Journal of Applied Analysis and Computation
Volume 9, Number 2, April 2019, 671–690
Website:http://jaac.ijournal.cn
DOI:10.11948/2156-907X.20180142

HOPF BIFURCATION IN A DIFFUSIVE PREDATOR-PREY MODEL WITH HERD BEHAVIOR AND PREY HARVESTING*

Heping Jiang[1,2] and Xiaosong Tang[3,†]

Abstract In this paper, the dynamics of a diffusive delayed predator-prey model with herd behavior and prey harvesting subject to the homogeneous Neumann boundary condition is considered. Firstly, choosing the harvesting term as a bifurcation parameter, then we obtain the existence and the stability of the equilibrium by analyzing the distribution of the roots of associated characteristic equation. Secondly, time delay is regarding as a bifurcation parameter, and the use of the normal form theory and center manifold theorem, the existence, stability and direction of bifurcating periodic solutions are all demonstrated detailly. Finally, summarizing some numerical simulations to illustrate the theoretical analysis.

Keywords Hopf bifurcation, predator-prey model, herd behavior, prey harvesting, delay.

MSC(2010) 35K57, 35B10.

1. Introduction

Predator-prey model has revealed some ordinary biological phenomenons and special relationship about biology in the real world, which can plays an very important role in problems about mathematics, and in many aspects, it has been totally accepted by many researchers. The Lotka-Volterra model is considered to be the simplest one, it has been modified in many different ways since 1920s the time when it was born. Especially, Rosenzweig and MacArthur improved its realism by introducing new factors prey dependent growth and the other is a nonlinear saturating uptake of prey by the predator (functional response). Nowadays, many models are mainly based on the Rosenzweig-MacArthur framework, but they are all amended by emphasizing some specific and real factors, such as the prey's autonomous inducible defenses or predator's adaptive foraging. As for population dynamics, the function

[†]the corresponding author. Email address: tangxs40@126.com(X. Tang)
[1]School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai 200240, China
[2]School of Mathematics and Statistics, Huangshan University, 245041, China
[3]School of Mathematics and Physics, Jinggangshan University, Ji'an 343009, China
[*]The authors were supported by the National Natural Science Foundation of China (No. 11701208, No. 11761038), the Postdoctoral Science Foundation of China (No.2018M632093), the key project of Provincial Excellent Talents in University of Anhui Province (No.gxyqZD2018077), and the Natural Science Foundation of Huangshan University (No.2017xkjq001).

response of predator to prey is embodied in the change in the density, which means that if the density of prey changes and the density of predator will changes too.

In more recent time, a new study that is aimed at making interaction be a more elaborated social is running, in which the individuals of one population gather together with the other one shows a much more individualistic behavior [1,2,19–23,30]. the authors who work on this field have thought out a new kind of predator-prey model with the fact that the predator-prey interactions usually occur mainly through the perimeter of the herd [2], and the new model can be vividly described by the following ordinary differential equations

$$\begin{cases} \frac{dX(t)}{dt} = r\left(1-\frac{X(t)}{K}\right)X(t) - \alpha\sqrt{X(t)}Y(t), \\ \frac{dY(t)}{dt} = -sY(t) + c\alpha\sqrt{X(t)}Y(t), \end{cases}$$

$X(t)$ is the prey density and $Y(t)$ is the predator density, the parameter r is viewed as the growth rate of the prey, K is its carrying capacity, and s denotes the death rate of the predator in the absence of prey, the parameter α is viewed as the search efficiency of $Y(t)$ for $X(t)$, the parameter c is biomass conversion or consumption rate. This model is usually also known as the predator-prey model along with herb behavior, and it has already been proved that there is the existence of the possibility of sustained limit cycles and more surprising founding is that the solution behavior near the origin is much more subtle and interesting than the classical one.

In terms of human survival needs, the use of biological resources and harvesting of populations are common in fishery, forestry, and wildlife management. At the same time, there is also a wide range of interests in using the biological economic models to detailly understand the scientific management of same renewable resources which has a close relation with the optimal management of these renewable resources. It is known to us all that a harvesting in preys can have a indirect but great influence on the predators' population, because of the sharp reduction of food population in the area. There are three different and basic types of harvesting being reported in the literature: (a) Constant harvesting, $h(x) = h$, which means that there is a fixed number of individuals per unit of time [4]. (b) Proportional harvesting $h(x) = Ex$. (c) Holling type II harvesting $h(x) = \frac{qEx}{m_1E+m_2x}$, the x is represented as the population of harvesting (prey or predator). For example, the authors worked with a model with Holling type II harvesting in prey in [7,8,12,28,29] and with Holling type II harvesting in predator in [9,10,31].

Based on the work of [7,8] and [19,20] , we propose a new model as following

$$\begin{cases} \frac{\partial X(x,t)}{\partial t} = r\left(1-\frac{X(x,t)}{K}\right)X(x,t) - \alpha\sqrt{X(x,t)}Y(x,t) - \frac{qEX(x,t)}{m_1E+m_2X(x,t)} \\ \qquad\qquad + d_1\Delta X(x,t), \\ \frac{\partial Y(x,t)}{\partial t} = -sY(x,t) + c\alpha\sqrt{X(x,t)}Y(x,t) + d_2\Delta Y(x,t). \end{cases} \tag{1.1}$$

By setting

$$u = \frac{1}{K}X,\ v = \frac{\alpha}{r\sqrt{K}}Y,\ \tilde{t} = rt,\ \tilde{x} = x$$

and dropping the bars for the sake of simplicity, then the system (1.1) can be transformed into

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = (1-u(x,t))\,u(x,t) - \sqrt{u(x,t)}v(x,t) - \frac{hu(x,t)}{\eta+u(x,t)} + d_1\Delta u(x,t), \\ \frac{\partial v(x,t)}{\partial t} = \gamma v(x,t)\left(-\beta+\sqrt{u(x,t)}\right) + d_2\Delta v(x,t), \end{cases} \tag{1.2}$$

where

$$\gamma = \frac{1}{c\alpha\sqrt{K}},\ \beta = \frac{s}{c\alpha\sqrt{K}},\ h = \frac{qE}{m_2},\ \eta = \frac{m_1 E}{m_2 K}.$$

The system (1.2) with homogeneous Neumann boundary conditions is as follows

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = (1-u(x,t))\,u(x,t) - \sqrt{u(x,t)}v(x,t) - \frac{hu(x,t)}{\eta+u(x,t)} + d_1\Delta u(x,t), \\ \frac{\partial v(x,t)}{\partial t} = \gamma v(x,t)\left(-\beta+\sqrt{u(x,t)}\right) + d_2\Delta v(x,t), \\ u_x(0,t) = u_x(\pi,t) = v_x(0,t) = v_x(\pi,t) = 0, t \geq 0, \\ u(x,t) = \phi(x,t), v(x,t) = \psi(x,t) \geq 0, x \in [0,\pi], \end{cases} \tag{1.3}$$

where $u(t)$ and $v(t)$ are the representations of the prey and predator densities, respectively, at time t. $\beta\gamma$ is viewed as the death rate of the predator in the absence of prey, γ is thought of as the conversion or the consumption rate of prey to predator.

Time delay plays an quite important roles in the realistic model. The consumption of prey will affect the number of predators to some extent in the later time. Initially, differential equations which are come with time delays are all elaborated in ordinary differential equations [14, 28]. In more recent years, authors pay more attention to partial differential system, this kind of diffusion is taken into consideration [13, 15–17, 21–24, 26, 27, 30]. The most of those authors mainly concentrated on studying the delay effect of the reaction-diffusion system, and seriousing investigate the stability/instability of the coexistence equilibrium and associated with Hopf bifurcation [3,12,16,19,20,29]. Hence, we will continue to study the dynamics of following system

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = (1-u(x,t))\,u(x,t) - \sqrt{u(x,t)}v(x,t) - \frac{hu(x,t)}{\eta+u(x,t)} + d_1\Delta u(x,t), \\ \frac{\partial v(x,t)}{\partial t} = \gamma v(x,t)\left(-\beta+\sqrt{u(x,t-\tau)}\right) + d_2\Delta v(x,t), \\ u_x(0,t) = u_x(\pi,t) = v_x(0,t) = v_x(\pi,t) = 0, t \geq 0, \\ u(x,t) = \phi(x,t), v(x,t) = \psi(x,t) \geq 0, x \in [0,\pi]\times[-\tau,0], \end{cases} \tag{1.4}$$

where the parameter τ is nonnegative, and represent the delay effect.

This paper is mainly aimed at considering the delay-induced Hopf bifurcation for the system (1.4) with the use of the normal form and the center manifold theory. The paper is going to be conclued as follows. In Section 2, the parameter h is regarded as a bifurcation parameter, and we consider about the Hopf bifurcation of the local system of (1.3). In Section 3, we firstly start at investigating the existence of the delay-induce Hopf bifurcation for the widely diffusive predator-prey model. Secondly, we carefully calculate the normal form on the center manifold just in order to further discuss the dynamical behavior around the delay-induced Hopf bifurcation value. Thirdly, we presented some accurate numerical simulations to precisely illustrate and expand our theoretical results. In Section 4, we end this paper with some discussions.

2. Stability and bifurcation analysis

2.1. Stability and bifurcation analysis for the system (1.3) without diffusion

In order to further study the complex dynamics of the system (1.3), we firstly begin with discussing the dynamics of system (1.3) without diffusion as following

$$\begin{cases} \frac{du(t)}{dt} = (1-u(t))\,u(t) - \sqrt{u(t)}v(t) - \frac{hu(t)}{\eta+u(t)}, \\ \frac{dv(t)}{dt} = \gamma v(t)\left(-\beta + \sqrt{u(t)}\right). \end{cases} \tag{2.1}$$

Proposition 2.1. *For the system* (2.1),
(a) The trivial equilibrium point $E_0(0,0)$*;*
(b) When $2\sqrt{h}-1 < \eta < min\{1,h\}$*, the semi-trivial equilibrium point* $E_1\left(u_1,0\right)$ *exists, where* $u_1 = \frac{(1-\eta)-\sqrt{(1-\eta)^2-4(h-\eta)}}{2}$*;*
(c) When $2\sqrt{h}-1 < \eta < min\{1,h\}$ *or* $h < \eta$*, the semi-trivial equilibrium point* $E_2\left(u_2,0\right)$ *exsits, where* $u_2 = \frac{(1-\eta)+\sqrt{(1-\eta)^2-4(h-\eta)}}{2}$*;*
(d) When $h < \left(1-\beta^2\right)\left(\eta+\beta^2\right)$*, the unique positive equilibrium point* $E^*\left(u^*,v^*\right)$ *exsits, where* $u^* = \beta^2, v^* = \beta\left(1-\beta^2\right) - \frac{h\beta}{\eta+\beta^2}$.

The linearization of (2.1) is

$$\begin{pmatrix} \frac{du(t)}{dt} \\ \frac{dv(t)}{dt} \end{pmatrix} = A \begin{pmatrix} u(t) \\ v(t) \end{pmatrix},\ A = \begin{pmatrix} a_{11}\ a_{12} \\ a_{21}\ a_{22} \end{pmatrix},$$

where the matrix A is indeterminate at $E_0(0,0)$.

The characteristic equaion is

$$\lambda^2 + T_0\lambda + D_0 = 0, \tag{2.2}$$

where

$$T_0 = -(a_{11}+a_{22}),\ D_0 = a_{11}a_{22} - a_{12}a_{21}.$$

At the equilibria $E_j, j = 1,2$,

$$a_{11} = \frac{u_j\left(1-\eta-2u_j\right)}{\eta+u_j},\ a_{12} = -\sqrt{u_j},\ a_{21} = 0,\ a_{22} = \gamma\left(-\beta+\sqrt{u_j}\right)\left(u_j > 0\right).$$

If

$$\left(1-\eta-2u_j\right)\left(-\beta+\sqrt{u_j}\right) < 0,$$

then $D_0 > 0$, the equilibria E_j is unstable.

If

$$\left(1-\eta-2u_j\right)\left(-\beta+\sqrt{u_j}\right) > 0,$$

then $D_0 < 0$, the equilibria E_j is locally asymptotically stable if $T_0 > 0$, and the equilibria E_j is unstable if $T_0 < 0$.

According to above the series of discussion, we can obtain the results as follows.

Proposition 2.2. *For the system* (2.1),
(a) $E_1(u_1, 0)$ *is unstable;*
(b) $E_2(u_2, 0)$ *is locally asymptotically stable if* $h < \eta\left(1+\beta^2\right) - \beta^2\left(1-\beta^2\right)$, *and* $E_2(u_2, 0)$ *is a saddle point if* $h > \eta\left(1+\beta^2\right) - \beta^2\left(1-\beta^2\right)$.

At the equilibrium E^*,

$$\begin{aligned} a_{11} &= \frac{1}{2} - \frac{3}{2}\beta^2 - \frac{h\eta}{2(\eta+\beta^2)^2} + \frac{h\beta^2}{2(\eta+\beta^2)},\ a_{12} = -\beta, \\ a_{21} &= \frac{\gamma}{2}\left(1-\beta^2 - \frac{h}{\eta+\beta^2}\right),\ a_{22} = 0. \end{aligned} \tag{2.3}$$

Since

$$T_0 = -\left(\frac{1}{2} - \frac{3}{2}\beta^2 - \frac{h\eta}{2\left(\eta+\beta^2\right)^2} + \frac{h\beta^2}{2(\eta+\beta^2)}\right),\ D_0 = \frac{\gamma\beta}{2}\left(1-\beta^2 - \frac{h}{\eta+\beta^2}\right).$$

In order to better study the stability of the positive equilibrium E^* for system (2.1), we present the mathmatical relation between the parameters h and β that appearing in the known equations. Denote

$$h_0(\eta, \beta) := \frac{\left(\eta+\beta^2\right)^2\left(1-3\beta^2\right)}{\eta\left(1-\beta^2\right) - \beta^4} > 0,$$

then $T_0(\eta, \beta, h_0(\eta, \beta)) = 0$.

In the following content, we analyze the existence of Hopf bifurcation at the interior equilibrium E^* by choosing the parameter h as the bifurcation parameter. In fact, h can also be regarded as the harvesting rate of prey, and plays an important role in determining the stability of the interior equilibrium and influencing the existence of Hopf bifurcation.

If we choose to consider h as a bifurcation parameter, then (2.2) has a pair of opposite purely imaginary eigenvalues $\omega = \pm\sqrt{D_0}$ when the value of h is $h = h_0$. Therefore, the system (2.1) has a very small amplitude nonconstant periodic solution which is bifurcated from the positive E^* when the parameter h crosses through h_0 if the transversal condition is satisfied.

Let $\lambda(h) = \alpha(h) + i\omega(h)$ be the root of (2.2), then

$$\alpha(h) = \frac{1}{2}T_0(h),\ \omega(h) = -\frac{1}{2}\sqrt{4D_0(h) - T_0^2(h)}.$$

Hence, $\alpha(h_0) = 0$ and

$$\alpha'(h_0) = -\frac{\eta\left(1-\beta^2\right) - \beta^4}{2\left(\eta+\beta^2\right)^2} \neq 0,\ \eta \neq \frac{\beta^4}{1-\beta^2}. \tag{2.4}$$

Which we can imply that the sysytem (2.1) will undergo Hopf bifurcation at E^* as the parameter h passes through the h_0, when the transversal condition (2.4) holds.

Proposition 2.3. *If* $h < \left(1-\beta^2\right)\left(\eta+\beta^2\right)$ *and the parameters* γ, η, β *are all positive. Then for the system* (2.1),
(a) The positive equilibrium point $E^*(u^*, v^*)$ *is locally asymptotically stable when* $h > h_0 = \frac{\left(\eta+\beta^2\right)^2\left(1-3\beta^2\right)}{\eta(1-\beta^2)-\beta^4}$, *and* $E^*(u^*, v^*)$ *is unstable when* $h < h_0\frac{\left(\eta+\beta^2\right)^2\left(1-3\beta^2\right)}{\eta(1-\beta^2)-\beta^4}$;

(b) The system (2.1) *undergoes a Hopf bifurcation at the positive equilibrium* $E^*(u^*,v^*)$ *when* $h = h_0 = \frac{(\eta+\beta^2)^2(1-3\beta^2)}{\eta(1-\beta^2)-\beta^4}$.

For the system (2.1), we can obtain the Hopf bifurcation line $H_0 : h_0 = \frac{(\eta+\beta^2)^2(1-3\beta^2)}{\eta(1-\beta^2)-\beta^4}$, and the stability region $D = \left\{(\beta,h)|h_0 < h < \left(1-\beta^2\right)\left(\eta+\beta^2\right)\right\}$ of the positive equilibrium $E^*(u^*,v^*)$. Fixed $\gamma = 2$, we depict the stability regions for the positive equilibrium E^* in the $\beta - h$ plane, $(A) : \eta = 0.5 < 1$ and $(B) : \eta = 1.2 > 1$, which is showed Fig.1.

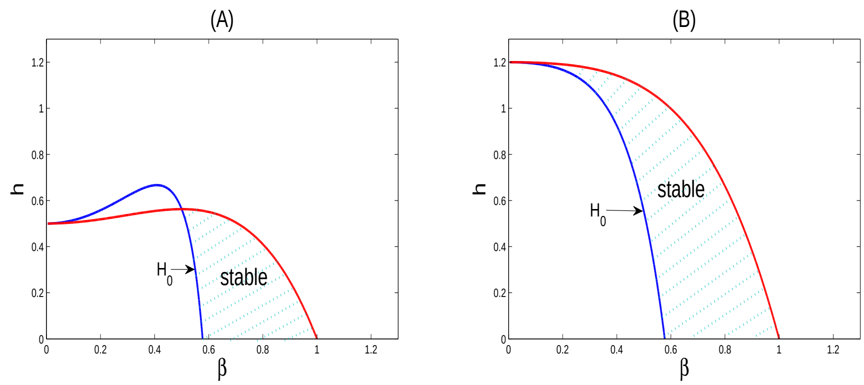

Figure 1. The stability region in the $\beta - h$ plane, (A): $\eta = 0.5 < 1$; (B): $\eta = 1.2 > 1$.

Next, we will continue to study the delay-include Hopf bifurcation for the diffusive predator-prey model.

2.2. Spatial-temporal dynamics for the diffusive model

Let

$$f^{(1)}(u,v) = (1-u(x,t))\,u(x,t) - \sqrt{u(x,t)}v(x,t) - \frac{hu(x,t)}{\eta+u(x,t)},$$

$$f^{(2)}(u,v) = \gamma v(x,t)\left(-\beta + \sqrt{u(x,t-\tau)}\right).$$

The linearization of (1.4) at the positive equilibrium E^* is

$$\begin{pmatrix} \frac{\partial u(x,t)}{\partial t} \\ \frac{\partial v(x,t)}{\partial t} \end{pmatrix} = D\Delta \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} + A_0 \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} + A_1 \begin{pmatrix} u(x,t-\tau) \\ v(x,t-\tau) \end{pmatrix}, \tag{2.5}$$

with

$$D\Delta = \begin{pmatrix} d_1\Delta & 0 \\ 0 & d_2\Delta \end{pmatrix},\ A_0 = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix},\ A_1 = \begin{pmatrix} 0 & 0 \\ a_{21} & 0 \end{pmatrix},$$

where $a_{11},\ a_{12},\ a_{21},\ a_{22}$ are given in (2.3).

Hence, the characteristic equation of (2.5) is

$$det\left(\lambda I - M_k - A_0 - A_1 e^{-\lambda\tau}\right) = 0, \tag{2.6}$$

where I is the 2×2 identity matrix and $M_k = -k^2 diag\{d_1, d_2\}, k \in N_0$, which imply that

$$\lambda^2+[(d_1+d_2)k^2-(a_{11}+a_{22})]\lambda+d_1d_2k^4-(a_{11}d_2+a_{22}d_1)k^2+(a_{11}a_{22}-a_{12}a_{21}e^{-\lambda\tau}) = 0. \tag{2.7}$$

When $\tau = 0$,

$$\lambda^2 + T_k\lambda + D_k = 0, \tag{2.8}$$

where

$$T_k = (d_1 + d_2)k^2 - (a_{11} + a_{22}),$$
$$D_k = d_1d_2k^4 - (a_{11}d_2 + a_{22}d_1)\, k^2 + (a_{11}a_{22} - a_{12}a_{21})\,. \tag{2.9}$$

So, we obtain a long series of Hopf bifurcation lines H_k

$$h_k = \left[2(d_1 + d_2)k^2 + \left(1 - 3\beta^2\right)\right] \frac{\left(\eta + \beta^2\right)^2}{\eta\,(1 - \beta^2) - \beta^4},\ k = 1, 2, 3, \cdots.$$

Obviously, the value of h_k increases as k increases when the value of $\eta > \frac{\beta^4}{1-\beta^2}$, this is imply that $h_0 < h_1 < h_2 < h_3 < \cdots$.

When $\tau \neq 0$, Assume that $\lambda = i\omega$, and then substitute $i\omega$ into (2.7), we can obtain

$$\begin{aligned} &-\omega^2 + i[(d_1 + d_2)k^2 - (a_{11} + a_{22})]\omega + d_1d_2k^4 - (a_{11}d_2 + a_{22}d_1)k^2 \\ &+ (a_{11}a_{22} - a_{12}a_{21}e^{-i\omega\tau}) = 0. \end{aligned} \tag{2.10}$$

Separating the real parts and imaginary parts, we have

$$\begin{cases} -\omega^2 + d_1d_2k^4 - (a_{11}d_2 + a_{22}d_1)k^2 + a_{11}a_{22} - a_{12}a_{21}\cos\omega\tau = 0, \\ \left[(d_1 + d_2)k^2 - (a_{11} + a_{22})\right]\omega + a_{12}a_{21}\sin\omega\tau = 0, \end{cases} \tag{2.11}$$

which is equivalent to

$$\omega^4 + P_k\omega^2 + Q_k = 0, \tag{2.12}$$

where

$$P_k = \left[d_1k^2 - a_{11}\right]^2 + \left[d_2k^2 - a_{22}\right]^2,$$
$$Q_k = D_k\left[d_1d_2k^4 - (a_{11}d_2 + a_{22}d_1)\, k^2 + (a_{11} + a_{22})\right].$$

For $0 < k < N_1$, there is an unique positive root ω_k of (2.12) is

$$\omega_k = \sqrt{\frac{-P_k + \sqrt{P_k^2 - 4Q_k}}{2}}. \tag{2.13}$$

By (2.11), we can obtain

$$\tau_k^j = \tau_k^0 + \frac{2\pi j}{\omega_k}, \tau_k^0 = \frac{1}{\omega_k}\arccos\frac{-\omega^2 + d_1d_2k^4 - (a_{11}d_2 + a_{22}d_1)k^2 + a_{11}a_{22}}{a_{12}a_{21}} \tag{2.14}$$

for $k \in \{0, 1, 2, \cdots, N_1\}$.

Lemma 2.1. *Assume that $h < \left(1 - \beta^2\right)\left(\eta + \beta^2\right)$ hold, then*

$$\tau_{N_1}^j \geq \tau_{k+1}^j \geq \tau_k^j \geq \cdots \geq \tau_1^j \geq \tau_0^j,$$

for $j \in N_0$.

Lemma 2.2. *If the condition $h < \left(1-\beta^2\right)\left(\eta+\beta^2\right)$ holds. $J_k < 0, D_k > 0$ for any $k \in N_0$. Then (2.6) has a pair has a pair of purely imaginary roots $i\omega_k$ for each $k \in \{0,1,2,\cdots,N_1\}$ and at the same time (2.6) has no purely imaginary roots for $k \geq N_1+1$.*

Let $\lambda(\tau) = \alpha(\tau) + i\delta(\tau)$ be the roots of (2.6) near $\tau = \tau_k^j$ satisfying $\alpha\left(\tau_k^j\right) = 0, \delta\left(\tau_k^j\right) = \omega_k$. Then we can have the following transversality condition.

Lemma 2.3. *For $k \in \{0,1,2,\cdots,N_1\}$ and $J \in N_0$, $\frac{dRe(\lambda)}{d\tau}|_{\tau=\tau_k^j} > 0$.*

Proof. Differentiating two sides of (2.6), we get

$$Re\left(\frac{d\lambda}{d\tau}\right)^{-1} = Re\left[\frac{(2\lambda+T_k)e^{\lambda\tau}}{-a_{12}a_{21}\lambda} - \frac{\tau}{\lambda}\right].$$

Thus, by (2.11) and (2.13), we have

$$\begin{aligned} Re\left(\left(\frac{d\lambda}{d\tau}\right)^{-1}\right)|_{\tau=\tau_k^j} &= Re\left[\frac{(2\lambda+T_k)e^{\lambda\tau}}{-a_{12}a_{21}\lambda} - \frac{\tau}{\lambda}\right]|_{\tau=\tau_k^j} \\ &= Re\left[\frac{(2i\omega_k+T_k)e^{i\omega_k\tau_k^j}}{-ia_{12}a_{21}\omega_k} - \frac{\tau_k^j}{i\omega_k}\right] \\ &= \frac{T_k\sin\omega_k\tau_k^j - 2\omega_k\cos\omega_k\tau_k^j}{a_{12}a_{21}\omega_k} = \frac{\omega_k^2+P_k}{(a_{12}a_{21})^2} > 0. \end{aligned}$$

□

Clearly,

$$\tau_k^0 = \min_{j\in N_0}\left\{\tau_k^j\right\}, k \in \{0,1,2,\cdots,N_1\},$$

and from Lemma 2.1, we alrealy know that $\tau_0^0 = \min\left\{\tau_k^j : 0 \leq k \leq N_1, j \in N_0\right\}$. Denote $\tau_* = \tau_0^0$. Let $\lambda(\tau) = \alpha(\tau)+i\delta(\tau)$ be the pair of roots of (2.6) near $\tau = \tau_k^j$ satisfying $\alpha\left(\tau_k^j\right) = 0$ and $\delta\left(\tau_k^j\right) = \omega_k$. Then we can have the following transversality condition.

Theorem 2.1. *Assume that the condition the parameter $h < \left(1-\beta^2\right)\left(\eta+\beta^2\right)$ hold. ω_k and τ_k^j is defined by (2.13) and (2.14), respectively. Denote the minimum value of the critical values of delay by $\tau_* = \min_{k,j}\left\{\tau_k^j\right\}$.*

(a) The positive equilibrium $E^*(u^*,v^*)$ of system (1.4) is asymptotically stable for $\tau \in (0,\tau_*)$ and unstable for $(\tau_*,+\infty)$;

(b) System (1.4) undergoes Hopf bifurcations near the positive equilibrium $E^*(u^*,v^*)$ at τ_k^j for $k \in \{0,1,2,\cdots,N_1\}$ and $j \in N_0$.

3. Normal form of Hopf bifurcation for the diffusive model

3.1. Normal form of the harvesting rate-induce Hopf bifurcation for a diffusive model

For $U_1 = (u_1, v_1)^T, U_2 = (u_2, v_2)^T \in X$, define the inner product

$$[U_1, U_2] = \int_0^{\pi} (u_1 u_2 + v_1 v_2) dx,$$

where $X = \left\{(u, v) \in W^{2,2}(0, \pi) | \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0 \ at \ x = 0, \pi \right\}$.

We denote $h^* = h_j$ and then introduce a new parameter $\varepsilon \in R$ by setting $h = h^* + \varepsilon$ such that $\varepsilon = 0$ is obviously being the bifurcation value. Rewrite the positive equilibrium as a parameter-dependent form $E^*_\varepsilon(u^*(\varepsilon), v^*(\varepsilon))$ with

$$u^*(\varepsilon) = \beta^2, \ v^* = \beta\left(1 - \beta^2\right) - \frac{(h^* + \varepsilon)\beta}{\eta + \beta^2}.$$

Setting $\tilde{u}(., t) = u(., t) - u^*(\varepsilon), \tilde{v}(., t) = v(., t) - v^*(\varepsilon), \tilde{U}(t) = (\tilde{u}(., t), \tilde{v}(., t))$ and then dropping the tides for the simplification of notation, system (1.3) can be written just as the equation

$$\frac{dU(t)}{dt} = D\Delta U + L_0(U) + f(U, \varepsilon),$$

where

$$D\Delta U = \begin{pmatrix} d_1 \Delta u \\ d_2 \Delta v \end{pmatrix}, \ L_0(U) = \begin{pmatrix} a_{11} u \ a_{12} v \\ a_{21} u \ a_{22} v \end{pmatrix},$$

$$f(U, \varepsilon) = \sum_{i+j+i \geq 2} \frac{1}{i!j!l!} f_{ijl} u^i v^j \varepsilon^l, f_{ijl} = \left(f^{(1)}_{ijl}, f^{(2)}_{ijl}\right)^T,$$

with $f^{(n)}_{ijl} = \frac{\partial^{i+j+l} \tilde{f}^{(n)}(0,0,0)}{\partial u^i \partial v^j \partial \varepsilon^l}, n = 1, 2,$ and

$$\begin{aligned}\tilde{f}^{(1)}(u, v, \varepsilon) = &\left(1 - (u + u^*(\varepsilon))\right)(u + u^*(\varepsilon)) - \sqrt{u + u^*(\varepsilon)}\,(v + v^*(\varepsilon)) \\ &- \frac{(h^* + \varepsilon)(u + u^*(\varepsilon))}{\eta + (u + u^*(\varepsilon))},\end{aligned}$$

$$\tilde{f}^{(2)}(u, v, \varepsilon) = \gamma(v + v^*(\varepsilon))\left(-\beta + \sqrt{u + u^*(\varepsilon)}\right). \tag{3.1}$$

By a direct computation, we obtain $f_{020} = f_{120} = f_{030} = 0$.

We assume that there do exist a $k \in N_0$ such that $\Delta_k = 0$ with $h = h^*$ has a pair of purely imaginary roots $\pm i\omega_k$ and the remaining roots of characteristic equation (2.8) will have nonzero real parts, where

$$\omega_k = \sqrt{d_1 d_2 k^4 - d_2\left(\frac{1}{2} - \frac{3}{2}\beta^2 - \frac{h^*\eta}{2(\eta + \beta^2)^2} + \frac{h^*\beta^2}{2(\eta + \beta^2)}\right)k^2 + \frac{\beta\gamma}{2}\left(1 - \beta^2 - \frac{h^*}{\eta + \beta^2}\right)}.$$

In term of $\mathcal{M}_k p_k = i\omega_k p_k$ and $\mathcal{M}_k^T q_k = i\omega_k q_k$, we choose p_k and q_k such that $\left\langle q_k^T, p_k \right\rangle = 1$, where

$$p_k = \begin{pmatrix} 1 \\ \frac{i\omega_k + d_1 k^2 - a_{11}}{a_{12}} \end{pmatrix}, \ q_k = D \begin{pmatrix} 1 \\ \frac{i\omega_k + d_1 k^2 - a_{11}}{a_{21}} \end{pmatrix},$$

with $D = \left[1 + \frac{\left(i\omega_k + d_1 k^2 - a_{11}\right)^2}{a_{12}a_{21}} \right]^{-1}$.

By (3.1) and a very direct computation, we get

$$\frac{1}{2} f_2(U, \varepsilon) = f_{101} u\varepsilon + f_{011} v\varepsilon + \frac{1}{2} f_{020} u^2 + f_{110} uv + \frac{1}{2} f_{020} v^2,$$

Then

$$\begin{aligned} \frac{1}{2} f_2(z, 0, \varepsilon) =& \frac{1}{2} f_2(\Phi_k z \gamma_k(x), 0) \\ =& f_{101} \left(p_{k1} z_1 \varepsilon + \bar{p}_{k1} z_2 \varepsilon\right) \gamma_k(x) + f_{011} \left(p_{k2} z_1 \varepsilon + \bar{p}_{k2} z_2 \varepsilon\right) \gamma_k(x) \\ &+ \frac{1}{2} \left(A_{k20} z_1^2 + A_{k11} z_1 z_2 + A_{k02} z_2^2\right) \gamma_k^2(x), \end{aligned}$$

where

$$\begin{aligned} &A_{k20} = f_{200} p_{k1}^2 + 2 f_{110} p_{k1} p_{k2}, \ A_{k02} = \bar{A}_{k20}, \\ &A_{k11} = 2 f_{200} |p_{k1}|^2 + 4 f_{110} Re\{p_{k1} \bar{p}_{k2}\}. \end{aligned}$$

Thus, we obtain

$$\frac{1}{2} g_2^1(z, 0, \varepsilon) = \frac{1}{2} Proj_{KerM_2^1} f_2^1(z, 0, \varepsilon) = \begin{pmatrix} B_{k1} z_1 \varepsilon \\ \bar{B}_{k1} z_2 \varepsilon \end{pmatrix},$$

where $B_{k1} = q_k^T \left(f_{101} p_{k1} + f_{011} p_{k2}\right)$.

The calculation of $Proj_S f_3^1(z, 0, 0)$.

$$\int_0^\pi \gamma_k^4(x) dx = \begin{cases} \frac{1}{\pi}, k = 0, \\ \frac{3}{2\pi}, k \neq 0, \end{cases}$$

it is easy to verify that

$$\frac{1}{3!} Proj_S f_3^1(z, 0, 0) = \begin{pmatrix} B_{k21} z_1^2 z_2 \\ \bar{B}_{k21} z_1 z_2^2 \end{pmatrix},$$

where

$$B_{k21} = \begin{cases} \frac{1}{2\pi} b_{k21}, k = 0, \\ \frac{3}{4\pi} b_{k21}, k \neq 0, \end{cases}$$

with $b_{k21} = q_k^T \left(f_{300} p_{k1} |p_{k1}|^2 + f_{210} \left(p_{k1}^2 \bar{p}_{k2} + 2 p_{k2} |p_{k1}|^2\right)\right)$.

The calculation of $Proj_S\left[\left(D_z f_2^1\right)(z,0,0)U_2^1(z,0)\right]$.

$$f_2^1(z,0,0) = \Psi_k\left(A_{k20}z_1^2 + A_{k11}z_1z_2 + A_{k02}z_2^2\right)\int_0^\pi \gamma_k^3(x)dx,$$

There is a straightforward calculation which shows that

$$\begin{aligned} U_2^1(z,0) &= \left(M_2^1\right)^{-1} f_2^1(z,0,0) \\ &= \frac{\int_0^\pi \gamma_k^3(x)dx}{i\omega_k}\begin{pmatrix} q_k^T\left(A_{k20}z_1^2 - A_{k11}z_1z_2 - \frac{1}{3}A_{k02}z_2^2\right) \\ \bar{q}_k^T\left(\frac{1}{3}A_{k20}z_1^2 + A_{k11}z_1z_2 - A_{k02}z_2^2\right)\end{pmatrix}, \end{aligned}$$

and then

$$\frac{1}{3!}Proj_S\left[\left(D_z f_2^1\right)U_2^1\right](z,0,0) = \begin{pmatrix} C_{k21}z_1^2z_2 \\ \bar{C}_{k21}z_1z_2^2\end{pmatrix},$$

with

$$C_{k21} = \begin{cases} \frac{1}{6\pi}c_{k21}, k=0, \\ 0, k\neq 0,\end{cases}$$

where

$$c_{k21} = \frac{i}{\omega_k}\left(\left(q_k^TA_{k20}\right)\left(q_k^TA_{k11}\right) - \left|q_k^TA_{k11}\right|^2 - \frac{2}{3}\left|q_k^TA_{k02}\right|^2\right).$$

The calculation of $Proj_S\left[\left(D_w f_2^1\right)(z,0,0)U_2^2(z,0)\right]$.

$$\frac{1}{3!}Proj_S\left(D_w f_2^1\right)(z,0,0)(h) = \begin{pmatrix} D_{k21}z_1^2z_2 \\ \bar{D}_{k21}z_1z_2^2\end{pmatrix},$$

with

$$C_{k21} = \begin{cases} \frac{1}{3\sqrt{\pi}}E_{(0,0)}, k=0, \\ \frac{1}{3\sqrt{\pi}}E_{(k,0)} + \frac{1}{3\sqrt{2\pi}}E_{(k,2k)}, k\neq 0,\end{cases}$$

where, for $j = 0, 2k$,

$$\begin{aligned} E_{(k,j)} =& q_k^T\Big((f_{200}p_{k1} + f_{110}p_{k2})h_{kj11}^{(1)} + (f_{110}p_{k1} + f_{020}p_{k2})h_{kj11}^{(2)} \\ &+ (f_{200}\bar{p}_{k1} + f_{110}\bar{p}_{k2})h_{kj20}^{(1)} + (f_{110}\bar{p}_{k1} + f_{020}\bar{p}_{k2})h_{kj20}^{(2)}\Big). \end{aligned}$$

In order to obtain D_{k21}, we compute h_{kj20} and h_{kj11} as follow:

$$\begin{cases} h_{0020} = \frac{1}{\sqrt{\pi}}(2i\omega I_2 - \mathcal{M}_0)^{-1}\left(A_{020} - q_0^TA_{020}p_0 - \bar{q}_0^TA_{020}\bar{p}_0\right), \\ h_{0020} = -\frac{1}{\sqrt{\pi}}\left(A_{011} - q_0^TA_{011}p_0 - \bar{q}_0^TA_{011}\bar{p}_0\right), k=0, j=0,\end{cases}$$

and

$$\begin{cases} h_{kj20} = c_{kj}(2i\omega I_2 - \mathcal{M}_0)^{-1}A_{020}, \\ h_{0020} = -c_{kj}A_{011}, k\neq 0, j = 0, 2k,\end{cases}$$

Thus, the normal form on the center manifold for the critical values h^* of Hopf bifurcations has the following form

$$\dot{z} = B_k z + \begin{pmatrix} B_{k1} z_1 \varepsilon \\ \bar{B}_{k1} z_2 \varepsilon \end{pmatrix} + \begin{pmatrix} B_{k2} z_1^2 z_2 \\ \bar{B}_{k2} z_1 z_2^2 \end{pmatrix} + O\left(|z|\varepsilon^2 + |z^4|\right),$$

where

$$B_{k2} = B_{k21} + \frac{3}{2}(C_{k21} + D_{k21}) = \begin{cases} \frac{1}{2\pi} b_{021} + \frac{1}{4\pi} c_{021} + \frac{1}{2\sqrt{\pi}} E_{(0,0)}, k = 0, \\ \frac{3}{4\pi} b_{k21} + \frac{1}{2\sqrt{\pi}} E_{(k,0)} + \frac{1}{2\sqrt{2\pi}} E_{(k,2k)}, k \neq 0, \end{cases}$$

which can be written down in real coordinates w through the change of variables $z_1 = w_1 - iw_2, z_2 = w_1 + iw_2$. Then transforming to polar coordinates $w_1 = \rho\cos\xi, w_2 = \rho\sin\xi$, this normal form becomes

$$\begin{cases} \dot{\rho} = \nu_{k1}\rho\varepsilon + \nu_{k2}\rho^3 + O\left(\varepsilon^2\rho + |(\rho,\varepsilon)|^4\right), \\ \dot{\xi} = -\omega_k + O(|(\rho,\varepsilon)|), \end{cases}$$

with $\nu_{k1} = Re(B_{k1}), \nu_{k2} = Re(B_{k2})$.

It is well know to us that the sign of $\nu_{k1}\nu_{k2}$ determines the direction of the bifurcation (supercritical if $\nu_{k1}\nu_{k2} < 0$, subcritical $\nu_{k1}\nu_{k2} > 0$),and the sign of ν_{k2} determines the stability of the nontrivial periodic orbits (stable if $\nu_{k2} < 0$, unstable if $\nu_{k2} > 0$).

3.2. Normal form of the delay-induce Hopf bifurcation for a diffusive model

In this subsection, we shall study on the directions, stability and the period of bifurcating periodic solutions by moderately applying the normal formal theory and the center manifold theory of partial functional differential equations which are presented in [5, 6, 18, 25]. Fixed $j \in N_0, 0 \leq k \leq N_1$, we denote $\tau^* = \tau_k^j$, and introduce a new parameter $\varepsilon \in \mathbb{R}$ by setting $\varepsilon = \tau - \tau^*$ such that $\varepsilon = 0$ is the Hopf bifurcation value obviously.

Setting $\tilde{u}(.,t) = u(.,\tau t) - u^*, \tilde{v}(.,t) = v(.,\tau t) - v^*, \tilde{U}(t) = (\tilde{u}(.,\tau t), \tilde{v}(.,\tau t))$ and $\mathcal{C} = C([-1,0],X)$, then dropping the tides for simplification of notation, then system (1.4) can be written as follows

$$\frac{dU(t)}{dt} = \tau D_0 \Delta U(t) + L(\tau)(U_t) + \tilde{F}(U_t,\tau),\ \varphi = (\varphi_1,\varphi_2)^T,$$

where

$$D_0\Delta U = \begin{pmatrix} d_1\Delta u \\ d_2\Delta v \end{pmatrix},\ L(\tau)\varphi = \begin{pmatrix} a_{11}\varphi_1(0) & a_{12}\varphi_2(0) \\ a_{21}\varphi_1(-1) & a_{22}\varphi_2(0) \end{pmatrix},$$

$$\tilde{F}(\varphi,\varepsilon) = \tau \begin{pmatrix} f^{(1)}(\tau) \\ f^{(2)}(\tau) \end{pmatrix} = \begin{pmatrix} \sum_{i+j+l\geq 2} \frac{1}{i!j!l!} f_{ijl}^{(1)} \varphi_1^i(0)\varphi_2^j(0)\varphi_1^l(-1) \\ \sum_{i+j+l\geq 2} \frac{1}{i!j!l!} f_{ijl}^{(2)} \varphi_1^i(0)\varphi_2^j(0)\varphi_1^l(-1) \end{pmatrix},$$

with $f_{ijl}^{(n)} = \frac{\partial^{i+j+l}\tilde{f}^{(n)}(u^*,v^*,u^*)}{\partial u^i \partial v^j \partial w^l}, n = 1, 2$, and

$$\tilde{f}^{(1)}(u,v,w) = (u+u^*)(1-u-u^*) - \sqrt{u+u^*}(v+v^*) - \frac{h(u+u^*)}{\eta+(u+u^*)},$$

$$\tilde{f}^{(2)}(u,v,w) = \gamma(v+v^*)\left(-\beta+\sqrt{w+u^*}\right).$$

By direct computation, we can obtain $f_{020} = f_{210} = f_{120} = f_{030} = 0$.

Setting $\tau = \tau^* + \varepsilon$, $\Lambda_0 = \{-i\tau^*\omega^*, i\tau^*\omega^*\}$,

$$\frac{dU(t)}{dt} = \tau^* D_0 \Delta U(t) + L(\tau^*)(U_t),$$

$$\frac{dU(t)}{dt} = \tau D_0 \Delta U(t) + L(\tau)(U_t) + \tilde{F}(U_t,\varepsilon),\ \varphi = (\varphi_1,\varphi_2)^T,$$

$$\tilde{F}(U_t,\varepsilon) = \varepsilon D_0 \Delta\varphi(0) + L(\varepsilon)(\varphi) + f(\varphi,\tau^*+\varepsilon),\ for\ \varphi \in \mathcal{C}.$$

The eigenvalues of $\tau^* D\Delta$ on X are $\mu_k^i = -d_i\tau^* k^2, i = 1,2, k \in N_0$.

$$\beta_k^1 = \begin{pmatrix} \gamma_k(x) \\ 0 \end{pmatrix},\ \beta_k^2 = \begin{pmatrix} 0 \\ \gamma_k(x) \end{pmatrix},\ \gamma_k(x) = \frac{\cos kx}{\|\cos kx\|_{2,2}},\ k \in N_0.$$

$$\mathbf{B}_k = span\left\{< v(.), \beta_k^i > \beta_k^i | v \in \mathcal{C}, i = 1,2\right\}, z_t(\theta) \in \mathcal{C} = C\left([-1,0], R^2\right),$$

$$z_t^T(\theta)\begin{pmatrix} \beta_k^1 \\ \beta_k^2 \end{pmatrix} \in \mathbf{B}_k.$$

Then linear PDE restricted on $\mathbf{B}_k$ is equivalent to the FDE on $\mathcal{C}=C\left([-1,0],R^2\right)$,

$$\dot{z}(t) = \begin{pmatrix} \mu_k^1 & 0 \\ 0 & \mu_k^2 \end{pmatrix} z(t) + L(\tau^*)(z_t).$$

When $\tau = \tau^*$, define $\eta(\theta) \in BV([-1,0],R)$, such that

$$\mu_k\varphi(0) + L(\tau^*)\varphi = \int_{-1}^{0} d\eta(\theta)\varphi(\theta),$$

and the adjiont bilnear form on $\mathcal{C}^* \times \mathcal{C}, \mathcal{C}^* = C\left([0,1], R^{2*}\right)$ as follows

$$< \psi(s), \phi(\theta) >= \psi(0)\phi(0) - \int_{-1}^{0}\int_{0}^{\theta} \psi(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi,$$

$$\Phi_k = \left(pe^{i\omega^*\tau^*\theta}, \bar{p}e^{-i\omega^*\tau^*\theta}\right), \Psi_k = col\left(q^T e^{-i\omega^*\tau^* s}, \bar{q}^T e^{i\omega^*\tau^* s}\right),$$

where $< \Phi_k, \Psi_k >= I_2$, and

$$p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{i\omega^*+d_1k^2-a_{11}}{a_{12}} \end{pmatrix}, q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = D\begin{pmatrix} 1 \\ \frac{i\omega^*+d_1k^2-a_{11}}{a_{21}} e^{i\omega^*\tau^*} \end{pmatrix},$$

with $D = \left(1 + \tau^* \left(i\omega^* + d_1 k^2 - a_{11}\right) + \frac{\left(i\omega^* + d_1 k^2 - a_{11}\right) e^{i\omega^* \tau^*}}{a_{12} a_{21}}\right)^{-1}$.

$$\dot{z} = Bz + \begin{pmatrix} A_{k1} z_1 \varepsilon \\ \overline{A_{k1}} z_2 \varepsilon \end{pmatrix} + \begin{pmatrix} A_{k2} z_1^2 z_2 \\ \overline{A_{k2}} z_1 z_2^2 \end{pmatrix} + O(|z|\varepsilon^2 + |z|^4),$$

where

$$A_{k1} = -k^2 \left(d_1 q_1 p_1 + d_2 q_2 p_2\right) + i\omega^* q^T p,$$

and

$$A_{k2} = \frac{i}{2\omega^* \tau^*} \left(B_{k20} B_{k11} - 2|B_{k11}|^2 - \frac{1}{3}|B_{k02}|^2\right) + \frac{1}{2}\left(B_{k21} + D_{k21}\right),$$

with

$$B_{k20} = \begin{cases} \frac{\tau^*}{\sqrt{\pi}} \left(c_1 q_1 + c_2 q_2\right), & k = 0, \\ 0, & k \neq 0, \end{cases} \qquad B_{k11} = \begin{cases} \frac{\tau^*}{\sqrt{\pi}} \left(c_3 q_1 + c_4 q_2\right), & k = 0, \\ 0, & k \neq 0. \end{cases}$$

$$B_{k02} = \begin{cases} \frac{\tau^*}{\sqrt{\pi}} \left(\overline{c_1} q_1 + \overline{c_2} q_2\right), & k = 0, \\ 0, & k \neq 0, \end{cases} \qquad B_{k21} = \begin{cases} \frac{\tau^*}{\pi} c_5, & k = 0, \\ \frac{3\tau^*}{2\pi} c_5, & k \neq 0. \end{cases}$$

$$c_1 = f_{200}^{(1)} p_1^2 + 2 f_{110}^{(1)} p_1 p_2, \; c_2 = f_{002}^{(2)} p_1^2 e^{-2i\omega^*\tau^*} + 2 f_{110}^{(2)} p_1 p_2 e^{-i\omega^*\tau^*},$$

$$c_3 = f_{200}^{(1)} |p_1|^2 + 2 f_{110}^{(1)} Re\{p_1 \bar{p_2}\}, \; c_4 = f_{002}^{(2)} |p_1|^2 + 2 f_{110}^{(2)} Re\left\{p_1 \bar{p_2} e^{-i\omega^*\tau^*}\right\},$$

$$\begin{aligned} c_5 = & q_1 \Big(f_{300}^{(1)} p_1 |p_1|^2 + f_{210}^{(1)} \left(p_1^2 \bar{p_2} + 2|p_1|^2 p_2\right) \Big) + q_2 \Big(f_{003}^{(2)} p_1 |p_1|^2 e^{-i\omega^*\tau^*} \\ & + f_{012}^{(2)} \left(p_1^2 \bar{p_2} e^{-2i\omega^*\tau^*} + 2|p_1|^2 p_2\right) \Big), \end{aligned}$$

and

$$D_{k21} = \begin{cases} E_0, & k = 0, \\ E_0 + \frac{\sqrt{2}}{2} E_{2k}, & k \neq 0, \end{cases}$$

$$E_j = \frac{2\tau^*}{\sqrt{\pi}} \begin{pmatrix} F_1 h_{j11}^{(1)}(0) + \overline{F_1} h_{j20}^{(1)}(0) + F_2 h_{j11}^{(2)}(0) + \overline{F_2} h_{j20}^{(2)}(0) \\ F_3 h_{j11}^{(1)}(-1) + \overline{F_3} h_{j20}^{(1)}(-1) + F_4 h_{j11}^{(2)}(0) + \overline{F_4} h_{j20}^{(2)}(0) \end{pmatrix},$$

where

$$F_1 = f_{200}^{(1)} p_1 + f_{110}^{(1)} p_2, \; F_2 = f_{110}^{(1)} p_1, \; F_3 = f_{011}^{(2)} p_2 + f_{002}^{(2)} p_1, \; F_4 = f_{011}^{(2)} p_1 e^{-i\omega^*\tau^*},$$

where $h_{k20}(\theta)$ and $h_{k11}(\theta)$ are both determined by the following equations

$$\begin{cases} \dot{h}_{k20}(\theta) - 2i\omega^*\tau^* h_{k20}(\theta) = \Phi_k(\theta) \begin{pmatrix} B_{k20} \\ \bar{B}_{k20} \end{pmatrix}, \\ \dot{h}_{k20}(0) - L(\tau^*)(h_{k20}) = \tau^* c_{kj} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \end{cases} \quad \begin{cases} \dot{h}_{k11}(\theta) = 2\Phi_k(\theta) \begin{pmatrix} B_{k11} \\ \bar{B}_{k11} \end{pmatrix}, \\ \dot{h}_{k11}(0) - L(\tau^*)(h_{k11}) = 2\tau^* c_{kj} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \end{cases}$$

with

$$c_{kj} = \begin{cases} \frac{1}{\sqrt{\pi}}, & j = k = 0, \\ \frac{1}{\sqrt{\pi}}, & j = 0, k \neq 0, \\ \frac{1}{\sqrt{2\pi}}, & j = 2k \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Through the change of variables $z_1 = \omega_1 - i\omega_2, z_2 = \omega_1 + i\omega_2$ and $\omega_1 = \rho_1 \cos\phi, \omega_2 = \rho_1 \sin\phi$, then the normal form becomes the follows polar coordinate system

$$\begin{cases} \dot{\rho} = \kappa_{k1}\alpha\rho + \kappa_{k2}\rho^3 + O\left(\alpha^2\rho + |(\rho,\alpha)|^4\right), \\ \dot{\phi} = -\omega^*\tau^* + O(|(\rho,\alpha)|), \end{cases}$$

where $\kappa_{k1} = ReA_{k1}, \kappa_{k2} = ReA_{k2}$. Thus, from [18], we can know the results that the sign of $\kappa_{k1}\kappa_{k2}$ determines the direction of the bifurcation and the sign of κ_{k2} determines the stability of the nontrivial periodic orbits and we have following results.

Theorem 3.1. *(a) When $\kappa_{k1}\kappa_{k2} < 0$, it comes out that the Hopf bifurcation that the system undergoes at the critical value $\tau = \tau^*$ is showed a supercritical bifurcation. Moreover, if $\kappa_{k2} < 0$, then the bifurcating periodic solution is stable; if $\kappa_{k2} > 0$, then the bifurcating periodic solution will be unstable.*
(b) When $\kappa_{k1}\kappa_{k2} > 0$, the Hopf bifurcation that the system undergoes at the critical value $\tau = \tau^*$ is a subcritical bifurcation. Moreover, if $\kappa_{k2} < 0$, then the bifurcating periodic solution is showed to be stable; if $\kappa_{k2} > 0$, then the bifurcating periodic solution is showed to be unstable.

Next, we will present some exact numerical simulations and dynamical analysis for Hopf bifurcation of the systems (1.3), (1.4) and (2.1).

4. Numerical Simulations

In this section, by using a kind of mathematical software named Matlab, we give out some numerical simulations to support and extend our analytical results.

4.1. Harvesting rate-induce Hopf bifurcation

For the system (2.1), choosing $\gamma = 2, \eta = 0.5 < 1, \beta = 0.55$, according to the simple calculation, we can obtain the Hopf bifurcation value $h_0 = 0.3016$. Which implied that a stable periodic orbit is created around E^* when the parameter h cross through the critical value 0.3016, which are depicted by Fig.2 and Fig.3.

For the system (1.3), choosing $d_1 = 0.02, d_2 = 1, \gamma = 2, \eta = 0.5 < 1, \beta = 0.55$, and according to simple calculation, we can obtain the Hopf bifurcation value $h_0 = 0.3016$. So, we also get the values $\kappa_{k1} = 0.0767, \kappa_{k2} = -0.2785$. This implies that a famlily stable spatially homogenous periodic solutions will bifurcate from the positive equilibrium E^* when the parameter h cross through the critical value 0.3016, which are depicted by Fig.4 and Fig.5.

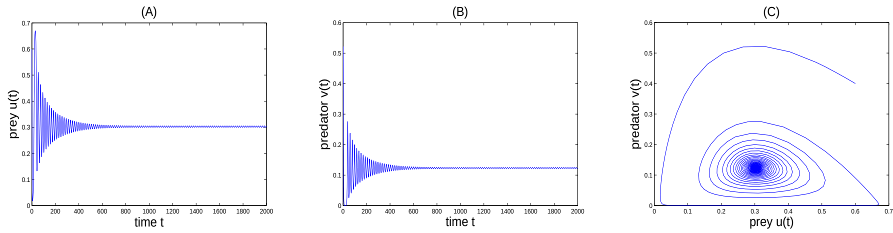

Figure 2. The trajectory graphs and phase portrait of (2.1), which shows the positive equilibrium (u^*, v^*) is asymptotically stable when $h = 0.38 > 0.3016$. Here we set some parameter values $\gamma = 2, \eta = 0.5, \beta = 0.55$, and the initial value is $u_0 = 0, 6, v_0 = 0.4$.

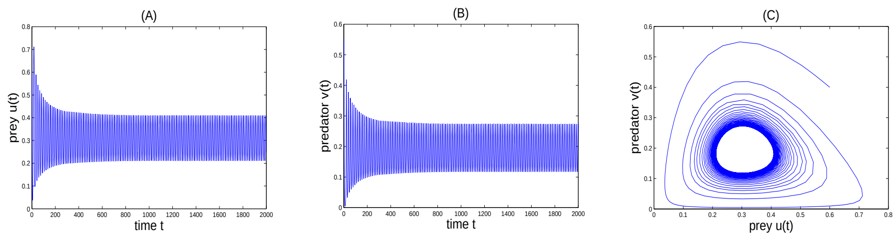

Figure 3. The trajectory graphs and phase portrait of (2.1), there exists a stable periodic orbit that bifurcating from the positive equilibrium (u^*, v^*) when the parameter $h = 0.28 < 0.3016$. Here we set these parameter values $\gamma = 2, \eta = 0.5, \beta = 0.55$, and the initial value is $u_0 = 0, 6, v_0 = 0.4$.

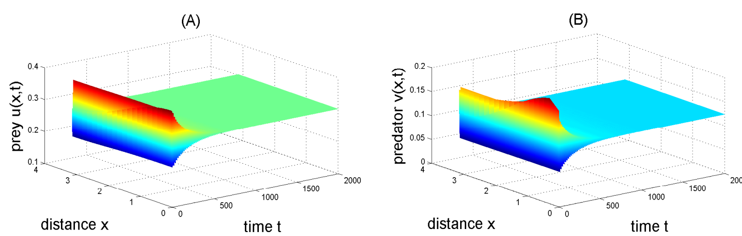

Figure 4. The positive equilibrium (u^*, v^*) of (1.3) is asymptotically stable when the value of the parameteris $h = 0.38 > 0.3016$. Here we set these parameter values $d_1 = 0.02, d_2 = 1, \gamma = 2, \eta = 0.5, \beta = 0.55$, and the initial values is $u(x, 0) = u^* + 0.05 \cos x, v(x, 0) = v^* + 0.05 \cos x$.

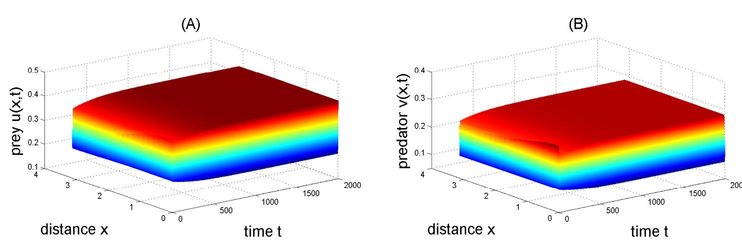

Figure 5. There exists stable spatially homogenous periodic solutions that bifurcating from the positive equilibrium (u^*, v^*) of (1.3) when $h = 0.28 < 0.3016$. Here we set parameter values $d_1 = 0.02, d_2 = 1, \gamma = 2, \eta = 0.5, \beta = 0.55$, and the initial values is $u(x, 0) = u^* + 0.05 \cos x, v(x, 0) = v^* + 0.05 \cos x$.

4.2. Delay-induce Hopf bifurcation

For the system (1.4) without diffusion, we choose the set parameter values $\gamma = 2, \eta = 0.5, \beta = 0.7, h = 0.3$ and $(u_0, v_0) = (0.3, 0.2)$. Then, a series of accurate calculations show that $(u^*, v^*) = (0.49, 0.1449)$ and $\tau_* = 1.7348$. Hence, $(0.49, 0.1449)$ is locally stable when the value of $\tau \in [0, \tau_*)$. When τ crosses through the critical value τ_*, $(0.49, 0.1449)$ will loses its stability and at the same time Hopf bifurcation occurs, a family of stable periodic solutions are bifurcating from $(0.49, 0.1449)$, which are depicted by Fig.6 and Fig.7.

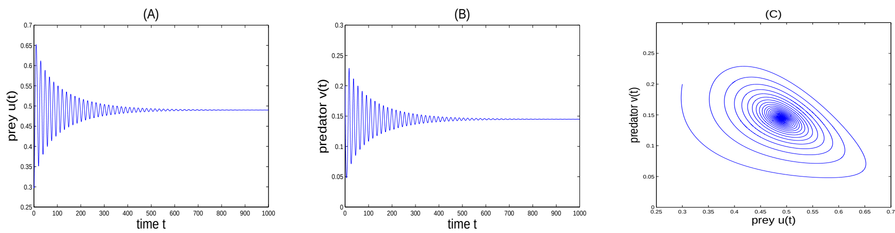

Figure 6. The trajectory graphs and phase portrait of (1.4) without diffusion. The positive equilibrium (u^*, v^*) is asymptotically stable when the parameter $\tau = 1.68 < 1.7348$. Here we set parameter values $\gamma = 2, \eta = 0.5, \beta = 0.7, h = 0.3$ and the initial value is $u_0 = 0.3, v_0 = 0.2$.

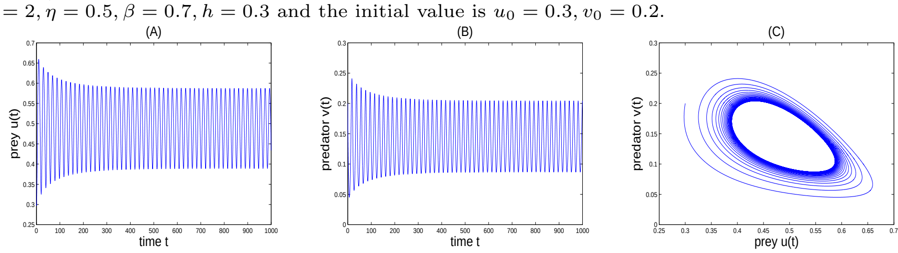

Figure 7. The trajectory graphs and phase portrait of (1.4) without diffusion. There exists a stable periodic orbit that bifurcating from the positive equilibrium (u^*, v^*) when the parameter $\tau = 1.8 > 1.7348$. Here we set parameter values $\gamma = 2, \eta = 0.5, \beta = 0.7, h = 0.3$ and the initial value is $u_0 = 0.3, v_0 = 0.2$.

For the system (1.4), we choose to set the values $d_1 = 0.02, d_2 = 1, \gamma = 2, \eta = 0.5, \beta = 0.7, h = 0.3$. Then, a series of exact calculations show that $(u^*, v^*) = (0.49, 0.1449)$, and the values $\tau_* = 1.7348, \kappa_{k1} = 0.0880, \kappa_{k2} = -9.4706$. Hence, $(0.49, 0.1449)$ is locally stable when the parameter $\tau \in [0, \tau_*)$. When τ crosses through the critical value τ_*, $(0.49, 0.1449)$ will loses its stability and Hopf bifurcation will occurs, a family of stable spatially homogenous periodic solutions are bifurcating from $(0.49, 0.1449)$, which are depicted by Fig.8 and Fig.9.

5. Discussion and conclusion

In this paper, we mainly propose a delayed diffusive predator-prey model with the herd behavior and the prey harvesting subject to the homogeneous Neumann boundary conditions.This model shows pretty rich and varied dynamics. We mainly study about the dynamics of model with nonlinear harvesting in prey, and investigate the special effect of time delay on the model.

In order to detaily investigate the influence of harvesting rate, we choose to make harvesting term as a bifurcation parameter, and we conclude the existence of periodic solutions which are near positive constant equilibrium, this conclusions

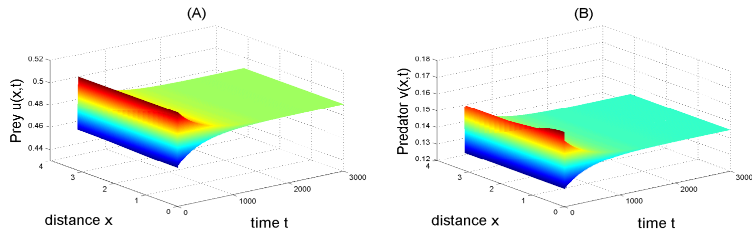

Figure 8. The positive equilibrium (u^*, v^*) of (1.4) is asymptotically stable when $\tau = 1.68 < 1.7348$. Here we choose parameter values $d_1 = 0.02, d_2 = 1, \gamma = 2, \eta = 0.5, \beta = 0.7, h = 0.3$ and set the initial values $u(x, 0) = u^* + 0.01 \cos x, v(x, 0) = v^* + 0.01 \cos x$.

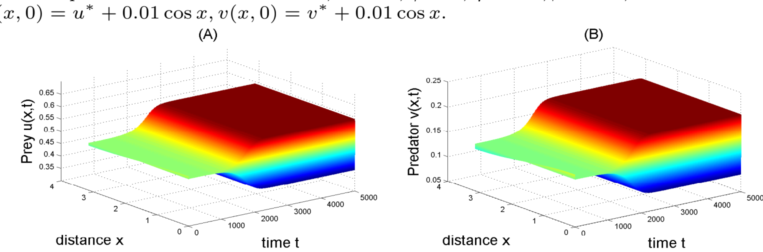

Figure 9. There exists stable spatially homogenous periodic solutions which are bifurcating from the positive equilibrium (u^*, v^*) of (1.4) when $\tau = 1.8 > 1.7348$. Here we choose parameter values $d_1 = 0.02, d_2 = 1, \gamma = 2, \eta = 0.5, \beta = 0.7, h = 0.3$ and set the initial values is $u(x, 0) = u^* + 0.01 \cos x, v(x, 0) = v^* + 0.01 \cos x$.

shows us that proper harvesting rate can create the periodic changes of prey and predator, which is really one of the most exciting features in the ecosystem. Especially, we obtain a critical values for the Hopf bifurcation, and we present the following interesting conclusions, as for the system (2.1), a positive constant equilibrium is given already, which is locally asymptotically stable when the parameter $h > h_0$, and a stable periodic solutions will bifurcate from the constant equilibrium E^*, when the harvesting term h decreasing crosses through the critical value h_0. For the system (1.4), our results present the fact that delay can induce very complex dynamics, and a positive constant equilibrium E^* is showed to be locally asymptotically stable when the parameter τ is less than the critical value τ_*, and a stable periodic solutions will bifurcate from the constant equilibrium E^*, when the delay term τ increase and it crosses through the critical value τ_*, which means that a stable and spatially homogeneous periodic solutions will occur at the critical value of time delay τ_*. These conclusions show us that the critical value can greatly affect the stability of the positive constant equilibrium, some other numerical simulations are carried out to accurately depict our theoretical analysis.

Acknowledgements

The authors would like to thank the referee for his or her careful reading and some comments on improving the presentation of this paper.

References

[1] V. Ajraldi, M. Pittavino, E. Venturino, *Modeling herd behavior in population systems*, Nonlinear Anal. RWA, 2011, 12, 2319–2333.

[2] P. A. Braza, *Predator-prey dynamics with square root functional responses,* Nonlinear Anal. RWA, 2012, 13, 1837–1843.

[3] S. Chen, J. Shi, *Global attractivity of equilibrium in Gierer-Meinhardt system with activator production saturation and gene expression time delays,* Nonlinear Anal. RWA, 2013, 14, 1871–1886.

[4] X. Chang, J. Wei, *Hopf bifurcation and optimal control in a diffusive predator-prey system with time delay and prey harvesting,* Nonlinear Anal. Model. Cont., 2012, 17(4), 379–409.

[5] T. Faria, *Normal forms and Hopf bifurcation for partial differential equations with delay,* Trans. Amer. Math. Soc., 2000, 352, 2217–2238.

[6] T. Faria, *Stability and bifurcation for a delayed predator-prey model and the effect of diffusion*, J. Math. Anal. Appl., 2001, 254, 433–463.

[7] R. P. Gupta, M. Banerjee, P. Chandra, *Bifurcation analysis and control of Leslie-Gower predator-prey model with Michaelis-Menten type prey-harvesting*, Differ. Equ. Dyn. Syst., 2012, 20, 339–366.

[8] R. P. Gupta, P. Chandra, *Bifurcation analysis of modified Leslie-Gower predator-prey model with Michaelis-Menten type prey harvesting*, J. Math. Anal. Appl., 2013, 398, 278–295.

[9] R. P. Gupta, P. Chandra, M. Banerjee, *Dynamical complexity of a predator-prey model with nonlinear predator harvesting,* Discrete Contin. Dynam. Syst. Ser. B, 2015, 20(2), 423–443.

[10] D. Hu, H. Cao, *Stability and bifurcation analysis in a predator-prey system with Michaelis-Menten type predator harvesting,* Nonlinear Anal. RWA, 2017, 33, 58–82.

[11] C. A. Klausmeier, *Regular and irregular patterns in semiarid vegetation,* Science, 1999, 285, 838–838.

[12] Y. Li, M. Wang, *Hopf bifurcation and global stability of a delayed predator-prey model with prey harvesting*, Comput. Math. Appl., 2015, 69, 398–410.

[13] J. D. Murray, *Mathematical Biology II,* Springer-Verlag, Heidelberg, 2002.

[14] S. Ruan, *On nonlinear dynamics of predator-prey models with discrete delay,* Math. Model. Nat. Phenom., 2009, 4(2), 140–188.

[15] Y. Song, X. Zou, *Bifurcation analysis of a diffusive ratio-dependent predator-prey model*, Nonlinear Dyn., 2014, 78, 49–70.

[16] Y. Song, X. Tang, *Stability, steady-state bifurcations, and Turing patterns in a predator-prey model with herd behavior and prey-taxis*, Stud. Appl. Math., 2017, 139(3), 371-404.

[17] Y. Song, H. Jiang, Q. Liu, Y. Yuan, *Spatiotemporal dynamics of the diffusive mussel-algae model near Turing-Hopf bifurcation*, SIAM J. Appl. Dyn. Syst., 2017, 16(4), 2030–2062.

[18] Y. Song, Y. Peng, X. Zou, *Persistence, stability and Hopf bifurcation in a diffusive ratio-dependent predator-prey model with delay,* Int. J. of Bifurcation Chaos, 2014, 24, 1450093.

[19] X. Tang, Y. Song, *Stability, Hopf bifurcations and spatial patterns in a delayed diffusive predator-prey model with herd behavior,* Appl. Math. Comput., 2015, 254, 375–391.

[20] X. Tang, H. Jiang, Z. Deng, T. Yu, *Delay induced subcritical Hopf bifurcation in a diffusive predator-prey model with herd behavior and hyperbalic mortality*, J. Appl. Anal. Comput., 2017, 7(4), 1385–1401.

[21] X. Tang and Y. Song, *Cross-diffusion induced spatiotemporal patterns in a predator-prey model with herd behavior,* Nonlinear Anal. RWA, 2015, 24, 36–49.

[22] X. Tang, Y. Song and T. Zhang, *Turing-Hopf bifurcation analysis of a predator-prey model with herd behavior and cross-diffusion*, Nonlinear Dyn., 2016, 86(1), 73–89.

[23] X. Tang and Y. Song, *Bifurcation analysis and Turing instability in a diffusive predator-prey model with herd behavior and hyperbolic mortality*, Chaos Solitons Fract., 2015, 81, 303–314.

[24] J. Wang, J. Shi, J. Wei, *Dyanmics and pattern formation in a diffusive predator-prey system with strong Allee effect in prey,* J. Differ. Equ., 2011, 251, 1276–1304.

[25] J. Wu, *Theory and Applications of Partial Functional Differential Equations,* New York: Springer- Verlag, 1996.

[26] R. Yang, J. Wei, *Stability and bifurcation analysis of a diffusive prey-predator system in Holling type III with a prey refuge,* Nonlinear Dyn., 2015, 79, 631–646.

[27] F. Yi, J. Wei, J. Shi, *Bifurcation and spatio-temporal patterns in a homogeneous diffusive predator-prey system,* J. Differ. Equ., 2009, 246, 1944–1977.

[28] R. Yuan, W. Jiang, Y. Wang, *Saddle-node-Hopf bifurcation in a modified Leslie-Gower predator-prey model with time-delay and prey harvesting,* J. Math. Anal. Appl., 2015, 422, 1072–1090.

[29] R. Yuan, Z. Wang, W. Jiang, *Global Hopf bifurcation of a delayed diffusive predator-prey model with Michaelis-Menten-type prey harvesting*, Appl. Anal., 2016, 95(2), 444–466.

[30] S. Yuan, C. Xu, T. Zhang, *Spatial dynamics in a predator-prey model with herd behavior,* Chaos, 2013, 23, 0331023.

[31] C. Zhu, L. Kong, *Bifurcations analysis of Leslie-Gower predator-prey models with nonlinear predator-harvesting,* Discrete Contin. Dynam. Syst. Ser. S, 2017, 10(5), 1187–1206.