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EXISTENCE OF TIME PERIODIC SOLUTIONS FOR THE 3-D VISCOUS PRIMITIVE EQUATIONS OF LARGE-SCALE DRY ATMOSPHERE*

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Abstract In this paper, we consider the existence of time periodic solutions of the 3-D viscous primitive equations of large-scale dry atmosphere. We used the Galerkin method. Firstly, by Leray-Schauder fixed point theorem, we prove the existence of approximate solutions of the primitive equations, then we show the convergence of the approximate solutions, and we also get the uniqueness to the primitive equations.

Keywords Primitive equations, time periodic solutions, navier-stokes equation.

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1. Introduction

The atmospheric primitive equations have been studied since the last century, and they are derived from the Navier-Stokes equations. Refer literature [21], we know the primitive atmospheric equations which including hydrodynamic, thermodynamic, state equation with the Coriolis force, it was first introduced by Richardson in 1922. In 1992, Lions, Temam and Wang in [17] got a new expression of the primitive equations of dry atmosphere by the introduction of viscosity and some technical treatment. In 1997, Chou Jifan and Li Jianping in [18] researched asymptotic behavior of solutions for the initial boundary value problem of the dry and moist atmosphere equations. From 2006 to 2011, Guo Bolin and Huang Daiwen in [6–9, 12] continue discussed the initial boundary value problem of the new expression of primitive equations. They proved the existence of weak solutions and trajectory attractors for the dry and moist atmospheric equations in geophysics; show the existence, uniqueness and long-time behavior of global strong solutions and got the global existence of the smooth solutions for the problem. In 2015, Hirotada Honda and Atusi Tani in [14] showed some boundedness of solutions for the primitive equations of the atmosphere and the ocean. In the same year, Chongsheng Cao, Slim Ibrahim, Kenji Nakanishi and Edriss S. Titi in [1] proved the finite-time blowup for the inviscid primitive equations of oceanic and atmospheric dynamics. In 2017, Hong Mingli in [13] proved the global well-posedness of the 3-D viscous

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primitive equations of the large-scale ocean, the 3-D viscous primitive equations of the large-scale ocean are similar to the 3-D viscous primitive equations of large-scale dry atmosphere.

The purpose of writing this article is to survey the existence of time periodic solutions of the 3-D viscous primitive equations of large-scale dry atmosphere. Our main results are Theorem 5.1 and Theorem 5.2. To prove the time periodic solutions of the primitive equations, we use the well-known Galerkin method which used to prove the existence of time periodic solutions and weak solutions for many systems, such as Navier-Stokes equations, Schrödinger-Boussinesq equation, quantum equation and pseudo-parabolic equation. So motivated by the ideas in [4, 5, 10, 11, 15, 16, 19, 20], we can accomplished this paper.

The paper is made as follows. In section 2, we are divided into three parts, in the first part we introduced the 3-D viscous primitive equations of large-scale dry atmosphere in the pressure coordinate system, the second part we presented some function spaces, the last part we gave some useful lemmas. In section 3, we proved the existence of approximate solutions. In section 4, in order to show the convergence of the approximate solution, we made a series of estimates. In last section, we proved the existence and uniqueness of time periodic solutions.

2. Preliminaries

2.1. The 3-D viscous primitive equations of large-scale dry atmosphere

The non-dimensional form of three-dimensional viscous primitive equations of large-scale dry atmosphere in the pressure coordinate system are

$$\frac{\partial \vec{v}}{\partial t} + \nabla_{\vec{v}}\vec{v} + \omega\frac{\partial \vec{v}}{\partial \xi} + \frac{f}{R_0}k \times \vec{v} + \mathrm{grad}\Phi - \frac{1}{Re_1}\Delta\vec{v} - \frac{1}{Re_2}\frac{\partial^2 \vec{v}}{\partial \xi^2} = 0, \tag{2.1}$$

$$\mathrm{div}\vec{v} + \frac{\partial \omega}{\partial \xi} = 0, \tag{2.2}$$

$$\frac{\partial \Phi}{\partial \xi} + \frac{bP}{p}T = 0, \tag{2.3}$$

$$\frac{\partial T}{\partial t} + \nabla_{\vec{v}}T + \omega\frac{\partial T}{\partial \xi} - \frac{bP}{p}\omega - \frac{1}{Rt_1}\triangle T - \frac{1}{Rt_2}\frac{\partial^2 T}{\partial \xi^2} = Q_1, \tag{2.4}$$

where $\vec{v} = (v_\theta, v_\varphi)$ is the horizontal velocity, ω is the vertical velocity in p-coordinate system, Φ is the geopotential, T is the temperature, and $\vec{v}, \omega, \Phi, T$ are all the unknown functions, $f = 2\cos\theta$, R_0 is Rossby number, k is vertical unit vector, Re_1, Re_2 are the positive constants which stand for the horizontal and vertical Reynolds numbers, Rt_1, Rt_2 are the products obtained by multiplying the horizontal and vertical Prandtl numbers by Reynolds numbers respectively, P stand for an approximate value of pressure at the surface of the earth, p_0 is pressure of the upper atmosphere and $p_0 > 0$, the variable ξ satisfying $p = (P - p_0)\xi + p_0 (0 < p_0 \le p \le P)$, Q_1 is given functions on $S^2 \times (0, 1)$, b is a positive constant. The definitions of $\nabla_{\vec{v}}\vec{v}$, $\triangle\vec{v}$, $\triangle T$, $\nabla_{\vec{v}}T$, $\mathrm{div}\vec{v}$, $\mathrm{grad}\Phi$ will be given in later.

The space domain of (2.1)-(2.4) is $\Omega = S^2 \times (0,1)$, S^2 is two-dimensional unit sphere. The boundary value conditions are given by

$$\xi = 1(p = P): \frac{\partial \vec{v}}{\partial \xi} = 0, \omega = 0, \frac{\partial T}{\partial \xi} = \alpha_s(\Theta - T), \tag{2.5}$$

$$\xi = 0(p = P_0): \frac{\partial \vec{v}}{\partial \xi} = 0, \omega = 0, \frac{\partial T}{\partial \xi} = 0, \tag{2.6}$$

where α_s is a positive constant, Θ is the given temperature on the surface of the earth. For simplicity and without loss generality, we assume that $\Theta = 0$.

Integrating (2.2) and using (2.5) we have

$$\omega(t; \theta, \varphi, \xi) = \int_\xi^1 \operatorname{div}\vec{v}(t; \theta, \varphi, \xi')d\xi'. \tag{2.7}$$

By $\omega|_{\xi=0} = 0$, we obtain

$$\int_0^1 \operatorname{div}\vec{v}d\xi = 0. \tag{2.8}$$

Suppose that Φ_s is a certain unknown function at the isobaric surface $\xi = 1$. Integrating (2.3), we get

$$\Phi(t; \theta, \varphi, \xi) = \Phi_s(t; \theta, \varphi) + \int_\xi^1 \frac{bP}{p} T d\xi'. \tag{2.9}$$

Thus, (2.1)-(2.4) can be rewritten as

$$\begin{aligned} &\frac{\partial \vec{v}}{\partial t} + \nabla_{\vec{v}}\vec{v} + \left(\int_\xi^1 \operatorname{div}\vec{v}d\xi'\right)\frac{\partial \vec{v}}{\partial \xi} + \frac{f}{R_0} k \times \vec{v} + \operatorname{grad}\Phi_s \\ &\quad + \int_\xi^1 \frac{bP}{p}\operatorname{grad}T d\xi' - \frac{1}{Re_1}\Delta\vec{v} - \frac{1}{Re_2}\frac{\partial^2 \vec{v}}{\partial \xi^2} = 0, \end{aligned} \tag{2.10}$$

$$\frac{\partial T}{\partial t} + \nabla_{\vec{v}}T + \left(\int_\xi^1 \operatorname{div}\vec{v}d\xi'\right)\frac{\partial T}{\partial \xi} - \frac{bP}{p}\left(\int_\xi^1 \operatorname{div}\vec{v}d\xi'\right) - \frac{1}{Rt_1}\triangle T - \frac{1}{Rt_2}\frac{\partial^2 T}{\partial \xi^2} = Q_1, \tag{2.11}$$

$$\int_0^1 \operatorname{div}\vec{v}d\xi = 0, \tag{2.12}$$

boundary value conditions of equations (2.10)-(2.12) are

$$\xi = 1: \frac{\partial \vec{v}}{\partial \xi} = 0, \frac{\partial T}{\partial \xi} = -\alpha_s T, \tag{2.13}$$

$$\xi = 0: \frac{\partial \vec{v}}{\partial \xi} = 0, \frac{\partial T}{\partial \xi} = 0, \tag{2.14}$$

and equations (2.10)–(2.14) are described in references [12].

In present paper, the problem we considered is as follows. Let the given function Q_1 be periodic in t with the period W, then we try to prove the existence and uniqueness of time periodic solutions of the three-dimensional viscous primitive equations of large-scale dry atmosphere (2.1)-(2.4) with the same period W, under the critical smallness assumption, i.e., $K \equiv \sup\limits_{0\le t\le W} \|Q_1\|_{L^N(\Omega)}$ is sufficiently small. Since we simplified (2.1)-(2.4) to (2.10)-(2.14) , so under the above assumption, we just need to prove the solutions U of equations (2.10)-(2.11) are periodic with W, i.e.,

$$U(t+W) = (\vec{v}(x,t+W), T(x,t+W)) = (\vec{v}(x), T(x)) = U(t) \quad x \in \Omega, t \in R^1. \quad (2.15)$$

2.2. Some function spaces

Now we introduce some function spaces and operator we need and define our working spaces for the problem(2.10)-(2.15).

For a two-dimensional unit sphere, the position vector of arbitrary point with coordinates (θ,φ) is $\vec{R} = \sin\theta\cos\varphi\vec{i} + \sin\theta\sin\varphi\vec{j} + \cos\theta\vec{k}$. Let $e_\theta, e_\varphi, e_\xi$ be the unit vectors in θ, φ, ξ directions of the space domain Ω respectively,

$$e_\theta = \frac{\partial}{\partial\theta}, e_\varphi = \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}, e_\xi = \frac{\partial}{\partial\xi}.$$

$L^p(\Omega) := \{h; h : \Omega \to R, \int_\Omega |h|^p < +\infty\}$ with the norm $|h|_p = \left(\int_\Omega |h|^p d\Omega\right)^{\frac{1}{p}}$, $1 \le p < \infty$. And we define $\int_\Omega \cdot d\Omega$ and $\int_{S^2} \cdot dS^2$ as $\int_\Omega \cdot$ and $\int_{S^2} \cdot$ respectively. $L^2(T\Omega|TS^2)$ is the first two components of L^2 vector fields on Ω with the norm $|\vec{v}|_2 = \left(\int_\Omega ({v_\theta}^2 + {v_\varphi}^2)\right)^{\frac{1}{2}}$, where $\vec{v} = (v_\theta, v_\varphi) : \Omega \to TS^2$. $C^\infty(S^2)$ is the function spaces for all smooth functions from $S^2 \to R$. $C^\infty(\Omega)$ is the function spaces for all smooth functions from $\Omega \to R$. $C^\infty(T\Omega|TS^2)$ is the first two components of smooth vector fields on Ω. $C_0^\infty(\Omega) := \{h; h \in C^\infty(\Omega), supph$ is a compact subset in $\Omega\}$. $C_0^\infty(T\Omega|TS^2) := \{\vec{v}; \vec{v} \in C^\infty(T\Omega|TS^2), supp\vec{v}$ is a compact subset in $\Omega\}$. $H^m(\Omega)$ is the Sobolev space of functions which are in $L^2(\Omega)$, together with all their covariant derivatives with respect to $e_\theta, e_\varphi, e_\xi$ of order up to m, with the norm $\|h\|_m = (\int_\Omega (\sum\limits_{1\le k\le m} \sum\limits_{i_j=1,2,3; j=1,\dots,k} |\nabla_{i_1} \cdots \nabla_{i_k} h|^2))^{\frac{1}{2}}$, where $\nabla_1 = \nabla_{e_\theta}$, $\nabla_2 = \nabla_{e_\varphi}$, $\nabla_3 = \nabla_{e_\xi} = \frac{\partial}{\partial\xi}$. $H^m(T\Omega|TS^2) = \{\vec{v}; \vec{v} = (v_\theta, v_\varphi) : \Omega \to TS^2, v_\theta, v_\varphi \in H^m(\Omega)\}$, which norm is similar to $H^m(\Omega)$, that is, in the above formula of norm, let $h = (v_\theta, v_\varphi) = v_\theta e_\theta + v_\varphi e_\varphi$.

Then we introduce some function space consisting of W-periodic functions. Let X be a Banach space. We denote by $C^k(W,X)$ the set of X-valued W-periodic functions on R^1 with continuous derivatives up to order k, and let us define the norm

$$|Q|_{C^k(W;X)} = \sup_{0\le t\le W}\{\sum_{i=0}^{k} |D_t^i Q(t)|_X\}.$$

We denote by $L^p(W,X)$ $(1 \le p \le \infty)$ the set of W-periodic X-valued measurable functions Q on R_1 such that

$$|Q|_{L^p(W;X)} = (\int_0^W |Q|_X^p dt)^{\frac{1}{p}} < +\infty \qquad (1 \le p < \infty),$$

$$|Q|_{L^\infty(W;X)} = \sup_{0\le t\le W} |Q|_X < +\infty.$$

We denote by $W^{k,p}(W;X)$ the set of functions Q which belong to $L^p(W;X)$ together with their derivatives up to order k, and in particular we write $H^k(W;X) = W^{k,2}(W;X)$ when X is a Hilbert space.

Next we introduce some operators, div is the horizontal divergence, $\nabla =$ grad is the horizontal gradient, $\nabla_{\vec{v}}$ is the horizontal covariant derivative and Δ is the horizontal Laplace-Beltrami operator. By direct computation we can obtain

$$\mathrm{div}\vec{v} = \mathrm{div}(v_\theta e_\theta + v_\varphi e_\varphi) = \frac{1}{\sin\theta}(\frac{\partial(v_\theta \sin\theta)}{\partial\theta} + \frac{\partial v_\varphi}{\partial\varphi}), \tag{2.16}$$

$$\nabla T = \mathrm{grad}T = \frac{\partial T}{\partial\theta}e_\theta + \frac{1}{\sin\theta}\frac{\partial T}{\partial\varphi}e_\varphi, \tag{2.17}$$

$$\mathrm{grad}\Phi_s = \frac{\partial\Phi_s}{\partial\theta}e_\theta + \frac{1}{\sin\theta}\frac{\partial\Phi_s}{\partial\varphi}e_\varphi, \tag{2.18}$$

$$\begin{aligned}\nabla_{\vec{v}}\vec{v}_1 =&(v_\theta\frac{\partial v_{1\theta}}{\partial\theta} + \frac{v_\varphi}{\sin\theta}\frac{\partial v_{1\theta}}{\partial\varphi} - v_\varphi v_{1\varphi}\cot\theta)e_\theta \\ &+ (v_\theta\frac{\partial v_{1\varphi}}{\partial\theta} + \frac{v_\varphi}{\sin\theta}\frac{\partial v_{1\varphi}}{\partial\varphi} + v_\varphi v_{1\theta}\cot\theta)e_\varphi,\end{aligned} \tag{2.19}$$

$$\nabla_{\vec{v}}T = v_\theta\frac{\partial T}{\partial\theta} + \frac{v_\varphi}{\sin\theta}\frac{\partial T}{\partial\varphi}, \tag{2.20}$$

$$\triangle T = \mathrm{div}(\mathrm{grad}T) = \frac{1}{\sin\theta}[\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial T}{\partial\theta}) + \frac{1}{\sin\theta}\frac{\partial^2 T}{\partial\varphi^2}], \tag{2.21}$$

$$\triangle\vec{v} = (\triangle v_\theta - \frac{2\cos\theta}{\sin^2\theta}\frac{\partial v_\varphi}{\partial\varphi} - \cot^2\theta\ v_\theta)e_\theta + (\triangle v_\varphi + \frac{2\cos\theta}{\sin^2\theta}\frac{\partial v_\theta}{\partial\varphi} - \cot^2\theta v_\varphi)e_\varphi, \tag{2.22}$$

where $\vec{v} = v_\theta e_\theta + v_\varphi e_\varphi$, $\vec{v}_1 = v_{1\theta}e_\theta + v_{1\varphi}e_\varphi \in C^\infty(T\Omega|TS^2)$, $T \in C^\infty(\Omega)$, $\Phi_s \in C^\infty(S^2)$.

Then we only give the calculation process of (2.22). e_θ and e_φ are unit vectors, thus e_θ and e_φ are both covariant basis and inverting basis, and we know

$$e_\theta = \frac{\partial\vec{R}}{\partial\theta} = \cos\theta\cos\varphi\vec{i} + \cos\theta\sin\varphi\vec{j} - \sin\theta\vec{k}, \quad e_\varphi = \frac{1}{\sin\theta}\frac{\partial\vec{R}}{\partial\varphi} = -\sin\varphi\vec{i} + \cos\varphi\vec{j}.$$

Since $\Gamma_{ijk} = e_{i,j}\cdot e_k$, where $e_{i,j} = \frac{\partial e_i}{\partial j}$ ($i,j,k = \theta$ or φ), we obtain

$$\frac{\partial e_\theta}{\partial\theta} = \Gamma_{\theta\theta\theta}e_\theta + \Gamma_{\theta\theta\varphi}e_\varphi = 0, \quad \frac{\partial e_\theta}{\partial\varphi} = \Gamma_{\theta\varphi\theta}e_\theta + \Gamma_{\theta\varphi\varphi}e_\varphi = \cot\theta e_\varphi,$$

$$\frac{\partial e_\varphi}{\partial\theta} = \Gamma_{\varphi\theta\theta}e_\theta + \Gamma_{\varphi\theta\varphi}e_\varphi = 0, \quad \frac{\partial e_\varphi}{\partial\varphi} = \Gamma_{\varphi\varphi\theta}e_\theta + \Gamma_{\varphi\varphi\varphi}e_\varphi = -\cot\theta e_\theta,$$

then we can get

$$\begin{aligned}\nabla\vec{v} &= e_\theta\frac{\partial(v_\theta e_\theta + v_\varphi e_\varphi)}{\partial\theta} + e_\varphi\frac{\partial(v_\theta e_\theta + v_\varphi e_\varphi)}{\partial\varphi} \\ &= e_\theta(\frac{\partial v_\theta}{\partial\theta}e_\theta + \frac{\partial v_\varphi}{\partial\theta}e_\varphi) + e_\varphi(\frac{1}{\sin\theta}\frac{\partial v_\theta}{\partial\varphi}e_\theta + \cot\theta v_\theta e_\varphi + \frac{1}{\sin\theta}\frac{\partial v_\varphi}{\partial\varphi}e_\varphi - \cot\theta v_\varphi e_\theta).\end{aligned}$$

Thus,

$$\triangle\vec{v} = \nabla\cdot\nabla\vec{v}$$

$$
\begin{aligned}
=& e_\theta\cdot\frac{\partial[e_\theta(\frac{\partial v_\theta}{\partial\theta}e_\theta+\frac{\partial v_\varphi}{\partial\theta}e_\varphi)+e_\varphi(\frac{1}{\sin\theta}\frac{\partial v_\theta}{\partial\varphi}e_\theta+\cot\theta v_\theta e_\varphi+\frac{1}{\sin\theta}\frac{\partial v_\varphi}{\partial\varphi}e_\varphi-\cot\theta v_\varphi e_\theta)]}{\partial\theta}\\
&+e_\varphi\cdot\frac{\partial[e_\theta(\frac{\partial v_\theta}{\partial\theta}e_\theta+\frac{\partial v_\varphi}{\partial\theta}e_\varphi)+e_\varphi(\frac{1}{\sin\theta}\frac{\partial v_\theta}{\partial\varphi}e_\theta+\cot\theta v_\theta e_\varphi+\frac{1}{\sin\theta}\frac{\partial v_\varphi}{\partial\varphi}e_\varphi-\cot\theta v_\varphi e_\theta)]}{\partial\varphi}\\
=&\frac{\partial^2 v_\theta}{\partial\theta^2}e_\theta+\frac{\partial^2 v_\varphi}{\partial\theta^2}e_\varphi+\cot\theta\frac{\partial v_\theta}{\partial\theta}e_\theta+\cot\theta\frac{\partial v_\varphi}{\partial\theta}e_\varphi+\frac{1}{\sin^2\theta}\frac{\partial^2 v_\theta}{\partial\varphi^2}e_\theta+\frac{2\cos\theta}{\sin^2\theta}\frac{\partial v_\theta}{\partial\varphi}e_\varphi\\
&-\cot^2\theta v_\theta e_\theta+\frac{1}{\sin^2\theta}\frac{\partial^2 v_\varphi}{\partial\varphi^2}e_\varphi-\frac{2\cos\theta}{\sin^2\theta}\frac{\partial v_\varphi}{\partial\varphi}e_\theta-\cot^2\theta v_\varphi e_\varphi\\
=&(\Delta v_\theta-\frac{2\cos\theta}{\sin^2\theta}\frac{\partial v_\varphi}{\partial\varphi}-\cot^2\theta v_\theta)e_\theta+(\triangle v_\varphi+\frac{2\cos\theta}{\sin^2\theta}\frac{\partial v_\theta}{\partial\varphi}-\cot^2\theta v_\varphi)e_\varphi.
\end{aligned}
$$

We complete the computation process of (2.22).

Last we define our working spaces, let

$$
\begin{aligned}
&\Theta_1:\{\vec v:\vec v\in C_0^\infty(T\Omega|TS^2),\frac{\partial\vec v}{\partial\xi}|_{\xi=0}=0,\frac{\partial\vec v}{\partial\xi}|_{\xi=1}=0,\int_0^1\text{div}\vec v d\xi=0\},\\
&\Theta_2:\{T:T\in C_0^\infty(\Omega),\frac{\partial T}{\partial\xi}|_{\xi=0}=0,\frac{\partial T}{\partial\xi}|_{\xi=1}=-\alpha_s T\}.
\end{aligned}
$$

V_1 = the closure of Θ_1 with respect to the norm $\|\cdot\|_1$, V_2 = the closure of Θ_2 with respect to the norm $\|\cdot\|_1$, H_1 = the closure of Θ_1 with respect to the norm $|\cdot|_2$, $V=V_1\times V_2$, $H=H_1\times L^2(\Omega)$. Inner products and norm in V_1 and V_2 are denoted by:

$$
\begin{aligned}
&(\vec v,\vec v_1)_{V_1}=\int_\Omega\left(\nabla_{e_\theta}\vec v\cdot\nabla_{e_\theta}\vec v_1+\nabla_{e_\varphi}\vec v\cdot\nabla_{e_\varphi}\vec v_1+\frac{\partial\vec v}{\partial\xi}\frac{\partial\vec v_1}{\partial\xi}\right),\ \|\vec v\|=(\vec v,\vec v)_{V_1}^{\frac12},\ \forall\vec v,\vec v_1\in V_1,\\
&(T,T_1)_{V_2}=\int_\Omega\left(\text{grad}T\cdot\text{grad}T_1+\frac{\partial T}{\partial\xi}\frac{\partial T_1}{\partial\xi}\right),\quad\|T\|=(T,T)_{V_2}^{\frac12},\quad\forall T,T_1\in V_2,\\
&(U,U_1)_H=(v_\theta,v_{1\theta})+(v_\varphi,v_{1\varphi})+(T,T_1),\quad(U,U_1)_V=(\vec v,\vec v_1)_{V_1}+(T,T_1)_{V_2},\\
&\|U\|=(U,U)_V^{\frac12},\quad|U|_2=(U,U)_H^{\frac12},\quad\forall U=(\vec v,T),U_1=(\vec v_1,T_1)\in V.
\end{aligned}
$$

$(\cdot,\cdot)$ represent the L^2 inner products in H_1 and $L^2(\Omega)$.

2.3. Some lemmas

Lemma 2.1 (see [17]). *Let $H_1^\perp$ be the orthogonal complement of H_1 in $L^2(T\Omega|TS^2)$. Then*

$$
\begin{aligned}
&H_1^\perp=\{\vec v\in L^2(T\Omega|TS^2)|\vec v=\text{grad}l,l\in H^1(S^2)\},\\
&H_1=\{\vec v\in L^2(T\Omega|TS^2)|\int_0^1\text{div}\vec v d\xi=0\},\\
&V_1=\{\vec v\in H_0^1(T\Omega|TS^2)|\int_0^1\text{div}\vec v d\xi=0\}.
\end{aligned}
$$

Lemma 2.2. *Let $\vec v=(v_\theta,v_\varphi),\vec v_1=(v_{1\theta},v_{1\varphi})\in C^\infty(T\Omega|TS^2)$ and $p\in C^\infty(S^2)$, then*

$$\int_{S^2}\text{div}\vec v=-\int_{S^2}\nabla p\cdot\vec v,\tag{2.23}$$

in particular,

$$\int_{S^2} \text{div}\vec{v} = 0; \tag{2.24}$$

$$\int_\Omega (-\triangle\vec{v}\cdot\vec{v}_1) = \int_\Omega (\nabla_{e_\theta}\vec{v}\cdot\nabla_{e_\theta}\vec{v}_1 + \nabla_{e_\varphi}\vec{v}\cdot\nabla_{e_\varphi}\vec{v}_1). \tag{2.25}$$

Proof. By using (2.16), (2.17) and Stokes Theorem we can get (2.23)(we can refer literature [23, 24]). From (2.19) and (2.22), by direct computation, we can obtain (2.25). □

Lemma 2.3 (see [12]). *For any $h \in C^\infty(S^2), \vec{v} \in C^\infty(T\Omega|TS^2)$, we have*

$$\int_{S^2} \nabla_{\vec{v}} h + \int_{S^2} h\text{div}\vec{v} = \int_{S^2} \text{div}(h\vec{v}) = 0.$$

Lemma 2.4 (see [17]). *Let $\vec{v}, \vec{v}_1 \in V_1$, $T \in V_2$. Then*

$$\int_\Omega \left(\nabla_{\vec{v}}\vec{v}_1 + \left(\int_\xi^1 \text{div}\vec{v} d\xi'\right)\frac{\partial \vec{v}_1}{\partial \xi}\right)\vec{v}_1 = 0, \tag{2.26}$$

$$\int_\Omega \left(\nabla_{\vec{v}}T + \left(\int_\xi^1 \text{div}\vec{v} d\xi'\right)\frac{\partial T}{\partial \xi}\right)T = 0, \tag{2.27}$$

$$\int_\Omega \left(\int_\xi^1 \frac{bP}{p}\text{grad}T d\xi'\cdot\vec{v} - \frac{bP}{p}\left(\int_\xi^1 \text{div}\vec{v} d\xi'\right)\cdot T\right) = 0. \tag{2.28}$$

Lemma 2.5 (Gagliardo-Nirenberg Inequality, see [22]). *Let Ω be an open, bounded domain of the Lipschitz class in R^n. Assume that $1 \le p \le \infty$, $1 \le q \le \infty$, $1 \le r$, $0 \le \theta \le 1$, and let $k - \frac{n}{p} \le \theta(m - \frac{n}{q}) + (1-\theta)\frac{n}{r}$, Then the following inequality hold*

$$\|D^k u\|_{L^p(\Omega)} \le c(\Omega)\|u\|^\theta_{W^{m,q}(\Omega)}\|u\|^{1-\theta}_{L^r(\Omega)}.$$

Lemma 2.6 (Minkowski Inequality). *Let (X,μ),(Y,ν) be two measurable spaces and $f(x,y)$ be a measurable function about $\mu\times\nu$ on $X\times Y$. If for a.e. $y \in Y$, $f(\cdot,y) \in L^p(X,\mu)$, $1 \le p \le \infty$, and $\int_Y \|f(\cdot,y)\|_{L^p(X,\mu)}d\nu(y) < \infty$, then*

$$\|\int_Y f(\cdot,y)d\nu(y)\|_{L^p(X,\mu)} \le \int_Y \|f(\cdot,y)\|_{L^p(X,\mu)}d\nu(y).$$

3. Approximate Solutions

In this section, we will prove the existence of approximate solution of (2.10)-(2.15). Now we let w_k be the completely orthonormal system in $L^2(\Omega)$ consisting of the eigenfunctions of the Stokes operator A, and denote the form of the approximate solution $U(\vec{v}_n, T_n)$ of the problem (2.10)-(2.15) as follows

$$\vec{v}_n = (\sum_{k=1}^n a_{kn}(t)w_k, \sum_{k=1}^n b_{kn}(t)w_k) = (v_{n\theta}, v_{n\varphi}) = v_{n\theta}e_\theta + v_{n\varphi}e_\varphi,$$

$$T_n = \sum_{k=1}^n c_{kn}(t)w_k.$$

We consider the system of nonlinear differential equations,

$$\begin{aligned}(\frac{\partial \vec{v}_n}{\partial t} + \nabla_{\vec{v}_n}\vec{v}_n + (\int_\xi^1 \mathrm{div}\vec{v}_n d\xi')\frac{\partial \vec{v}_n}{\partial \xi} + \frac{f}{R_0}k \times \vec{v}_n + \mathrm{grad}\Phi_s \\ + \int_\xi^1 \frac{bP}{p}\mathrm{grad}T_n d\xi' - \frac{1}{Re_1}\Delta\vec{v}_n - \frac{1}{Re_2}\frac{\partial^2 \vec{v}_n}{\partial \xi^2}, w_k) = 0,\end{aligned} \tag{3.1}$$

$$\begin{aligned}(\frac{\partial T_n}{\partial t} + \nabla_{\vec{v}_n}T_n + (\int_\xi^1 \mathrm{div}\vec{v}_n d\xi')\frac{\partial T_n}{\partial \xi} - \frac{bP}{p}(\int_\xi^1 \mathrm{div}\vec{v}_n d\xi') \\ - \frac{1}{Rt_1}\triangle T_n - \frac{1}{Rt_2}\frac{\partial^2 T_n}{\partial \xi^2}, w_k) = (Q_1, w_k),\end{aligned} \tag{3.2}$$

$$U_n(t+W) = (\vec{v}_n(x,t+W), T_n(x,t+W)) = (\vec{v}_n(x), T_n(x)) = U_n(t). \tag{3.3}$$

Let W'_n be the subspace of $L^2(\Omega)$ spanned by $w_1, w_2, \cdots w_n$. we know that for any $V_n = (\vec{u}_n, C_n) \in C^1(W, W'_n)$, there exists a unique W-periodic solution $U_n = (\vec{v}_n, T_n) \in C^1(W, W'_n)$ of the linear equations

$$\begin{aligned}&(\frac{\partial \vec{v}_n}{\partial t} + \frac{f}{R_0}k \times \vec{v}_n + \mathrm{grad}\Phi_s - \frac{1}{Re_1}\Delta\vec{v}_n - \frac{1}{Re_2}\frac{\partial^2 \vec{v}_n}{\partial \xi^2}, w_k) \\ =&(-\nabla_{\vec{u}_n}\vec{u}_n - (\int_\xi^1 \mathrm{div}\vec{u}_n d\xi')\frac{\partial \vec{u}_n}{\partial \xi} - \int_\xi^1 \frac{bP}{p}\mathrm{grad}C_n d\xi', w_k),\end{aligned} \tag{3.4}$$

$$\begin{aligned}&(\frac{\partial T_n}{\partial t} - \frac{1}{Rt_1}\triangle T_n - \frac{1}{Rt_2}\frac{\partial^2 T_n}{\partial \xi^2}, w_k) \\ =&(-\nabla_{\vec{u}_n}C_n - (\int_\xi^1 \mathrm{div}\vec{u}_n d\xi')\frac{\partial C_n}{\partial \xi} + \frac{bP}{p}(\int_\xi^1 \mathrm{div}\vec{u}_n d\xi') + Q_1, w_k).\end{aligned} \tag{3.5}$$

So we can see the mapping $F: V_n \to U_n$ is continuous and compact in $C^1(W, W'(n))$ Thus, we shall prove the existence of the solution of (3.1)-(3.3) by applying the Leray-schauder fixed point theorem. To apply the fixed point theorem it is only need to show the boundedness

$$\sup_{0 \le t \le W}(|\vec{v}_n(t)|_2^2 + |T_n(t)|_2^2) \le C, \tag{3.6}$$

for all possible solutions of (3.1)-(3.3) replaced by

$$\lambda\left(\nabla_{\vec{v}_n}\vec{v}_n + \left(\int_\xi^1 \mathrm{div}\vec{v}_n d\xi'\right)\frac{\partial \vec{v}_n}{\partial \xi} + \int_\xi^1 \frac{bP}{p}\mathrm{grad}T_n d\xi'\right)$$

and

$$\lambda\left(\nabla_{\vec{v}_n}T_n + \left(\int_\xi^1 \mathrm{div}\vec{v}_n d\xi'\right)\frac{\partial T_n}{\partial \xi} - \frac{bP}{p}\left(\int_\xi^1 \mathrm{div}\vec{v}_n d\xi'\right)\right)$$

instead of nonlinear terms

$$\nabla_{\vec{v}_n}\vec{v}_n + \left(\int_\xi^1 \mathrm{div}\vec{v}_n d\xi'\right)\frac{\partial \vec{v}_n}{\partial \xi} + \int_\xi^1 \frac{bP}{p}\mathrm{grad}T_n d\xi'$$

and

$$\nabla_{\vec{v}_n}T_n + \left(\int_\xi^1 \mathrm{div}\vec{v}_n d\xi'\right)\frac{\partial T_n}{\partial \xi} - \frac{bP}{p}\left(\int_\xi^1 \mathrm{div}\vec{v}_n d\xi'\right),$$

where C is a constant which independent of $\lambda (0 \leq \lambda \leq 1)$.

Actually, multiplying (3.1) by $a_{kn}(t)e_\theta, b_{kn}(t)e_\varphi$ and summing up over k respectively, we can get two equations, then add the two equations, we obtain

$$\begin{aligned}&(\vec{v}_{nt} + \frac{f}{R_0}k \times \vec{v}_n + \mathrm{grad}\Phi_s - \frac{1}{Re_1}\Delta \vec{v}_n - \frac{1}{Re_2}\frac{\partial^2 \vec{v}_n}{\partial \xi^2}, \vec{v}_n)\\ =&(-\lambda \nabla_{\vec{v}_n}\vec{v}_n - \lambda(\int_\xi^1 \mathrm{div}\vec{v}_n d\xi')\frac{\partial \vec{v}_n}{\partial \xi} - \lambda \int_\xi^1 \frac{bP}{p}\mathrm{grad}T_n d\xi', \vec{v}_n),\end{aligned} \tag{3.7}$$

using integration by parts, Lemma 2.1, Lemma 2.2 and Lemma 2.4, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|\vec{v}_n|_2^2 + \frac{1}{Re_1}\int_\Omega (|\nabla_{e_\theta}\vec{v}_n|^2 + |\nabla_{e_\varphi}\vec{v}_n|^2) + \frac{1}{Re_2}\int_\Omega \left|\frac{\partial \vec{v}_n}{\partial \xi}\right|^2\\ =&-\lambda \int_\Omega (\int_\xi^1 \frac{bP}{p}\mathrm{grad}T_n d\xi')\cdot \vec{v}_n.\end{aligned} \tag{3.8}$$

Multiplying (3.2) by $c_{nk}(t)$ and summing up over k, using integration by parts and Lemma 2.4, we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|T_n|_2^2 + \frac{1}{Rt_1}\int_\Omega |\nabla T_n|^2 + \frac{1}{Rt_2}\int_\Omega \left|\frac{\partial T_n}{\partial \xi}\right|^2 + \frac{\alpha_s}{Rt_2}|T_n|_{\xi=1}|_2^2\\ =&\lambda \int_\Omega \frac{bP}{p}T_n(\int_\xi^1 \mathrm{div}\vec{v}_n d\xi') + \int_\Omega Q_1 T_n.\end{aligned} \tag{3.9}$$

From (3.8)-(3.9) and Lemma 2.4, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(|\vec{v}_n|_2^2 + |T_n|_2^2) + \frac{1}{Re_1}\int_\Omega (|\nabla_{e_\theta}\vec{v}_n|^2 + |\nabla_{e_\varphi}\vec{v}_n|^2) + \frac{1}{Re_2}\int_\Omega |\frac{\partial \vec{v}_n}{\partial \xi}|^2\\ &+ \frac{1}{Rt_1}\int_\Omega |\nabla T_n|^2 + \frac{1}{Rt_2}\int_\Omega |\frac{\partial T_n}{\partial \xi}|^2 + \frac{\alpha_s}{Rt_2}|T_n|_{\xi=1}|_2^2 = \int_\Omega Q_1 T_n,\end{aligned} \tag{3.10}$$

using Young inequality, we obtain

$$\left|\int_\Omega Q_1 T_n\right| \leq c|Q_1|_2^2 + \varepsilon |T_n|_2^2, \tag{3.11}$$

and then by

$$T_n(t,\theta,\varphi,\xi) = -\int_\xi^1 \frac{\partial T_n}{\partial \xi'}d\xi' + T_n|_{\xi=1},$$

we obtain

$$\int_\Omega T_n^2(t,\theta,\varphi,\xi) = \int_\Omega (-\int_\xi^1 \frac{\partial T_n}{\partial \xi'}d\xi' + T_n|_{\xi=1})^2 \leq 2|\frac{\partial T_n}{\partial \xi}|_2^2 + 2|T_n|_{\xi=1}|_2^2, \tag{3.12}$$

thus, from (3.10)–(3.12) we have

$$\frac{d}{dt}(|\vec{v}_n|_2^2 + |T_n|_2^2) + c_1\|\vec{v}_n\|^2 + c_2\|T_n\|^2 + \frac{2\alpha_s}{Rt_2}|T_n|_{\xi=1}|_2^2 \leq cK^2, \tag{3.13}$$

where $c_1 = \min\{\frac{2}{Re_1}, \frac{2}{Re_2}\}$, $c_2 = \min\{\frac{2}{Rt_1}, \frac{2-2\varepsilon}{Rt_2}\}$, ε is a small enough positive constant.

Then using the periodicity of $\vec{v}_n, T_n$, integrating (3.13) over $[0, W]$ we get

$$\int_0^W (\|\vec{v}_n\|^2 + \|T_n\|^2 + |T_n|_{\xi=1}|_2^2) \leq cWK^2. \tag{3.14}$$

So by the first mean value theorems for definite integrals and (3.14), there exists $t^* \in [0, W]$ such that

$$\|\vec{v}_n(t^*)\|^2 + \|T_n(t^*)\|^2 + |T_n(t^*)|_{\xi=1}|_2^2 \leq cK^2, \tag{3.15}$$

poincaré inequality give that

$$|\vec{v}_n|_2^2 \leq \mu_1^{-1}\|\vec{v}_n\|^2, \quad |T_n|_2^2 \leq \mu_2^{-1}\|T_n\|^2,$$

integrating (3.13) again over $[t^*, t+W](t \in [0, W])$, we obtain

$$\begin{aligned}|\vec{v}_n|_2^2 + |T_n|_2^2 &\leq c(t+W-t^*)|Q_1|_2^2 + |\vec{v}_n(t^*)|_2^2 + |T_n(t^*)|_2^2 + \left|\int_0^W (\|\vec{v}_n\|^2 + \|T_n\|^2)\right| \\ &\leq c(\mu_1^{-1} + \mu_2^{-1} + 2W)K^2,\end{aligned}$$

thus

$$\sup_{0 \leq t \leq W} (|\vec{v}_n|_2^2 + |T_n|_2^2) \leq c(\mu_1^{-1} + \mu_2^{-1} + 2W)K^2 = M_1, \tag{3.16}$$

where M_1 is independent of n and λ. So we proved the $U_n = (\vec{v}_n, T_n) \in C^1(W, W_n')$ is the approximate solutions of (3.1)-(3.3).

4. Priori estimates

In this section, we will show the convergence of the approximate solutions. Our main result are Theorem 4.1, Theorem 4.2 and Theorem 4.3. First we should prove the following lemma.

Lemma 4.1. *Let $(\vec{v}_n(t), T_n(t))$ be the solutions of (3.1)-(3.3) given above. Set*

$$K_1 \equiv \int_0^W |Q_1|_3^3 dt, K_2 \equiv \int_0^W |Q_1|_4^4 dt,$$

then we have

$$\begin{aligned}&\sup_t |T_n|_3^3 \leq (\mu_2^{-1} + 2W)C(M_1, K, K_1) = M_2, \\ &\sup_t |T_n|_4^4 \leq (\mu_2^{-1} + 2W)C(M_2, K, K_2) = M_3, \\ &\sup_t |\vec{v}_n|_3^3 \leq (\mu_1^{-1} + 2W)C(M_1, K) = M_4, \\ &\sup_t |\vec{v}_n|_4^4 \leq (\mu_1^{-1} + 2W)C(M_4, K) = M_5,\end{aligned}$$

where $M_i (i = 2, 3, 4, 5)$ is independent of n.

Proof. Multiplying (3.2) by $c_{nk}(t)$ and summing up over k, we have

$$\begin{aligned}&(T_{nt} - \frac{1}{Rt_1}\triangle T_n - \frac{1}{Rt_2}\frac{\partial^2 T_n}{\partial \xi^2}, T_n) \\ =&(-\nabla_{\vec{v}_n} T_n - (\int_\xi^1 \mathrm{div}\vec{v}_n d\xi')\frac{\partial T_n}{\partial \xi} + \frac{bP}{p}(\int_\xi^1 \mathrm{div}\vec{v}_n d\xi') + Q_1, T_n),\end{aligned} \tag{4.1}$$

then multiplying (4.1) by $|T_n|$ and using integration by parts, we obtain

$$\begin{aligned}&\frac{d|T_n|_3^3}{dt}+\frac{2}{Rt_1}\int_\Omega|\nabla T_n|^2|T_n|+\frac{2}{Rt_2}\int_\Omega|\frac{\partial T_n}{\partial\xi}|^2|T_n|+\frac{\alpha_s}{Rt_2}|T_n|_{\xi=1}|_3^3\\&=-\int_\Omega[\nabla_{\vec{v}_n}T_n+(\int_\xi^1\mathrm{div}\vec{v}_nd\xi')\frac{\partial T_n}{\partial\xi}]|T_n|T_n+\int_\Omega Q_1|T_n|T_n\\&+\int_\Omega\frac{bP}{p}(\int_\xi^1\mathrm{div}\vec{v}_nd\xi')|T_n|T_n,\end{aligned}\tag{4.2}$$

with the method which get the (2.27), we find

$$\int_\Omega[\nabla_{\vec{v}_n}T_n+(\int_\xi^1\mathrm{div}\vec{v}_nd\xi')\frac{\partial T_n}{\partial\xi}]|T_n|T_n=0,\tag{4.3}$$

by applying Hölder inequality, Young inequality and Lemma 2.5, we have

$$\begin{aligned}\int_\Omega\frac{bP}{p}(\int_\xi^1\mathrm{div}\vec{v}_nd\xi')|T_n|T_n&\le c|\mathrm{div}\vec{v}_n|_2^2+c|T_n|_2^2\|T_n\|^2\\&\le c\int_\Omega(|\nabla_{e_\theta}\vec{v}_n|^2+|\nabla_{e_\varphi}\vec{v}_n|^2)+c|T_n|_2^2\|T_n\|^2,\end{aligned}\tag{4.4}$$

$$|\int_\Omega Q_1|T_n|T_n|\le|Q_1|_3|T_n|_3^2\le c|Q_1|_3^3+c|T_n|_2^6+c\|T_n\|^2,\tag{4.5}$$

so from (4.2)-(4.5), we get

$$\begin{aligned}&\frac{d|T_n|_3^3}{dt}+\frac{2}{Rt_1}\int_\Omega|\nabla T_n|^2|T_n|+\frac{2}{Rt_2}\int_\Omega|\frac{\partial T_n}{\partial\xi}|^2|T_n|+\frac{\alpha_s}{Rt_2}|T_n|_{\xi=1}|_3^3\\&\le c\int_\Omega(|\nabla_{e_\theta}\vec{v}_n|^2+|\nabla_{e_\varphi}\vec{v}_n|^2)+c(|T_n|_2^2+1)\|T_n\|^2+c|Q_1|_3^3+c|T_n|_2^6.\end{aligned}\tag{4.6}$$

Integrating (4.6) over [0,W], using the periodicity of T_n, (3.14) and (3.16), we obtain

$$\begin{aligned}&\int_0^W(\frac{2}{Rt_1}\int_\Omega|\nabla T_n|^2|T_n|+\frac{2}{Rt_2}\int_\Omega|\frac{\partial T_n}{\partial\xi}|^2|T_n|+\frac{\alpha_s}{Rt_2}|T_n|_{\xi=1}|_3^3)\\&\le C(M_1,K,K_1)W,\end{aligned}\tag{4.7}$$

so by (4.7), there exists $t^*\in[0,W]$ such that

$$\begin{aligned}&\frac{2}{Rt_1}\int_\Omega|\nabla T_n(t^*)|^2|T_n(t^*)|+\frac{2}{Rt_2}\int_\Omega|\frac{\partial T_n(t^*)}{\partial\xi}|^2|T_n(t^*)|+\frac{\alpha_s}{Rt_2}|T_n(t^*)|_{\xi=1}|_3^3\\&\le C(M_1,K,K_1),\end{aligned}\tag{4.8}$$

then, using poincaré inequality, we obtain

$$\begin{aligned}\int_\Omega|T(t^*)|^3&=||T(t^*)|^{\frac{3}{2}}|_2^2\le\mu_2^{-1}\||T(t^*)|^{\frac{3}{2}}\|^2\\&=\frac{4}{9}\mu_2^{-1}\int_\Omega(|\nabla T_n(t^*)|^2|T_n(t^*)|+|\frac{\partial T_n(t^*)}{\partial\xi}|^2|T_n(t^*)|),\end{aligned}\tag{4.9}$$

thus integrating (4.6) again over $[t^*, t+W](t \in [0,W])$, by (4.7)-(4.9) we have

$$\sup_t |T_n|_3^3 \le (\mu_2^{-1} + 2W)C(M_1, K, K_1) = M_2. \tag{4.10}$$

Multiplying (4.1) by $|T_n|^2$, applying integration by parts, we have

$$\begin{aligned} &\frac{d|T_n|_4^4}{dt} + \frac{3}{Rt_1}\int_\Omega |\nabla T_n|^2|T_n|^2 + \frac{3}{Rt_2}\int_\Omega |\frac{\partial T_n}{\partial \xi}|^2|T_n|^2 + \frac{\alpha_s}{Rt_2}|T_n|_{\xi=1}|_4^4 \\ =&-\int_\Omega [\nabla_{\vec{v}_n} T_n + (\int_\xi^1 \mathrm{div}\vec{v}_n d\xi')\frac{\partial T_n}{\partial \xi}]|T_n|^2 T_n + \int_\Omega Q_1|T_n|^2T_n \\ &+\int_\Omega \frac{bP}{p}(\int_\xi^1 \mathrm{div}\vec{v}_n d\xi')|T_n|^2T_n, \end{aligned} \tag{4.11}$$

similarly, with the method of getting the (2.27), we get

$$\int_\Omega [\nabla_{\vec{v}_n} T_n + (\int_\xi^1 \mathrm{div}\vec{v}_n d\xi')\frac{\partial T_n}{\partial \xi}]|T_n|^2T_n = 0, \tag{4.12}$$

by using Hölder inequality, Young inequality and Lemma 2.5, we obtain

$$\begin{aligned} \int_\Omega \frac{bP}{p}(\int_\xi^1 \mathrm{div}\vec{v}_n d\xi')|T_n|^2T_n &\le c|\mathrm{div}\vec{v}_n|_2^2 + c|T_n|_6^6 \\ &\le c\int_\Omega (|\nabla_{e_\theta}\vec{v}_n|^2 + |\nabla_{e_\varphi}\vec{v}_n|^2) + c|T_n|_3^4\|T_n\|^2, \end{aligned} \tag{4.13}$$

$$\left|\int_\Omega Q_1|T_n|^2T_n\right| \le |Q_1|_4|T_n|_4^3 \le c|Q_1|_4^4 + c|T_n|_3^2\|T_n\|^2, \tag{4.14}$$

from (4.11)-(4.14), we get

$$\begin{aligned} &\frac{d|T_n|_4^4}{dt} + \frac{3}{Rt_1}\int_\Omega |\nabla T_n|^2|T_n|^2 + \frac{3}{Rt_2}\int_\Omega |\frac{\partial T_n}{\partial \xi}|^2|T_n|^2 + \frac{\alpha_s}{Rt_2}|T_n|_{\xi=1}|_4^4 \\ \le& c\int_\Omega (|\nabla_{e_\theta}\vec{v}_n|^2 + |\nabla_{e_\varphi}\vec{v}_n|^2) + c(|T_n|_3^2 + |T_n|_3^4)\|T_n\|^2 + c|Q_1|_4^4. \end{aligned} \tag{4.15}$$

Using the periodicity of T_n, (3.14), (3.16) and (4.10), integrating (4.15) over [0,W], we obtain

$$\begin{aligned} &\int_0^W (\frac{3}{Rt_1}\int_\Omega |\nabla T_n|^2|T_n|^2 + \frac{3}{Rt_2}\int_\Omega |\frac{\partial T_n}{\partial \xi}|^2|T_n|^2 + \frac{\alpha_s}{Rt_2}|T_n|_{\xi=1}|_4^4) \\ \le& C(M_2, K, K_2)W, \end{aligned} \tag{4.16}$$

so by (4.16), there exists $t^* \in [0,W]$ such that

$$\begin{aligned} &\frac{3}{Rt_1}\int_\Omega |\nabla T_n(t^*)|^2|T_n(t^*)|^2 + \frac{3}{Rt_2}\int_\Omega |\frac{\partial T_n(t^*)}{\partial \xi}|^2|T_n(t^*)|^2 + \frac{\alpha_s}{Rt_2}|T_n(t^*)|_{\xi=1}|_4^4 \\ \le& C(M_2, K, K_2), \end{aligned} \tag{4.17}$$

using poincaré inequality, we obtain

$$\begin{aligned}\int_\Omega |T(t^*)|^4 &= ||T(t^*)|^2|_2^2 \le \mu_2^{-1}\||T(t^*)|^2\|^2\\ &= \frac{1}{4}\mu_2^{-1}\int_\Omega [|\nabla T_n(t^*)|^2|T_n(t^*)|^2 + |\frac{\partial T_n(t^*)}{\partial \xi}|^2|T_n(t^*)|^2],\end{aligned} \tag{4.18}$$

thus integrating (4.15) again over $[t^*, t+W](t \in [0,W])$, by (4.16)-(4.18) we have

$$\sup_t |T_n|_4^4 \le (\mu_2^{-1} + 2W)C(M_2, K, K_2) = M_3. \tag{4.19}$$

Multiplying (3.1) by $a_{kn}(t)e_\theta, b_{kn}(t)e_\varphi$ and summing up over k respectively, then summing up them, we get

$$\begin{aligned}&(\vec{v}_{nt} + \frac{f}{R_0}k \times \vec{v}_n + \mathrm{grad}\Phi_s - \frac{1}{Re_1}\Delta\vec{v}_n - \frac{1}{Re_2}\frac{\partial^2\vec{v}_n}{\partial\xi^2}, \vec{v}_n)\\ =&(-\nabla_{\vec{v}_n}\vec{v}_n - \left(\int_\xi^1 \mathrm{div}\vec{v}_n d\xi'\right)\frac{\partial\vec{v}_n}{\partial\xi} - \int_\xi^1 \frac{bP}{p}\mathrm{grad}T_n d\xi', \vec{v}_n),\end{aligned} \tag{4.20}$$

furthermore, multiplying (4.20) with $|\vec{v}_n|$, using integration by parts, we have

$$\begin{aligned}&\frac{1}{3}\frac{d|\vec{v}_n|_3^3}{dt} + \frac{1}{Re_1}\int_\Omega [(|\nabla_{e_\theta}\vec{v}_n|^2 + |\nabla_{e_\varphi}\vec{v}_n|^2)|\vec{v}_n| + \frac{4}{9}|\nabla_{e_\theta}|\vec{v}_n|^{\frac{3}{2}}|^2\\ &+ \frac{4}{9}|\nabla_{e_\varphi}|\vec{v}_n|^{\frac{3}{2}}|^2] + \frac{1}{Re_2}\int_\Omega (|\vec{v}_{n\xi}|^2|\vec{v}_n| + \frac{4}{9}|\partial_\xi|\vec{v}_n|^{\frac{3}{2}}|^2)\\ =&-\int_\Omega [\nabla_{\vec{v}_n}\vec{v}_n + (\int_\xi^1 \mathrm{div}\vec{v}_n d\xi')]|\vec{v}_n|\vec{v}_n - \int_\Omega \frac{bP}{p}(\int_\xi^1 \mathrm{grad}T_n d\xi')|\vec{v}_n|\vec{v}_n.\end{aligned} \tag{4.21}$$

Hölder inequality, Young inequality, Lemma 2.4 and Lemma 2.5 give that

$$\begin{aligned}&\frac{1}{3}\frac{d|\vec{v}_n|_3^3}{dt} + \frac{1}{Re_1}\int_\Omega [(|\nabla_{e_\theta}\vec{v}_n|^2 + |\nabla_{e_\varphi}\vec{v}_n|^2)|\vec{v}_n| + \frac{4}{9}|\nabla_{e_\theta}|\vec{v}_n|^{\frac{3}{2}}|^2\\ &+ \frac{4}{9}|\nabla_{e_\varphi}|\vec{v}_n|^{\frac{3}{2}}|^2] + \frac{1}{Re_2}\int_\Omega (|\vec{v}_{n\xi}|^2|\vec{v}_n| + \frac{4}{9}|\partial_\xi|\vec{v}_n|^{\frac{3}{2}}|^2)\\ \le&|\nabla T_n|_2^2 + |\vec{v}_n|_2^2\|\vec{v}_n\|^2.\end{aligned} \tag{4.22}$$

Using the periodicity of $\vec{v}_n$, (3.14) and (3.16), integrating (4.22) over [0,W] we get

$$\begin{aligned}&\int_0^W (\frac{1}{Re_1}\int_\Omega [(|\nabla_{e_\theta}\vec{v}_n|^2 + |\nabla_{e_\varphi}\vec{v}_n|^2)|\vec{v}_n| + \frac{4}{9}|\nabla_{e_\theta}|\vec{v}_n|^{\frac{3}{2}}|^2\\ &+ \frac{4}{9}|\nabla_{e_\varphi}|\vec{v}_n|^{\frac{3}{2}}|^2] + \frac{1}{Re_2}\int_\Omega (|\vec{v}_{n\xi}|^2|\vec{v}_n| + \frac{4}{9}|\partial_\xi|\vec{v}_n|^{\frac{3}{2}}|^2))\\ \le&C(M_1, K)W,\end{aligned}$$

so there exists $t^* \in [0,W]$ such that

$$\begin{aligned}&\frac{1}{Re_1}\int_\Omega [(|\nabla_{e_\theta}\vec{v}_n(t^*)|^2 + |\nabla_{e_\varphi}\vec{v}_n(t^*)|^2)|\vec{v}_n(t^*)| + \frac{4}{9}|\nabla_{e_\theta}|\vec{v}_n(t^*)|^{\frac{3}{2}}|^2\\ &+ \frac{4}{9}|\nabla_{e_\varphi}|\vec{v}_n(t^*)|^{\frac{3}{2}}|^2] + \frac{1}{Re_2}\int_\Omega (|\vec{v}_{n\xi}(t^*)|^2|\vec{v}_n(t^*)| + \frac{4}{9}|\partial_\xi|\vec{v}_n(t^*)|^{\frac{3}{2}}|^2)\\ \le&C(M_1, K),\end{aligned} \tag{4.23}$$

integrating (4.22) again over $[t^*, t+W](t \in [0,W])$, using poincaré inequality, we have

$$\sup_t |\vec{v}_n|_3^3 \leq (\mu_1^{-1} + 2W)C(M_1, K) = M_4. \tag{4.24}$$

Multiplying (4.20) with $|\vec{v}_n|^2$, we find

$$\begin{aligned}
&\frac{1}{4}\frac{d|\vec{v}_n|_4^4}{dt} + \frac{1}{Re_1}\int_\Omega [(|\nabla_{e_\theta}\vec{v}_n|^2 + |\nabla_{e_\varphi}\vec{v}_n|^2)|\vec{v}_n|^2 + \frac{1}{2}|\nabla_{e_\theta}|\vec{v}_n|^2|^2 \\
&+ \frac{1}{2}|\nabla_{e_\varphi}|\vec{v}_n|^2|^2] + \frac{1}{Re_2}\int_\Omega (|\vec{v}_{n\xi}|^2|\vec{v}_n|^2 + \frac{1}{2}|\partial_\xi|\vec{v}_n|^2|^2) \\
=& -\int_\Omega [\nabla_{\vec{v}_n}\vec{v}_n + (\int_\xi^1 \mathrm{div}\vec{v}_n d\xi')]|\vec{v}_n|^2\vec{v}_n - \int_\Omega \frac{bP}{p}(\int_\xi^1 \mathrm{grad}T_n d\xi')|\vec{v}_n|^2\vec{v}_n.
\end{aligned} \tag{4.25}$$

Use the similar way, we can easily get

$$\sup_t |\vec{v}_n|_4^4 \leq (\mu_1^{-1} + 2W)C(M_4, K) = M_5. \tag{4.26}$$

This completes the proof of Lemma 4.1. □

Theorem 4.1. *Let $(\vec{v}_n(t), T_n(t))$ be the solutions of (3.1)-(3.3) given above. Set*

$$K_3 \equiv \int_0^W |Q_{1\xi}|_2^2 dt,$$

we have

$$\begin{aligned}
&\sup_t |\vec{v}_{n\xi}(t)|_2^2 \leq (\mu_1^{-1} + 2W)C(M_1) = M_6, \\
&\sup_t |T_{n\xi}(t)|_2^2 \leq (\mu_2^{-1} + 2W)C(M_1, M_3, M_5, M_6, K, K_3) = M_7,
\end{aligned}$$

where $M_i (i = 6, 7)$ is independent of n.

Proof. Taking the derivative with respect to ξ of (4.20), we have

$$\begin{aligned}
&(\frac{\partial \vec{v}_{n\xi}}{\partial t} + \frac{f}{R_0}k \times \vec{v}_{n\xi} - \frac{1}{Re_1}\Delta\vec{v}_{n\xi} - \frac{1}{Re_2}\frac{\partial^2\vec{v}_{n\xi}}{\partial\xi^2}, \vec{v}_{n\xi}) \\
=&(-\nabla_{\vec{v}_n}\vec{v}_{n\xi} - \nabla_{\vec{v}_{n\xi}}\vec{v}_n - \left(\int_\xi^1 \mathrm{div}\vec{v}_n d\xi'\right)\frac{\partial\vec{v}_{n\xi}}{\partial\xi} + (\mathrm{div}\vec{v}_n)\frac{\partial\vec{v}_n}{\partial\xi} + \frac{bP}{p}\mathrm{grad}T_n, \vec{v}_{n\xi}),
\end{aligned} \tag{4.27}$$

applying integration by parts and Lemma 2.1 to Lemma 2.4, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\vec{v}_{n\xi}|_2^2 + \frac{1}{Re_1}\int_\Omega (|\nabla_{e_\theta}\vec{v}_{n\xi}|^2 + |\nabla_{e_\varphi}\vec{v}_{n\xi}|^2) + \frac{1}{Re_2}\int_\Omega \left|\frac{\partial\vec{v}_{n\xi}}{\partial\xi}\right|^2 \\
=& -\int_\Omega [\nabla_{\vec{v}_{n\xi}}\vec{v}_n - (\mathrm{div}\vec{v}_n)\frac{\partial\vec{v}_n}{\partial\xi}]\vec{v}_{n\xi} + \int_\Omega (\frac{bP}{p}\mathrm{grad}T_n)\vec{v}_{n\xi},
\end{aligned} \tag{4.28}$$

by using integration by parts, Young inequality, Hölder inequality, Poincaré inequality, Lemma 2.4 and Lemma 2.5 continue, we have

$$\begin{aligned}
&-\int_\Omega [\nabla_{\vec{v}_{n\xi}}\vec{v}_n-(\mathrm{div}\vec{v}_n)\frac{\partial \vec{v}_n}{\partial\xi}]\vec{v}_{n\xi}\\
\leq& c\int_\Omega |\vec{v}_n||\vec{v}_{n\xi}|(|\nabla_{e_\theta}\vec{v}_{n\xi}|^2+|\nabla_{e_\varphi}\vec{v}_{n\xi}|^2)^{\frac{1}{2}}\\
\leq& c|\vec{v}_n|_4|\vec{v}_{n\xi}|_2^{\frac{1}{4}}\|\vec{v}_{n\xi}\|^{\frac{3}{4}}(\int_\Omega(|\nabla_{e_\theta}\vec{v}_{n\xi}|^2+|\nabla_{e_\varphi}\vec{v}_{n\xi}|^2))^{\frac{1}{2}}\\
\leq& c|\vec{v}_n|_4^8\|\vec{v}_{n\xi}\|^2+\varepsilon\|\vec{v}_{n\xi}\|^2,
\end{aligned}\tag{4.29}$$

$$\int_\Omega(\frac{bP}{p}\mathrm{grad}T_n)\vec{v}_{n\xi}=-\int_\Omega\frac{bP}{p}T_n\mathrm{div}\vec{v}_{n\xi}\leq c|T_n|_2^2+\varepsilon\|\vec{v}_{n\xi}\|^2,\tag{4.30}$$

let ε enough small, from (4.28)-(4.30), by (3.16) and Lemma 4.1, seeing that $C(M_5)<1$, we have

$$\frac{d}{dt}|\vec{v}_{n\xi}|_2^2+\|\vec{v}_{n\xi}\|^2\leq C(M_5)\|\vec{v}_{n\xi}\|^2+C(M_1)\leq C(M_1).\tag{4.31}$$

Using the periodicity of $\vec{v}_n$, by a similar discussion we can find

$$\int_0^W\|\vec{v}_{n\xi}\|^2\leq C(M_1)W.\tag{4.32}$$

$$\sup_t|\vec{v}_{n\xi}(t)|_2^2\leq(\mu_1^{-1}+2W)C(M_1)=M_6.\tag{4.33}$$

Taking the derivative with respect to ξ of (4.1), we have

$$\begin{aligned}
&(\frac{\partial T_{n\xi}}{\partial t}-\frac{1}{Rt_1}\triangle T_{n\xi}-\frac{1}{Rt_2}\frac{\partial^2T_{n\xi}}{\partial\xi^2},T_{n\xi})\\
=&(-\nabla_{\vec{v}_n}T_{n\xi}-\nabla_{\vec{v}_{n\xi}}T_n-\left(\int_\xi^1\mathrm{div}\vec{v}_nd\xi'\right)\frac{\partial T_{n\xi}}{\partial\xi}+(\mathrm{div}\vec{v}_n)\frac{\partial T_n}{\partial\xi}\\
&-\frac{bP(P-p_0)}{p^2}\left(\int_\xi^1\mathrm{div}\vec{v}_nd\xi'\right)-\frac{bP}{p}\mathrm{div}\vec{v}_n+Q_{1\xi},T_{n\xi}),
\end{aligned}\tag{4.34}$$

using integration by parts, we find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|T_{n\xi}|_2^2+\frac{1}{Rt_1}\int_\Omega|\nabla T_{n\xi}|^2+\frac{1}{Rt_2}\int_\Omega\left|\frac{\partial T_{n\xi}}{\partial\xi}\right|^2-\frac{1}{Rt_2}\int_{S^2}(T_\xi|_{\xi=1}\cdot T_{\xi\xi}|_{\xi=1})\\
=&-\int_\Omega[\nabla_{\vec{v}_n}T_{n\xi}+(\int_\xi^1\mathrm{div}\vec{v}_nd\xi')\frac{\partial T_{n\xi}}{\partial\xi}]T_{n\xi}-\int_\Omega[\nabla_{\vec{v}_{n\xi}}T_n-(\mathrm{div}\vec{v}_n)\frac{\partial T_n}{\partial\xi}]T_{n\xi}\\
&-\int_\Omega[\frac{bP}{p}\mathrm{div}\vec{v}_n+\frac{bP(P-p_0)}{p^2}(\int_\xi^1\mathrm{div}\vec{v}_nd\xi')]T_{n\xi}+\int_\Omega Q_{1\xi}T_{n\xi}.
\end{aligned}\tag{4.35}$$

By using Hölder inequality, Young inequality, poincaré inequality and Lemma 2.5, we obtain

$$\begin{aligned}
&\Big|\int_\Omega [\nabla_{\vec{v}_{n\xi}} T_n - (\mathrm{div}\vec{v}_n)\frac{\partial T_n}{\partial \xi}]T_{n\xi}\Big| \\
\leq & c\int_\Omega [(|\nabla_{e_\theta}\vec{v}_{n\xi}|^2 + |\nabla_{e_\varphi}\vec{v}_{n\xi}|^2)^{\frac{1}{2}}|T_n||T_{n\xi}| + |\vec{v}_{n\xi}||T_n||\nabla T_{n\xi}| + |\vec{v}_n||T_{n\xi}||\nabla T_{n\xi}|] \\
\leq & \varepsilon(|T_{n\xi\xi}|_2^2 + |\nabla T_{n\xi}|_2^2) + c|\vec{v}_{n\xi\xi}|_2^2 + c\int_\Omega (|\nabla_{e_\theta}\vec{v}_{n\xi}|^2 + |\nabla_{e_\varphi}\vec{v}_{n\xi}|^2) + c|T_n|_4^8|\vec{v}_{n\xi}|_2^2 \\
& + c\mu_2^{-1}(|T_n|_4^8 + |\vec{v}_n|_4^8)\|T_{n\xi}\|^2,
\end{aligned} \tag{4.36}$$

$$\begin{aligned}
&\Big|\int_\Omega [\frac{bP}{p}\mathrm{div}\vec{v}_n + \frac{bP(P-p_0)}{p^2}(\int_\xi^1 \mathrm{div}\vec{v}_n d\xi')]T_{n\xi}\Big| \\
\leq & \int_\Omega \frac{bP}{p}|\vec{v}_n||\nabla T_{n\xi}| + \int_\Omega |\frac{bP(P-p_0)}{p^2}||\int_\xi^1 \mathrm{div}\vec{v}_n d\xi'||T_{n\xi}| \\
\leq & \varepsilon|\nabla T_{n\xi}|_2^2 + \varepsilon\mu_2^{-1}\|T_{n\xi}\|^2 + c\|\vec{v}_n\|^2.
\end{aligned} \tag{4.37}$$

Taking the trace on $\xi = 1$ of equation (3.2), we have

$$T_{n\xi\xi}|_{\xi=1} = \frac{\partial T_n|_{\xi=1}}{\partial t} + (\nabla_{\vec{v}_n} T_n)|_{\xi=1} - \triangle T_n|_{\xi=1} - Q_1|_{\xi=1}, \tag{4.38}$$

by boundary value conditions and (4.38), we find

$$\begin{aligned}
& -\frac{1}{Rt_2}\int_{S^2}(T_{n\xi}|_{\xi=1}T_{n\xi\xi}|_{\xi=1}) \\
= & \frac{\alpha_s}{Rt_2}\int_{S^2} T_n|_{\xi=1}[\frac{\partial T_n|_{\xi=1}}{\partial t} + (\nabla_{\vec{v}_n}T_n)|_{\xi=1} - \triangle T_n|_{\xi=1} - Q_1|_{\xi=1}] \\
= & \frac{\alpha_s}{Rt_2}(\frac{1}{2}\frac{d|T_n(\xi=1)|_2^2}{dt} + |\nabla T_n(\xi=1)|_2^2) + \frac{\alpha_s}{Rt_2}\int_{S^2} T_n|_{\xi=1}[(\nabla_{\vec{v}_n}T_n)|_{\xi=1} - Q_1|_{\xi=1}],
\end{aligned} \tag{4.39}$$

integration by parts give that

$$\begin{aligned}
& -\frac{\alpha_s}{Rt_2}\int_{S^2} T_n|_{\xi=1}[(\nabla_{\vec{v}_n}T_n)|_{\xi=1} - Q_1|_{\xi=1}] \\
\leq & c|T_n(\xi=1)|_4^4 + c\|\vec{v}_{n\xi}\|^2 + c\|\vec{v}_n\|^2 + c|T_n(\xi=1)|_2^2 + c|Q_1(\xi=1)|_2^2.
\end{aligned} \tag{4.40}$$

From (4.35)-(4.40), let ε enough small, seeing that Lemma 4.1 and $C(M_3, M_5) < 1$, we have

$$\begin{aligned}
& \frac{1}{2}\frac{d(|T_{n\xi}|_2^2 + \frac{\alpha_s}{Rt_2}|T_n(\xi=1)|_2^2)}{dt} + c\|T_{n\xi}\|^2 + \frac{\alpha_s}{Rt_2}|\nabla T_n(\xi=1)|_2^2 \\
\leq & C(M_3, M_5)\|T_{n\xi}\|^2 + c|T_n|_4^8|\vec{v}_{n\xi}|_2^2 + c\|\vec{v}_{n\xi}\|^2 + c\|\vec{v}_n\|^2 + c|T_n(\xi=1)|_4^4 \\
& + c|T_n(\xi=1)|_2^2 + c|Q_1(\xi=1)|_2^2 + c|Q_{1\xi}|_2^2 \\
\leq & c|T_n|_4^8|\vec{v}_{n\xi}|_2^2 + c\|\vec{v}_{n\xi}\|^2 + c\|\vec{v}_n\|^2 + c|T_n(\xi=1)|_4^4 + c|T_n(\xi=1)|_2^2 \\
& + c|Q_1(\xi=1)|_2^2 + c|Q_{1\xi}|_2^2.
\end{aligned} \tag{4.41}$$

Because of the periodicity of T_n, (3.14), (4.16), Lemma 4.1 and (4.32)-(4.33), we obtain

$$\sup_t |T_{n\xi}(t)|_2^2 \leq (\mu_2^{-1} + 2W)C(M_1, M_3, M_5, M_6, K, K_3) = M_7. \tag{4.42}$$

□

Lemma 4.2. *Let $(\vec{v}_n(t), T_n(t))$ be the solutions of (3.1)-(3.3) given above. We have*

$$\begin{aligned}
&\sup_t |\vec{v}_{n\xi}(t)|_3^3 \leq (\mu_1^{-1} + 2W)C(M_1, M_3, M_5, M_6) = M_8,\\
&\sup_t |\vec{v}_{n\xi}(t)|_4^4 \leq (\mu_1^{-1} + 2W)C(M_1, M_3, M_5, M_8) = M_9,\\
&\sup_t |T_{n\xi}(t)|_3^3 \leq (\mu_2^{-1} + 2W)C(M_1, M_3, M_5, M_6, M_7, M_9, K_3) = M_{10},\\
&\sup_t |T_{n\xi}(t)|_4^4 \leq (\mu_2^{-1} + 2W)C(M_1, M_3, M_5, M_6, M_7, M_8, M_{10}, K, K_3) = M_{11},
\end{aligned}$$

where $M_i (i = 8, 9, 10, 11)$ is independent of n.

Proof. Multiplying (4.27) with $\vec{v}_{n\xi}$, applying integration by parts we have

$$\begin{aligned}
&\frac{1}{3}\frac{d}{dt}|\vec{v}_{n\xi}|_3^3 + \frac{1}{Re_1}\int_\Omega [(|\nabla_{e_\theta}\vec{v}_{n\xi}|^2 + |\nabla_{e_\varphi}\vec{v}_{n\xi}|^2)|\vec{v}_{n\xi}| + \frac{4}{9}|\nabla_{e_\theta}|\vec{v}_{n\xi}|^{\frac{2}{3}}|^2\\
&+ \frac{4}{9}|\nabla_{e_\varphi}|\vec{v}_{n\xi}|^{\frac{2}{3}}|^2] + \frac{1}{Re_2}\int_\Omega (|\frac{\partial \vec{v}_{n\xi}}{\partial \xi}|^2|\vec{v}_{n\xi}|^2 + \frac{4}{9}|\partial_\xi|\vec{v}_{n\xi}|^{\frac{2}{3}}|^2)\\
&= -\int_\Omega [\nabla_{\vec{v}_{n\xi}}\vec{v}_n - (\mathrm{div}\vec{v}_n)\frac{\partial \vec{v}_n}{\partial \xi}]|\vec{v}_{n\xi}|\vec{v}_{n\xi} + \int_\Omega (\frac{bP}{p}\mathrm{grad}T_n)|\vec{v}_{n\xi}|\vec{v}_{n\xi}.
\end{aligned} \tag{4.43}$$

Making use of Lemma 2.5, Hölder and Young inequality, we obtain

$$\begin{aligned}
&-\int_\Omega (\nabla_{\vec{v}_{n\xi}}\vec{v}_n - (\mathrm{div}\vec{v}_n)\frac{\partial \vec{v}_n}{\partial \xi})|\vec{v}_{n\xi}|\vec{v}_{n\xi}\\
\leq& c\int_\Omega |\vec{v}_n||\vec{v}_{n\xi}|^2(|\nabla_{e_\theta}\vec{v}_{n\xi}|^2 + |\nabla e_\varphi \vec{v}_{n\xi}|^2)^{\frac{1}{2}}\\
\leq& c|\vec{v}_n|_4||\vec{v}_{n\xi}|^{\frac{3}{2}}|_4[\int_\Omega (|\nabla_{e_\theta}\vec{v}_{n\xi}|^2 + |\nabla_{e_\varphi}\vec{v}_{n\xi}|^2)|\vec{v}_{n\xi}|]^{\frac{1}{2}}\\
\leq& c|\vec{v}_n|_4(\int_\Omega |\vec{v}_{n\xi}|^3)^{\frac{1}{8}}\||\vec{v}_{n\xi}|^{\frac{3}{2}}\|^{\frac{3}{4}}[\int_\Omega (|\nabla_{e_\theta}\vec{v}_{n\xi}|^2 + |\nabla_{e_\varphi}\vec{v}_{n\xi}|^2)|\vec{v}_{n\xi}|]^{\frac{1}{2}}\\
\leq& \varepsilon[\||\vec{v}_{n\xi}|^{\frac{3}{2}}\|^2 + \int_\Omega (|\nabla_{e_\theta}\vec{v}_{n\xi}|^2 + |\nabla_{e_\varphi}\vec{v}_{n\xi}|^2)|\vec{v}_{n\xi}|] + c|\vec{v}_n|_4^8(|\vec{v}_{n\xi}|_2^6 + \|\vec{v}_n\|^2),
\end{aligned} \tag{4.44}$$

$$\int_\Omega (\frac{bP}{p}\mathrm{grad}T_n)|\vec{v}_{n\xi}|\vec{v}_{n\xi} \leq c|T_n|_4^4 + c|\vec{v}_{n\xi}|_2^2 + \varepsilon\int_\Omega (|\nabla_{e_\theta}\vec{v}_{n\xi}|^2 + |\nabla e_\varphi \vec{v}_{n\xi}|^2)|\vec{v}_{n\xi}|. \tag{4.45}$$

From (4.43)-(4.45), let ε enough small, we have

$$\begin{aligned}
&\frac{1}{3}\frac{d}{dt}|\vec{v}_{n\xi}|_3^3 + \frac{1}{Re_1}\int_\Omega [(|\nabla_{e_\theta}\vec{v}_{n\xi}|^2 + |\nabla_{e_\varphi}\vec{v}_{n\xi}|^2)|\vec{v}_{n\xi}| + \frac{4}{9}|\nabla_{e_\theta}|\vec{v}_{n\xi}|^{\frac{2}{3}}|^2\\
&+ \frac{4}{9}|\nabla_{e_\varphi}|\vec{v}_{n\xi}|^{\frac{2}{3}}|^2] + \frac{1}{Re_2}\int_\Omega (|\frac{\partial \vec{v}_{n\xi}}{\partial \xi}|^2|\vec{v}_{n\xi}| + \frac{4}{9}|\partial_\xi|\vec{v}_{n\xi}|^{\frac{2}{3}}|^2)\\
&\leq c|\vec{v}_n|_4^8(|\vec{v}_{n\xi}|_2^6 + c\|\vec{v}_{n\xi}\|^2) + c|T_n|_4^4 + c|\vec{v}_{n\xi}|_2^2.
\end{aligned} \tag{4.46}$$

Using the periodicity of $\vec{v}_n$, (4.32), Lemma 4.1, Theorem 4.1, we get

$$\sup_t |\vec{v}_{n\xi}(t)|_3^3 \le (\mu_1^{-1} + 2W)C(M_1, M_3, M_5, M_6) = M_8. \tag{4.47}$$

Similarly, multiplying (4.27) with $|\vec{v}_n|^2$, applying integration by parts, Hölder inequality, Young inequality and Lemma 2.1 to Lemma 2.5 we have

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}|\vec{v}_{n\xi}|_4^4 + \frac{1}{Re_1}\int_\Omega [(|\nabla_{e_\theta}\vec{v}_{n\xi}|^2 + |\nabla_{e_\varphi}\vec{v}_{n\xi}|^2)|\vec{v}_{n\xi}|^2 + \frac{1}{2}|\nabla_{e_\theta}|\vec{v}_{n\xi}|^2|^2 \\
&+ \frac{1}{2}|\nabla_{e_\varphi}|\vec{v}_{n\xi}|^2|^2] + \frac{1}{Re_2}\int_\Omega (|\frac{\partial \vec{v}_{n\xi}}{\partial \xi}|^2|\vec{v}_{n\xi}|^2 + \frac{1}{2}|\partial_\xi|\vec{v}_{n\xi}|^2|^2) \\
\le& c(1 + |\vec{v}_n|_4^8)|\vec{v}_{n\xi}|_3^2\|\vec{v}_{n\xi}\|^2 + c|T_n|_4^4,
\end{aligned} \tag{4.48}$$

so using the periodicity of $\vec{v}_n$, (4.32), (4.47) and Lemma 4.1, we can get

$$\sup_t |\vec{v}_{n\xi}(t)|_4^4 \le (\mu_1^{-1} + 2W)C(M_1, M_3, M_5, M_8) = M_9. \tag{4.49}$$

Multiplying (4.34) with $|T_{n\xi}|$, using integration by parts, we have

$$\begin{aligned}
&\frac{1}{3}\frac{d|T_{n\xi}|_3^3}{dt} + \frac{2}{Rt_1}\int_\Omega |\nabla T_{n\xi}|^2|T_{n\xi}| + \frac{2}{Rt_2}\int_\Omega |\frac{\partial T_{n\xi}}{\partial \xi}|^2|T_{n\xi}| \\
&- \frac{1}{Rt_2}\int_{S^2}[(T_{n\xi}|T_{n\xi}|)|_{\xi=1}T_{n\xi\xi}|_{\xi=1}] \\
=& -\int_\Omega [\nabla_{\vec{v}_n}T_{n\xi} + (\int_\xi^1 \mathrm{div}\vec{v}_n d\xi')\frac{\partial T_{n\xi}}{\partial \xi}]T_{n\xi}|T_{n\xi}| \\
&- \int_\Omega [\nabla_{\vec{v}_{n\xi}}T_n - (\mathrm{div}\vec{v}_n)\frac{\partial T_n}{\partial \xi}]T_{n\xi}|T_{n\xi}| + \int_\Omega Q_{1\xi}T_{n\xi}|T_{n\xi}| \\
&- \int_\Omega [\frac{bP}{p}\mathrm{div}\vec{v}_n + \frac{bP(P-p_0)}{p^2}(\int_\xi^1 \mathrm{div}\vec{v}_n d\xi')]T_{n\xi}|T_{n\xi}|.
\end{aligned} \tag{4.50}$$

Using integration by parts, Hölder inequality, Minkowski inequality, poincaré inequality, Young inequality and Lemma 2.1 from Lemma 2.5, we have

$$\begin{aligned}
&\frac{1}{3}\frac{d(|T_{n\xi}|_3^3 + \frac{\alpha_s^2}{Rt_2}|T_n(\xi=1)|_3^3)}{dt} + \frac{2}{Rt_1}\int_\Omega |\nabla T_{n\xi}|^2|T_{n\xi}| \\
&+ \frac{2}{Rt_2}\int_\Omega |\frac{\partial T_{n\xi}}{\partial \xi}|^2|T_{n\xi}| + \frac{\alpha_s^2}{Rt_2}\int_{S^2}(|\nabla T_n|^2|T_n|)|_{\xi=1} \\
\le& c(1 + |\vec{v}_n|_4^8)|T_{n\xi}|_2^{\frac{3}{2}}\|T_{n\xi}\|^{\frac{3}{2}} + |T_{n\xi}|_2^{\frac{10}{3}} + |\vec{v}_{n\xi}|_4^2\|T_n\|^2 + c|Q_{1\xi}|_2^3 + c|\vec{v}_n|_4^2|T_{n\xi}|_2 \\
&+ c\|\vec{v}_{n\xi}\|^2 + c\|\vec{v}_n\|^2 + c|T_n(\xi=1)|_6^6 + c|T_n(\xi=1)|_4^4 + c|Q_{1\xi}(\xi=1)|_2^2.
\end{aligned} \tag{4.51}$$

Similarly, multiplying (4.34) with $|T_{n\xi}|^2$, we have

$$\begin{aligned}
&\frac{1}{4}\frac{d(|T_{n\xi}|_4^4 + \frac{\alpha_s^3}{Rt_2}|T_n(\xi=1)|_4^4)}{dt} + \frac{2}{Rt_1}\int_\Omega |\nabla T_{n\xi}|^2|T_{n\xi}|^2 \\
&+ \frac{2}{Rt_2}\int_\Omega |\frac{\partial T_{n\xi}}{\partial \xi}|^2|T_{n\xi}|^2 + \frac{\alpha_s^3}{Rt_2}\int_{S^2}(|\nabla T_n|^2|T_n|^2)|_{\xi=1} \\
\le& c(1 + |\vec{v}_n|_4^8)|T_{n\xi}|_3^2\|T_{n\xi}\|^2 + \|T_n\|^2 + |\vec{v}_{n\xi}|_4^{\frac{20}{3}} + \||\vec{v}_{n\xi}|^2\|^2 \\
&+ \|T_{n\xi}\|^2|T_{n\xi}|_2^{58} + c|Q_{1\xi}|_2^4 + c|\vec{v}_n|_4^4 + c\|\vec{v}_{n\xi}\|^2 + c\|\vec{v}_n\|^2 \\
&+ c|T_n(\xi=1)|_8^8 + c|T_n(\xi=1)|_6^6 + c|Q_{1\xi}(\xi=1)|_2^2.
\end{aligned} \tag{4.52}$$

Thus using the periodicity of T_n, (3.14), (3.16), (4.32), Lemma 4.1 and Theorem 4.1, we easily get:

$$\sup_t |T_{n\xi}(t)|_3^3 \leq (\mu_2^{-1} + 2W)C(M_1, M_3, M_5, M_6, M_7, M_9, K_3) = M_{10},$$
$$\sup_t |T_{n\xi}(t)|_4^4 \leq (\mu_2^{-1} + 2W)C(M_1, M_3, M_5, M_6, M_7, M_8, M_{10}, K, K_0, K_3) = M_{11},$$

where we also use $\int_0^W |T_n(\xi = 1)|_6^6 \leq c$, $\int_0^W |T_n(\xi = 1)|_8^8 \leq c$. We can get them by making L^6, L^8 estimates about T like Lemma 4.1. This completes the proof of Lemma 4.2. □

Theorem 4.2. *Let $(\vec{v}_n(t), T_n(t))$ be the solutions of (3.1)-(3.3) given above. We have*

$$\begin{aligned}&\sup_t(|\nabla_{e_\theta}\vec{v}_n|_2^2 + |\nabla_{e_\varphi}\vec{v}_n|_2^2 + |\nabla T_n|_2^2)\\ \leq&(\mu_1^{-2} + \mu_2^{-2} + 2W)C(M_1, M_5, M_9, M_{11}, K) = M_{12},\end{aligned}$$

where M_{12} is independent of n.

Proof. Considering (3.1) we see,

$$\begin{aligned}&(\frac{\partial \vec{v}_n}{\partial t} + \nabla_{\vec{v}_n}\vec{v}_n + \left(\int_\xi^1 \mathrm{div}\vec{v}_n d\xi'\right)\frac{\partial \vec{v}_n}{\partial \xi} + \frac{f}{R_0}k \times \vec{v}_n + \mathrm{grad}\Phi_s\\ &+ \int_\xi^1 \frac{bP}{p}\mathrm{grad}T_n d\xi' - \frac{1}{Re_1}\Delta\vec{v}_n - \frac{1}{Re_2}\frac{\partial^2\vec{v}_n}{\partial\xi^2}, -\triangle\vec{v}_n) = 0,\end{aligned} \tag{4.53}$$

using integration by parts, we get

$$\begin{aligned}&\frac{1}{2}\frac{d\int_\Omega(|\nabla_{e_\theta}\vec{v}_n|^2 + |\nabla_{e_\varphi}\vec{v}_n|^2)}{dt} + \frac{1}{Re_1}\int_\Omega |\triangle\vec{v}_n|^2 + \frac{1}{Re_2}\int_\Omega(|\nabla_{e_\theta}\vec{v}_{n\xi}|^2 + |\nabla_{e_\varphi}\vec{v}_{n\xi}|^2)\\ =&\int_\Omega(\nabla_{\vec{v}_n}\vec{v}_n + (\int_\xi^1 \mathrm{div}\vec{v}_n)\frac{\partial\vec{v}_n}{\partial\xi})\cdot\triangle\vec{v}_n + \int_\Omega(\int_\xi^1 \frac{bP}{p}\mathrm{grad}T_n d\xi')\cdot\triangle\vec{v}_n\\ &+ \int_\Omega(\frac{f}{R_0}\vec{k}\times\vec{v}_n + \mathrm{grad}\Phi_s)\cdot\triangle\vec{v}_n,\end{aligned} \tag{4.54}$$

applying Hölder inequality, Young inequality, Lemma 2.5, we find

$$\begin{aligned}|\int_\Omega \nabla_{\vec{v}_n}\vec{v}_n\cdot\triangle\vec{v}_n| \leq& \int_\Omega |\vec{v}_n|(|\nabla_{e_\theta}\vec{v}_n|^2 + |\nabla_{e_\varphi}\vec{v}_n|^2)^{\frac{1}{2}}|\triangle\vec{v}_n|\\ \leq& c(\int_\Omega |\vec{v}_n|^4)^{\frac{1}{2}}(\int_\Omega(|\nabla_{e_\theta}\vec{v}_n|^2 + |\nabla_{e_\varphi}\vec{v}_n|^2)^2)^{\frac{1}{2}} + \varepsilon|\triangle\vec{v}_n|_2^2\\ \leq& c|\vec{v}_n|_4^2(\int_\Omega(|\nabla_{e_\theta}\vec{v}_n|^2 + |\nabla_{e_\varphi}\vec{v}_n|^2))^{\frac{1}{4}}(\int_\Omega(|\triangle\vec{v}_n|^2 + |\nabla_{e_\theta}\vec{v}_{n\xi}|^2\\ &+ |\nabla_{e_\varphi}\vec{v}_{n\xi}|^2))^{\frac{3}{4}} + \varepsilon|\triangle\vec{v}_n|_2^2\\ \leq& c|\vec{v}_n|_4^8\int_\Omega(|\nabla_{e_\theta}\vec{v}_n|^2 + |\nabla_{e_\varphi}\vec{v}_n|^2) + \varepsilon\int_\Omega(|\triangle\vec{v}_n|^2 + |\nabla_{e_\theta}\vec{v}_{n\xi}|^2\\ &+ |\nabla_{e_\varphi}\vec{v}_{n\xi}|^2) + \varepsilon|\triangle\vec{v}_n|_2^2,\end{aligned} \tag{4.55}$$

using Hölder inequality, Minkowski inequality, Young inequality and Lemma 2.5, we have

$$
\begin{aligned}
&|\int_\Omega(\int_\xi^1 \mathrm{div}\vec{v}_n d\xi')\frac{\partial\vec{v}_n}{\partial\xi}\cdot\triangle\vec{v}_n|\\
\le&\int_{S^2}[\int_0^1(|\nabla_{e_\theta}\vec{v}_n|^2+|\nabla_{e_\varphi}\vec{v}_n|^2)^{\frac{1}{2}}d\xi\int_0^1|\vec{v}_{n\xi}||\triangle\vec{v}_n d\xi|]\\
\le&\int_{S^2}[(\int_0^1(|\nabla_{e_\theta}\vec{v}_n|^2+|\nabla_{e_\varphi}\vec{v}_n|^2)d\xi)^{\frac{1}{2}}(\int_0^1|\vec{v}_{n\xi}|^2d\xi)^{\frac{1}{2}}(\int_0^1|\triangle\vec{v}_n|^2d\xi)^{\frac{1}{2}}]\\
\le&[\int_{S^2}(\int_0^1(|\nabla_{e_\theta}\vec{v}_n|^2+|\nabla_{e_\varphi}\vec{v}_n|^2)d\xi)^2]^{\frac{1}{4}}|\vec{v}_{n\xi}|_4|\triangle\vec{v}_n|_2\\
\le&|\vec{v}_{n\xi}|_4^2[\int_{S^2}(\int_0^1(|\nabla_{e_\theta}\vec{v}_n|^2+|\nabla_{e_\varphi}\vec{v}_n|^2)d\xi)^2]^{\frac{1}{2}}+\varepsilon|\triangle\vec{v}_n|_2^2\\
\le&|\vec{v}_{n\xi}|_4^2(\int_\Omega(|\nabla_{e_\theta}\vec{v}_n|^2+|\nabla_{e_\varphi}\vec{v}_n|^2))^{\frac{1}{4}}(\int_\Omega(|\nabla_{e_\theta}\vec{v}_{n\xi}|^2+|\nabla_{e_\varphi}\vec{v}_{n\xi}|^2+|\triangle\vec{v}_n|^2))^{\frac{3}{4}}+\varepsilon|\triangle\vec{v}_n|_2^2\\
\le&|\vec{v}_{n\xi}|_4^4\int_\Omega(|\nabla_{e_\theta}\vec{v}_n|^2+|\nabla_{e_\varphi}\vec{v}_n|^2)+\varepsilon\int_\Omega(|\nabla_{e_\theta}\vec{v}_{n\xi}|^2+|\nabla_{e_\varphi}\vec{v}_{n\xi}|^2+|\triangle\vec{v}_n|^2)+\varepsilon|\triangle\vec{v}_n|_2^2,
\end{aligned}
\tag{4.56}
$$

$$
\begin{aligned}
|\int_\Omega(\int_\xi^1\frac{bP}{p}\mathrm{grad}T_n d\xi')\cdot\triangle\vec{v}_n|\le&\int_{S^2}[\int_0^1|\nabla T_n|d\xi\int_0^1|\triangle\vec{v}_n|d\xi]\\
\le&(\int_{S^2}(\int_0^1|\nabla T_n|d\xi)^2)^{\frac{1}{2}}(\int_{S^2}(\int_0^1|\triangle\vec{v}_n|d\xi)^2)^{\frac{1}{2}}\\
\le&c|\nabla T_n|_2^2+\varepsilon|\triangle\vec{v}_n|_2^2.
\end{aligned}
\tag{4.57}
$$

From (4.54)-(4.57), let ε enough small, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d\int_\Omega(|\nabla_{e_\theta}\vec{v}_n|^2+|\nabla_{e_\varphi}\vec{v}_n|^2)}{dt}+\frac{1}{Re_1}\int_\Omega|\triangle\vec{v}_n|^2\\
&+\frac{1}{Re_2}\int_\Omega(|\nabla_{e_\theta}\vec{v}_{n\xi}|^2+|\nabla_{e_\varphi}\vec{v}_{n\xi}|^2)\\
\le&c(|\vec{v}_n|_4^8+|\vec{v}_{n\xi}|_4^4)\int_\Omega(|\nabla_{e_\theta}\vec{v}_n|^2+|\nabla_{e_\varphi}\vec{v}_n|^2)+c|\nabla T_n|_2^2.
\end{aligned}
\tag{4.58}
$$

Considering (3.2) we see,

$$
\begin{aligned}
&(\frac{\partial T_n}{\partial t}+\nabla_{\vec{v}_n}T_n+\left(\int_\xi^1\mathrm{div}\vec{v}_n d\xi'\right)\frac{\partial T_n}{\partial\xi}-\frac{bP}{p}\left(\int_\xi^1\mathrm{div}\vec{v}_n d\xi'\right)\\
&-\frac{1}{Rt_1}\triangle T_n-\frac{1}{Rt_2}\frac{\partial^2 T_n}{\partial\xi^2},-\triangle T_n)=(Q_1,-\triangle T_n),
\end{aligned}
\tag{4.59}
$$

using integration by parts, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d|\nabla T_n|_2^2}{dt}+\frac{1}{Rt_1}|\triangle T_n|_2^2+\frac{1}{Rt_2}(|\nabla T_{n\xi}|_2^2+\alpha_s|\nabla T_n(\xi=1)|_2^2)\\
=&\int_\Omega(\nabla_{\vec{v}_n}T_n+\left(\int_\xi^1\mathrm{div}\vec{v}_n d\xi'\right)\frac{\partial T_n}{\partial\xi})\triangle T_n-\int_\Omega\frac{bP}{p}\left(\int_\xi^1\mathrm{div}\vec{v}_n d\xi'\right)\triangle T_n
\end{aligned}
$$

$$-\int_\Omega Q_1\triangle T_n. \tag{4.60}$$

Similar to (4.58) we have

$$\begin{aligned}&\frac{1}{2}\frac{d|\nabla T_n|_2^2}{dt}+\frac{1}{Rt_1}|\triangle T_n|_2^2+\frac{1}{Rt_2}(|\nabla T_{n\xi}|_2^2+\alpha_s|\nabla T_n(\xi=1)|_2^2)\\ \le& c(|\vec{v}_n|_4^8+|T_{n\xi}|_4^4)|\nabla T_n|_2^2+\int_\Omega(|\nabla_{e_\theta}\vec{v}_n|^2+|\nabla_{e_\varphi}\vec{v}_n|^2)+c|Q_1|_2^2,\end{aligned} \tag{4.61}$$

merging (4.58) and (4.61), we find

$$\begin{aligned}&\frac{1}{2}\frac{d[\int_\Omega(|\nabla_{e_\theta}\vec{v}_n|^2+|\nabla_{e_\varphi}\vec{v}_n|^2)+|\nabla T_n|_2^2]}{dt}+\frac{1}{Re_1}|\triangle\vec{v}_n|_2^2+\frac{1}{Rt_1}|\triangle T_n|_2^2\\ &+\frac{1}{Re_2}\int_\Omega(|\nabla_{e_\theta}\vec{v}_{n\xi}|^2+|\nabla_{e_\varphi}\vec{v}_{n\xi}|^2)+\frac{1}{Rt_2}(|\nabla T_{n\xi}|_2^2+\alpha_s|\nabla T_n(\xi=1)|_2^2)\\ \le& c(1+|\vec{v}_n|_4^8+|\vec{v}_{n\xi}|_4^4)\int_\Omega(|\nabla_{e_\theta}\vec{v}_n|^2+|\nabla_{e_\varphi}\vec{v}_n|^2)\\ &+c(1+|\vec{v}_n|_4^8+|T_{n\xi}|_4^4)|\nabla T_n|_2^2+c|Q_1|_2^2.\end{aligned} \tag{4.62}$$

Hence, with the periodicity of $\vec{v}_n, T_n$, (3.14), Lemma 4.1 and Lemma 4.2, we obtain

$$\begin{aligned}&\int_0^W[\frac{1}{Re_1}|\triangle\vec{v}_n|_2^2+\frac{1}{Re_2}\int_\Omega(|\nabla_{e_\theta}\vec{v}_{n\xi}|^2+|\nabla_{e_\varphi}\vec{v}_{n\xi}|^2)\\ &+\frac{1}{Rt_1}|\triangle T_n|_2^2+\frac{1}{Rt_2}(|\nabla T_{n\xi}|_2^2+\alpha_s|\nabla T_n(\xi=1)|_2^2)]\\ \le& C(M_1,M_5,M_9,M_{11},K)W,\end{aligned} \tag{4.63}$$

$$\begin{aligned}&\sup_t(|\nabla_{e_\theta}\vec{v}_n|_2^2+|\nabla_{e_\varphi}\vec{v}_n|_2^2+|\nabla T_n|_2^2)\\ \le&(\mu_1^{-2}+\mu_2^{-2}+2W)C(M_1,M_5,M_9,M_{11},K)=M_{12}.\end{aligned} \tag{4.64}$$

□

Theorem 4.3. *Let $(\vec{v}_n, T_n(t))$ be the solutions of (3.1)-(3.3) given above. Set*

$$K_4\equiv\int_0^W|Q_{1t}|_2^2dt,$$

we have

$$\sup_t(|\vec{v}_{nt}(t)|_2^2+|T_{nt}(t)|_2^2)\le(\mu_1^{-1}+\mu_2^{-1}+2W)C(M_1,M_3,M_5,M_9,M_{11},K,K_4).$$

Proof. Considering (3.1) we see,

$$\begin{aligned}&(\frac{\partial\vec{v}_n}{\partial t}+\nabla_{\vec{v}_n}\vec{v}_n+\left(\int_\xi^1\mathrm{div}\vec{v}_nd\xi'\right)\frac{\partial\vec{v}_n}{\partial\xi}+\frac{f}{R_0}k\times\vec{v}_n+\mathrm{grad}\Phi_s\\ &+\int_\xi^1\frac{bP}{p}\mathrm{grad}T_nd\xi'-\frac{1}{Re_1}\Delta\vec{v}_n-\frac{1}{Re_2}\frac{\partial^2\vec{v}_n}{\partial\xi^2},\vec{v}_{nt})=0,\end{aligned} \tag{4.65}$$

similar to (4.58), using integration by parts, Hölder inequality, Minkowski inequality, Young inequality and Lemma 2.5, we obtain

$$\begin{aligned}
&|\vec{v}_{nt}|_2^2+\frac{1}{Re_1}\frac{d}{dt}\int_\Omega(|\nabla_{e_\theta}\vec{v}_n|^2+|\nabla_{e_\varphi}\vec{v}_n|^2)+\frac{1}{Re_2}\frac{d}{dt}\int_\Omega|\frac{\partial\vec{v}_n}{\partial\xi}|^2\\
\leq& c(|\vec{v}_n|_4^8+|\vec{v}_{n\xi}|_4^8)\int_\Omega(|\nabla_{e_\theta}\vec{v}_n|^2+|\nabla_{e_\varphi}\vec{v}_n|^2)+|\nabla T_n|_2^2\\
&+c\int_\Omega(|\nabla_{e_\theta}\vec{v}_{n\xi}|^2+|\nabla_{e_\varphi}\vec{v}_{n\xi}|^2+|\triangle\vec{v}_n|^2),
\end{aligned}\tag{4.66}$$

similarly to (4.66), we also can get

$$\begin{aligned}
&|T_{nt}|_2^2+\frac{1}{Rt_1}\frac{d}{dt}\int_\Omega|\nabla T_n|^2+\frac{1}{Rt_2}\frac{d}{dt}\int_\Omega|\frac{\partial T_n}{\partial\xi}|^2+\frac{\alpha_s}{Rt_2}\frac{d}{dt}|T_n|_{\xi=1}|_2^2\\
\leq& c(|\vec{v}_n|_4^8+|T_{n\xi}|_4^8)|\nabla T_n|_2^2+c\int_\Omega(|\triangle T_n|^2+|\nabla T_{n\xi}|)\\
&+c\int_\Omega(|\nabla_{e_\theta}\vec{v}_n|^2+|\nabla_{e_\varphi}\vec{v}_n|^2)+|Q_1|_2^2,
\end{aligned}\tag{4.67}$$

thus, adding (4.66) and (4.67), using the periodicity of $\vec{v}_n, T_n$, integrating it over $[0,W]$ we have

$$\int_0^W(|\vec{v}_{nt}|_2^2+|T_{nt}|_2^2)\leq C(M_1,M_3,M_5,M_9,M_{11},K).\tag{4.68}$$

Taking the derivative with respect to t of (4.20), applying integration by parts, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\vec{v}_{nt}|_2^2+\frac{1}{Re_1}\int_\Omega(|\nabla_{e_\theta}\vec{v}_{nt}|^2+|\nabla_{e_\varphi}\vec{v}_{nt}|^2)+\frac{1}{Re_2}\int_\Omega|\frac{\partial\vec{v}_{nt}}{\partial\xi}|^2\\
=&-\int_\Omega(\nabla_{\vec{v}_{nt}}\vec{v}_n+(\int_\xi^1\operatorname{div}\vec{v}_{nt}d\xi')\frac{\partial\vec{v}_n}{\partial\xi})\vec{v}_{nt}-\int_\Omega(\int_\xi^1\frac{bP}{p}\operatorname{grad}T_{nt}d\xi')\vec{v}_{nt},
\end{aligned}\tag{4.69}$$

employing integration by parts, Hölder inequality, Young inequality and Lemma 2.5, we have

$$\begin{aligned}
&-\int_\Omega(\nabla_{\vec{v}_{nt}}\vec{v}_n+(\int_\xi^1\operatorname{div}\vec{v}_{nt}d\xi')\frac{\partial\vec{v}_n}{\partial\xi})\vec{v}_{nt}\\
\leq& c\int_\Omega(|\vec{v}_n|+|\vec{v}_{n\xi}|)|\vec{v}_{nt}|(|\nabla_{e_\theta}\vec{v}_{nt}|^2+|\nabla_{e_\varphi}\vec{v}_{nt}|^2)^{\frac{1}{2}}\\
\leq&\varepsilon\|\vec{v}_{nt}\|^2+c(|\vec{v}_n|_4^8+|\vec{v}_{n\xi}|_4^8)|\vec{v}_{nt}|_2^2,
\end{aligned}\tag{4.70}$$

$$-\int_\Omega(\int_\xi^1\frac{bP}{p}\operatorname{grad}T_{nt}d\xi')\vec{v}_{nt}\leq\varepsilon\int_\Omega(|\nabla_{e_\theta}\vec{v}_{nt}|^2+|\nabla_{e_\varphi}\vec{v}_{nt}|^2)+|T_{nt}|_2^2,\tag{4.71}$$

So from (4.69)-(4.71), we get

$$\begin{aligned}
&\frac{d|\vec{v}_{nt}|_2^2}{dt}+\|\vec{v}_{nt}\|^2\\
\leq& c(|\vec{v}_n|_4^8+|\vec{v}_{n\xi}|_4^8)|\vec{v}_{nt}|_2^2+|T_{nt}|_2^2+\varepsilon\|\vec{v}_{nt}\|^2+\varepsilon\int_\Omega(|\nabla_{e_\theta}\vec{v}_{nt}|^2+|\nabla_{e_\varphi}\vec{v}_{nt}|^2).
\end{aligned}\tag{4.72}$$

Taking the derivative with respect to t of (4.1), similarly, we obtain

$$\begin{aligned}&\frac{d|T_{nt}|_2^2}{dt}+\|T_{nt}\|^2\\ \le& c(1+|T_n|_4^8+|T_{n\xi}|_4^8)|T_{nt}|_2^2+c|T_n|_4^8|\vec{v}_{nt}|_2^2+c|Q_{1t}|_2^2+\varepsilon\|T_{nt}\|^2+\varepsilon\|\vec{v}_{nt}\|^2,\end{aligned} \tag{4.73}$$

note that (4.72) and (4.73), let ε enough small, we have

$$\begin{aligned}&\frac{d}{dt}(|\vec{v}_{nt}|_2^2+|T_{nt}|_2^2)+\|\vec{v}_{nt}\|^2+\|T_{nt}\|^2\\ \le& c(1+|\vec{v}_n|_4^8+|\vec{v}_{n\xi}|_4^8+|T_n|_4^8)|\vec{v}_{nt}|_2^2+c(1+|T_n|_4^8+|T_{n\xi}|_4^8)|T_{nt}|_2^2+c|Q_{1t}|_2^2.\end{aligned} \tag{4.74}$$

Therefore, using the periodicity of $\vec{v}_n, T_n$, Lemma 4.1, Lemma 4.2 and (4.68) we obtain

$$\begin{aligned}&\sup_t(|\vec{v}_{nt}(t)|_2^2+|T_{nt}(t)|_2^2)\\ \le&(\mu_1^{-1}+\mu_2^{-1}+2W)C(M_1,M_3,M_5,M_9,M_{11},K,K_4).\end{aligned} \tag{4.75}$$

□

5. W-periodic solutions

Theorem 5.1. *Let $Q_1 \in L^\infty(W, H^1(\Omega))(W > 0)$, there exists a constant $C_0 = C_0(N) > 0(N = 1, 2...)$, if*

$$K \equiv \sup_{0\le t\le W} \|Q_1\|_{L^N}(\Omega) \le C_0,$$

the problem(2.10)-(2.15) has W-periodic solution $(\vec{v}, T)$, it satisfies

$$(\vec{v}, T) \in L^\infty(W; V) \cap H^1(W, H).$$

Proof. In section 4, we get the $\vec{v}_n, T_n$ estimate in H^1 and $\vec{v}_{nt}, T_{nt}$ estimate in L^2, use of compactness theorem, we know there exists a subsequence $(\vec{v}_n, T_n)$ lending to $(\vec{v}, T)$ in such a way

$$\begin{gathered}(\vec{v}_n, T_n) \to (\vec{v}, T) \text{ weakly}^* \text{ in } L^\infty(W; V),\\ (\vec{v}_n, T_n) \to (\vec{v}, T) \text{ strongly in } L^\infty(W; H),\\ (\vec{v}_{nt}, T_{nt}) \to (\vec{v}_t, T_t) \text{ weakly}^* \text{ in } L^\infty(W; H).\end{gathered}$$

By the above estimate we know that the nonliner terms are well defined. If $n \to \infty$, uniformly in t, we have

$$\begin{aligned}|\nabla_{\vec{v}_n}\vec{v}_n-\nabla_{\vec{v}}\vec{v}|_2 &\le |(\vec{v}_n-\vec{v})\nabla\vec{v}_n|_2+|(\nabla\vec{v}_n-\nabla\vec{v})\vec{v}|_2\\ &\le |(\vec{v}_n-\vec{v})|_4|\nabla\vec{v}_n|_4+|(\nabla\vec{v}_n-\nabla\vec{v})|_4|\vec{v}|_4\\ &\le |(\vec{v}_n-\vec{v})|_4|\nabla\vec{v}_n|_4+|(\nabla\vec{v}_n-\nabla\vec{v})|_2^{\frac14}\|(\nabla\vec{v}_n-\nabla\vec{v})\|^{\frac34}|\vec{v}|_4\\ &\to 0,\end{aligned}$$

$$
\begin{aligned}
&|(\int_\xi^1 \text{div}\vec{v}_n d\xi')\frac{\partial \vec{v}_n}{\partial \xi} - (\int_\xi^1 \text{div}\vec{v} d\xi')\frac{\partial \vec{v}}{\partial \xi}|_2 \\
&\le |(\int_\xi^1 (\text{div}\vec{v}_n - \text{div}\vec{v}) d\xi')\frac{\partial \vec{v}_n}{\partial \xi}|_2 + |(\int_\xi^1 \text{div}\vec{v} d\xi')(\frac{\partial \vec{v}_n}{\partial \xi} - \frac{\partial \vec{v}}{\partial \xi})|_2 \\
&\le |\nabla_{e_\theta}\vec{v}_n + \nabla_{e_\varphi}\vec{v}_n - \nabla_{e_\theta}\vec{v} - \nabla_{e_\varphi}\vec{v}|_2^{\frac{1}{4}} \|\nabla_{e_\theta}\vec{v}_n + \nabla_{e_\varphi}\vec{v}_n - \nabla_{e_\theta}\vec{v} - \nabla_{e_\varphi}\vec{v}\|^{\frac{3}{4}} |\frac{\partial \vec{v}_n}{\partial \xi}|_4 \\
&+ |\frac{\partial \vec{v}_n}{\partial \xi} - \frac{\partial \vec{v}}{\partial \xi}|_4 |\int_\xi^1 \text{div}\vec{v} d\xi'|_4 \\
&\to 0,
\end{aligned}
$$

$$
|\int_\xi^1 \frac{bP}{p}\text{grad}T_n d\xi' - \int_\xi^1 \frac{bP}{p}\text{grad}T d\xi'|_2 \to 0.
$$

Similarly, we get

$$
\begin{gathered}
|\nabla_{\vec{v}_n}T_n - \nabla_{\vec{v}}T|_2 \to 0, \\
|(\int_\xi^1 \text{div}\vec{v}_n d\xi')\frac{\partial T_n}{\partial \xi} - (\int_\xi^1 \text{div}\vec{v} d\xi')\frac{\partial T}{\partial \xi}|_2 \to 0, \\
|\frac{bP}{p}\int_\xi^1 \text{div}\vec{v}_n d\xi' - \frac{bP}{p}\int_\xi^1 \text{div}\vec{v} d\xi'|_2 \to 0.
\end{gathered}
$$

Consequently, we see that

$$
\begin{aligned}
(\frac{\partial \vec{v}}{\partial t} + \nabla_{\vec{v}}\vec{v} + \left(\int_\xi^1 \text{div}\vec{v} d\xi'\right)\frac{\partial \vec{v}}{\partial \xi} + \frac{f}{R_0}k \times \vec{v} + \text{grad}\Phi_s \\
+ \int_\xi^1 \frac{bP}{p}\text{grad}T d\xi' - \frac{1}{Re_1}\Delta\vec{v} - \frac{1}{Re_2}\frac{\partial^2 \vec{v}}{\partial \xi^2}, w_j) = 0,
\end{aligned}
$$

$$
(\frac{\partial T}{\partial t} + \nabla_{\vec{v}}T + \left(\int_\xi^1 \text{div}\vec{v} d\xi'\right)\frac{\partial T}{\partial \xi} - \frac{bP}{p}\left(\int_\xi^1 \text{div}\vec{v} d\xi'\right) - \frac{1}{Rt_1}\triangle T - \frac{1}{Rt_2}\frac{\partial^2 T}{\partial \xi^2}, w_j) = (Q_1, w_j).
$$

So we get

$$
\begin{aligned}
\frac{\partial \vec{v}}{\partial t} + \nabla_{\vec{v}}\vec{v} + \left(\int_\xi^1 \text{div}\vec{v} d\xi'\right)\frac{\partial \vec{v}}{\partial \xi} + \frac{f}{R_0}k \times \vec{v} + \text{grad}\Phi_s \\
+ \int_\xi^1 \frac{bP}{p}\text{gradT} d\xi' - \frac{1}{Re_1}\Delta\vec{v} - \frac{1}{Re_2}\frac{\partial^2 \vec{v}}{\partial \xi^2} = 0,
\end{aligned}
$$

$$
\frac{\partial T}{\partial t} + \nabla_{\vec{v}}T + \left(\int_\xi^1 \text{div}\vec{v} d\xi'\right)\frac{\partial T}{\partial \xi} - \frac{bP}{p}\left(\int_\xi^1 \text{div}\vec{v} d\xi'\right) - \frac{1}{Rt_1}\triangle T - \frac{1}{Rt_2}\frac{\partial^2 T}{\partial \xi^2} = Q_1.
$$

Thus, the proof of Theorem 5.1 is complete. □

Theorem 5.2. *The solution of (2.10)-(2.15) given in theorem5.1 is unique.*

Proof. Let $(\vec{v}_1, \vec{v}_1)$ and $(\vec{v}_2, \vec{v}_2)$ be two W-periodic solutions of problem (2.10)-(2.15) with corresponding geopotentials Φ_{1s}, Φ_{2s} respectively. Define $\vec{v} = \vec{v}_1 - \vec{v}_2$, $T = T_1 - T_2$, $\Phi_s = \Phi_{s1} - \Phi_{s2}$, then $\vec{v}, T, \Phi_s$ satisfy the following system

$$
\begin{aligned}
\frac{\partial \vec{v}}{\partial t} + \nabla_{\vec{v}_1}\vec{v} + \nabla_{\vec{v}}\vec{v}_2 + \left(\int_\xi^1 \text{div}\vec{v}_1 d\xi'\right)\frac{\partial \vec{v}}{\partial \xi} + \left(\int_\xi^1 \text{div}\vec{v} d\xi'\right)\frac{\partial \vec{v}_2}{\partial \xi} + \frac{f}{R_0}k \times \vec{v} \\
+ \text{grad}\Phi_s + \int_\xi^1 \frac{bP}{p}\text{gradT} d\xi' - \frac{1}{Re_1}\Delta\vec{v} - \frac{1}{Re_2}\frac{\partial^2 \vec{v}}{\partial \xi^2} = 0,
\end{aligned}
\tag{5.1}
$$

$$
\begin{aligned}
&\frac{\partial T}{\partial t}+\nabla_{\vec{v}_1}T+\nabla_{\vec{v}}T_2+\left(\int_\xi^1 \mathrm{div}\vec{v}_1 d\xi'\right)\frac{\partial T}{\partial \xi}+\left(\int_\xi^1 \mathrm{div}\vec{v} d\xi'\right)\frac{\partial T_2}{\partial \xi}\\
&-\frac{bP}{p}\left(\int_\xi^1 \mathrm{div}\vec{v} d\xi'\right)-\frac{1}{Rt_1}\triangle T-\frac{1}{Rt_2}\frac{\partial^2 T}{\partial \xi^2}=0,
\end{aligned}
\tag{5.2}
$$

$$
\xi=1:\frac{\partial \vec{v}}{\partial \xi}=0, \frac{\partial T}{\partial \xi}=-\alpha_s T, \quad \xi=0:\frac{\partial \vec{v}}{\partial \xi}=0, \frac{\partial T}{\partial \xi}=0. \tag{5.3}
$$

Taking the inner product of (5.1) with $\vec{v}$, using integration by parts, Hölder inequality, Young inequality and Minkowski inequality, Lemma 2.1 to Lemma 2.5, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d|\vec{v}|_2^2}{dt}+\frac{1}{Re_1}\int_\Omega(|\nabla_{e_\theta}\vec{v}|^2+|\nabla_{e_\varphi}\vec{v}|^2)+\frac{1}{Re_2}\int_\Omega|\vec{v}_\xi|^2\\
\le&\varepsilon\|\vec{v}\|^2+c(|\vec{v}_2|_4^8+|\vec{v}_{2\xi}|_4^8)|\vec{v}|_2^2-\int_\Omega(\int_\xi^1\frac{bP}{p}\mathrm{grad}T d\xi')\cdot\vec{v}.
\end{aligned}
\tag{5.4}
$$

Taking the inner product of (5.2) with T, similarly, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d|T|_2^2}{dt}+\frac{1}{Rt_1}\int_\Omega|\nabla T|^2+\frac{1}{Rt_2}\int_\Omega|T_\xi|^2+\frac{\alpha_s}{Rt_2}|T(\xi=1)|_2^2\\
\le&\varepsilon\|\vec{v}\|^2+\varepsilon\|T\|^2+c(|T|_2^2+|\vec{v}|_2^2)|T_2|_4^8+c|T_{2\xi}|_4^8|T|_2^2+\int_\Omega\frac{bP}{p}(\int_\xi^1\mathrm{div}\vec{v}d\xi')T.
\end{aligned}
\tag{5.5}
$$

Choosing ε small enough, we obtain

$$
\begin{aligned}
&\frac{d(|\vec{v}|_2^2+|T|_2^2)}{dt}+\|\vec{v}\|^2+\|T\|^2\\
\le&c(|\vec{v}_2|_4^8+|\vec{v}_{2\xi}|_4^8+|T_2|_4^8)|\vec{v}|_2^2++c(|T_{2\xi}|_4^8+|T_2|_4^8)|T|_2^2,
\end{aligned}
\tag{5.6}
$$

poincaré inequality can give that

$$
\frac{d(|\vec{v}|_2^2+|T|_2^2)}{dt}\le(C(M_3,M_5,M_9,M_{11})-1)\mu(|\vec{v}|_2^2+|T|_2^2), \tag{5.7}
$$

since $C(M_3,M_5,M_9,M_{11})<1$, it follows

$$
|\vec{v}|_2^2+|T|_2^2\le(|\vec{v}|_2^2+|T|_2^2)(0)\exp(-Lt), \quad t\in(0,\infty).
$$

Because $(\vec{v},T)$ is W-periodic in t, so for any positive integer N and for any $t\ge 0$, we have

$$
(|\vec{v}|_2^2+|T|_2^2)(t)=(|\vec{v}|_2^2+|T|_2^2)(t+NW).
$$

Hence, it follows

$$
|\vec{v}|_2^2+|T|_2^2\le(|\vec{v}|_2^2+|T|_2^2)(0)\exp(-LNt),
$$

which implies $|\vec{v}|_2^2+|T|_2^2=0$. The proof of theorem 5.2 is complete. □

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