\{h_n\}$ are T-periodic sequences. Typical representative of superlinear nonlinearities is

$$f_n(u) = c_n|u|^q u, \quad c_n \neq 0, \ q > 0,$$

where $\{c_n\}$ is T-periodic. Consider (2.2), we suppose that the nonlinearity is gauge invariant, i.e.,

$$f_n(e^{i\theta}u) = e^{i\theta}f_n(u), \quad \theta \in \mathbb{R},$$

and, in addition, $f_n(u) \geq 0$ for $u \geq 0$ for each $n \in \mathbb{Z}$. Now let us perform the standard reduction of the standing wave problem for (2.2) to a stationary problem. Making use of the standing wave Ansatz

$$\psi_n = u_n e^{-i\omega t}, \quad \lim_{|n|\to\infty} \psi_n = 0,$$

where u_n is a real valued sequence and $\omega \in \mathbb{R}$ is the temporal frequency, equation (2.2) becomes

$$-\Delta u_n + v_n u_n - \omega u_n = \sigma f_n(u_n),\ n \in \mathbb{Z}, \quad \lim_{|n|\to\infty} u_n = 0. \tag{2.3}$$

Assume that $f_n(0) = 0$ for $n \in \mathbb{Z}$, then $\{u_n\} = \{0\}$ is a solution of (2.1), which is called the trivial solution. As usual, if $u = \{u_n\} \neq \{0\}$, then u is called a nontrivial homoclinic solution. Clearly, discrete solitons of (2.2) correspond to the homoclinic solutions of (2.3), and (2.3) is a special form of (2.1) with $a_n = -1, b_n = v_n + 2$. Therefore, we will study the existence of the nontrivial homoclinic solutions of (2.1).

Since the operator L is bounded and self-adjoint in the space l^2 of two-sided infinite sequences, we consider (2.1) as a nonlinear equation in l^2. The spectrum $\sigma(L)$ of L has a band structure, i.e., $\sigma(L)$ is a union of a finite number of closed intervals [93]. Thus the complement $\mathbb{R}\backslash\sigma(L)$ consists of a finite number of open intervals called spectral gaps and two of them are semi-infinite. We fix a spectral gap denoted by (α, β). In this section, we consider the homoclinic solutions of (2.1) in l^2 for the general case where $\omega \in (\alpha, \beta)$.

We first establish the variational setting associated with (2.1). On the Hilbert space l^2, we consider the functional

$$J(u) = \frac{1}{2}(Lu - \omega u, u) - \sigma \sum_{n=-\infty}^{\infty} F_n(u_n)$$

where $(\cdot, \cdot)$ is the inner product in l^2, and

$$F_n(u) = \int_0^u f_n(s)ds$$

is the primitive function of $f_n(u)$. The corresponding norm in l^2 is denoted by $\|\cdot\|$. Then $J \in C^1(l^2, \mathbb{R})$ and

$$\langle J'(u), v\rangle = (Lu - \omega u, v) - \sigma \sum_{n=-\infty}^{\infty} f_n(u_n)v_n, \quad u, v \in l^2. \tag{2.4}$$

Equation (2.4) implies that (2.1) is the corresponding Euler-Lagrange equation for J. Therefore, we have reduced the problem of finding a nontrivial homoclinic solution of (2.1) to that of seeking a nonzero critical point of the functional J on l^2.

In 2006, Pankov [68] studied (2.1) when ω belongs to some spectral gap (α, β) of the Jacobi operator L [93]. By using the linking theorem [75, 95] in combination with periodic approximations, Pankov obtained the following result [68].

Theorem 2.1 ([68]). *The nonlinearity $f_n(u)$ is supposed to satisfy the following assumptions.*
(H_1^P) *$f_n(u)$ is continuous in $u \in \mathbb{R}$ and depends periodically in n with period T.*
(H_2^P) *There exist $p > 2$ and $c > 0$ such that $0 \leq f_n(u) \leq c|u|^{p-1}$ near $u = 0$.*
(H_3^P) *There exists $\mu > 2$ such that $0 < \mu F_n(u) \leq f_n(u)u$ for $u \neq 0$.*
Suppose that either $\sigma = 1$ and $\beta \neq \infty$, or $\sigma = -1$ and $\alpha \neq -\infty$. Then the equation (2.1) has a nontrivial exponentially decaying solution. If either $\sigma = +1$ and $\beta = \infty$, or $\sigma = -1$ and $\alpha = -\infty$, then there is no nontrivial solution in l^2.

In 2010, by using a variant linking theorem [77], Yang et al. [96] proved the existence of solutions in l^2 for equation (2.1) with ω being a lower bound of a finite spectral gap.

Theorem 2.2 ([96]). *Suppose $\sigma = 1$. Assume further that*
(V_1) *$\omega \in \sigma(L)$ and there exists $\rho > 0$ such that $(\omega, \rho] \cap \sigma(L) = \emptyset$.*
(H_1^Y) *The function $f_n(u)$ is continuous in $u \in \mathbb{R}$ and depends periodically in n with period T and $f_n(u) = o(u)$ as $u \to 0$.*
(H_2^Y) *There exist $p > 2$ and $c > 0$ such that $|f_n(u)| \leq c(1 + |u|^{p-1})$.*
(H_3^Y) *$\frac{F_n(u)}{|u|^2} \to \infty$ as $|u| \to \infty$.*
(H_4^Y) *$\frac{f_n(u)}{|u|}$ is strictly increasing on $(-\infty, 0)$ and $(0, \infty)$.*
Then (2.1) has at least one solution in l^2.

In 2013, by taking advantage of the classical linking theorem combined with an approximation technique, Zhou et al. [110] considered the existence of homoclinic solutions in (2.1) with superlinear nonlinearity when ω belongs to some spectral gap (α, β) of L. Interestingly, the classical AR superlinear condition is improved by a general superlinear one for this difficult strongly indefinite problem.

Theorem 2.3 ([110]). *Assume that $\omega \in (\alpha, \beta)$, $f_n(u)$ is continuous in u, $f_{n+T}(u) = f_n(u)$ for any $n \in \mathbb{Z}$ and $u \in \mathbb{R}$, $f_n(u) = o(u)$ as $u \to 0$. And for each $n \in \mathbb{Z}$, the following conditions hold.*
(H_1^Z) *The function $f_n(u)u > 0$ for $u \neq 0$ and $\lim_{|u|\to\infty} f_n(u)/u = \infty$.*
(H_2^Z) *$f_n(u)u - 2F_n(u) > 0$ for $u \neq 0$, $f_n(u)u - 2F_n(u) \to \infty$ as $|u| \to \infty$, and*

$$\limsup_{u\to 0} \frac{f_n^2(u)}{f_n(u)u - 2F_n(u)} = p_n < \infty.$$

If $\sigma = 1$ and $\beta \neq \infty$, then equation (2.1) has at least one nontrivial solution u in l^2. Moreover, the solution decays exponentially at infinity. That is, there exist two positive constants C and τ such that

$$|u_n| \leq Ce^{-\tau|n|},\ n \in \mathbb{Z}.$$

Other related results of (2.1) for the case when f_n is a superlinear nonlinearity, we refer to [13, 15, 36, 37, 62, 65, 69, 74, 81, 85, 87].

In 2008, Pankov-Rothos [73] considered a special form of (2.1) with $a_n = -1$ and $b_n = 2$ when the nonlinearity $f_n(u) = f(u)$ is asymptotically linear at ∞ for $n \in \mathbb{Z}$. Nehari manifolds were employed to establish the existence of homoclinic solutions.

Theorem 2.4 ([73]). *Suppose that $a_n = -1$, $b_n = 2$, $f_n(u) = f(u)$ and the nonlinearity $f(u)$ satisfies the following assumptions, in which $F(u) = \int_0^u f(s)ds$.*

(h_1^P) *$f(u) = o(u)$ as $u \to 0$.*
(h_2^P) *$\lim_{|u|\to\infty}(f(u)/u) = l < \infty$.*
(h_3^P) *$f \in C^1(\mathbb{R})$ and $f(u)u < f'(u)u^2$ for $u \neq 0$.*
(h_4^P) *$(1/2)f(u)u - F(u) \to \infty$ as $|u| \to \infty$.*
Assume either $\sigma = 1$, $\omega < 0$ and $l + \omega > 0$, or $\sigma = -1$, $\omega > 4$ and $-l + \omega < 4$. Then equation (2.1) has at least one nontrivial ground-state solution u in l^2. Moreover, the solution decays exponentially at infinity. That is, there exist two positive constants C and τ such that

$$|u_n| \leq Ce^{-\tau|n|},\ n \in \mathbb{Z}.$$

In 2010, Zhou-Yu [109] obtained a new sufficient condition on the existence of homoclinic solutions of (2.1) by using the mountain pass lemma [75,95] in combination with periodic approximations. Moreover, they proved that it is also necessary in some special cases. Interestingly, an original approximation technique has been used to overcome the loss of compactness.

Theorem 2.5 ([109]). *Assume that $\sigma = 1$, $\omega \in (-\infty, \beta)$, $f_n(u)$ is continuous in u, $f_{n+T}(u) = f_n(u)$ for any $n \in \mathbb{Z}$ and $u \in \mathbb{R}$, $f_n(u) = o(u)$ as $u \to 0$. And the following conditions hold.*
$(H1^Z)$ *$f_n(u)/u$ is strictly increasing in $(0, \infty)$ and strictly decreasing in $(-\infty, 0)$. Moreover, $\lim_{|u|\to\infty} f_n(u)/u = d_n < \infty$.*
$(H2^Z)$ *$f_n(u)u - 2F_n(u) \to \infty$ as $|u| \to \infty$, and*

$$\limsup_{u\to 0} \frac{f_n^2(u)}{f_n(u)u - 2F_n(u)} = p_n < \infty.$$

If $d_n > \beta - \omega$, then equation (2.1) has at least one nontrivial solution u in l^2. Moreover, the solution decays exponentially at infinity. That is, there exist two positive constants C and τ such that

$$|u_n| \leq Ce^{-\tau|n|},\ n \in \mathbb{Z}.$$

In 2016, Chen et al. [16] considered (2.1) when the nonlinearities $f_n(u)$ are asymptotically linear as $|u| \to \infty$. In the two different cases (ω is a spectral endpoint of L, or it belongs to a finite spectral gap of L), they obtained the existence of nontrivial solitons of this equation by using a variant of the generalized weak linking theorem [77].

Theorem 2.6 ([16]). *Suppose that $\sigma = 1$, $f_n(u)$ is continuous in u, $f_{n+T}(u) = f_n(u)$ for any $n \in \mathbb{Z}$ and $u \in \mathbb{R}$, $f_n(u) = o(u)$ as $u \to 0$.*
(G_1^C) *$\lim_{|u|\to\infty} f_n(u)/u = V_n < \infty$.*
(G_2^C) *$(1/2)f_n(u)u - F_n(u) > 0$ if $u \neq 0$ and $F_n(u) > 0$.*
(1) Assume that $\omega \notin \sigma(L)$ and ω belongs to a finite spectral gap (α, β), $V_n > \inf \sigma(L|_{E_1^+}) - \omega$ where E_1^+ is the positive spectral subspace of $L - \omega$ in l^2, and (G_1^C) and (G_2^C) hold. Assume further the condition holds:
(G_3^C) *There is $\zeta \in (0, \frac{\eta}{2})$ where $\eta = \min\{|\alpha - \omega|, |\beta - \omega|\}$ such that*

$$\frac{f_n(u)}{u} \geq \frac{\eta}{2} - \zeta \Rightarrow \frac{1}{2}f_n(u)u - F_n(u) > \zeta.$$

Then equation (2.1) has at least one nontrivial solution u in l^2. Moreover, the solution decays exponentially at infinity. That is, there exist two positive constants C and τ such that

$$|u_n| \leq Ce^{-\tau|n|},\ n \in \mathbb{Z}.$$

(2) Assume that $\omega \in \sigma(L)$ and there exists $\rho > 0$ such that $(\omega, \rho] \cap \sigma(L) = \emptyset$, and (G_1^C) and (G_2^C) hold. Assume further the condition holds:
(G_4^C) $u \to f_n(u)/|u|$ is strictly increasing in $(0, \infty)$ and $(-\infty, 0)$.
If $V_n > \inf \sigma(L|_{E_2^+}) - \omega$ where E_2^+ satisfies $l^2 = E_2^- \bigoplus E_2^+$, corresponding to the decomposition of $\sigma(L-\omega)$ into $(-\infty, 0] \cap \sigma(L-\omega)$ and $[\rho - \omega, \infty) \cap \sigma(L-\omega)$, then equation (2.1) has at least a nontrivial solution u in l^2.

Other related results of (2.1) for the case when f_n is a asymptotically linear nonlinearity, we refer to [14, 15, 23, 39, 48, 62, 64, 70, 80, 85, 111, 112].

In 2016, by using critical point theory [76, 84] in combination with periodic approximations, Lin-Zhou [52] obtained some new sufficient conditions on the nonexistence and existence of homoclinic solutions for (2.1). Their novel results are necessary in some sense, and extend and improve many existing ones for some special cases. This was the first time to consider the homoclinic solutions of this class of difference equations with mixed nonlinearities: The nonlinear terms can mix superlinear nonlinearities with asymptotically linear ones at both ∞ and 0.

Theorem 2.7 ([52]). *Assume that $\sigma = 1$, $\beta \neq \infty$, $f_n(u)$ is continuous in u, $f_{n+T}(u) = f_n(u)$ for $n \in \mathbb{Z}$ and $u \in \mathbb{R}$. And for $n \in \mathbb{Z}$, $f_n(u)$ and $F_n(u) = \int_0^u f_n(s)ds$ satisfy the following conditions:*
(F_1^L) $\lim_{u\to 0} \frac{f_n(u)}{u} = \delta_n < \infty$.
(F_2^L) $\lim_{|u|\to\infty} \frac{f_n(u)}{u} = d_n \leq \infty$.
(F_3^L) $F_n(u) \geq 0$ for $u \in \mathbb{R}$, $f_n(u)u - 2F_n(u) > 0$ for $u \neq 0$, and $f_n(u)u - 2F_n(u) \to \infty$ as $|u| \to \infty$.

If $\delta_n < \min\{\omega - \alpha, \beta - \omega\}$ and $\beta - \omega < d_n$ for $n \in \mathbb{Z}$, then (2.1) has at least one nontrivial solution u in l^2. Moreover, if $\delta_n = 0$ for $n \in \mathbb{Z}$, then the solution decays exponentially at infinity. That is, there exist two positive constants C and τ such that

$$|u_n| \leq Ce^{-\tau|n|}, \quad n \in \mathbb{Z}.$$

2.2. Unbounded potential case

In contrast to the periodic case of (2.1), it is of interest to consider the non-periodic case of (2.1). Specially, since it has been studied by many authors, we will review some progress on the following equation:

$$-\Delta u_n + \nu_n u_n - \omega u_n = f_n(u_n), \quad n \in \mathbb{Z}, \tag{2.5}$$

where $\Delta\psi_n = \psi_{n+1} + \psi_{n-1} - 2\psi_n$ is the discrete Laplacian in one spatial dimension, the given sequence $\{\nu_n\}$ is assumed to be non-periodic, and $f_n(u)$ is a non-periodic function in n of nonlinearities. As we know, periodic assumptions are very important in the study of (2.1) since periodicity is used to control the lack of compactness due to the fact that (2.1) is defined on $\mathbb{Z}$. But non-periodic equations are quite different from the ones described in periodic cases. Note that the domain $\mathbb{Z}$ is unbounded. Thus, to overcome the loss of compactness caused by the unboundedness of the domain $\mathbb{Z}$, one can use the following assumption:

$(V1)$ *The discrete potential $V = \{\nu_n\}$ satisfies*

$$\lim_{|n|\to\infty} \nu_n = \infty.$$

Then condition $(V1)$ implies that the spectrum $\sigma(-\Delta+V)$ is discrete and consists of simple eigenvalues accumulating at ∞ [105]. Now we can assume that

$$\lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \to \infty$$

are all eigenvalues of $H = -\Delta + V$. Obviously, the operator H is an unbounded self-adjoint operator in l^2. There is no harm in assuming that $\lambda_1 > 0$, we define the space

$$E = \{\ u \in l^2 : H^{\frac{1}{2}}u \in l^2\}.$$

Then E is a Hilbert space equipped with the norm

$$\|u\| = \|H^{\frac{1}{2}}u\|_{l^2}.$$

On the Hilbert space E, we consider the functional

$$J(u) = \frac{1}{2}((H-\omega)u, u)_{l^2} - \sum_{n\in\mathbb{Z}} F_n(u_n)$$

where

$$F_n(u) = \int_0^u f_n(s)ds$$

is the primitive function of $f_n(u)$. Standard arguments show that the functional J satisfies

$$\langle J'(u), v\rangle = ((H-\omega)u, v)_{l^2} - \sum_{n\in\mathbb{Z}} f_n(u_n)v_n, \quad u, v \in E. \tag{2.6}$$

Equation (2.6) implies that (2.5) is the corresponding Euler-Lagrange equation for J. Therefore, we have reduced the problem of finding a nontrivial homoclinic solution of (2.5) to that of seeking a nonzero critical point of the functional J on E.

In 2008, Zhang-Pankov [105] obtained the existence of non-trivial solutions for a special superlinear case of (2.5) with $f_n(u) = \gamma_n|u|^{p-2}u\,(p > 2)$ and $\omega < \lambda_1$. The results in [105] were further extended by Zhang and his coworkers to the ones in [71, 72, 99, 100, 104, 106] by using critical point theory. For example, in 2009, Zhang [99] showed the existence of a nontrivial homoclinic solution of (2.5) .

Theorem 2.8 ([99]). *Assume that $(V1)$ hold and the nonlinearity $f_n(u) = \gamma_n f(u)$ for any $n \in \mathbb{Z}$, $f(u) = o(u)$ as $u \to 0$ and the following conditions hold.*
(F_1^Z) *There exists a positive constant $\overline{\gamma}$ such that $0 < \gamma_n \le \overline{\gamma}$ for any $n \in \mathbb{Z}$.*
(F_2^Z) *There are two positive constants C_1, C_2 and $2 < p < \infty$ such that*

$$|f(u)| \le C_1(1 + |u|^{p-1}),$$

and

$$|f(u) - f(v)| \le C_2(1 + |u|^{p-2} + |v|^{p-2})|u - v|.$$

(F_3^Z) *There is a $2 < q < \infty$ such that*

$$0 < (q-1)f(u)u \le f'(u)u^2,\ u \ne 0.$$

If $\omega < \lambda_1$, then equation (2.5) has at least one nontrivial solution u in E. Moreover, there exist two positive constants C and τ such that

$$|u_n| \le Ce^{-\tau|n|},\ n \in \mathbb{Z}.$$

In 2014, Zhou-Ma [108] obtained some new multiplicity results of nontrivial homoclinic solutions of (2.5) by using the fountain theorem [95,113]. This is a novel and interesting result in which infinitely many high energy homoclinic solutions were obtained.

Theorem 2.9 ([108]). *Assume that* $(V1)$ *holds and the nonlinearity* $f_n(u) = \gamma_n f(u)$ *for any* $n \in \mathbb{Z}$, $f(u) = o(u)$ *as* $u \to 0$ *and the following conditions hold.*
(A_1) *There exist two positive constants* $\underline{\gamma}$ *and* $\bar{\gamma}$, *such that for any* $n \in \mathbb{Z}$,

$$\underline{\gamma} \leq \gamma_n \leq \bar{\gamma}.$$

(f_1^{ZM}) $f \in C(\mathbb{R}, \mathbb{R})$, *and there exists* $a > 0$, $p \in (2, \infty)$ *such that*

$$|f(u)| \leq a(1 + |u|^{p-1}), \quad \text{for all } u \in \mathbb{R}.$$

(f_2^{ZM}) $\lim\limits_{|u|\to\infty} F(u)/u^2 = \infty$, *where* $F(u)$ *is the primitive function of* $f(u)$, *i.e.,*

$$F(u) = \int_0^u f(t)dt.$$

(f_3^{ZM}) $f(u)/u$ *is increasing in* $u > 0$ *and decreasing in* $u < 0$.

If $\omega < \lambda_1$ *and the nonlinearity* $f(u)$ *is odd in* u, *then equation (2.5) has infinitely many solutions* $\{u^{(k)}\}_{k=1}^{\infty}$ *in* E *satisfying*

$$\frac{1}{2}(Hu^{(k)}, u^{(k)}) - \frac{1}{2}\omega(u^{(k)}, u^{(k)}) - \sum_{n\in\mathbb{Z}} \gamma_n F(u_n^{(k)}) \to \infty \text{ as } k \to \infty.$$

Moreover, there exist two positive constants C *and* τ *such that*

$$|u_n| \leq Ce^{-\tau|n|}, \ n \in \mathbb{Z}.$$

In 2016, by using the variant weak linking theorem [77], Chen-Schechter [17] obtained the existence of non-trivial homoclinic solutions of (2.5) in the following three cases:
(H_1) $\lambda_{k_0} - \omega = a < 0 < b = \lambda_{k_0+1} - \omega$ for some $k_0 \geq 1$ (the indefinite case).
(H_2) $\omega < \lambda_1$ (the positive definite case).
(H_3) $\omega = \lambda_{k_0'}$ for some $k_0' \geq 1$ (ω is an eigenvalue of H).

Theorem 2.10 ([17]). *Assume that the following conditions hold for the nonlinearities* f_n, $n \in \mathbb{Z}$.
(f_1^{CS}) $f_n \in C(\mathbb{R}, \mathbb{R})$, $|f_n(u)| \leq c(1 + |u|^{p-1})$, *for some* $c > 0$ *and* $p > 2$, $u \in \mathbb{R}$.
(f_2^{CS}) $F_n(u) = \int_0^u f_n(s)ds \geq \frac{1}{2}au^2$, *here the constant* a *is defined in* (H_1), $u \in \mathbb{R}$.
(f_3^{CS}) $|f_n(u)| \leq \gamma|u|$ *if* $|u| < \delta$ *for some* $0 < \gamma < b$ *and* $\delta > 0$, $u \in \mathbb{R}$.
(f_4^{CS}) $\lim\limits_{|u|\to\infty} F_n(u)/u^2 = \infty$ *and* $F_n(u) \geq -W_n$ *for some* $W = (W_n)_{n\in\mathbb{Z}} \in l^1$, $u \in \mathbb{R}$.
(f_5^{CS}) $F_n(u + l) - F_n(u) - rf_n(u)l + \frac{(r-1)^2}{2} f_n(u)u \geq -W_n$, $r \in [0, 1]$, $u \in \mathbb{R}$.

Assume If $(V1)$, (f_1^{CS})-(f_5^{CS}) *and* (H_1) *(or* (H_2), *or* (H_3)*) hold, then equation (2.5) has at least one non-trivial solution* u *in* E *Moreover, if* $f_n(u) = o(u)$ *as* $u \to 0$, *then there exist two positive constants* C *and* τ *such that*

$$|u_n| \leq Ce^{-\tau|n|}, \ n \in \mathbb{Z}.$$

Other related results for (2.5) without periodic assumptions can be found in [12, 38, 42, 43, 51, 57, 63, 66].

One should mention that the existing results of homoclinic solutions of DNLS equations mainly focus on the case with periodic coefficients or unbounded potential. However, the existence of homoclinic solutions of DNLS equations with bounded and non-periodic coefficients, especially asymptotically periodic coefficients, is still an open problem. Such an unsolved problem needs a further study.

3. Discrete Hamiltonian systems

In 2006, Ma-Guo [60] first used critical point theory [75, 95] to study the existence of nontrivial homoclinic solutions of the following discrete Hamiltonian system:

$$\triangle[p(t)\triangle u(t-1)] + q(t)u(t) = f(t, u(t)), \quad t \in \mathbb{Z}, \tag{3.1}$$

where $\triangle$ defined by $\triangle u(t) = u(t+1) - u(t)$ is the forward difference operator. To prove the Palais-Smale condition on the unbounded domain, an original embedding lemma was given. Moreover, an unbounded sequence of homoclinic solutions were first obtained by invoking the symmetric mountain pass theorem.

Theorem 3.1 ([60]). *Assume that the following conditions hold.*
(f_1^M) $\lim_{x\to 0} \frac{f(t,x)}{x} = 0$ *uniformly for* $t \in \mathbb{Z}$.
(f_2^M) *There exists a constant* $\beta > 2$ *such that*

$$xf(t,x) \le \beta \int_0^x f(t,s)ds < 0$$

for all $(t,x) \in \mathbb{Z} \times \mathbb{R}\backslash\{0\}$.
(p) $p(t) > 0$ *for all* $t \in \mathbb{Z}$.
(q) $q(t) < 0$ *for all* $t \in \mathbb{Z}$ *and* $\lim_{|t|\to\infty} q(t) = -\infty$.

Suppose (p), (q), (f_1^M), *and* (f_2^M) *are satisfied. Then there exist a homoclinic solution* u *of equation (3.1) emanating from* 0 *such that*

$$0 < \sum_{t=-\infty}^{\infty} \left[\frac{1}{2}p(t)(\triangle u(t-1))^2 - \frac{1}{2}q(t)(u(t))^2 + F(t,u(t))\right] < \infty.$$

Equation (3.1) was considered in [60] without any periodicity assumptions on $p(t)$, $q(t)$, and f, providing that $f(t,x)$ grows superlinearly both at origin and at infinity. In 2007, Ma-Guo [61] further extended the result in [60] to the periodic case by applying the mountain pass theorem relying on Ekeland' s variational principle and the diagonal method.

These pioneer works of [60, 61] on the existence of nontrivial homoclinic solutions of discrete Hamiltonian systems have attracted a great deal of attentions. To generalize the results in [60, 61] for the scalar case to the N-dimensional case, many authors have considered the nontrivial homoclinic solution of the following discrete Hamiltonian systems:

$$\triangle[p(n)\triangle u(n-1)] - L(n)u(n) = \nabla W(n, u(n)), \quad n \in \mathbb{Z}, \tag{3.2}$$

where $u \in \mathbb{R}^N$, $p, L : \mathbb{Z} \to \mathbb{R}^{N\times N}$, and $W : Z \times \mathbb{R}^N \to \mathbb{R}$ for some integer $N \in \mathbb{Z}$.

We first consider the periodic case of equation (3.2). In 2009, equation (3.2) with periodicity assumptions was considered in [26]. Based on the critical point theory, some sufficient conditions for the existence of subharmonic solutions and homoclinic solutions were obtained by Deng-Cheng-Shi [26]. The obtained results extended the results of [61] by relaxing the assumptions on the sign of the potential.

Theorem 3.2 ([26]). *Assume that $W(n,x) = b(n)V(x)$ for $n \in \mathbb{Z}$ and $x \in \mathbb{R}^N$. For a given integer T, assume that $p(\cdot)$ is a T-periodic, $N \times N$ real symmetric positive definite matrix function, $L(\cdot)$, $b(\cdot)$, and $V(\cdot)$ satisfy the following conditions:*
(L) $L(\cdot)$ is a T-periodic and $N \times N$ real symmetric matrix function, and there exists $a > 0$ such that

$$L(n)x \cdot x \geq a|x|^2, \quad x \in \mathbb{R}^N,\ n \in \mathbb{Z}.$$

(B) $b(\cdot)$ is T-periodic, and there exists $n_0 \in \mathbb{Z}$ such that $b(n_0) > 0$.
(V_1) $V \in C^1(\mathbb{R}^N, \mathbb{R}), V(x) \geq V(0) = 0, |\nabla V(x)| = o|x|$ as $x \to 0$.
(V_2) There exist $a_1 > 0, a_2 \geq 0, \alpha > 2$ such that

$$V(x) \geq a_1|x|^\alpha - a_2, \quad x \in \mathbb{R}^N.$$

(V_3) There exist $\beta > 2$ and $a_3 \in \left[0, \frac{a(\beta-2)}{2}\right]$ such that

$$|\nabla V(x) \cdot x - \beta V(x)| \leq \bar{B}^{-1} a_3 |x|^2, \quad x \in \mathbb{R}^N$$

where $\bar{B} = \max_{n\in\mathbb{Z}}\{|b(n)|\}$.

Then equation (3.2) has at least one nontrivial homoclinic solution.

If $p(n)$ and $L(n)$ are T-periodic $N \times N$ real symmetric matrices, it is easy to check that the operator A given as

$$(Au)(n) = \triangle[p(n)\triangle u(n-1)] - L(n)u(n), \quad n \in \mathbb{Z}, \tag{3.3}$$

is a bounded self-adjoint operator in $l^2(\mathbb{Z}, \mathbb{R}^N)$. By the Floquet Theorem, we can see that A has only continuous spectrum $\sigma(A)$, which is a union of bounded closed intervals.

In 2015, based on a generalized linking theorem for the strongly indefinite functionals [49], Zhang [102] studied the existence of homoclinic solutions of equation (3.2) where $p(n)$, $L(n)$ and $W(n,x)$ are T-periodic in n, and 0 lies in a gap of the spectrum $\sigma(A)$ of A with weak superquadratic conditions.

Theorem 3.3 ([102]). *Assume that p, L and W satisfy the following conditions.*
(PL) $p(n)$ and $L(n)$ are T-periodic $N \times N$ real symmetric matrices, and

$$\sup[\sigma(A)] \cap (-\infty, 0)] < 0 < \inf[\sigma(A)] \cap (0, \infty)].$$

$(W1)$ $W(n,x)$ is continuously differentiable in x for every $n \in \mathbb{Z}$, $W(n,0) = 0$, $W(n,x) \geq 0$ and $W(n,x)$ is T-periodic in n.
$(W2)$ $\nabla W(n,x) = o(|x|)$ as $|x| \longrightarrow 0$ uniformly for $n \in \mathbb{Z}$.
$(W3)$ $\lim_{|x|\longrightarrow 0} \frac{|W(n,x)|}{|x|^2} = \infty$ for all $n \in \mathbb{Z}$.
$(W4)$ $\widetilde{W}(n,x) = \frac{1}{2}(\nabla W(n,x), x) - W(n,x) > 0$, $(n,x) \in \mathbb{Z} \times (\mathbb{R}^N \setminus \{0\})$, and there exist $c_1 > 0$ and $R_0 > 0$ such that

$$|\nabla W(n,x)| \leq c_1 |x| \widetilde{W}(n,x), \quad (n,x) \in \mathbb{Z} \times \mathbb{R}^N,\ |x| \geq R_0.$$

Then system (3.2) possesses a nontrivial homoclinic solution.

Recently, inspired by Tang-Lin-Yu [92], Condition $(W3)$ was substantially improved in [103] by the following much weaker condition
$(W3)'$ *there is some integer* n_0 *such that* $\lim_{|x|\longrightarrow 0}\frac{|W(n,x)|}{|x|^2}=\infty$ *just for* $n\in\{n_0-1,n_0,n_0+1\}$.

Other related results of equation (3.2) with periodicity assumptions can be found in [8, 24, 88, 94, 101].

Now we turn to consider the non-periodic case of equation (3.2). In 2008, Deng-Cheng [25] considered equation (3.2) without any periodicity assumptions. Based on critical point theory, they obtained some sufficient conditions for the existence of homoclinic solution and extended the results in [60] by relaxing the assumptions on the sign of the potential.

Theorem 3.4 ([25]). *Assume that* $W(n,x)=b(n)V(x)$ *for* $n\in\mathbb{Z}$ *and* $x\in\mathbb{R}^N$, *and* $p(n)$ *is a* $N\times N$ *real symmetric positive definite matrix. Moreover* $L(n)$, $b(n)$, *and* $V(x)$ *satisfy the following:*
(A) $L(n)$ *is a* $N\times N$ *real symmetric matrix, and there exists* $a>0$ *such that*

$$L(n)x\cdot x\geq a|x|^2, \forall x\in\mathbb{R}^N, \forall n\in\mathbb{Z}.$$

(B) $b(n)$ *is a bounded real number for each* $n\in\mathbb{Z}$, *and there exists* $n_0\in\mathbb{Z}$ *such that* $b(n_0)>0$ *and* $\sum_{n\in\mathbb{Z}}|b(n)|^2<\infty$.
(V_1) $V\in C^1(\mathbb{R}^N,\mathbb{R})$, $V(0)=0$, *and* $|\nabla V(x)|=o|x|$ *as* $x\to 0$.
(V_2) *There exist* $\mu>2$, *and* $a_1>0$ *such that*

$$V(x)\geq a_1|x|^\mu, \forall x\in\mathbb{R}^N.$$

(V_3) *There exist* $\alpha>2$ *and* $a_2\in\left[0,\frac{a(\alpha-2)}{2}\right)$, $r_1>0$ *such that*

$$|\nabla V(x)\cdot x-\alpha V(x)|\leq \bar{B}^{-1}a_2|x|^2, \forall |x|\geq r_1, x\in\mathbb{R}^N,$$

where $\bar{B}=\sup_{n\in\mathbb{Z}}\{|b(n)|\}$.

Then equation (3.2) has at least one homoclinic solution.

By using Symmetric Mountain Pass Theorem [75, 95], in 2011, Lin-Tang [50] established some existence criteria to guarantee equation (3.2) has infinitely many homoclinic solutions, where $p(n)$, $L(n)$ and $W(n,x)$ are non-periodic in n.

Theorem 3.5 ([50]). *Assume that* $p(n)$ *is a real symmetric positive definite matrix for all* $n\in\mathbb{Z}$, *where* L *and* W *satisfy the following assumptions:*
(L) $L(n)$ *is a real symmetric positive definite matrix for all* $n\in\mathbb{Z}$ *and there exists a function* $l:\mathbb{Z}\to(0,\infty)$ *such that* $l(n)\to\infty$ *as* $|n|\to\infty$ *and*

$$(L(n)x,x)\geq l(n)|x|^2, \forall (n,x)\in\mathbb{Z}\times\mathbb{R}^N.$$

$(W1)$ $W(n,x)=W_1(n,x)-W_2(n,x)$, *for every* $n\in\mathbb{Z}$, W_1 *and* W_2 *are continuously differentiable in* x, *and there is a bounded set* $\mathbb{J}\subset\mathbb{Z}$ *such that*

$$W_2(n,x)\geq 0, \forall (n,x)\in\mathbb{J}\times\mathbb{R}^N, |x|\leq 1,$$

and

$$\frac{1}{l(n)}|\nabla W(n,x)|=o(|x|)$$

as $x \to 0$ *uniformly in* $n \in \mathbb{Z}\backslash\mathbb{J}$.
$(W2)$ *There is a constant* $\mu > 2$ *such that*

$$0 < \mu W_1(n,x) \leq (\nabla W_1(n,x), x), \forall (n,x) \in \mathbb{Z} \times (\mathbb{R}^N \backslash \{0\}).$$

$(W3)$ $W_2(n,0) \equiv 0$ *and there is a constant* $\varrho \in (2,\mu)$ *such that*

$$(\nabla W_2(n,x), x) \leq \varrho W_2(n,x), \forall (n,x) \in \mathbb{Z} \times \mathbb{R}^N.$$

$(W4)$ $W(n,-x) = W(n,x), \forall (n,x) \in \mathbb{Z} \times \mathbb{R}^N$.
Then there exists an unbounded sequence of homoclinic solutions for system (3.2).

Other related results for the non-periodic case of equation (3.2) can respectively be seen from [11] with asymptotically quadratic potentials, [89,91] with subquadratic potentials, and [9,10,18,21,35,59,90,107] with superquadratic potentials.

4. Discrete p-Laplacian equations and difference equations containing both advance and retardation

Using critical point theory, some scholars established sufficient conditions on the existence of homoclinic solutions for discrete p-Laplacian equations and difference equations containing both advance and retardation. We view some progress in this direction.

4.1. Discrete p-Laplacian equations

Cabada-Li-Tersian [7] in 2010 studied the existence of homoclinic solutions for the p-Laplacian difference equation with periodic coefficients:

$$\triangle\phi_p(\triangle u(k-1)) - V(k)u(k)|u(k)|^{q-2} + \lambda f(k,u(k)) = 0, \quad k \in \mathbb{Z}. \tag{4.1}$$

Here the p-Laplacian operator ϕ_p is defined as $\phi_p(t) = |t|^{p-2}t$ for all $t \in \mathbb{R}$ and $p > 1$.

Theorem 4.1 ([7]). *Assume that the following hypotheses are satisfied:*
(F_1) *The function* $f(k,t)$ *is continuous in* $t \in \mathbb{R}$ *and* T*-periodic in* k.
(F_2) *The potential function* $F(k,t)$ *of* $f(k,t)$,

$$F(k,t) = \int_0^t f(k,s)ds$$

satisfies the Rabinowitz's type condition: There exist $\mu > p \geqslant q > 1$ *and* $s > 0$ *such that*

$$\mu F(k,t) \leqslant t f(k,t), \quad k \in \mathbb{Z}, \quad t \neq 0,$$
$$F(k,t) > 0, \quad \forall k \in \mathbb{Z}, \quad \textit{for } t \geqslant s > 0.$$

(F_3) $f(k,t) = o(|t|^{q-1})$ *as* $|t| \to 0$.
Suppose that the function $V : \mathbb{Z} \to \mathbb{R}$ *is positive and* T*-periodic and the functions* $f(\cdot,\cdot) : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ *satisfy assumptions* $(F_1) - (F_3)$*. Then, for each* $\lambda > 0$*, equation (4.1) has a nonzero homoclinic solution* $u \in \ell^q$*. Moreover, given a nontrivial solution* u *of equation (4.1), there exist* $k_\pm$ *two integer numbers such that for all* $k > k_+$ *and* $k < k_-$*, the sequence* $u(k)$ *is strictly monotone.*

In 2013, Liu-Zhang-Shi [55] considered the following second order p-Laplacian difference equation containing both advanced and retarded arguments:

$$\triangle\left(\phi_p(\triangle(t-1))\right) - q(t)\phi_p(u(t)) = f\left(t, u(t+1), u(t), u(t-1)\right) \quad t \in \mathbb{Z}. \tag{4.2}$$

By using the critical point theory, they obtained the existence of a nontrivial homoclinic solution. The proof is based on the mountain pass lemma in combination with periodic approximations.

Theorem 4.2 ([55]). *Assume that the following hypotheses are satisfied:*
(r) $q(t) > 0$, *for all* $t \in \mathbb{Z}$.
(F_1) *There exists a functional* $F(t, v_1, v_2) \in C^1(\mathbb{Z} \times \mathbb{R}^2, \mathbb{R})$ *with* $F(t, v_1, v_2) \leqslant 0$ *and it satisfies*

$$F(t+T, v_1, v_2) = F(t, v_1, v_2),$$
$$\frac{\partial F(t-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(t, v_1, v_2)}{\partial v_2} = f(t, v_1, v_2, v_3),$$
$$\lim_{\rho \to 0} \frac{F(t, v_1, v_2)}{\rho^p} = 0 \quad \rho = \sqrt{v_1^2 + v_2^2},$$
$$\lim_{r \to 0} \frac{f(t, v_1, v_2, v_3)}{\phi_p(v_2)} = 0 \quad r = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

(F_2) *There exists a constant* $\beta > p$ *such that*

$$\frac{\partial F(t, v_1, v_2)}{\partial v_1} v_1 + \frac{\partial F(t, v_1, v_2)}{\partial v_2} v_2 \leqslant \beta F(t, v_1, v_2) < 0,$$

for all $(t, v_1, v_2) \in \mathbb{Z} \times \mathbb{R}^2 \backslash \{(0,0)\}$.

Then equation (4.2) has a nontrivial homoclinic solution.

Using critical point theory in combination with periodic approximations, in 2014, Kuang [47] established sufficient conditions on the existence of homoclinic solutions for higher-order periodic difference equations with p-Laplacian:

$$(-1)^n \triangle^n \left[r(k)\phi_p(\triangle^n u(k-n))\right] + q(k)\phi_p(u(k)) = f\left(k, u(k)\right), \ k \in \mathbb{Z}. \tag{4.3}$$

The results in [47] provide rather weaker conditions to guarantee the existence of homoclinic solutions and considerably improve some existing ones even for some special cases.

Theorem 4.3 ([47]). *Assume that the following hypotheses are satisfied:*
(r) $r(k) > 0$ *and* $r(k+T) = r(k)$ *for all* $k \in \mathbb{Z}$.
(q) $q(k) > 0$ *and* $q(k+T) = q(k)$ *for all* $k \in \mathbb{Z}$.
(f) $f(k, u)$ *is continuous in* u *and* T*-periodic in* k, *and* $F(k, u) = \int_0^u f(k, s) ds$ *for* $u \in \mathbb{R}$.
(F_1) *there exist positive constants* δ_1 *and* $a_1 < q_*$ *such that*

$$|f(k, u) \leqslant a_1 |u|^{p-1}, \quad \text{for all } k \in \mathbb{Z}, \quad |u| \leqslant \delta_1.$$

(F_2) $f(k, u)u - pF(k, u) > 0$ *for all* $k \in \mathbb{Z}$ *and* $u \in \mathbb{R}\backslash\{0\}$.
(F_3) $f(k, u)u - pF(k, u) \to \infty$ *as* $|u| \to \infty$.
(F_4) *there exist constants* $\rho_1 > 0$, $c_1 > \frac{q^* + r^* 2^n (\sqrt{2}c)^{pn}}{p}$ *and* b_1 *such that*

$$F(k, u) \geqslant c_1 |u|^p + b_1 \quad \text{for all } k \in \mathbb{Z}, \quad |u| \geqslant \rho_1.$$

Then equation (4.3) has at least a nontrivial homoclinic solution.

In 2013, Iannizzotto-Tersian [40] dealt with nontrivial homoclinic solutions of the following discrete p-Laplacian equation:

$$-\triangle\phi_p(\triangle u(k-1)) + a(k)\phi_p(u(k)) = \lambda f\left(k, u(k)\right), \quad k \in \mathbb{Z}, \tag{4.4}$$

involving a coercive weight function and a positive parameter λ. By means of critical point theory, they proved the existence of at least two nontrivial homoclinic solutions for λ big enough.

Theorem 4.4 ([40]). *Assume that the following hypotheses are satisfied:*
(A) $a(k) \geqslant a_0 > 0$ *for all* $k \in \mathbb{Z}$, $a(k) \to \infty$ *as* $|k| \to \infty$.
(F_1) $\lim_{t\to 0} \frac{|f(k,t)|}{|t|^p} = 0$ *uniformly for* $k \in \mathbb{Z}$.
(F_2) $\sup_{|t|\leqslant T} |F(\cdot,t)| \in \ell^1$ *for all* $T > 0$, *where* $F(\cdot,t) = \int_0^t f(\cdot,s)ds$ *for* $t \in \mathbb{R}$.
(F_3) $\limsup_{|t|\to\infty} \frac{F(k,t)}{|t|^p} \leqslant 0$ *uniformly for all* $k \in \mathbb{Z}$.
(F_4) $F(h,b) > 0$ *for some* $h \in \mathbb{Z}$, $b \in \mathbb{R}$.

Then for all $\lambda > 0$ *big enough problem (4.4) admits at least two nonzero homoclinic solutions. Moreover, whenever* $u : \mathbb{Z} \to \mathbb{R}$ *is a nontrivial solution of problem (4.4), there exist* $k_\pm \in \mathbb{Z}$ *such that both sequences* $(u(k))_{k\leqslant k_-}$ *and* $(u(k))_{k\geqslant k+}$ *are strictly monotone.*

In 2017, Stegliński [83] obtained conditions under which the following nonlinear second-order difference equation:

$$-\triangle(a(k)\phi_p(\triangle u(k-1))) + b(k)\phi_p(u(k)) = \lambda f\left(k, u(k)\right), \quad k \in \mathbb{Z}, \tag{4.5}$$

has infinitely many homoclinic solutions. A variant of the fountain theorem [95,113] is utilized in the proof of the main results.

Theorem 4.5 ([83]). *Assume that the following hypotheses are satisfied:*
(B) $b(k) \geqslant b_0 > 0$ *for all* $k \in \mathbb{Z}$, $b(k) \to \infty$ *as* $|k| \to \infty$;
(H_1) $f(k,-t) = -f(k,t)$ *for all* $k \in \mathbb{Z}, t \in \mathbb{R}$;
(H_2) *there exist* $d > 0$ *and* $q > p$ *such that* $|F(k,t)| \leqslant d(|t|^p + |t|^q)$ *for all* $k \in \mathbb{Z}, t \in \mathbb{R}$;
(H_3) $\lim_{t\to 0} \frac{|f(k,t)|}{|t|^{p-1}} = 0$ *uniformly for* $k \in \mathbb{Z}$;
(H_4) $\lim_{|t|\to\infty} \frac{f(k,t)t}{|t|^p} = \infty$ *for all* $k \in \mathbb{Z}$;
(H_5) *there exists* $\sigma \geqslant 1$ *such that* $\sigma G(k,t) \geqslant F(k,st)$ *for* $k \in \mathbb{Z}, t \in \mathbb{R}$, *and* $s \in [0,1]$, *where* $F(k,t)$ *is the primitive function of* $f(k,t)$, *that is* $F(k,t) = \int_0^t f(k,s)ds$ *for* $k \in \mathbb{Z}, t \in \mathbb{R}$, *and* $G(k,t) = f(k,t)t - pF(k,t)$.
Then, for any $\lambda > 0$, *the problem has a sequence* $\{u_n(k)\}$ *of solutions such that* $J_\lambda(u_n) \to \infty$ *as* $n \to \infty$, *where*

$$J_\lambda(u) = \frac{1}{p}\sum_{k\in\mathbb{Z}} [|a(k)||\triangle u(k-1)|^p + b(k)|u(k)|^p] - \lambda \sum_{k\in\mathbb{Z}} F(k,u(k)).$$

Other related results of nontrivial homoclinic solutions for discrete p-Laplacian equation can be seen in [5,22,30,34,44,53,54,78,82,83]. Recently, nontrivial results of homoclinic solutions with discrete p-Laplacian have been first extended to the ones with ϕ -Laplacian [53,54]. However, there exist only a few results of homoclinic solutions with discrete ϕ-Laplacian [53, 54]. This is an important aspect of homoclinic solutions for discrete equations and one that requires further investigation.

4.2. Difference equations containing both advances and retardations

We first consider the existence of nontrivial homoclinic solutions for periodic difference equations containing both advances and retardations. In 2009, by using the mountain pass theorem [75, 95], Fang-Zhao [28] obtained a sufficient condition for the existence of nontrivial homoclinic solutions for fourth-order difference equations:

$$\triangle^4 u(t-2) - q(t)u(t) = f\left(t, u(t+1), u(t), u(t-1)\right), \ t \in \mathbb{Z}. \tag{4.6}$$

Theorem 4.6 ([28]). *Assume that the following hypotheses are satisfied:*
(F_1) *$f(t,u,v,w) \in C(\mathbb{R}^4, \mathbb{R})$, and there exists a positive integer T such that*

$$f(t+T,u,v,w) = f(t,u,v,w), \quad q(t+T) = q(t), \quad q(t) < 0, \quad \forall t \in \mathbb{Z}.$$

(F_2) $\lim_{\rho \to 0} \frac{f(t,u,v,w)}{\rho} = 0$ *uniformly for $t \in \mathbb{Z}$, where $\rho = \sqrt{u^2+v^2+w^2}$.*
(F_3) *There exist a constant $\beta > 2$ and a functional $F(t,u,v) \in C^1(\mathbb{R}^3, \mathbb{R})$ with $F(t+T,u,v) = F(t,u,v)$, such that*

$$F_2'(t-1,v,w) + F_3'(t,u,v) = f(t,u,v,w), \quad \forall (t,u,v,w) \in \mathbb{Z} \times \mathbb{R}^3,$$

and

$$F_2'(t,u,v) + F_3'(t,u,v) \leqslant \beta F(t,u,v) < 0, \quad \forall (t,u,v) \in \mathbb{Z} \times \mathbb{R}^2 \backslash \{0\}.$$

(F_4) *There exist constants $a_0 > 0$, $a_1 > 0$, such that*

$$F(t,u,v) \leqslant -a_0 \rho_0^\beta, \quad \textit{for} \quad \rho_0 \geqslant 1;$$

$$F(t,u,v) \geqslant -a_1 \rho_0^\beta, \quad \textit{for} \quad \rho_0 \leqslant 1;$$

$$\lim_{\rho_0 \to 0} \frac{F_2'(t,u,v)}{\rho_0} = 0, \ \lim_{\rho_0 \to 0} \frac{F_3'(t,u,v)}{\rho_0} = 0, \ \lim_{\rho_0 \to 0} \frac{F(t,u,v)}{\rho_0^2} = 0 \quad \textit{uniformly for } t \in \mathbb{Z},$$

where $\rho_0 = \sqrt{u^2+v^2}$.

Then there exists a nontrivial homoclinic solution u of equation (4.6) emanating from 0 such that $\sum_{t=-T}^{T} |u(t)| > 0$.

In 2009, Yu-Shi-Guo [97] discussed for the first time how to use the critical point theory to study the existence of a nontrivial homoclinic solution for nonlinear difference equations:

$$Lu(t) - \omega u(t) = f\left(t, u(t+T), u(t), u(t-T)\right), \ t \in \mathbb{Z}, \tag{4.7}$$

containing both long-range advance and long-range retardation without any periodic assumptions. Here the operator L is a second-order difference operator given by

$$Lu(t) = a(t-1)u(t-1) + b(t)u(t) + a(t)u(t+1),$$

where $a(t)$, $b(t)$ are real valued for each $t \in \mathbb{Z}$ and $\omega \in \mathbb{R}$, $f \in C(\mathbb{R}^4, \mathbb{R})$.

Theorem 4.7 ([97]). *Assume that the following hypotheses are satisfied:*
(L) *$a(t) \neq 0, b(t) - |a(t-1)| - |a(t)| > \omega$, for all $t \in \mathbb{Z}$ and*

$$\lim_{|t| \to \infty} \left(b(t) - |a(t-1)| - |a(t)|\right) = \infty.$$

(F_1) *There exists a functional* $F(t,v_1,v_2) \in C^1(\mathbb{Z}\times\mathbb{R}^2,\mathbb{R})$ *with* $F(t,v_1,v_2) \geqslant 0$, *and it satisfies*

$$\frac{\partial F(t-T,v_2,v_3)}{\partial v_2} + \frac{\partial F(t,v_1,v_2)}{\partial v_2} = f(t,v_1,v_2,v_3),$$

$$\lim_{\rho\to 0} \frac{F(t,v_1,v_2)}{\rho^2} = 0 \quad \text{uniformly for } t\in\mathbb{Z},\ \rho = \sqrt{v_1^2+v_2^2},$$

$$\lim_{r\to 0} \frac{f(t,v_1,v_2,v_3)}{v_2} = 0 \quad \text{uniformly for } t\in\mathbb{Z},\ r = \sqrt{v_1^2+v_2^2+v_3^2}.$$

(F_2) *There exists a constant* $\beta > 2$ *such that*

$$0 < \beta F(t,v_1,v_2) \leqslant \frac{\partial F(t,v_1,v_2)}{\partial v_1}v_1 + \frac{\partial F(t,v_1,v_2)}{\partial v_2}v_2,$$

for all $(t,v_1,v_2) \in \mathbb{Z}\times\mathbb{R}^2\backslash\{(0,0)\}$.

Then equation (4.7) has a nontrivial homoclinic solution.

Chen-Tang in 2011 [19] considered the existence of homoclinic solutions for $2n$th-order nonlinear difference equations containing both many advances and retardations:

$$\triangle^n\left(r(t-n)\triangle^n u(t-n)\right) - q(t)u(t) = f\left(t,u(t+n),\cdots,u(t),\cdots,u(t-n)\right),\ t\in\mathbb{Z}, \tag{4.8}$$

where $\triangle$ is the forward difference operator.

Theorem 4.8 ([19]). *Assume that the following hypotheses are satisfied:*
(r) *For every* $t\in\mathbb{Z}, r(t)>0$.
(q) *For every* $t\in\mathbb{Z}, q(t)>0$, *and* $\lim_{|t|\to\infty} q(t) = \infty$.
(F_1) *There exists a functional* $F(t,x_n,\cdots,x_0)$ *which is continuously differentiable in the variable from* x_n *to* x_0 *for every* $t\in\mathbb{Z}$ *and satisfy*

$$\sum_{i=-n}^{0} F'_{2+n+i}(t+i,x_{n+i},\ldots,x_i) = f(t,x_n,x_{n-1},\ldots,x_0,x_{-1},\ldots,x_{-n})$$

and

$$|f(t,x_n,x_{n-1},\ldots,x_0,x_{-1},\ldots,x_{-n})| = o\left(\left(\sum_{i=-n}^{n} x_i^2\right)^{\frac12}\right), \quad \text{as} \left(\sum_{i=-n}^{n} x_i^2\right)^{\frac12} \to 0,$$

$$|F(t,x_n,\ldots,x_0)| = o\left(\sum_{i=0}^{n} x_i^2\right), \quad \text{as} \sum_{i=0}^{n} x_i^2 \to 0$$

uniformly in $t\in\mathbb{Z}\backslash J$.
(F_2) $F(t,x_n,\cdots,x_0) = W(t,x_0) - H(t,x_n,\cdots,x_0)$, *for every* $t\in\mathbb{Z}$, W,H *are continuously differentiable in* x_0 *and* $x_n,\cdots,x_0$, *respectively. Moreover, there is a bounded set* $J\in\mathbb{Z}$ *such that*

$$H(t,x_n,\cdots,x_0) \geqslant 0.$$

(F_3) *There exists a constant* $\mu > 2$ *such that*

$$0 < \mu W(t, x_0) \leqslant W_2'(t, x_0)x_0, \forall (t, x_0) \in \mathbb{Z} \times \mathbb{R}\backslash\{0\}.$$

(F_4) $H(t, 0, \cdots, 0)$ *and there exists a constant* $\varrho \in (2, \mu)$ *such that*

$$\sum_{i=-n}^{0} H'_{2+n+i}(t, x_n, \ldots, x_0)x_{-i} \leqslant \varrho H(t, x_n, \cdots, x_0).$$

(F_5) *there exists a constant* b *such that*

$$H(t, x_n, \cdots, x_0) \leqslant b\gamma^{\varrho}, \quad \textit{for } t \in \mathbb{Z}, \quad \gamma > 1,$$

where $\gamma = \left(\sum_{i=0}^{n} x_i^2\right)^{\frac{1}{2}}$.

Then equation (4.8) possesses at least one nontrivial homoclinic solution.

Other related results of nontrivial homoclinic solutions for difference equations containing both advanced and retarded arguments can be seen in [20, 56, 79].

Acknowledgements. The authors are grateful to the anonymous referees for their useful suggestions which improve the contents of this article.

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